Journal of Applied Nonlinear Dynamics 5(1) (2016) 93–104

Journal of Applied Nonlinear Dynamics

https://lhscientificpublishing.com/Journals/JAND-Default.aspx

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Global Dynamics of a Three Species Predator-Prey Competition Model with Holling type II Functional Response on a Circular Domain

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Submission Info

Communicated by J.A T. Machado
Received 10 June 2015
Accepted 15 July 2015
Available online 1 January 2016

Keywords

Predator-prey
Top predator
Local and global stability
Pattern formation
Circular domain

Abstract

This paper is devoted to the study of a three species ecosystem model consisting of a prey, a predator and a top predator. This model is given by a reaction diffusion system defined on a circular spatial domain and incorporates the Holling type II and a modified Leslie-Gower functional response. The aim of this paper is to investigate theoretically and numerically the asymptotic behavior of the interior equilibrium of the model. The conditions of boundedness, existence of a positively invariant and attracting set are proved. Sufficient conditions of local/global stability of the positive steady state are established. In the end, we present a numerical evidence of time evolution of the pattern formation.

1 Introduction and Mathematical Model

In the last few decades, the dynamic relationship between predator and its prey has long been and will continue to be one of the dominant themes in both ecology and mathematical modeling. One of the oldest and well known mathematical model which describing the interaction between predator and prey populations was introduced by A. Lotka 1925 [1] and V. Volterra 1927 [2], governed by the following differential equations

$$\begin{cases} \dot{u}(t) = u(t)(m_1 - n_1 v(t)), \\ \dot{v}(t) = v(t)(-m_2 + n_2 u(t)), \end{cases} \tag{1}$$

where u and v represent the population densities of prey and predator at time t, respectively, m_1, n_1, m_2 and n_2 are positive constants, which stand for the prey growth rate in the absence of the predators, the capture rate of prey by per predator, the constant death rate in the absence of prey and the rate at which each predator converts captured prey into predator births, respectively. They showed that, predator-prey systems permanently oscillate for any initial condition if the prey growth rate is constant and

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ISSN 2164 – 6457, eISSN 2164 – 6473/$-see front materials
DOI : 10.5890/JAND.2016.03.007

the predator functional response is linear. The dynamical behavior of interacting species with different functional response has been extensively investigated in terms of boundedness of solutions, existence of an attracting set, local/global stability of equilibria and bifurcations (see, for example, [3–6]).

In [4], the authors considered a reaction diffusion predator-prey model defined on a square domain which incorporates Holling type II and a modified Leslie-Gower functional response. They have proved the local/global stability, occurrence of bifurcations and patterns formations. In [7], the authors considered the same model defined on a circular domain which is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{r\theta}u + u(1-u) - \dfrac{av}{u+e_1}u = \Delta_{r\theta}u + f(u,v) & (r,\theta)\in\Omega,\ t>0, \\ \dfrac{\partial v}{\partial t} = \delta\Delta_{r\theta}v + b(1-\dfrac{v}{u+e_2})v = \delta\Delta_{r\theta}v + g(u,v) & (r,\theta)\in\Omega,\ t>0, \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = 0 \text{ for } r=R \text{ (radial derivative).} \end{cases} \tag{2}$$

This two species food chain model describes a prey population u which serves as food for a predator v, where $f(u,v)$ and $g(u,v)$ are the local activity (in the absence of diffusion), Ω is a disc domain and $\Delta_{r\theta}$ is the Laplacian operator in polar coordinates. The parameters a, b, e_1 and e_2 are assumed positive values. Boundedness of the system, existence of an attracting set, local/global stability, occurrence of Hopf and Turing bifurcations and patterns formation are studied.

In [8], the authors studied a reaction diffusion system of predator-prey model which is based on the modified Leslie-Gower model with Beddington-DeAngelis functional responses on a circular domain below

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1\Delta_{r\theta}u + (a_1 - b_1 u - \dfrac{c_1 v}{d_1 u + d_2 v + k_1})u & (r,\theta)\in\Omega,\ t>0, \\ \dfrac{\partial v}{\partial t} = D_2\Delta_{r\theta}v + (a_2 - \dfrac{c_2 v}{u+k_2})v & (r,\theta)\in\Omega,\ t>0. \end{cases} \tag{3}$$

Recently, many researchers have studied the pattern formation for different models of three interacting species in discrete and continuous cases and most of them have considered a food chain model with diffusion and have investigated the local/global stability in the spatio-temporal system defined on a square domain (see [9–13]).

In the present paper, we consider a three-species food chain model consisting of prey, intermediate predator and top-predator, modeled by a reaction-diffusion system defined on a circular spatial domain and incorporates the Holling type *II* and a modified Leslie-Gower functional response. One of the well known methods in biology or ecology plays a crucial role in regulating the balance of the ecosystem and controlling the dynamics of species is the introduction of a population further called “top-predator”. However, the impact of this introduction should previously be studied in order to minimize adverse effects. The first species denoted by U is the only food source of the second V. As well, intermediate predator V is the only prey of a top-predator W. Local interactions between species U and V are modeled by Lotka-Volterra type scheme and the interactions between species W and V has been modeled by Leslie-Gower scheme [14, 15]. The spatio-temporal system for the three components species can be written as follows (see [16]):

$$\begin{cases} \dfrac{\partial U(T,x,y)}{\partial T} = D_1\Delta U(T,x,y) + (a_0 - b_0 U(T,x,y) - \dfrac{v_0 V(T,x,y)}{U(T,x,y)+d_0})U(T,x,y), \\ \dfrac{\partial V(T,x,y)}{\partial T} = D_2\Delta V(T,x,y) + (-a_1 + \dfrac{v_1 U(T,x,y)}{U(T,x,y)+d_0} - \dfrac{v_2 W(T,x,y)}{V(T,x,y)+d_2})V(T,x,y), \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{\partial W(T,x,y)}{\partial T} = D_3 \Delta W(T,x,y) + (c_3 - \dfrac{v_3 W(T,x,y)}{V(T,x,y)+d_3})W(T,x,y), \\ \dfrac{\partial U}{\partial n} = \dfrac{\partial V}{\partial n} = \dfrac{\partial W}{\partial n} = 0, \\ U(0,x,y) = U_0(x,y) \geq 0,\ V(0,x,y) = V_0(x,y) \geq 0,\ W(0,x,y) = W_0(x,y) \geq 0, \end{cases}$$

where $U(T,x,y)$, $V(T,x,y)$ and $W(T,x,y)$ are the densities of prey, intermediate predator and top-predator, respectively, at time T and position (x,y) defined on a circular domain Ω with radius R (i.e. $\Omega = \{(x,y) \in \mathbf{R}^2 / x^2 + y^2 < R^2\}$. The three species are assumed to diffuse at rates D_i (i = 1, 2, 3). The parameters a_0, b_0, v_0, d_0, a_1, v_1, v_2, d_2, c_3, v_3 and d_3 are assumed to be positive constants and are defined as follows: a_0 is the growth rate of the prey U, b_0 measures the mortality due to the competition between individuals of the species U, v_0 is the maximum extent that the rate of reduction by individual U can reach, d_0 measures the protection that the species U and V benefit through the environment, a_1 represents the death rate of V in the absence of U, v_1, v_2 and v_3 are the the maximum value that the rate of reduction by the individual of U, V and W can reach respectively, d_2 is the value of V for which the rate of elimination by individual V becomes $\frac{v_2}{2}$, c_3 describes the growth rate of W, assuming that there is the same number of males and females and d_3 represents the residual loss caused by high scarcity of prey V of the specie W. The vector n is an outward unit normal vector to the smooth boundary $\partial\Omega$. The homogeneous Neumann boundary conditions mean that the system is self contained and there is no flux across the boundary $\partial\Omega$.

The first model proposed in this topic is given by a system of ordinary differential equations as follows (see [17]):

$$\begin{cases} \dfrac{\partial U}{\partial T} = a_0 U - b_0 U^2 - \dfrac{v_0 V U}{U+d_0}, \\ \dfrac{\partial V}{\partial T} = -a_1 V + \dfrac{v_1 U V}{U+d_1} - \dfrac{v_2 W V}{V+d_2}, \\ \dfrac{\partial W}{\partial T} = c_3 W^2 - \dfrac{v_3 W^2}{V+d_3}, \end{cases} \tag{5}$$

where U, V and W represent the population densities at time T, a_0, b_0, v_0, d_0, a_1, v_1, d_1, v_2, d_2, c_3, v_3 and d_3 are model parameters assumed to be positive. Based on the studies presented in [17, 18], our main contribution in this paper is to generalize the results presented in [7, 8] for two species to three-species reaction-diffusion system defined on a circular domain.

The organization of the remaining part of this work is as follows. In Section 2, we show the boundedness of solutions. In section 3, we prove the existence of the equilibrium points and their stability. Section 4 is devoted to the global stability of the nontrivial steady state. In section 5, we give some numerical simulations and we end our work by a conclusion.

2 Boundedness of solutions

Considering system (4) and writing x and y in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we get $\Gamma = \{(r,\theta) : 0 < r < R, 0 \leq \theta < 2\pi\}$ (R is the radius of the disc), $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$.
Without loss of generalities we denote also $u(t,x,y) = u(t,r\cos(\theta),r\sin(\theta)) = u(t,r,\theta)$,
$v(t,x,y) = v(t,r\cos(\theta),r\sin(\theta)) = v(t,r,\theta)$ and $w(t,x,y) = w(t,r\cos(\theta),r\sin(\theta)) = w(t,r,\theta)$ as the densities of prey, predator and top predator in polar coordinates, respectively. Therefore, the Laplacian operator in polar coordinates is given by

$$\Delta_{r\theta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{6}$$

To simplify system (4) we introduce the following transformations:

$$U=\frac{a_0}{b_0}u,\ V=\frac{a_0^2}{b_0v_0}v,\ W=\frac{a_0^3}{b_0v_0v_2}w,\ T=\frac{t}{a_0},\ r=\frac{r^{'}}{a_0},\ \theta=\theta^{'},$$

and

$$a=\frac{b_0d_0}{a_0},\ b=\frac{a_1}{a_0},\ c=\frac{v_1}{a_0},\ d=\frac{d_2v_0b_0}{a_0^2},\ p=\frac{c_3a_0^2}{v_0b_0v_2},\ q=\frac{v_3}{v_2},\ s=\frac{d_3v_0b_0}{a_0^2},\ \delta_1=\frac{D_1}{a_0},\ \delta_2=\frac{D_2}{a_0},\ \delta_3=\frac{D_3}{a_0}.$$

In polar coordinates, the spatio-temporal system (4) is written as follows:

$$\begin{cases}\frac{\partial u(t,r,\theta)}{\partial t}=\delta_1\Delta_{r\theta}u(t,r,\theta)+(1-u(t,r,\theta)-\frac{v(t,r,\theta)}{u(t,r,\theta)+a})u(t,r,\theta), & \forall(r,\theta)\in\Gamma,\ t>0\\ \frac{\partial v(t,r,\theta)}{\partial t}=\delta_2\Delta_{r\theta}v(t,r,\theta)+(-b+\frac{cu(t,r,\theta)}{u(t,r,\theta)+a}-\frac{w(t,r,\theta)}{v(t,r,\theta)+d})v(t,r,\theta), & \forall(r,\theta)\in\Gamma,\ t>0\\ \frac{\partial w(t,r,\theta)}{\partial t}=\delta_3\Delta_{r\theta}w(t,r,\theta)+(p-\frac{qw(t,r,\theta)}{v(t,r,\theta)+s})w(t,r,\theta) & \forall(r,\theta)\in\Gamma,\ t>0\\ \partial_r u(.,r,\theta)=\partial_r v(.,r,\theta)=\partial_r w(.,r,\theta)=0 \text{ for } r=R \text{ (radial derivative)}\\ u(0,r,\theta)=u_0(r,\theta)\geq 0,\ v(0,r,\theta)=v_0(r,\theta)\geq 0,\ w(0,r,\theta)=w_0(r,\theta)\geq 0.\end{cases}\tag{7}$$

The following result gives the boundedness of solutions for system (7).

Theorem 1. *Let* Θ *be a set defined as follows:*

$$\Theta\equiv[0,1]\times[0,1+a]\times[0,\frac{p}{q}(1+a+s)].\tag{8}$$

Then,
i) Θ *is positively invariant region,*
ii) *All solutions of (7) starting in* Θ *are ultimately bounded and eventually enter the attracting set* Θ*.*

Proof: From equation (7), we have

$$\begin{cases}\frac{\partial u(t,r,\theta)}{\partial t}=\delta_1\Delta_{r\theta}u(t,r,\theta)+(1-u(t,r,\theta))u(t,r,\theta)\leq(1-u)u,\\ \frac{\partial u(0,r,\theta)}{\partial n}=0,\\ u(0,r,\theta)=u_0(r,\theta)\leq u_{01}=\max\limits_{(r,\theta)\in\overline{\Gamma}}u_0(r,\theta),\end{cases}\tag{9}$$

and

$$\begin{cases}\frac{\partial v(t,r,\theta)}{\partial t}=\delta_2\Delta_{r\theta}v+(-b+\frac{cu}{u+a}-\frac{w}{v+d})v\leq(\frac{c}{1+a}-b)v,\\ \frac{\partial v(0,r,\theta)}{\partial n}=0,\\ v(0,r,\theta)=v_0(r,\theta)\leq v_{01}=\max\limits_{(r,\theta)\in\overline{\Gamma}}v_0(r,\theta),\end{cases}\tag{10}$$

and

$$\begin{cases}\frac{\partial w(t,r,\theta)}{\partial t}=\delta_3\Delta_{r\theta}w+(p-\frac{qw}{v+s})w\leq p(1-\frac{w}{\frac{p}{q}(1+a+s)})w,\\ \frac{\partial w(0,r,\theta)}{\partial n}=0,\\ w(0,r,\theta)=w_0(r,\theta)\leq w_{01}=\max\limits_{(r,\theta)\in\overline{\Gamma}}w_0(r,\theta).\end{cases}\tag{11}$$

From equations (9)-(11) and by applying the comparison principle, we have $u(t,r,\theta) \le u_1 \le 1$, $v(t,r,\theta) \le v_1 \le 1$ and $w(t,r,\theta) \le w_1 \le 1$ such that: $\limsup_{t\to+\infty} u_1(t) = 1$, $\limsup_{t\to+\infty} v_1(t) = 1+a$ and $\limsup_{t\to+\infty} w_1(t) = \frac{p}{q}(1+a+s)$, where u_1, v_1 and w_1 are solutions of the following equations, respectively,

$$\begin{cases} \dfrac{du(t,r,\theta)}{dt} = (1-u_1)u_1, \\ u_1(0) = u_{01} = \max\limits_{(r,\theta)\in\overline{\Gamma}} u_0(r,\theta) \le 1, \end{cases} \tag{12}$$

$$\begin{cases} \dfrac{dv(t,r,\theta)}{dt} = (\dfrac{c}{1+a} - b)v_1, \\ v_1(0) = v_{01} = \max\limits_{(r,\theta)\in\overline{\Gamma}} v_0(r,\theta) \le 1, \end{cases} \tag{13}$$

and

$$\begin{cases} \dfrac{dw(t,r,\theta)}{dt} = p(1 - \dfrac{w}{\dfrac{p}{q}(1+a+s)})w, \\ w_1(0) = w_{01} = \max\limits_{(r,\theta)\in\overline{\Gamma}} w_0(r,\theta) \le 1. \end{cases} \tag{14}$$

Then, we deduce the result.

3 Analysis of Temporal System

In this section, we will study the behavior of system (7) in the absence of diffusion, (i.e., $\delta_1 = \delta_2 = \delta_3 = 0$).

3.1 Equilibria and Local Stability

Without diffusion, system (7) becomes

$$\begin{cases} \dfrac{du(t)}{dt} = (1 - u(t) - \dfrac{v(t)}{u(t)+a})u(t), \\ \dfrac{dv(t)}{dt} = (-b + \dfrac{cu(t)}{u(t)+a} - \dfrac{w(t)}{v(t)+d})v(t), \\ \dfrac{dw(t)}{dt} = (p - \dfrac{qw(t)}{v(t)+s})w(t). \end{cases} \tag{15}$$

Let $E = (u,v,w)^T$ and

$$F(E) = \begin{pmatrix} f(u,v,w) \\ g(u,v,w) \\ h(u,v,w) \end{pmatrix} = \begin{pmatrix} (1-u-\dfrac{v}{u+a})u \\ (-b+\dfrac{cu}{u+a} - \dfrac{w}{v+d})v \\ (p - \dfrac{qw}{v+s})w \end{pmatrix}.$$

Then, system (15) takes the following form:

$$\frac{dE}{dt} = F(E). \tag{16}$$

By computation, system (16) has four trivial equilibrium points $E_0 = (0,0,0)$, $E_1 = (1,0,0)$, $E_2 = (0,0,\frac{sp}{q})$, $E_3 = (1,0,\frac{sp}{q})$ and a positive nontrivial one $E^* = (u^*,v^*,w^*)$ which exists if and only if the following inequality is satisfied

$$qc > bq + p \text{ and } qc - bq - p > a(bq+p) \tag{17}$$

such that

$$u^* = \frac{a(bq+p)}{qc-bq-p},\ v^* = (1-u^*)(u^*+a) \text{ and } w^* = \frac{p(v^*+s)}{q}. \tag{18}$$

By linearizing system (16) around the equilibrium point E^*, we obtain the associated Jacobian Matrix J defined by

$$J(E^*) = \begin{pmatrix} 1-2u^* - \dfrac{av^*}{(u^*+a)^2} & -\dfrac{u^*}{u^*+a} & 0 \\ \dfrac{acv^*}{(u^*+a)^2} & \dfrac{cu^*}{u^*+a} - b - \dfrac{dw^*}{(v^*+d)^2} & -\dfrac{v^*}{v^*+d} \\ 0 & \dfrac{q(w^*)^2}{u^*+a} - b - \dfrac{dw^*}{(v^*+d)^2} & -\dfrac{2qw^*}{v^*+d} \end{pmatrix}. \tag{19}$$

Evaluating the Jacobian Matrix J at E_0, E_1, E_2 and E_3 and calculating the corresponding eigenvalues, we have the following result:

Theorem 2. i) *The steady state E_0 is unstable,*
ii) *If $ab > c-b$, then the steady state E_1 is asymptotically stable and it is unstable elsewhere,*
iii) *The steady state E_2 is a saddle point,*
iv) *If $b + \frac{sp}{qd} > \frac{c}{1+a}$, then the equilibrium point $E_3 = (1, 0, \frac{sp}{q})$ is asymptotically stable and is a saddle point elsewhere.*

In the following, we prove the stability of the positive equilibrium $E^* = (u^*, v^*, w^*)$.

Theorem 3. *If the condition (17) holds and the following inequalities are satisfied*

$$\frac{a+1}{qc} > \frac{2a}{qc-bq-p}$$

and

$$b + \frac{dp((1-u^*)(u^*+a)+s)}{q((1-u^*)(u^*+a)+d)^2} > \frac{cu^*}{u^*+a} \tag{20}$$

and

$$\frac{p^2((1-u^*)(u^*+a)+s)^2}{q(u^*+a)} > b + \frac{dp((1-u^*)(u^*+a)+s)}{q((1-u^*)(u^*+a)+d)^2},$$

then, the nontrivial equilibrium point $E^* = (u^*, v^*, w^*)$ is asymptotically stable.

Proof: Writing $F_E(E^*)$ as

$$F_E(E^*) = J(E^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

According to (18), a direct computation yields

$$\begin{cases} a_{11} = 1-2u^* - \dfrac{a(1-u^*)}{(u^*+a)},\ a_{12} = -\dfrac{u^*}{u^*+a}, a_{13} = 0, \\ a_{21} = \dfrac{ac(1-u^*)}{(u^*+a)},\ a_{22} = \dfrac{cu^*}{u^*+a} - b - \dfrac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2},\ a_{23} = -\dfrac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d}, \\ a_{31} = 0,\ a_{32} = \dfrac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a} - b - \dfrac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2},\ a_{33} = -\dfrac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}, \end{cases} \tag{21}$$

where

$$u^* = \frac{a(bq+p)}{qc-bq-p}.$$

The characteristic polynomial of $F_E(E^*)$ can be written as:

$$\varphi(\lambda) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3, \tag{22}$$

where
$(1-u^*)(u^*+a) = \frac{aqc(qc-(bq+p)(a+1))}{(qc-bq-p)^2}$, $\frac{(1-u^*)}{(u^*+a)} = \frac{qc-(bq+p)(a+1)}{aqc}$ and $\frac{u^*}{(u^*+a)} = \frac{bq+p}{qc}$.
Hence

$$\begin{aligned}
B_1 =& -\mathrm{tr}(L_E(E^*))\\
=& -(a_{11}+a_{22}+a_{33})\\
=& u^*(2-\frac{a}{u^*+a})+b+\frac{dp(qc-bq-p)^2(aqc(qc-(bq+p)(a+1))+s(qc-bq-p)^2)}{q(aqc(qc-(bq+p)(a+1))+d(qc-bq-p)^2)}\\
&+\frac{2paqc(qc-(bq+p)(a+1))}{aqc(qc-(bq+p)(a+1))+d(qc-bq-p)^2} > 0,\\
B_2 =& a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{23}a_{32}-a_{12}a_{21}\\
=& (-\frac{cu^*}{u^*+a}+b+\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})(2u^*-\frac{(bq+p)(a+1)}{qc}+\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d})\\
&+(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d})(2u^*-\frac{(bq+p)(a+1)}{qc})+(\frac{ac(1-u^*)u^*}{(u^*+a)^2})\\
&+(\frac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a}-b-\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})(\frac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d}) > 0,\\
B_3 =& -\det(L_E(E^*))\\
=& a_{12}a_{21}a_{33}+a_{11}a_{23}a_{32}-a_{11}a_{22}a_{33}\\
=& (\frac{ac(1-u^*)u^*}{(u^*+a)^2})(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d})\\
&+(\frac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a}-b-\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})\\
&\times(2u^*-\frac{(bq+p)(a+1)}{qc})(\frac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d})+(-\frac{cu^*}{u^*+a}+b+\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})\\
&\times(2u^*-\frac{(bq+p)(a+1)}{qc})(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}) > 0,
\end{aligned}$$

$$\begin{aligned}
B_1B_2-B_3 =& a_{11}^2(-a_{22}-a_{33})+a_{22}^2(-a_{11}-a_{33})+a_{33}^2(-a_{11}-a_{22})\\
&+a_{12}a_{21}(a_{11}+a_{22})+a_{32}a_{23}(a_{33}+a_{22})-2a_{11}a_{22}a_{33}\\
=& [a_{11}^2-a_{32}a_{23}](-a_{22}-a_{33})+[a_{33}^2-a_{12}a_{21}](-a_{11}-a_{22})+a_{22}^2(-a_{11}-a_{33})-2a_{11}a_{22}a_{33}\\
=& (-\frac{cu^*}{u^*+a}+b+\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}+\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d})\\
&[a_{11}^2+(\frac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d})\times(\frac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a}-b-\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})]\\
&+a_{22}^2(2u^*-\frac{(bq+p)(a+1)}{qc}+\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d})
\end{aligned}$$

$$+[a_{33}^2+(\frac{acu^*(1-u^*)}{(u^*+a)^2})](2u^*-\frac{(bq+p)(a+1)}{qc}-\frac{cu^*}{u^*+a}+b+\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})$$
$$+2(2u^*-\frac{(bq+p)(a+1)}{qc})(-\frac{cu^*}{u^*+a}+b+\frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2})(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d})$$
$$> 0.$$

Using the condition (20), we have $B_1B_2-B_3>0$. By applying the Routh-Hurwitz criteria, we deduce the result.

4 Global Stability of the Nontrivial Steady State with Diffusion

In this section, we study the global stability of the homogeneous non-trivial equilibrium $E^*=(u^*,v^*,w^*)$ with diffusion terms.

Theorem 4. *Suppose that the condition (17) holds and the following inequalities are satisfied*

$$b<c,\ 1-a<u^*<\frac{ab}{c-b}\ \text{ and }\ v^*<d(c-b). \tag{23}$$

Then, the homogeneous non-trivial steady state (u^*,v^*,w^*) is globally asymptotically stable for system (7).

Proof: The proof is based on the positive definite Lyapunov function. Let

$$\int_\Gamma f(\rho)d\rho=\int_0^R\int_0^{2\pi}f(r,\theta)d\theta dr$$

and

$$Z(u,v,w)=z_1(u)+z_2(v)+z_3(w),$$

where

$$z_1(u)=\int_{u^*}^u\frac{\eta-u^*}{\eta}d\eta,\ z_2(v)=\int_{v^*}^v\frac{(\eta-v^*)(\eta+d)}{\eta}d\eta \text{ and } z_3(w)=\int_{w^*}^w\frac{\eta-w^*}{\eta}d\eta. \tag{24}$$

Let

$$\begin{aligned}\Psi(u,v,w)=&\int_\Gamma Z(u(t,\rho),v(t,\rho),w(t,\rho))d\rho \\ =&\int_\Gamma z_1(u(t,\rho))d\rho+\int_\Gamma z_2(v(t,\rho))d\rho+\int_\Gamma z_3(w(t,\rho))d\rho \\ =&\int_\Gamma z_1(u(t,\rho))d\rho+\int_\Gamma z_2(v(t,\rho))d\rho+\int_\Gamma z_3(w(t,\rho))d\rho,\end{aligned} \tag{25}$$

where Ψ is positive for all $(u,v,w)\in\mathbb{R}^3_{*+}$ and $\Psi(u^*,v^*,w^*)=0$.By differentiating Ψ with respect to time t, we have

$$\begin{aligned}\frac{d\Psi}{dt} &= \int_\Gamma(\frac{\partial Z}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial Z}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial Z}{\partial w}\frac{\partial w}{\partial t})d\rho\\ &= \int_\Gamma(\frac{\partial Z}{\partial u}(\delta_1\Delta u+f(u,v,w))+\frac{\partial Z}{\partial v}(\delta_2\Delta v+g(u,v,w))+\frac{\partial Z}{\partial w}(\delta_3\Delta w+h(u,v,w)))d\rho\\ &= \int_\Gamma(\delta_1\frac{\partial Z}{\partial u}\Delta u+\delta_2\frac{\partial Z}{\partial v}\Delta v+\delta_3\frac{\partial Z}{\partial w}\Delta w)d\rho+\int_\Gamma(\frac{\partial Z}{\partial u}f(u,v,w)+\frac{\partial Z}{\partial v}g(u,v,w)+\frac{\partial Z}{\partial w}h(u,v,w))d\rho\\ &= \int_\Gamma(\delta_1\frac{\partial Z}{\partial u}\Delta u+\delta_2\frac{\partial Z}{\partial v}\Delta v+\delta_3\frac{\partial Z}{\partial w}\Delta w)d\rho+\int_\Gamma \dot{Z}d\rho,\end{aligned}$$

where $\dot{Z}=\frac{\partial Z}{\partial t}$. From Green's identity we get

$$\begin{aligned}\int_\Gamma\frac{\partial Z}{\partial u}\Delta u d\rho &= \int_{\partial\Gamma}\frac{\partial Z}{\partial u}\frac{\partial u}{\partial\eta}-\int_\Gamma\nabla\frac{\partial Z}{\partial u}\nabla u d\rho\\ &= -\int_\Gamma\nabla\frac{\partial Z}{\partial u}\nabla u d\rho,\end{aligned}$$

$$\begin{aligned}\int_\Gamma\frac{\partial Z}{\partial v}\Delta v d\rho &= \int_{\partial\Gamma}\frac{\partial Z}{\partial v}\frac{\partial v}{\partial\eta}-\int_\Gamma\nabla\frac{\partial Z}{\partial v}\nabla v d\rho\\ &= -\int_\Gamma\nabla\frac{\partial Z}{\partial v}\nabla v d\rho\end{aligned}$$

and

$$\begin{aligned}\int_\Gamma\frac{\partial Z}{\partial w}\Delta w d\rho &= \int_{\partial\Gamma}\frac{\partial Z}{\partial w}\frac{\partial w}{\partial\eta}-\int_\Gamma\nabla\frac{\partial Z}{\partial w}\nabla w d\rho\\ &= -\int_\Gamma\nabla\frac{\partial Z}{\partial w}\nabla w d\rho,\end{aligned}$$

where $\nabla_{r\theta}u=(\frac{\partial u}{\partial r},\frac{1}{r}\frac{\partial u}{\partial\theta})$.
As $Z(u,v,w)=z_1(u)+z_2(v)+z_3(w)$ is written in a separable form, we have

$$z_1''(u)=\frac{u^*}{u}\geq 0,\ z_2''(v)=\frac{v+dv^*}{v}\geq 0 \text{ and } z_3''(v)=\frac{w^*}{w}\geq 0.$$

Therefore, the matrix

$$\begin{pmatrix}\frac{\partial^2 Z}{(\partial u)^2} & \frac{\partial^2 Z}{\partial u\partial v} & \frac{\partial^2 Z}{\partial u\partial w}\\ \frac{\partial^2 Z}{\partial v\partial u} & \frac{\partial^2 Z}{(\partial v)^2} & \frac{\partial^2 Z}{\partial v\partial w}\\ \frac{\partial^2 Z}{\partial w\partial u} & \frac{\partial^2 Z}{\partial w\partial v} & \frac{\partial^2 Z}{(\partial w)^2}\end{pmatrix}$$

is positive definite and we have

$$\int_\Gamma(\frac{\partial Z}{\partial u}\delta_1\Delta u+\delta_2\frac{\partial Z}{\partial v}\Delta v+\delta_3\frac{\partial Z}{\partial w}\Delta w)d\rho\leq 0.$$

As $\dot{Z}\leq 0$, we deduce that

$$\frac{d\Psi}{dt}\leq 0 \text{ for } b<c,\ 1-a<u^*<\frac{ab}{c-b} \text{ and } v^*<d(c-b).$$

Then, we have the result.

5 Numerical simulations

In this section, we give some numerical simulations of pattern formation resulting from spatial distribution of system (7) on the disc $\Gamma = \{(r,\theta) : 0 < r < R, 0 \leq \theta < 2\pi\}$. The Laplacian describing diffusion is calculated using finite difference schemes, that is, the derivatives are approached by differences over space steps (Δr) and an explicit Euler's method for the time integration with a time step size (Δt) with zero-flux on the boundary. In order to avoid numerical artifacts, the values of time (Δt) and space steps (Δr and $\Delta\theta$) have been chosen sufficiently small satisfying the CFL (Courant-Friedrichs-Levy) stability criterion for diffusion equation. For numerical simulations, the initial condition is a small perturbation in the vicinity of equilibrium point (u^*, v^*, w^*). These initial conditions have been chosen as:

$$\begin{aligned} u(0,r,\theta) &= u^*((r\cos\theta)^2 + (r\sin\theta)^2) < 50, \\ v(0,r,\theta) &= v^*((r\cos\theta)^2 + (r\sin\theta)^2) < 50, \\ w(0,r,\theta) &= w^*((r\cos\theta)^2 + (r\sin\theta)^2) < 50. \end{aligned}$$

The used parameters are summarized in Table 1. The time evolution of spatial distributions is observed

Table 1 Table of the used parameters and the corresponding pictures of patterns formations

	t	a_0	a_1	b_0	c_3	d_0	d_2	d_3	v_0	v_1	v_2	v_3
Fig.1(a)	0	0.5	0.4	0.36	0.2	0.3	0.4	0.4	0.4	0.8	0.4	0.6
Fig.1(b)	1200	0.5	0.4	0.36	0.2	0.3	0.4	0.4	0.4	0.8	0.4	0.6
Fig.1(c)	20000	0.5	0.4	0.36	0.2	0.3	0.4	0.4	0.4	0.8	0.4	0.6

in Fig.1, where the left figures are the evolution of the prey spatial distribution and the right figures are the top predators and the center ones are the predators. We observe that, for $t = 0$ we have spots patterns over the whole domain (see Fig.1 (a)). If we increase time t, these spots burst leading to an aperiodic spatial distribution of some domain. Then this aperiodicity spreads throughout the area and remains in time. After a while, the patterns exhibit a behavior that does not seem to change its characteristics anymore and the labyrinth spatial pattern arise and we obtain the spatio-temporal chaos. Thus, we have observed the patterns formation with respect to time (see Fig.1).

6 Conclusion

In this paper, we have considered a three-species food chain, namely, prey, predator and top predator, given by a reaction diffusion system incorporating Holling type II and a modified Leslie-Gower functional response defined in a circular domain. We have proved the conditions for boundedness, existence of a positively invariant and attracting set. By using Routh-Hurwitz criterion, we have showed that E^* is locally asymptotically stable for system (15) if some conditions are satisfied. By constructing a Lyapunov function, we have obtained a sufficient conditions for global stability of the positive equilibrium for system (7). By numerical simulations, we have plotted the nature of spatial patterns with respect to time (see Fig.1) which leads to the formation of the labyrinth spatial patterns (the formation of spatio-temporal chaos).

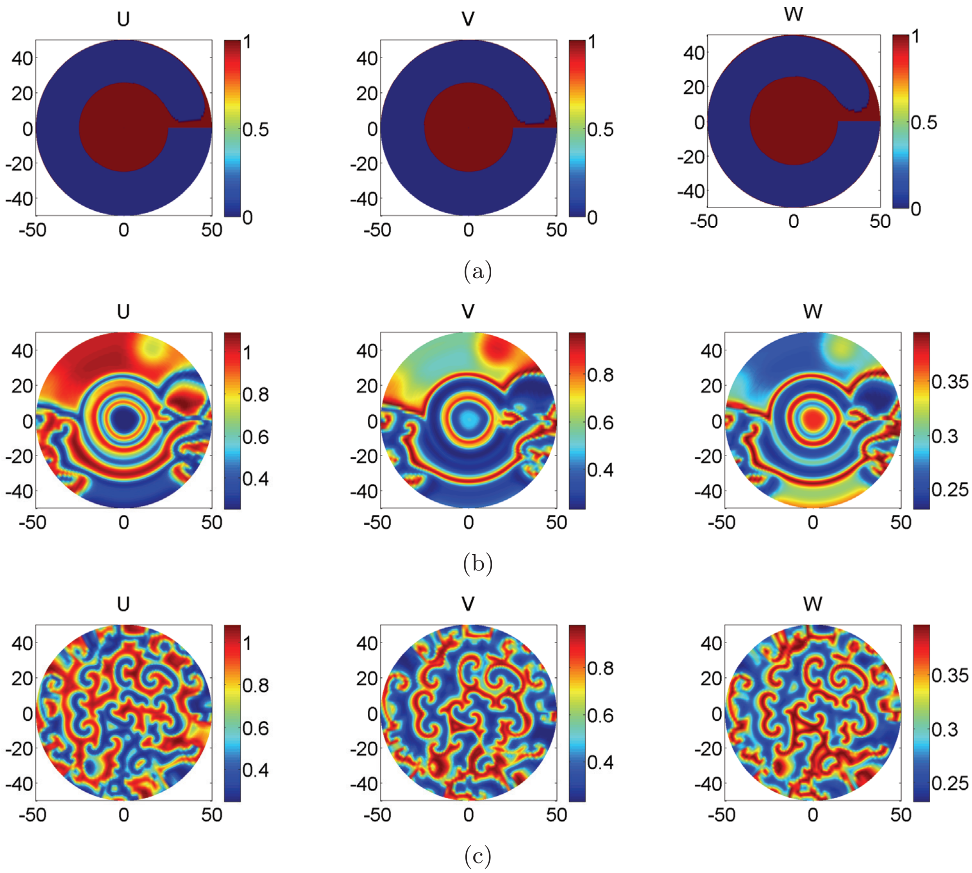

Fig. 1 Spatial distribution of prey (first column), predator (second column) and top predator (third column) are population densities of the spatial system (7). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$.

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