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Review on Finite Difference Method for Reaction-Diffusion Equation Defined on a Circular Domain

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Abstract

In this paper, a finite difference method for a non-linear reaction diffusion equation defined on a circular domain is presented. A simple second-order finite difference treatment of polar coordinate singularity for Laplacian operator, the centered difference approximations, the treatments for Neumann boundary problems are used to discretize this equation. By using this method, numerical solutions can be computed. In the end, we give two applications of reaction diffusion predator-prey models with modified Leslie-Gower and Holling type II functional responses.

1 Introduction

A reaction-diffusion equation is a partial differential equation which comprises reaction and diffusion terms:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \tag{1}$$

where $u = u(t,x)$ is a state variable and describes density/concentration of a substance or a population at position $x \in \Omega \subset \mathbb{R}^n$ and at time t, Ω is an open domain, Δ is the Laplacian operator and D is a diagonal matrix of diffusion coefficients.

This type of equations was introduced by Fisher [2] and Kolmogorov, Petrovsky and Piskunov [3] to describe the spreading of biological populations. Some of these equations can be solved analytically and numerically but for a large number of equations, the analytical solution is unknown and can be approximated by using numerical methods. In the literature, there exist many numerical methods for the resolution of different problems, here are some used to discretize a system of equations: the finite element method, finite difference method, finite volume method. In this review, we are interested in finite difference method.

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For the finite difference method (Thomée [15]), the domain is represented by a finite number of points $x_i = \Omega_h$ called nodes of the mesh and the solution is represented by a set of values u_i approaching $u(x_i)$. The method replaces the partial derivatives by differences or combinations of these punctual values of the function using truncated Taylor developments. The advantages of this method are its simplicity of writing and low computational cost.

The finite difference method (Richtmyer and Morton [6]; Hildebrand [7]) is widely used by many authors for approximating numerical solution of reaction-diffusion equations. Many authors (for example Ascher et al. [8]; Pao [9]; Jerome [16]; Li et al. [14]) also used this method to study stability and convergence result.

In [1], the author considered the following reaction-diffusion model defined:

$$\begin{cases} \dfrac{\partial u(t,x,y)}{\partial t} = D_1\Delta u(t,x,y) + f(u(t,x,y),v(t,x,y)) & (x,y)\in\Omega, t>0,\\ \dfrac{\partial v(t,x,y)}{\partial t} = D_2\Delta v(t,x,y) + g(u(t,x,y),v(t,x,y)) & (x,y)\in\Omega, t>0,\\ \dfrac{\partial u(t,x,y)}{\partial n} = \dfrac{\partial v(t,x,y)}{\partial n} = 0, & (x,y)\in\partial\Omega, \end{cases} \tag{2}$$

$u(t,x,y)$ and $v(t,x,y)$ represents the densities of populations, $\Omega = [0,L]\times[0,L]$ and Δu is the Laplacian operator

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

D_1 and D_2 are the diffusion coefficients, $f(u,v)$ and $g(u,v)$ model the local activity (absence of diffusion). Authors in [13] are numerically solved this system with appropriate functions f and g by using finite difference method on square domain and with Neumann boundary conditions.
In this review, we extend this method to a 2-D reaction diffusion system defined on a circular domain ($\Omega = \{(x,y)\in\mathbf{R}^2/x^2+y^2<R^2\}$) and with Neumann boundary conditions. To do that, we strive to linearize the reaction-diffusion system using the finite difference method [4] in polar coordinates. To apply this method to two dimensions, a simple division in reaction-diffusion equation defines the node at any point of the mesh of the circular domain.

The organization of the remaining part of the paper is as follows: In Section 2, we present a finite difference discretization for equation (2) given in polar coordinates. In Section 3, we apply this method to two component reaction diffusion predator-prey model defined on a disk. Then, we extend this result to three component reaction diffusion predator-prey model.

2 Discretization of reaction-diffusion equation defined on a disk domain

In this section, through the finite difference method and the principle of the numerical method used in [5], we solve numerically equation (2) defined on a disk domain $\Omega = \{(x,y)\in\mathbf{R}^2/x^2+y^2<R^2\}$.

As $(x,y)\in\Omega$, we can make the following change of variables:

$$\begin{cases} x = r\cos\theta,\\ y = r\sin\theta, \end{cases} \quad \text{where} \quad \begin{cases} r = \sqrt{x^2+y^2},\\ \theta = \tan^{-1}\left(\frac{y}{x}\right). \end{cases}$$

Without loss of generalities we also denote

$$\begin{cases} u(t,x,y) = u(t,r\cos(\theta),r\sin(\theta)) = u(t,r,\theta),\\ v(t,x,y) = v(t,r\cos(\theta),r\sin(\theta)) = v(t,r,\theta). \end{cases}$$

Therefore the Laplacian operator in polar coordinates is given by :

$$\Delta_{r\theta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{4}$$

and

$$\begin{cases} f(u(t,x,y),v(t,x,y)) = f(u(t,r,\theta),v(t,r,\theta)), \\ g(u(t,x,y),v(t,x,y)) = g(u(t,r,\theta),v(t,r,\theta)). \end{cases}$$

The Neumann boundary conditions in polar coordinates becomes:

$$\begin{cases} \dfrac{\partial u(t,x,y)}{\partial n}|_{\forall(x,y)\in\partial\Omega} = \partial_r u(t,r,\theta)|_{\text{ for } r=R \text{ Radial derivative}}, \\ \dfrac{\partial v(t,x,y)}{\partial n}|_{\forall(x,y)\in\partial\Omega}) = \partial_r v(t,r,\theta)|_{\text{ for } r=R \text{ Radial derivative}}. \end{cases}$$

Then system (2) can be written as

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = D_1\Delta_{r\theta}u(t,r,\theta) + f(u(t,r,\theta),v(t,r,\theta)) \text{ for } (r,\theta)\in\mathscr{D} \text{ and } t>0, \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = D_2\Delta_{r\theta}v(t,r,\theta) + g(u(t,r,\theta),v(t,r,\theta)) \text{ for } (r,\theta)\in\mathscr{D} \text{ and } t>0, \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = 0, \quad \text{for } r=R \text{ Radial derivative}, \end{cases} \tag{5}$$

where $\mathscr{D} = \{(r,\theta) : 0<r<R, 0\le\theta<2\pi\}$.

Equation (2) is written

$$\frac{\partial u(t,r,\theta)}{\partial t} = D_1(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}) + f(u(t,r,\theta),v(t,r,\theta)). \tag{6}$$

We see that, equation (6) has a singularity at the origin $r=0$, (see [11]). This singularity is due to the representation of the equation in polar coordinates. If f is regular enough, the solution itself is nonsingular at the origin. In order to have the desired regularity and accuracy, the classical finite difference scheme uses a uniformly integers grid with some conditions at the origin. This pole conditions act as a numerical boundary condition at the origin which is needed in finite difference scheme.

Considering the Neumann boundary conditions [12] and by discretization, we obtain the following approximation of equation (15). For $n=1,\dots,N$, with $N=\frac{T}{\Delta t}$, $i=1,\dots,P+1$, and $j=1,\dots,M+1$ we find $\{u_{i,j}^n, v_{i,j}^n\}$ such that

$$\begin{cases} \partial_n u_{i,j}^n = \Delta_{r_i\theta_j}u_{i,j}^n + f(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}), \\ \partial_n v_{i,j}^n = \delta\Delta_{r_i\theta_j}v_{i,j}^n + g(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}), \end{cases} \tag{7}$$

where $\overrightarrow{u_{i,j}^n} = (u_{i,j}^n, v_{i,j}^n)^T$ denotes the two-dimensional approximation at the point (r_i,θ_j,t_n) with $t_n = n\Delta t$. The approximations of the initial conditions are given as:

$$u_{i,j}^0 = u_0(r_i,\theta_j),\, v_{i,j}^0 = v_0(r_i,\theta_j).$$

We choose a grid such that the points are integers in azimuthal direction and half-integer in radial direction (see Fig. 1):

$$r_i = (i-\frac{1}{2})\Delta r,\, \theta_j = (j-1)\Delta\theta, \tag{8}$$

where

$$\Delta r = \frac{2}{2P+1},\ \Delta\theta = \frac{2\pi}{M}.$$

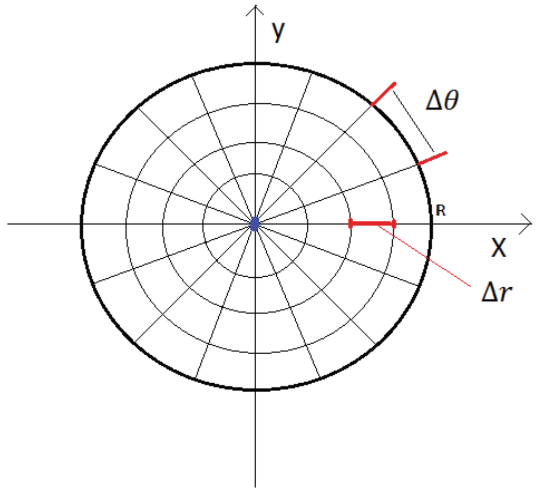


Fig. 1 The circular mesh.

For $i = 2,\ldots,P$ and $j = 1,\ldots,M$ and using the centered difference method to discretize the Laplacian operator, we have

$$\Delta_{r_i\theta_j}u^n_{i,j} \approx \frac{u^n_{i+1,j}+u^n_{i-1,j}-2u^n_{i,j}}{\Delta r^2}+\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2r_i\Delta r}+\frac{u^n_{i,j+1}+u^n_{i,j-1}-2u^n_{i,j}}{r_i^2\Delta\theta^2}. \tag{9}$$

From the Neumann boundary conditions (the flow is zero on the edge)

$$\frac{u^n_{P+1,j}-u^n_{P,j}}{\Delta r}=0, \tag{10}$$

so the numerical boundary values at $r = 1$, $u^n_{P+1,j}$ can be approximated by $u^n_{P,j}$, and $u^n_{i,0} = u^n_{i,M}$, $u^n_{i,1} = u^n_{i,M+1}$ since u is 2π periodic in θ. At $i = 1$, equation (9) becomes

$$\Delta_{r_1\theta_j}u^n_{1,j} \approx \frac{u^n_{2,j}+u^n_{0,j}-2u^n_{1,j}}{\Delta r^2}+\frac{u^n_{2,j}-u^n_{0,j}}{2r_1\Delta r}+\frac{u^n_{1,j+1}+u^n_{1,j-1}-2u^n_{1,j}}{r_1^2\Delta\theta^2} \tag{11}$$

since $r_1 = \frac{\Delta r}{2}$, the term $u^n_{0,j}$ is simplified and the equation (11) is written by

$$\Delta_{r_1\theta_j}u^n_{1,j} \approx \frac{2(u^n_{2,j}-u^n_{1,j})}{\Delta r^2}+\frac{u^n_{1,j+1}+u^n_{1,j-1}-2u^n_{1,j}}{r_1^2\Delta\theta^2}. \tag{12}$$

In order to approach $\partial_n u^n_{i,j}$, we use the implicit Euler method,

$$\partial_n u^n_{i,j} = \frac{u^n_{i,j}-u^{n-1}_{i,j}}{\Delta t}.$$

Finally we obtain the following equation

$$\begin{cases} \dfrac{u^n_{i,j}-u^{n-1}_{i,j}}{\Delta t} = D_1\left(\dfrac{u^n_{i+1,j}+u^n_{i-1,j}-2u^n_{i,j}}{\Delta r^2}+\dfrac{u^n_{i+1,j}-u^n_{i-1,j}}{2r_i\Delta r}+\dfrac{u^n_{i,j+1}+u^n_{i,j-1}-2u^n_{i,j}}{r_i^2\Delta\theta^2}\right)+f(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}), \\ \dfrac{v^n_{i,j}-v^{n-1}_{i,j}}{\Delta t} = D_2\left(\dfrac{v^n_{i+1,j}+v^n_{i-1,j}-2v^n_{i,j}}{\Delta r^2}+\dfrac{v^n_{i+1,j}-v^n_{i-1,j}}{2r_i\Delta r}+\dfrac{v^n_{i,j+1}+v^n_{i,j-1}-2v^n_{i,j}}{r_i^2\Delta\theta^2}\right)+f(\overrightarrow{v^n_{i,j}},\overrightarrow{v^{n-1}_{i,j}}). \end{cases} \tag{13}$$

3 Application

In this section, we apply the above method to 2-D two component reaction-diffusion predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response and then we extend this method to 2-D three reaction diffusion component predator-prey model.

3.1 Example of a predator-prey model of two species

Let us now consider the model with two component:

$$\begin{cases} \dfrac{\partial u(t,x,y)}{\partial t} = D_1\Delta u(t,x,y) + (a_1 - b_1 u(t,x,y) - \dfrac{c_1 v(t,x,y)}{d_1 u(t,x,y) + d_2 v(t,x,y) + k_1})u(t,x,y) \\ \dfrac{\partial v(t,x,y)}{\partial t} = D_2\Delta v(t,x,y) + (a_2 - \dfrac{c_2 v(t,x,y)}{u(t,x,y) + k_2})v(t,x,y). \end{cases} \tag{14}$$

This two species food chain model describes a prey population u which serves as food for a predator v. $u(t,x,y)$ and $v(t,x,y)$ represent population densities at time t and the position (x,y) defined on a circular domain Ω with radius R (i.e. $\Omega = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 < R^2\}$), $r_1, a_1, b_1, k_1, r_2, a_2$, and k_2 are positive parameters, a_1 is the growth rate of prey u, a_2 describes the growth rate of predator v, b_1 measures the strength of competition among individuals of species u, c_1 is the maximum value of the per capita reduction of u due to v, c_2 has a similar meaning to c_1, k_1 measures the extent protection to which environment provides to prey u, k_2 has a similar meaning to k_1 relatively to the predator v, d_1 and d_2 are two positive constants, D_1 and D_2 are the diffusions coefficients of the preys and the predators.

In polar coordinates model (14) is written as follows:

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = D_1\Delta_{r\theta} u(t,r,\theta) + f(u(t,r,\theta), v(t,r,\theta)) & \text{for } (r,\theta) \in \mathscr{D} \text{ and } t > 0, \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = D_2\Delta_{r\theta} v(t,r,\theta) + g(u(t,r,\theta), v(t,r,\theta)) & \text{for } (r,\theta) \in \mathscr{D} \text{ and } t > 0, \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = 0 & \text{for } r = R \text{ Radial derivative}, \end{cases} \tag{15}$$

where

$$\begin{cases} f(u(t,r,\theta), v(t,r,\theta)) = (a_1 - b_1 u(t,r,\theta) - \dfrac{c_1 v(t,r,\theta)}{d_1 u(t,r,\theta) + d_2 v(t,r,\theta) + k_1})u(t,r,\theta), \\ g(u(t,r,\theta), v(t,r,\theta)) = (a_2 - \dfrac{c_2 v(t,r,\theta)}{u(t,r,\theta) + k_2})v(t,r,\theta). \end{cases} \tag{16}$$

$u(t,r,\theta)$ and $v(t,r,\theta)$ represent the population densities at time t and the position (r,θ).
By computation, one can show that system (15) has four equilibrium points:

$$E_0 = (0,0), E_1 = (1,0), E_2 = (0,e_2), E^* = (u^*, v^*),$$

where

$$u^* = \frac{1 - a - e_1 + \sqrt{(a + e_1 - 1)^2 + 4(e_1 - ae_2)}}{2}, \tag{17}$$

and

$$v^* = u^* + e_2. \tag{18}$$

Next, we consider the disc domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 400\}$ and the boundary conditions are of the Neumann type. In order to avoid numerical artifacts, the values of the time (Δt) and space steps $(\Delta r$ and $\Delta\theta)$

have been chosen sufficiently small and satisfying the CFL (Courant-Friedrichs-Levy) stability criterion for reaction diffusion equation.

Therefore, we obtain the following system

$$\begin{cases} \partial_n u_{i,j}^n = D_1 \Delta_{r_i\theta_j} u_{i,j}^n + f(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) \\ \partial_n v_{i,j}^n = D_2 \Delta_{r_i\theta_j} v_{i,j}^n + g(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) \end{cases} \tag{19}$$

with

$$\begin{cases} f(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) = a_1 u_{i,j}^{n-1} - b_1 u_{i,j}^{n-1}|u_{i,j}^{n-1}| - \dfrac{c_1 v_{i,j}^{n-1}}{d_1|u_{i,j}^{n-1}| + d_2|v_{i,j}^{n-1}| + k_1} u_{i,j}^{n-1} \\ g(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) = a_2 v_{i,j}^{n-1} - \dfrac{c_2 v_{i,j}^{n-1}}{|u_{i,j}^{n-1}| + k_2} v_{i,j}^{n-1}. \end{cases} \tag{20}$$

The linear system associated to system (15) is

$$AZ = B$$

and the unknown vector $Z = \begin{pmatrix} \overrightarrow{u}^n \\ \overrightarrow{v}^n \end{pmatrix}$ is defined by

$$\overrightarrow{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{pmatrix}, \overrightarrow{v}^n = \begin{pmatrix} v_1^n \\ v_2^n \\ \vdots \\ \vdots \\ v_{M-1}^n \\ v_M^n \end{pmatrix}, \text{ with } u_j^n = \begin{pmatrix} u_{1,j}^n \\ u_{2,j}^n \\ \vdots \\ \vdots \\ u_{P-1,j}^n \\ u_{P,j}^n \end{pmatrix} \text{ and } v_j^n = \begin{pmatrix} v_{1,j}^n \\ v_{2,j}^n \\ \vdots \\ \vdots \\ v_{P-1,j}^n \\ v_{P,j}^n \end{pmatrix}$$

and the matrix A can be written as

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where

$$\begin{pmatrix} A_1 = I + D_1 \Delta t L \\ A_2 = I + D_2 \Delta t L \end{pmatrix}$$

in which I is the identity matrix and the size of the matrix L is $((P+1)\times(P+1))$ and written as follows:

$$\mathbf{L} = \begin{pmatrix} Q-2S & S & 0 & \dots & 0 & S \\ S & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & S \\ S & 0 & \dots & 0 & S & Q-2S \end{pmatrix},$$

where

$$\mathbf{Q} = \begin{pmatrix} -2 & 1+\lambda_1 & 0 & \dots & \dots & 0 \\ 1-\lambda_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1+\lambda_i & \ddots & \vdots \\ \vdots & \ddots & 1-\lambda_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -2 & 1+\lambda_{P-1} \\ 0 & \dots & \dots & 0 & 1-\lambda_P & 1+\lambda_P \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} \beta_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \beta_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & \beta_P \end{pmatrix}$$

with

$$\beta_i = \frac{1}{(i-0.5)^2\Delta\theta^2}, \lambda_i = \frac{1}{(i-0.5)}, i = 1, ..., P.$$

The known vector $B = \begin{pmatrix} \overrightarrow{u}^{n-1} + \Delta t \overrightarrow{f} \\ \overrightarrow{v}^{n-1} + \Delta t \overrightarrow{g} \end{pmatrix}$ is defined by

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ \vdots \\ B_{M-1} \\ B_M \end{pmatrix}, \ B_j = \begin{pmatrix} \Delta r^2(u_{1,j}^{n-1} + \Delta t f_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2(u_{P,j}^{n-1} + \Delta t f_{P,j}^{n-1}) \\ \Delta r^2(v_{1,j}^{n-1} + \Delta t g_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2(v_{P,j}^{n-1} + \Delta t g_{P,j}^{n-1}) \end{pmatrix}.$$

The initial conditions are small perturbation in the vicinity of equilibrium point (u^*, v^*) and are chosen as:

$$u_0(r_i, \theta_j) = u^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = u^* r_i^2 < 400, \tag{21}$$

$$v_0(r_i, \theta_j) = v^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = v^* r_i^2 < 400. \tag{22}$$

The values of the used parameters are given by

$$a_1 = 1, \ a_2 = 0.02, \ b_1 = 1, \ k_1 = 0.2, \ k_2 = 0.1, \ d_1 = 0.9, \ d_2 = 0.1, \ c_1 = 1.1,$$

$$c_2 = 0.02, \ D_1 = 1, \ D_2 = 1. \tag{23}$$

We suppose that the two species diffuse in the same way, (i.e. $D_1 = D_2$). In Fig. (2), the numerical solutions $u_{i,j}^n$ and $v_{i,j}^n$ of the predator-prey system are plotted. The left figures are the spatial distribution of prey population and on the right ones are of the predator population. We observe that the spatial distribution is a spiral wave type for system (15).

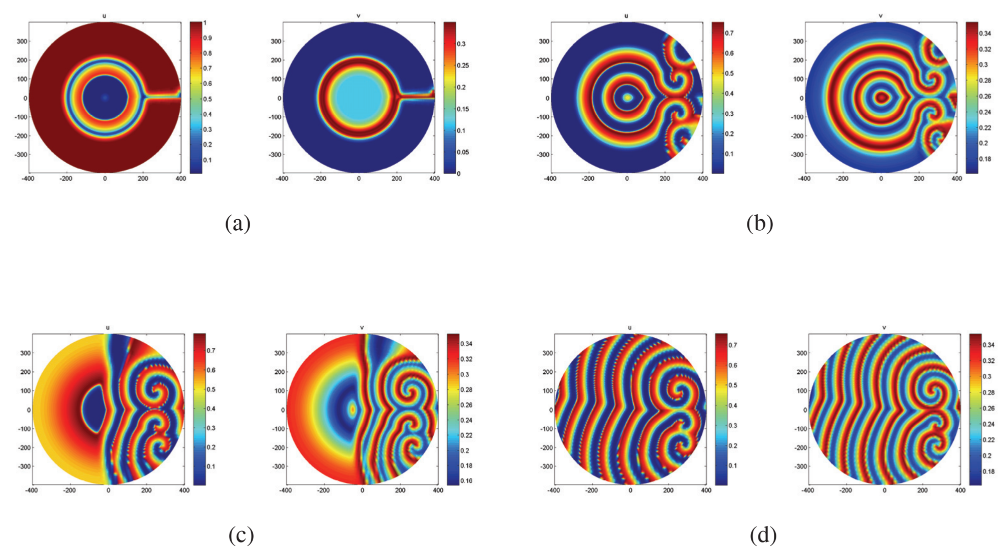


Fig. 2 Spatial distribution of species of system (15), for different values of time t, (a) t=100, (b) t=1000, (c) t=1800, (d) t=6000

3.2 Example of a three species predator-prey

In this example, we consider a three-species food chain model consisting of prey, intermediate predator and top-predator, modeled by a system of three reaction-diffusion equations defined on a circular spatial domain and incorporates the Holling type II and a modified Leslie-Gower functional response. The first species denoted U is the only food source of the second V. As well, intermediate predator V is the only prey of a top-predator W. Local interactions between species U and V are modeled by Lotka-Volterra type scheme and the interactions between species W and V has been modeled by Leslie-Gower scheme [17]. The spatio-temporal system can be written as follows (see [10]):

$$\begin{cases} \dfrac{\partial U(T,x,y)}{\partial T} = D_1\Delta U(T,x,y) + (a_0 - b_0 U(T,x,y) - \dfrac{v_0 V(T,x,y)}{U(T,x,y)+d_0})U(T,x,y), \\ \dfrac{\partial V(T,x,y)}{\partial T} = D_2\Delta V(T,x,y) + (-a_1 + \dfrac{v_1 U(T,x,y)}{U(T,x,y)+d_0} - \dfrac{v_2 W(T,x,y)}{V(T,x,y)+d_2})V(T,x,y), \\ \dfrac{\partial W(T,x,y)}{\partial T} = D_3\Delta W(T,x,y) + (c_3 - \dfrac{v_3 W(T,x,y)}{V(T,x,y)+d_3})W(T,x,y), \\ \dfrac{\partial U}{\partial n} = \dfrac{\partial V}{\partial n} = \dfrac{\partial W}{\partial n} = 0, \\ U(0,x,y) = U_0(x,y) \geq 0,\ V(0,x,y) = V_0(x,y) \geq 0,\ W(0,x,y) = W_0(x,y) \geq 0, \end{cases} \tag{24}$$

where $U(T,x,y)$, $V(T,x,y)$ and $W(T,x,y)$ are the densities of prey, intermediate predator and top-predator, respectively, at time T and position (x,y) defined on a circular domain Ω with radius R (i.e. $\Omega = \{(x,y) \in \mathbf{R}^2 / x^2 + y^2 < R^2\}$. The three species are assumed to diffuse at rates D_i (i = 1, 2, 3). The parameters a_0, b_0, v_0, d_0, a_1, v_1, v_2, d_2, c_3, v_3 and d_3 are assumed to be positive constants and are defined as follows: a_0 is the growth rate of the prey U, b_0 measures the mortality due to the competition between individuals of the species U, v_0 is the maximum extent that the rate of reduction by individual U can reach, d_0 measures the protection that the species U and V benefit through the environment, a_1 represents the death rate of V in the absence of U, v_1, v_2 and v_3 are the the maximum value that the rate of reduction by the individual of U, V and W can reach respectively, d_2 is the value of V for which the rate of elimination by individual V becomes $\frac{v_2}{2}$, c_3 describes the growth rate of W, assuming that there is the same number of males and females and d_3 represents the residual loss caused by high scarcity of prey V of the specie W.

Using the following transformations and by change of variables to polar coordinates:

$$U = \frac{a_0}{b_0}u,\ V = \frac{a_0^2}{b_0 v_0}v,\ W = \frac{a_0^3}{b_0 v_0 v_2}w,\ T = \frac{t}{a_0},$$

and

$$a = \frac{b_0 d_0}{a_0},\ b = \frac{a_1}{a_0},\ c = \frac{v_1}{a_0},\ d = \frac{d_2 v_0 b_0}{a_0^2},\ p = \frac{c_3 a_0^2}{v_0 b_0 v_2},\ q = \frac{v_3}{v_2},\ s = \frac{d_3 v_0 b_0}{a_0^2},\ \delta_1 = \frac{D_1}{a_0},\ \delta_2 = \frac{D_2}{a_0},\ \delta_3 = \frac{D_3}{a_0}$$

the spatio-temporal system (24) becomes

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = \delta_1 \Delta_{r\theta} u(t,r,\theta) + (1 - u(t,r,\theta) - \dfrac{v(t,r,\theta)}{u(t,r,\theta)+a})u(t,r,\theta), & \forall (r,\theta) \in \Gamma,\ t > 0 \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = \delta_2 \Delta_{r\theta} v(t,r,\theta) + (-b + \dfrac{cu(t,r,\theta)}{u(t,r,\theta)+a} - \dfrac{w(t,r,\theta)}{v(t,r,\theta)+d})v(t,r,\theta), & \forall (r,\theta) \in \Gamma,\ t > 0 \\ \dfrac{\partial w(t,r,\theta)}{\partial t} = \delta_3 \Delta_{r\theta} w(t,r,\theta) + (p - \dfrac{qw(t,r,\theta)}{v(t,r,\theta)+s})w(t,r,\theta) & \forall (r,\theta) \in \Gamma,\ t > 0 \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = \partial_r w(.,r,\theta) = 0 \text{ for } r = R \text{ (radial derivative)}, & \\ u(0,r,\theta) = u_0(r,\theta) \geq 0,\ v(0,r,\theta) = v_0(r,\theta) \geq 0,\ w(0,r,\theta) = w_0(r,\theta) \geq 0. & \end{cases} \tag{25}$$

$u(t,r,\theta)$, $v(t,r,\theta)$ and $w(t,r,\theta)$ represent the population densities at time t and the position $(r,\theta)\in\Gamma$, $\Gamma=\{(r,\theta):0<r<R,0\le\theta<2\pi\}$.

By computation, system (25) has four trivial equilibrium points $E_0=(0,0,0)$, $E_1=(1,0,0)$, $E_2=(0,0,\frac{sp}{q})$, $E_3=(1,0,\frac{sp}{q})$ and a positive nontrivial one $E^*=(u^*,v^*,w^*)$ which exists if and only if the following inequalities hold

$$qc>bq+p \text{ and } qc-bq-p>a(bq+p), \tag{26}$$

such that

$$u^*=\frac{a(bq+p)}{qc-bq-p},\ v^*=(1-u^*)(u^*+a) \text{ and } w^*=\frac{p(v^*+s)}{q}. \tag{27}$$

Therefore, we obtain the following system

$$\begin{cases}\partial_n u^n_{i,j}=\delta_1\Delta_{r_i\theta_j}u^n_{i,j}+f(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}),\\ \partial_n v^n_{i,j}=\delta_2\Delta_{r_i\theta_j}v^n_{i,j}+g(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}),\\ \partial_n w^n_{i,j}=\delta_3\Delta_{r_i\theta_j}w^n_{i,j}+h(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}),\end{cases} \tag{28}$$

with

$$\begin{cases}f(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}})=u^{n-1}_{i,j}-u^{n-1}_{i,j}|u^{n-1}_{i,j}|-\dfrac{v^{n-1}_{i,j}}{|u^{n-1}_{i,j}|+a}u^{n-1}_{i,j},\\ g(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}})=-bv^{n-1}_{i,j}+\dfrac{cu^{n-1}_{i,j}}{|u^{n-1}_{i,j}|+a}v^{n-1}_{i,j}-\dfrac{w^{n-1}_{i,j}}{|v^{n-1}_{i,j}|+d}v^{n-1}_{i,j},\\ h(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}})=pw^{n-1}_{i,j}-\dfrac{qw^{n-1}_{i,j}}{|v^{n-1}_{i,j}|+s}w^{n-1}_{i,j}.\end{cases} \tag{29}$$

The linear system associated with system (15) is given by

$$DH=C.$$

The unknown vector $H=\begin{pmatrix}\overrightarrow{u}^n\\ \overrightarrow{v}^n\\ \overrightarrow{w}^n\end{pmatrix}$ is defined by

$$\overrightarrow{u}^n=\begin{pmatrix}u^n_1\\u^n_2\\ \vdots\\ \vdots\\ u^n_{M-1}\\u^n_M\end{pmatrix},\ \overrightarrow{v}^n=\begin{pmatrix}v^n_1\\v^n_2\\ \vdots\\ \vdots\\ v^n_{M-1}\\v^n_M\end{pmatrix},\ \overrightarrow{w}^n=\begin{pmatrix}w^n_1\\w^n_2\\ \vdots\\ \vdots\\ w^n_{M-1}\\w^n_M\end{pmatrix},\ \text{with } u^n_j=\begin{pmatrix}u^n_{1,j}\\u^n_{2,j}\\ \vdots\\ \vdots\\ u^n_{P-1,j}\\u^n_{P,j}\end{pmatrix},\ v^n_j=\begin{pmatrix}v^n_{1,j}\\v^n_{2,j}\\ \vdots\\ \vdots\\ v^n_{P-1,j}\\v^n_{P,j}\end{pmatrix},\ w^n_j=\begin{pmatrix}w^n_{1,j}\\w^n_{2,j}\\ \vdots\\ \vdots\\ w^n_{P-1,j}\\w^n_{P,j}\end{pmatrix}$$

and the matrix A can be written as

$$D=\begin{pmatrix}D_1&0&0\\0&D_2&0\\0&0&D_3\end{pmatrix},$$

$$\begin{pmatrix}D_1=I+\delta_1\Delta tL\\ D_2=I+\delta_2\Delta tL\\ D_3=I+\delta_2\Delta tL\end{pmatrix}.$$

The known vector $C = \begin{pmatrix} \overrightarrow{u}^{n-1} + \Delta t \overrightarrow{f} \\ \overrightarrow{v}^{n-1} + \Delta t \overrightarrow{g} \\ \overrightarrow{w}^{n-1} + \Delta t \overrightarrow{h} \end{pmatrix}$ is defined by

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_{M-1} \\ C_M \end{pmatrix}, \ C_j = \begin{pmatrix} \Delta r^2 (u_{1,j}^{n-1} + \Delta t f_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2 (u_{P,j}^{n-1} + \Delta t f_{P,j}^{n-1}) \\ \Delta r^2 (v_{1,j}^{n-1} + \Delta t g_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2 (v_{P,j}^{n-1} + \Delta t g_{P,j}^{n-1}) \\ \Delta r^2 (w_{1,j}^{n-1} + \Delta t h_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2 (w_{P,j}^{n-1} + \Delta t h_{P,j}^{n-1}) \end{pmatrix}.$$

We simulate the spatial distributions of the three populations in the limited field $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 50\}$. The boundary conditions are of Neumann type, (i.e. there is no emigration or immigration of populations). The initial conditions are a small perturbation in the vicinity of equilibrium point (u^*, v^*, w^*) and are chosen as

$$u_0(r_i, \theta_j) = u^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = u^* r_i^2 < 50, \tag{30}$$

$$v_0(r_i, \theta_j) = v^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = v^* r_i^2 < 50,$$

$$w_0(r_i, \theta_j) = w^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = w^* r_i^2 < 50$$

and parameters values are:

$$a_0 = 0.5,\ a_1 = 0.4,\ b_0 = 0.36,\ d_0 = 0.3,\ d_2 = 0.4,\ d_3 = 0.4,\ v_0 = 0.4,\ v_1 = 0.8,\ v_2 = 0.4,\ v_3 = 0.6. \tag{31}$$

From Fig. (3), different types of dynamics are observed when the bifurcation parameter c_3 varies.

4 Conclusions

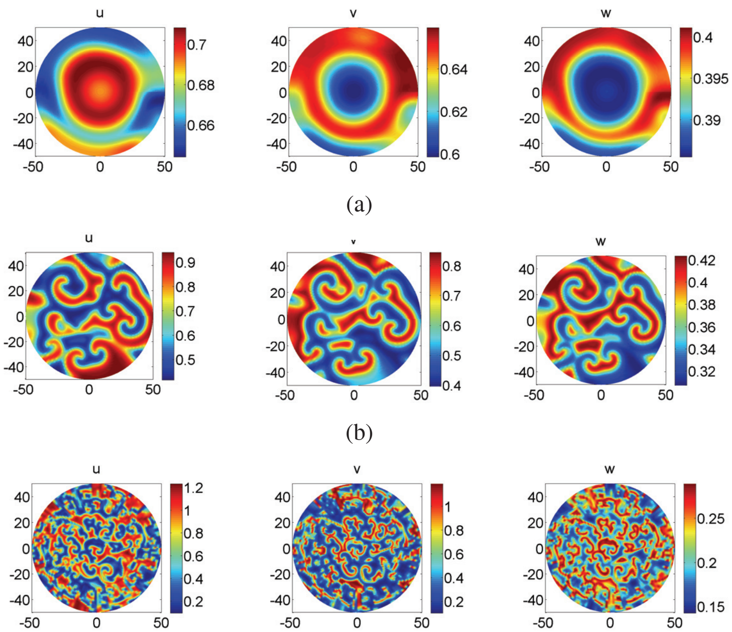
In this paper, we have considered a nonlinear reaction-diffusion equation defined on a circular domain with the Neumann boundary conditions. We used the implicit Euler scheme to approach the derivative in time and the finite difference method to approximate the Laplacian operator in polar-coordinates. So, we extract a linear system in the form $AX = B$ which is necessary for the numerical solution of such equation.

To provide efficiency of this method, we have presented two applications arising from mathematical ecology.

A MATLAB code was also developed with the assumptions that the values of the time step (Δt) and space steps $(\Delta r$ and $\Delta\theta)$ have been chosen sufficiently small (number of nodes on the radius and the perimeter is very large) and satisfying the CFL (Courant-Friedrichs-Levy) stability criterion for reaction diffusion equation.

We chose a set of fixed parameters and the initial conditions depends on the points of the grid on the radius and the numbers of nodes of the mesh on the radius and perimeter are set. Figure (2) represents the evolution of the spatial distribution of two species for different values of time. We observe from this figure that when increasing the value of time, the number of iterations in time increases (to calculate the solution), so the solution of the system converges to a stable state. Figure (3) represents the evolution of the spatial distribution when the control parameter varies.

Therefore, the advantages of this method are: the simplicity of implementation, effectiveness, ability to construct approximations to high orders. Other methods such as finite element and finite volume can often be interpreted as finite difference schemes in the case of regular mesh.



(a)

(b)

(c)

Fig. 3 Spatial distribution of prey (first column), predator (second column) and top predator (third column) are population densities of system (25). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$, for fixed time $t = 12000$ at different bifurcation parameter $c_3 = 0.23$ (a), c_3=0.22 (b), $c_3 = 0.15$ (c)

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