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GLOBAL DYNAMICS ON A CIRCULAR DOMAIN OF A DIFFUSION PREDATOR-PREY MODEL WITH MODIFIED LESLIE-GOWER AND BEDDINGTON-DEANGELIS FUNCTIONAL TYPE

Walid Abid

University of Tunis El Manar, National Engineering School of Tunis
Laboratory of Engineering Mathematics EPT, B.P 743, La Marsa 2078, Tunisia

Radouane Yafia

Ibn Zohr University, Polydisciplinary Faculty of Ouarzazate
B.P: 638, Ouarzazate, Morocco

M. A. Aziz-Alaoui

Normandie Univ, France; ULH, LMAH, F-76600 Le Havre; FR CNRS 3335, ISCN
25 rue Philippe Lebon 76600 Le Havre, France

Habib Bouhafa and Azgal Abichou

University of Tunis El Manar, National Engineering School of Tunis
Laboratory of Engineering Mathematics EPT, B.P 743, La Marsa 2078, Tunisia

Abstract. In this paper, we present a predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response. The model is governed by a two dimensional reaction diffusion system defined on a disc domain. The conditions of boundedness, existence of a positively invariant and attracting set are proved. Sufficient conditions of local and global stability of the positive steady state are established. In the end, we carry out some numerical simulations in order to illustrate our theoretical results and to interpret how biological processes affect spatio-temporal pattern formation.

1. **Introduction and mathematical model.** In the last decades, the dynamical problems of prey and predator associated with mathematical modelling becomes an important area of research in ecology. An important orientation in theoretical work on preypredator dynamics has been to acquire more realistic models, trying to keep to maximum the unavoidable increase in complexity of their mathematics. One of the oldest and well known mathematical model which describes the interaction between predator and prey was introduced by A. Lotka 1925 [13] and V. Volterra 1927 [18], this model is well known by Lotka-Volterra mathematical model. It consists of two differential equations with a simple correspondence between prey consumption and predator production. The link between the dynamics of the two species in predator-prey model is based on a linear functional response type, which represents the number of prey consumed per predator per unit of time. Many

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authors have proposed many improvements of Lotka-Volterra model by introducing different functional responses type: Holling type $II-III$ [1, 12, 20, 21] and Hassel-Varley type [17, 3, 10] and Beddington-DeAngelis type [19, 9, 2, 24, 7] and references therein . Recently, a modified Leslie-Gower functional response is introduced by Aziz Alaoui et al. (see [4, 5]) and the corresponding reaction diffusion model is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \Delta u + (a_1 - b_1 u - \frac{c_1 v}{u+k_1})u \\ \frac{\partial v}{\partial t} = D_2 \Delta v + (a_2 - \frac{c_2 v}{u+k_2})v \end{cases} \tag{1}$$

This two-species food chain model describes a prey population u which serves as food for a predator v, with $u(0) \geq 0$, $v(0) \geq 0$,.

$u(x, y, t)$ and $v(x, y, t)$ are densities of prey and predator, respectively at time t and position (x, y). $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator. The parameters $c_1, a_1, b_1, k_1, c_2, k_2$, and k_2 are positive.

a_1 is the growth rate of preys u, a_2 describes the growth rate of predators v, b_1 measures the strength of competition among individuals of species u, c_1 is the maximum value of the per capita reduction of u due to v, c_2 has a similar meaning to c_1, k_1 measures the extent to which environment provides protection to prey u, k_2 has a similar meaning to k_1 relatively to the predator v and D_1, D_2 are the terms diffusions of the preys and the predators.

Let us mention that the first equation of this system is the same as in the volterra-Lotka prey-predator model and the second is absolutely not standard. Thus, in the absence of predators the prey population is assumed to grow according to the Malthusian law.

The Leslie-Gower type model is characterized by the assumption that the reduction in predator population has a reciprocal relationship with per capita availability of its preferred food, the carrying capacity of the predator v environment is proportional to the number of prey u, that is, it depends on the available resources. He stresses the fact that there are upper limits to the rates of increase of both prey u and predator v, which are not recognized in the LotkaVolterra model. In the case of continuous time, it is $\frac{\partial v}{\partial t} = a_2 v \left(1 - \frac{v}{\mu u}\right)$, in which the growth of predator population takes logistic form ($\frac{\partial v}{\partial t} = a_2 v \left(1 - \frac{v}{K}\right)$).

It is assumed that $K_v = K(u) = \mu u$, i.e., the carrying capacity is proportional to the prey abundance, just as it is assumed in May- Holling-Tanner model, (μ is the conversion factor of prey into predators). The term $\frac{v}{\mu u}$ is called the Leslie-Gower term. We will suppose that the predators have an alternative food when the quantity of prey u decreases, that is $K(u) = \mu u + \alpha$ (modified Leslie Gower model). In the absence of prey u i.e. $u = 0$, then $K(u) = \alpha$, and the predator v becomes generalist since it has an alternative food. The equation above is written as $\frac{\partial v}{\partial t} = a_2 v \left(1 - \frac{v}{\mu u + \alpha}\right)$, therefore $\frac{\partial v}{\partial t} = v \left(a_2 - \left(\frac{a_2}{\mu}\right) \left(\frac{v}{(u + \frac{\alpha}{\mu})}\right)\right)$ which is the second equation of system (1).

In [4, 5], the authors study the boundedness and global stability of the system (1) by using de Lyapunov functional and prove the occurrence of Turing and Hopf bifurcation.

In [8], authors give assumptions where the solutions are bounded for the system (1), existence of a positively invariant and attracting set and the global stability (see [23]) and the existence and uniqueness of limit cycle.

In [22], the authors show the existence of the periodic travelling waves via Hopf bifurcation Theorem by considering the diffusion as a parameter of bifurcation.

Although much progress has been seen in the study of predator-prey problem which incorporates the Holling type II and a modified Leslie-Gower functional response.

The Beddington-DeAngelis functional response was introduced by DeAngelis [9] as a solution of the these observed problems. Beddington [2] offered the same form of a functional response for describing parasite-host interactions. The Beddington-DeAngelis functional response had an extra term in the denominator which models mutual interference between predators.

In [7], Chen and Wang have considered a predator-prey system with the Beddington-DeAngelis functional response in the two equations. The authors show the dissipation, persistence, and stability of nonnegative constant steady states, as well as the existence and while bifurcation (Local and Global) of non constant positive steady states of the associated system.

In this paper we will incorporate the Beddington-DeAngelis functional response into model (1). The model under consideration is a $2-D$ reaction diffusion model which is based on the modified Leslie-Gower model with Beddington-DeAngelis functional responses:

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t} = D_1\Delta u(t,x,y) + (a_1 - b_1u(t,x,y) - \frac{c_1v(t,x,y)}{d_1u(t,x,y)+d_2v(t,x,y)+k_1})u(t,x,y) \\ \frac{\partial v(t,x,y)}{\partial t} = D_2\Delta v(t,x,y) + (a_2 - \frac{c_2v(t,x,y)}{u(t,x,y)+k_2})v(t,x,y) \end{cases} \tag{2}$$

$u(t,x,y)$ and $v(t,x,y)$ represent population densities at time t and space (x,y) defined on a circular domain (or disc domain) with radius R (i.e $\Omega = \{(x,y) \in \mathbb{R}^2/x^2+y^2 < R^2\}$), $r_1, a_1, b_1, k_1, r_2, a_2$, and k_2 are model parameters assuming only positive values, a_1 is the growth rate of preys u, a_2 describes the growth rate of predators v, b_1 measures the strength of competition among individuals of species u, c_1 is the maximum value of the per capita reduction of u due to v, c_2 has a similar meaning to c_1, k_1 measures the extent to which environment provides protection to prey u, k_2 has a similar meaning to k_1 relatively to the predator v, d_1 and d_2 are two positive constants, D_1 and D_2 are the terms diffusions of the preys and the predators.

In [19], the corresponding ordinary differential equations is studied in terms of local and global stability.

In this work, we prove the positivity and boundedness of solutions and we give the attracting positive region. Also, we showe the local and global stability of the positive steady state. The result of pattern formation is given via numerical simulations.

The current paper is organized as follows: In section 2, we recall some results on the model without diffusion. In section 3, we prove the existence of the equilibrium points and the local stability of the nontrivial steady state and the boundedness of solutions. Section 4 is devoted to the global stability of the nontrivial steady state. In section 5, we illustrate our results by numerical simulations. We end this work by a conclusion.

2. **Preliminaries.**

2.1. **Asymptotic behavior of ODE system.**

In this subsection we recall some results on the asymptotic behavior of the system without diffusion of system (2)

$$\begin{cases} \frac{du}{dt} = \left(a_1 - b_1 u - \frac{c_1 v}{d_1 u + d_2 v + k_1}\right) u \\ \\ \frac{dv}{dt} = \left(a_2 - \frac{c_2 v}{u + k_2}\right) v \end{cases} \tag{3}$$

System (3) has three trivial equilibrium points $E_0 = (0,0)$, $E_1 = (\frac{a_1}{b_1}, 0)$, $E_2 = (0, \frac{a_2 k_2}{c_2})$ and one nontrivial positive equilibrium point $E^* = (u^*, v^*)$ which corresponds to the coexistence of the two species, where

$$u^* = \frac{-B + \sqrt{B^2 + 4AC}}{2A}, \tag{4}$$

and

$$B = c_1 a_2 + b_1 c_2 k_1 + b_1 d_2 k_2 a_2 - a_1 d_1 c_2 - a_1 d_2 a_2,$$
$$A = b_1 d_2 a_2 + d_1 b_1 c_2,$$
$$C = k_1 a_1 c_2 + a_1 a_2 d_2 k_2 - c_1 a_2 k_2,$$

and

$$v^* = \frac{a_2}{c_2}(u_* + k_2), \tag{5}$$

The condition $k_1 a_1 c_2 + a_1 a_2 d_2 k_2 > c_1 a_2 k_2$ ensures the positivity of $E_* = (u_*, v_*)$. By linearizing system (3) around $E^* = (u^*, v^*)$, we obtain the following Jacobian matrix

$$G(E_*) = \begin{pmatrix} \frac{(a_1 d_1 - k_1 b_1) u^* - 2 b_1 d_1 {u^*}^2 - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1} & -\frac{c_1 u^* (k_1 + d_1 u^*)}{(d_1 u^* + d_2 v^* + k_1)^2} \\ \\ \frac{a_2^2}{c_2} & -a_2 \end{pmatrix}$$

From the jacobian matrix we deduce that, E_0 is an unstable node and E_1 is a saddle point. By a simple computation, the eigenvalues associated to E_2 are $\mu_1 = a_1 - \frac{c_1 a_2 k_2}{k_1 c_2 + d_2 a_2 k_2}$ and $\mu_2 = -a_2 < 0$, therefore we conclude that:

i) If $k_1 a_1 c_2 + a_1 a_2 d_2 k_2 < c_1 a_2 k_2$, E_2 is locally asymptotically stable.
ii) If $k_1 a_1 c_2 + a_1 a_2 d_2 k_2 > c_1 a_2 k_2$, then the equilibrium E_2 is a saddle point. Its stable manifold is v-axis.
iii) If $k_1 a_1 c_2 + a_1 a_2 d_2 k_2 = c_1 a_2 k_2$, a local bifurcation appears.

In the next, we give results on the stability of the positive equilibrium E^*.

If $q(u^*) > 0$ is asymptotically stable and unstable if $q(u^*) < 0$, where

$$q(u) = z_1 u^2 + z_2 u + z_3. \tag{6}$$

and

$$z_1 = 2 b_1 d_1 c_2 + b_1 d_2 a_2,$$
$$z_2 = a_2^2 d_2 + a_2 d_1 c_2 + k_1 b_1 c_2 - a_1 d_1 c_2,$$
$$z_3 = a_2^2 d_2 k_2 + b_1 d_2 k_2 a_2 + k_1 a_2 c_2.$$

It is obvious that if $(a_2^2 d_2 + a_2 d_1 c_2 + k_1 b_1 c_2 \geq a_1 d_1 c_2)$ or if $(a_2^2 d_2 + a_2 d_1 c_2 + k_1 b_1 c_2 < a_1 d_1 c_2)$ and $z_2^2 - 4 z_1 z_3 < 0$, we have $q(u) > 0$.

If $(a_2^2 d_2 + a_2 d_1 c_2 + k_1 b_1 c_2 < a_1 d_1 c_2)$ and $z_2^2 - 4 z_1 z_3 > 0$, then $q(u)$ has two positive roots θ_1 and θ_2 such that $0 < \theta_1 < \theta_2 < \frac{a_1}{b_1}$where

$$\theta_{1,2} = \frac{-z_2 \pm \sqrt{z_2^2 - 4 z_1 z_3}}{z_1^2}. \tag{7}$$

Therefore, we conclude taht if $0 < u < \theta_1$ or $\theta_2 < u < \frac{a_1}{b_1}$, we have $q(u) > 0$.

By against if $\theta_1 < u < \theta_2$, we get $q(u) < 0$.

Then, we have the following result:

Proposition 1.
- *If* $0 < u^* < \theta_1$ *or* $\theta_2 < u^* < \frac{a_1}{b_1}$, *then* $E^* = (u^*, v^*)$ *is asymptotically stable.*
- *If* $(a_2^2 d_2 + a_2 d_1 c_2 + k_1 b_1 c_2 < a_1 d_1 c_2)$ *and* $\theta_1 < u^* < \theta_2$, *then* $E^* = (u^*, v^*)$ *is unstable for system* (3).
- *If* $a_1 d_1 < k_1 b_1$, *we have* $tr(G(E^*)) < 0$ *and* $det(G(E^*)) > 0$, *then the positive equilibrium* $E^* = (u^*, v^*)$ *is locally asymptotically stable.*

Where

$$\begin{aligned}
tr(G(E^*)) &= \frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 {u^*}^2 - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1} - a_2,\\
det(G(E^*)) &= -a_2 \frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 {u^*}^2 - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1} + \frac{a_2^2 c_1 u^* (k_1 + d_1 u^*)}{c_2 (d_1 u^* + d_2 v^* + k_1)^2}\\
&= -a_2 \frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 {u^*}^2 - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1}\\
&+ \frac{a_2^2 u^* (k_1 + d_1 u^*)(a_1 - b_1 u^*)}{c_2 (d_1 u^* + d_2 v^* + k_1) v^*}\\
&= \frac{a_2 u^*}{d_1 u^* + d_2 v^* + k_1} (2b_1 d_1 u^* - (a_1 d_1 - k_1 b_1) + \frac{d_1 d_2 (u^* + k_2)}{c_2}\\
&+ \frac{(k_1 + d_1 u^*)(a_1 - b_1 u^*)}{(u^* + k_2)})\\
&= \frac{a_2 u^*}{c_2 (d_1 u^* + d_2 v^* + k_1)(u^* + k_2)} ((b_1 d_1 c_2 + b_1 d_2 a_2) {u^*}^2\\
&+ 2(b_1 d_2 a_2 k_2 + b_1 d_1 k_2 c_2) u^* + b_1 d_2 a_2 k_2^2 + k_1 a_1 c_2\\
&+ k_1 b_1 k_2 c_2 - a_1 d_1 k_2 c_2).
\end{aligned}$$

3. **Equilibrium points and stability.** In this section, we consider the reaction diffusion system of the two species defined on a circular domain with Neumann boundary conditions (which means that, there are no flux of the two species on the boundary of the circular domain Ω), where $\Omega = \{(x, y) : x^2 + y^2 < R^2\}$. Writing x and y in polar coordinates $x = rcos\theta$ and $y = rsin\theta$ and define $\Gamma = \{(r, \theta) : 0 < r < R,\ 0 \le \theta < 2\pi\}$ (R the radius of the disk) where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$.

Without loss of generalities we denote $u(t, x, y) = u(t, rcos(\theta), rsin(\theta)) = u(t, r, \theta)$ and $v(t, x, y) = v(t, rcos(\theta), rsin(\theta)) = v(t, r, \theta)$ are the densities of prey and predator respectively in polar coordinates. At $t = 0$ $u(0, r, \theta) = u_0(r, \theta) \ge 0, v(0, r, \theta) = v_0(r, \theta) \ge 0$. Therefore the Laplacian operator in polar coordinates is given by:

$$\Delta_{r\theta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{8}$$

Then, the spatio-temporal system (2) in polar coordinates is written as follows:

$$\begin{cases}
\frac{\partial u(t,r,\theta)}{\partial t} = D_1 \Delta_{r\theta} u(t, r, \theta) + f(u(t, r, \theta), v(t, r, \theta)) & \forall (r, \theta) \in \Gamma, t > 0\\
\frac{\partial v(t,r,\theta)}{\partial t} = D_2 \Delta_{r\theta} v(t, r, \theta) + g(u(t, r, \theta), v(t, r, \theta)) & \forall (r, \theta) \in \Gamma, t > 0\\
\frac{\partial u(t,r,\theta)}{\partial n} = \frac{\partial v(t,r,\theta)}{\partial n} = 0, & \forall (r, \theta) \in \partial\Gamma
\end{cases} \tag{9}$$

where

$$\begin{cases}
f(u(t, r, \theta), v(t, r, \theta)) = (a_1 - b_1 u(t, r, \theta) - \frac{c_1 v(t,r,\theta)}{d_1 u(t,r,\theta) + d_2 v(t,r,\theta) + k_1}) u(t, r, \theta),\\
g(u(t, r, \theta), v(t, r, \theta)) = (a_2 - \frac{c_2 v(t,r,\theta)}{u(t,r,\theta) + k_2}) v(t, r, \theta),
\end{cases} \tag{10}$$

A steady state (u_e, v_e) of (9) is a solution of the following system

$$\begin{cases} D_1\Delta_{r\theta}u_e(t,r,\theta) + f(u_e(t,r,\theta), v_e(t,r,\theta)) = 0 \\ D_2\Delta_{r\theta}v_e(t,r,\theta) + g(u_e(t,r,\theta), v_e(t,r,\theta)) = 0 \end{cases} \tag{11}$$

The trivial steady states (belonging to the boundary of int $\mathbb{R}^2_+$, i.e. at which one or more of populations has zero density or is extinct) are in the following forms:

$$E_0 = (0,0), E_1 = (\frac{a_1}{b_1}, 0), E_2 = (0, \frac{a_2k_2}{c_2}) \tag{12}$$

$\mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2, u_0 \geq 0, v_0 \geq 0\}$.

We will investigate the asymptotic behavior of orbits starting in the positive cone $int\mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2, u_0 > 0, v_0 > 0\}$.

Theorem 3.1. *Let Θ be the set defined by*

$$\Theta = \{(u,v) \in \mathbb{R}^2_+,\ 0 \leq u \leq \frac{a_1}{b_1},\ 0 \leq v \leq \frac{a_2}{b_1c_2}(a_1 + b_1k_2)\}$$

i) Θ is positively invariant region.
ii) All solutions of (9) initiating in Θ are ultimately bounded with respect to $\mathbb{R}^2_+$ and eventually enter the attracting set Θ.

For this demonstration we use Lemma (3.2) of chen [6],

Lemma 3.2. *if $a > 0, b > 0$ and $\frac{du}{dt} \geq u(a - bu)$ when $t \geq t_0$ and $u(t_0) > 0$, we have*

$$\liminf_{t\longrightarrow+\infty} u(t) \geq \frac{a}{b}$$

if $a > 0, b > 0$ and $\frac{du}{dt} \leq u(a - bu)$ when $t \geq t_0$ and $u(t_0) > 0$, we have

$$\limsup_{t\longrightarrow+\infty} u(t) \leq \frac{a}{b}$$

Proof. Since $(u(t), v(t))$ is a positive solution of $(9)_1$, we have

$$\begin{cases} \frac{\partial u(t,r,\theta)}{\partial t} \leq (a_1 - b_1u(t,r,\theta))u(t,r,\theta) \\ \frac{\partial u(.,R,\theta)}{\partial r} = 0 \\ u(0,r,\theta) = u_0(r,\theta) \leq \max\limits_{(r,\theta)\in\overline{\mathcal{D}}} u_0(r,\theta) \end{cases} \tag{13}$$

from Lemma (3.2), we deduce that

$$\limsup_{t\longrightarrow+\infty} u(t) \leq \frac{a_1}{b_1}. \tag{14}$$

thus, for any $\epsilon > 0$, there exists $T > 0$ such that

$$u(t) \leq \frac{a_1}{b_1} + \epsilon \ for\ t \geq T \tag{15}$$

From equation $(9)_2$, we have

$$\begin{cases} \frac{\partial v(t,r,\theta)}{\partial t} \leq (a_2 - \frac{c_2v(t,r,\theta)}{u(t,r,\theta)+k_2})v(t,r,\theta) \\ \frac{\partial v(.,R,\theta)}{\partial r} = 0 \\ v(0,r,\theta) = v_0(r,\theta) \leq \max\limits_{(r,\theta)\in\overline{\mathcal{D}}} v_0(r,\theta) \end{cases} \tag{16}$$

From Lemma (3.2) and the expression (15), we have

$$\frac{\partial v}{\partial t} \leq (a_2 - \frac{c_2v}{\frac{a_1}{b_1} + \epsilon + k_2})v,\ for\ t \geq T.$$

then we deduce

$$\limsup_{t \longrightarrow +\infty} v(t) \leq \frac{a_2}{\frac{c_2}{\frac{a_1}{b_1}+\epsilon+k_2}} = \frac{a_2(\frac{a_1}{b_1}+\epsilon+k_2)}{c_2}$$

by $\epsilon \longrightarrow 0$, we obtain

$$\limsup_{t \longrightarrow +\infty} v(t) \leq \frac{a_2}{b_1 c_2}(a_1 + b_1 k_2)$$

which completes the proof. □

Theorem 3.3. *Suppose $k_1a_1c_2 + a_1a_2d_2k_2 > c_1a_2k_2$, system (9) have a unique non-trivial positive steady state $E^* = (u^*, v^*)$, where u^* is given in equation (4). If $h_1h_3 < 0$ and $0 \leq k_1 \leq k_{1+}$ are satisfied, the non-trivial equilibrium point E^* is asymptotically stable.*

Where

$$k_{1+} = \frac{-h_2 + \sqrt{h_2^2 - 4h_1h_3}}{2h_1}$$

h_1, h_2 and h_3 are the coefficients of a quadratic equation in k_1.

Proof. To study the asymptotic stability of the non-trivial steady state; one need to linearize around it.

Let

$$(u(r,\theta,t), v(r,\theta,t)) = E^* + S(t,r,\theta) = E^* + (S_1(t,r,\theta), S_2(t,r,\theta))$$

By linearizing system (9) around E^*, we obtain the following the variational equation:

$$\frac{\partial S}{\partial t} = D\Delta_{r\theta}S + MS \tag{17}$$

Where $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$

and $M = G(E^*) = \begin{pmatrix} \frac{(a_1d_1-k_1b_1)u^*-2b_1d_1u^{*2}-b_1d_2u^*v^*}{d_1u^*+d_2v^*+k_1} & -\frac{u^*(k_1+d_1u^*)(a_1-b_1u^*)}{(d_1u_*+d_2v_*+k_1)v^*} \\ \frac{a_2^2}{c_2} & -a_2 \end{pmatrix}$

$$S(t,r,\theta) = \sum_{i=0}^{\infty} h_n(t)\Phi_n^{\lambda_{n,m}}(r,\theta), \tag{18}$$

$h_n(t) \in \mathbb{R}^2$, $\Phi_n^{\lambda_{nm}}(r,\theta)$ is the eigenfunction of $\Delta_{r\theta}$,With λ_{nm} are of scalar as $0 = \lambda_0 < \lambda_1 < \lambda_2 < ...$ the problem associated with the operator $-\Delta$ with Neumann boundary condition.

From equations (17) and (18), we obtain

$$\frac{dh_n(t)}{dt} = K_n h_n(t) \tag{19}$$

where $K_n = (M - \lambda_{n,m}D)$.

To show that the homogeneous equilibrium (u^*, v^*) is stable it suffices to show that each $h_n(t)$ decreases towards zero. This means that all the eigenvalues each

K_n are of negative real part. By direct calculation, we obtain

$$\begin{aligned} det(\eta I - K_n) &= (\eta + \lambda_{nm} D_1 - \frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 u^{*2} - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1}) \\ &\times (\eta + \lambda_{nm} D_2 + a_2) + (\frac{a_2^2 u^*(k_1 + d_1 u^*)(a_1 - b_1 u^*)}{(d_1 u^* + d_2 v^* + k_1)c_2 v^*}). \\ &= \eta^2 + \Omega_1 \eta + \Omega_2. \end{aligned}$$

where

$$\Omega_1 = \lambda_{nm}(D_1 + D_2) - (\frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 u^{*2} - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1}) + a_2.$$

and

$$\begin{aligned} \Omega_2 &= (\lambda_{nm} D_1 - \frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 u^{*2} - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1})(\lambda_{nm} D_2 + a_2) \\ &+ (\frac{a_2^2 u^*(k_1 + d_1 u^*)(a_1 - b_1 u^*)}{(d_1 u_* + d_2 v_* + k_1)c_2 v^*}). \end{aligned}$$

Then

$$\eta_\pm = \frac{-\Omega_1 \pm \sqrt{\Omega_1^2 - 4\Omega_2}}{2}$$

thus, $Re(\eta_\pm)$ is negative if $\Omega_1 > 0$ and $\Omega_2 > 0$.

As $\lambda_{nm} > 0$ and $\delta > 0$ and $a_2 > 0$, we need only to have

$$\frac{(a_1 d_1 - k_1 b_1)u^* - 2b_1 d_1 u^{*2} - b_1 d_2 u^* v^*}{d_1 u^* + d_2 v^* + k_1} < 0$$

which imply that

$$u^*(2b_1 d_1 + \frac{b_1 d_2 a_2}{c_2}) \geq a_1 d_1 - k_1 b_1 - \frac{b_1 d_2 a_2 k_2}{c_2}.$$

From formula of u^* given in (4), we have $\sqrt{\Delta} \geq m$ where Δ is defined by

$$\begin{aligned} \Delta &= (c_1 a_2 + b_1 c_2 k_1 + b_1 d_2 k_2 a_2 - a_1 d_1 c_2 - a_1 d_2 a_2)^2 \\ &+ 4(b_1 d_2 a_2 + d_1 b_1 c_2)(k_1 a_1 c_2 + a_1 a_2 d_2 k_2 - c_1 a_2 k_2). \end{aligned}$$

and

$$\begin{aligned} m &= c_1 a_2 + b_1 c_2 k_1 + b_1 d_2 k_2 a_2 - a_1 d_1 c_2 - a_1 d_2 a_2 \\ &+ \frac{2(b_1 d_2 a_2 + d_1 b_1 c_2)(a_1 d_1 - k_1 b_1 - \frac{b_1 d_2 a_2 k_2}{c_2})}{2b_1 d_1 + \frac{b_1 d_2 a_2}{c_2}}. \end{aligned}$$

a simple computation gives us

$$\Delta - m^2 = h_1 k_1^2 + h_2 k_1 + h_3$$

which is positive if $k_1 \leq k_{1+} = \frac{-h_2 + \sqrt{h_2^2 - 4h_1 h_3}}{2h_1}$, and we must have $k_{1+} > 0$ which is satisfied if $h_1 h_3 < 0$.

Then $Re(\eta_\pm) < 0$ for $k_1 \leq k_{1+}$, therefore we deduce the results.

Where

$$h_1 = \left(\frac{4}{(2b_1 d_1 + \frac{b_1 d_2 a_2}{c_2})}\right)(b_1^2 c_2(a_2 d_2 b_1 + d_1 b_1 c_2)) - b_1^2 \left(\frac{2(a_2 d_2 b_1 + d_1 b_1 c_2)}{2b_1 d_1 + \frac{b_1 d_2 a_2}{c_2}}\right)^2.$$

$$\begin{aligned}
h_2 &= 4(a_2d_2b_1+d_1b_1c_2)(a_1c_2) - \left(\frac{4(a_2d_2b_1+d_1b_1c_2)}{(2b_1d_1+\frac{b_1d_2a_2}{c_2})}\right)\\
&\times \left(-b_1(c_1a_2+b_1d_2k_2a_2-a_1d_1c_2-a_1d_2a_2)+b_1c_2(a_1d_1-\frac{b_1d_2a_2k_2}{c_2})\right)\\
&+ 2b_1\left(a_1d_1-\frac{b_1d_2a_2k_2}{c_2}\right)\left(\frac{2(a_2d_2b_1+d_1b_1c_2)}{2b_1d_1+\frac{b_1d_2a_2}{c_2}}\right)^2.
\end{aligned}$$

$$\begin{aligned}
h_3 &= 4(a_2d_2b_1+d_1b_1c_2)(a_1a_2d_2k_2-c_1a_2k_2) - \frac{4(a_2d_2b_1+d_1b_1c_2)}{(2b_1d_1+\frac{b_1d_2a_2}{c_2})}\\
&\times (c_1a_2+b_1d_2k_2a_2-a_1d_1c_2-a_1d_2a_2)(a_1d_1-\frac{b_1d_2a_2k_2}{c_2})\\
&- \left(\frac{2(a_2d_2b_1+d_1b_1c_2)}{2b_1d_1+\frac{b_1d_2a_2}{c_2}}\right)^2\left(a_1d_1-\frac{b_1d_2a_2k_2}{c_2}\right)^2.
\end{aligned}$$

□

4. **Global stability of the non-trivial steady state.** In this section, we study the global stability of the homogeneous non-trivial steady state $E^* = (u^*, v^*)$. For this we assume the following conditions.

Theorem 4.1. *Let*
i)

$$k_1a_1c_2 + a_1a_2d_2k_2 > c_1a_2k_2$$

ii)

$$\frac{\frac{a_2}{b_1c_2}(a_1+b_1k_2)+\epsilon}{2k_2} + \frac{k_1+d_1u^*}{2k_1} < 1$$

and

$$\frac{c_1(2d_1v^*+k_1+d_1u^*)}{2k_1} + \frac{c_1(\frac{a_2}{b_1c_2}(a_1+b_1k_2)+\epsilon)}{2k_2} < b_1(d_1u^*+d_2v^*+k_1)$$

If i) and ii) are satisfied, then the steady state E^* is globally asymptotically stable.

Proof. The proof is based on the positive definite Lyapunov function.

The hypothesis $k_1a_1c_2 + a_1a_2d_2k_2 > c_1a_2k_2$ ensures the existence of the non-trivial positive steady state E^*.

$$V(u,v) = Q_1(u,v) + Q_2(u,v)$$

where

$$Q_1(u,v) = (d_1u^*+d_2v^*+k_1)(u-u^*-u^*\ln(\frac{u}{u^*}))$$

and

$$Q_2(u,v) = \frac{c_1(u^*+k_2)}{c_2}(v-v^*-v^*\ln(\frac{v}{v^*})).$$

By Theorem (3.1), there exists $T>0$ such that

$$0 < v(t) \le \frac{a_2}{b_1c_2}(a_1+b_1k_2)+\epsilon, \quad \textit{for } t \ge T \tag{20}$$

Calculating the derivative of V along the solution of system, we have

$$\begin{aligned}
\frac{dV}{dt} &= (d_1u^* + d_2v^* + k_1)(u - u^*)\left(a_1 - b_1u - \frac{c_1v}{d_1u + d_2v + k_1}\right)\\
&+ \frac{c_1(u^* + k_2)}{c_2}(v - v^*)\left(a_2 - \frac{c_2v}{u + k_2}\right)\\
&= \left(-b_1(u - u^*) + \frac{c_1v^*}{d_1u^* + d_2v^* + k_1} - \frac{c_1v}{d_1u + d_2v + k_1}\right)\\
&\times (d_1u^* + d_2v^* + k_1)(u - u^*) + \frac{c_1(u^* + k_2)}{c_2}(v - v^*)\left(\frac{c_2v^*}{u^* + k_2} - \frac{c_2v}{u + k_2}\right)\\
&= \left(\frac{c_1d_1v^*}{d_1u + d_2v + k_1} - b_1(d_1u^* + d_2v^* + k_1)\right)(u - u^*)^2 - c_1(v - v^*)^2\\
&+ \left(\frac{c_1v}{u + k_2} - \frac{c_1(d_1u^* + k_1)}{d_1u + d_2v + k_1}\right)(u - u^*)(v - v^*)\\
&\leq \left(\frac{c_1d_1v^*}{d_1u + d_2v + k_1} - b_1(d_1u^* + d_2v^* + k_1)\right)(u - u^*)^2 - c_1(v - v^*)^2\\
&+ \left(\frac{c_1v}{u + k_2} + \frac{c_1(d_1u^* + k_1)}{d_1u + d_2v + k_1}\right)\frac{(u - u^*)^2 + (v - v^*)^2}{2}\\
&\leq \left(\frac{c_1d_1v^*}{k_1} - b_1(d_1u^* + d_2v^* + k_1)\right)(u - u^*)^2 - c_1(v - v^*)^2\\
&+ \left(\frac{c_1v}{k_2} + \frac{c_1(d_1u^* + k_1)}{k_1}\right)\frac{(u - u^*)^2 + (v - v^*)^2}{2}\\
&= \left(\frac{c_1d_1v^*}{k_1} - b_1(d_1u^* + d_2v^* + k_1) + \frac{c_1v}{2k_2} + \frac{c_1(d_1u^* + k_1)}{2k_1})\right)(u - u^*)^2\\
&+ \left(-c_1 + \frac{c_1v}{2k_2} + \frac{c_1(d_1u^* + k_1)}{2k_1}\right)(v - v^*)^2\\
&= \left(-b_1(d_1u^* + d_2v^* + k_1) + \frac{c_1v}{2k_2} + \frac{c_1(d_1u^* + k_1) + 2c_1d_1v^*}{2k_1})\right)(u - u^*)^2\\
&+ c_1\left(-1 + \frac{v}{2k_2} + \frac{(d_1u^* + k_1)}{2k_1}\right)(v - v^*)^2\\
&\leq \left(-b_1(d_1u^* + d_2v^* + k_1) + \frac{c_1(\frac{a_2}{b_1c_2}(a_1 + b_1k_2) + \epsilon)}{2k_2} + \frac{c_1(\alpha u^* + k_1) + 2c_1d_1v^*}{2k_1})\right)\\
&\times (u - u^*)^2 + c_1\left(-1 + \frac{(\frac{a_2}{b_1c_2}(a_1 + b_1k_2) + \epsilon)}{2k_2} + \frac{(d_1u^* + k_1)}{2k_1}\right)(v - v^*)^2.
\end{aligned}$$

According to the hypothesis of Theorem (4.1), we have $\frac{dV}{dt} \leq 0$ and we deduce the result. □

5. **Numerical analysis.** In this section, via Matlab software we simulate the distribution of the two populations prey and predator with respect to time and their densities. For that we consider the disc domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 400\}$ and the boundary conditions are of the Neumann type and the initial conditions are a small perturbations in the vicinity of equilibrium point (u^*, v^*). These initial conditions have been chosen as follows:

$$u(0, r, \theta) = u^*((rcos\theta)^2 + (rsin\theta)^2) < 400 \tag{21}$$

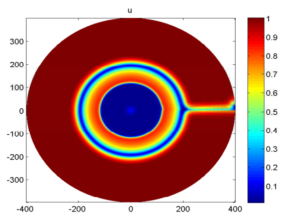
$$v(0, r, \theta) = v^*((rcos\theta)^2 + (rsin\theta)^2) < 400 \tag{22}$$

In the next we consider the following parameters values:

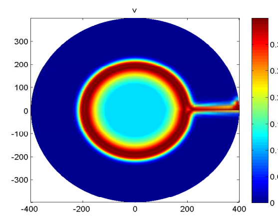
$$a_1 = 1,\ a_2 = 0.02,\ b_1 = 1,\ k_1 = 0.2,\ k_2 = 0.1,\ d_1 = 0.9,\ d_2 = 0.1,\ c_1 = 1.1,$$
$$c_2 = 0.02,\ D_1 = 1,\ D_2 = 1. \tag{23}$$

Biological systems are in general far-from-equilibrium systems. The self-organization phenomena can effectuate via symmetry-breaking instabilities. Also, structures such as target patterns and spiral waves are considered in a large set of chemical and biological system.

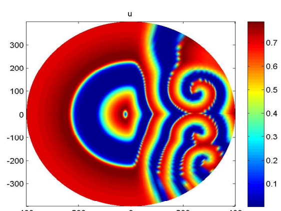
We suppose that the two species diffuse in the same way, (i.e. $D_1 = D_2$). In Fig.(1), we observe the spatial distribution of spiral waves types for system (1).



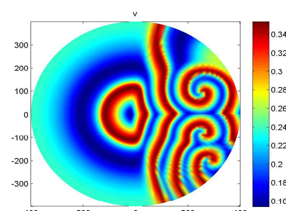
Spatial distribution of prey at $t = 100$



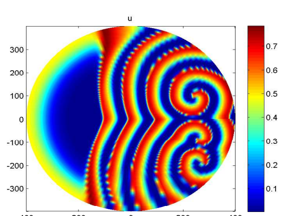
Spatial distribution of predator at $t = 100$



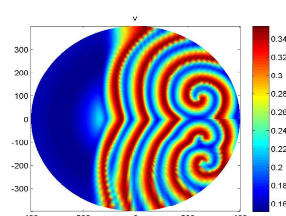
Spatial distribution of prey at $t = 2400$



Spatial distribution of predator at $t = 2400$



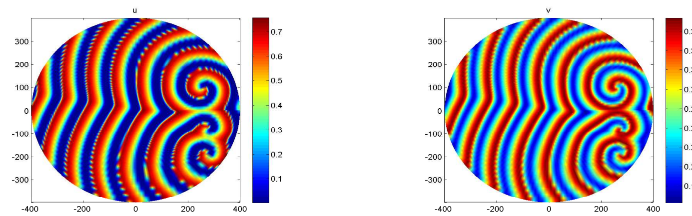
Spatial distribution of prey at $t = 3500$



Spatial distribution of predator at $t = 3500$

In what follows, we plot for different values of a_2 the curves of densities of prey and predator with respect to time. Let us consider the following parameters values:

$$a_1 = 1,\ b_1 = 1,\ k_1 = 0.2,\ k_2 = 0.1,\ d_1 = 0.9,\ d_2 = 0.1,\ c_1 = 1.1,\ c_2 = 0.02,$$



Spatial distribution of prey at $t = 6000$

Spatial distribution of predator at $t = 6000$

Figure 1: Spatial distribution of species for $D_1 = D_2$ and time varying, the other parameters are given in (23).

$$D_1 = 1,\ D_2 = 1,\ t = 500. \tag{24}$$

and the initial conditions are given by the expressions (21) and (22). Firstly from the

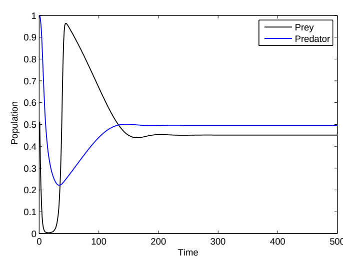


Figure 2: Stable behavior of prey-predator populations with time for control parameter $a_2 = 0.018$, the other parameters are given in (24).

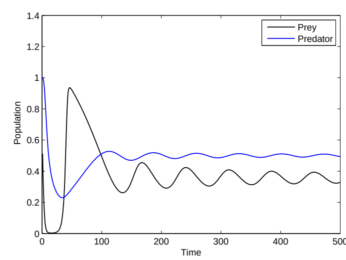


Figure 3: Solution curves of prey-predator populations with the time for control parameter $a_2 = 0.022$, the other parameters are given in (24).

Figure (2), we observe that populations of prey-predator converge to their steady states with the passage of time and $E^* = (u^*, v^*)$ is locally asymptotically stable for system (9). If we increase the value of the control parameter $a_2 = 0.022$ then Figure (3) show that equilibrium $E^* = (u^*, v^*)$ loses its stability and becomes unstable.

For better studying the properties of the population dynamics as a whole, we estimate the species size of prey and predator by,

$$U(t) = \int_0^R \int_0^{2\pi} u(t, r, \theta) dr d\theta \qquad \text{and} \qquad V(t) = \int_0^R \int_0^{2\pi} v(t, r, \theta) dr d\theta \tag{25}$$

Then we plot the oscillations of the solutions when the parameter a_2 varies. Taking the following parameters values:

$$a_1 = 1,\ b_1 = 0.1,\ k_1 = 0.2,\ k_2 = 0.1,\ d_1 = 0.9,\ d_2 = 0.1,\ c_1 = 1.1,\ c_2 = 0.02,$$
$$D_1 = 1,\ D_2 = 1,\ t = 2200. \tag{26}$$

and the initial conditions are given as in (21) and (22) and we let a transition time fairly large so that the quantities of the species U and V fall within the domain of attraction.

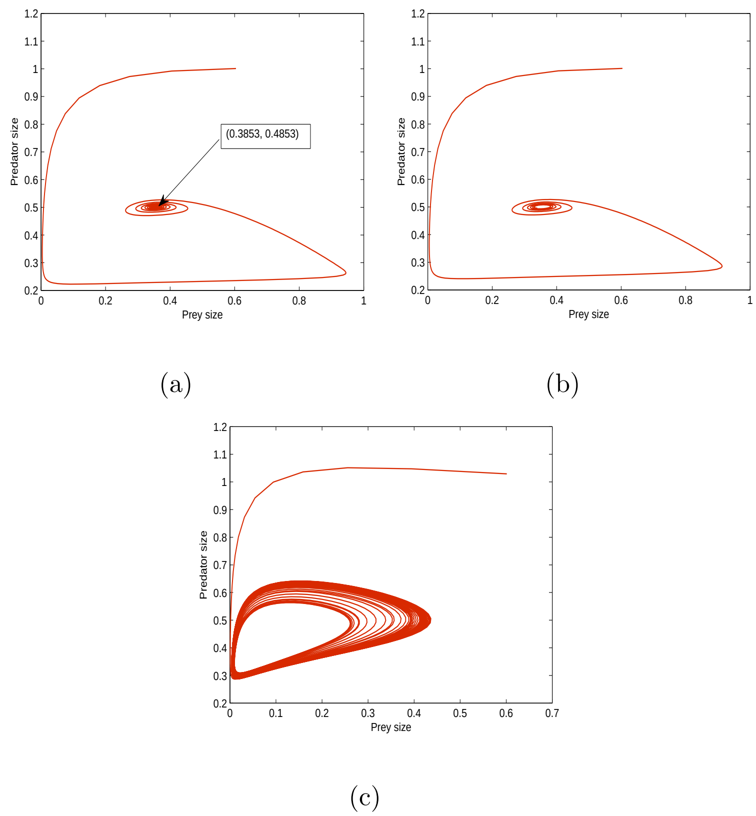


Figure 4: Phase plane trajectories of prey-predator populations, for system (9), for different values of a_2: (a) $a_2 = 0.022$, (b) $a_2 = 0.0221$, (c) $a_2 = 0.05$ the other parameters are given in (26).

6. **Conclusion.** This paper presents the spatio-temporal dynamics of a predator-prey model with Beddington-DeAngelis functional response with diffusion on a circular domain. We have proved the existence of positive equilibriums and their local stability. By constructing a Lyapunov function, we obtain a sufficient conditions for global stability of the positive equilibrium. By numerical simulation, we obtain a spatial distribution of spiral waves types (see Fig.(1)). Also, we plot the behavior of solutions when the parameter a_2 varies.

Fig .(4)(a) shows that the system presents an attractor focus in (U, V) for $a_2 = 0.022$, if we increase the value of a_2 the equilibrium $E^* = (u^*, v^*)$ loses its stability (see Fig.(4)(b)) and becomes unstable (see Fig (4)(c)) and oscillating solutions appear. Our aim in the next works is to study the same model in terms of Turing instability and the optimal harvest policy.

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E-mail address: abidwalid007@yahoo.fr
E-mail address: yafia1@yahoo.fr
E-mail address: aziz.alaoui@univ-lehavre.fr
E-mail address: HabibBouhafa@yahoo.fr
E-mail address: azgal.abichou@ept.rnu.tn