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Instability and Pattern Formation in Three-Species Food Chain Model via Holling Type II Functional Response on a Circular Domain

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This paper is devoted to the study of food chain predator–prey model. This model is given by a reaction–diffusion system defined on a circular spatial domain, which includes three-state variables namely, prey and intermediate predator and top predator and incorporates the Holling type II and a modified Leslie–Gower functional response. The aim of this paper is to investigate theoretically and numerically the asymptotic behavior of the interior equilibrium of the model. The local and global stabilities of the positive steady-state solution and the conditions that enable the occurrence of Hopf bifurcation and Turing instability in the circular spatial domain are proved. In the end, we carry out numerical simulations to illustrate how biological processes can affect spatiotemporal pattern formation in a disc spatial domain and different types of spatial patterns with respect to different time steps and diffusion coefficients are obtained.

Keywords: Predator–prey model; local and global stability; Hopf bifurcation; diffusion driven instability; pattern formation; chaos; circular domain.

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1. Introduction

In the last decades, the dynamical problems of preys and predators associated with mathematical modeling have become an important area of research in ecology. One of the oldest and well known mathematical model which describes the interaction between two species of predator and prey was introduced by Lotka [1925] and Volterra [1927], this model is well known as Lotka–Volterra mathematical model. It consists of two differential equations with a simple correspondence between prey consumption and predator production. The link between the dynamics of the two species in predator–prey models is based on the trophic function, which describes the number of prey consumed per predator per unit time for given number of preys and predators.

The mathematical modeling of species interactions has been extensively investigated for two species in food chains, based on systems with a modified version of the Leslie–Gower scheme. The necessary and sufficient conditions for diffusion driven instability which leads to the formation of spatial patterns, have been derived and very interesting patterns have also been observed from the numerical simulations see [Nindjin *et al.*, 2006; Nindjin & Aziz-Alaoui, 2008; Letellier & Aziz-Alaoui, 2002; Jia & Jiang, 2011; Aziz-Alaoui & Daher, 2003; Yafia *et al.*, 2008; Wang & Wu, 2008; Letellier *et al.*, 2002] and references therein.

In [Nindjin *et al.*, 2006; Camara & Aziz-Alaoui, 2008], the authors studied a prey–predator model given by a reaction–diffusion system incorporating Holling type II and a modified Leslie–Gower functional response defined in a square domain. They have demonstrated the qualitative analysis in terms of local and global stability, bifurcations and patterns formation.

In [Abid *et al.*, 2014], the authors considered the same model defined on a circular domain which is given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{r\theta}u + u(1-u) - \dfrac{av}{u+e_1}u = \Delta_{r\theta}u + f(u,v) & (r,\theta) \in \Omega, \;\; t > 0 \\ \dfrac{\partial v}{\partial t} = \delta\Delta_{r\theta}v + b\left(1 - \dfrac{v}{u+e_2}\right)v = \delta_{r\theta}\Delta v + g(u,v) & (r,\theta) \in \Omega, \;\; t > 0 \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0 \end{cases} \tag{1}$$

where $f(u,v)$ and $g(u,v)$ model the local activity (absence of diffusion), with Ω being a disk domain and $\Delta_{r\theta}$ the Laplacian operator in polar coordinates. The model parameters a, b, e_1 and e_2 are assumed to be positive values. These parameters are defined as follows: a is the maximum value of the per capita reduction of u due to v, b describes the growth rate of predators v, e_1 measures the extent to which the environment provides protection to prey u, e_2 has a similar meaning to e_1 relatively to the predator v and δ is the diffusion coefficient of predator. $\frac{\partial u}{\partial \eta}$ and $\frac{\partial v}{\partial \eta}$ are respectively the normal derivatives of u and v on $\partial\Omega$. The local and global stabilities, the conditions for Hopf and Turing bifurcation in the spatial domain and the patterns formation are studied.

Recently many researchers have studied the formation of patterns for different three-species interacting discrete or continuous systems and most of the authors have considered a food chain model with diffusion and investigated the diffusion driven instability in the spatial system defined on a rectangular domain, see [Hong & Murray, 2003; Maionchi *et al.*, 2006; Wu *et al.*, 2010; Zhao & Lv, 2009; Shen & You, 2010; Baghel & Dhar, 2012a, 2012b; Araujo & de Aguiar, 2007; Wang, 2004].

In this paper, we consider a reaction–diffusion model with three species, prey, intermediate predator and top-predator. One of the well known methods in biology or ecology which plays a crucial role in regulating the balance of the ecosystem and also control the dynamics of species, is the introduction of a further population called "top predator". However, the impact of this introduction should be previously studied in order to minimize adverse effects. Mathematical modeling provides a reasonable solution at this step. The first species denoted U is the only food source of the second V, and the intermediate predator V is the only prey of a top-predator W. Local interactions between species U and V are modeled by the Lotka–Volterra type scheme (the predator population dies out exponentially in the absence of its prey), the interaction between species W and its prey V has been modeled by the Leslie–Gower scheme [Leslie & Gower, 1960; Leslie, 1948] (the loss in predator population is proportional to the reciprocal of per capita availability of its most favorite food). While the model studied here is mainly based on a modified version of Leslie–Gower regime. The diffusion term describes the ability to move in a domain of $\mathbb{R}^2$ and the model is given by a system of three differential equations with diffusion as follows:

$$\begin{cases} \dfrac{\partial U(T,x,y)}{\partial T} = D_1 \Delta U(T,x,y) + \left(a_0 - b_0 U(T,x,y) - \dfrac{v_0 V(T,x,y)}{U(T,x,y)+d_0}\right) U(T,x,y), \\ \dfrac{\partial V(T,x,y)}{\partial T} = D_2 \Delta V(T,x,y) + \left(-a_1 + \dfrac{v_1 U(T,x,y)}{U(T,x,y)+d_0} - \dfrac{v_2 W(T,x,y)}{V(T,x,y)+d_2}\right) V(T,x,y), \\ \dfrac{\partial W(T,x,y)}{\partial T} = D_3 \Delta W(T,x,y) + \left(c_3 - \dfrac{v_3 W(T,x,y)}{V(T,x,y)+d_3}\right) W(T,x,y), \\ \dfrac{\partial U}{\partial n} = \dfrac{\partial V}{\partial n} = \dfrac{\partial W}{\partial n} = 0, \\ U(0,x,y) = U_0(x,y) \geq 0, \quad V(0,x,y) = V_0(x,y) \geq 0, \quad W(0,x,y) = W_0(x,y) \geq 0, \end{cases} \tag{2}$$

$U(T,x,y)$ is the density of prey species, $V(T,x,y)$ the density of intermediate predator species and $W(T,x,y)$ the density of top-predator species at time T and position (x,y) defined on a circular domain (or disc domain) with radius R (i.e. $\Omega = \{(x,y) \in \mathbf{R}^2/x^2+y^2 < R^2\}$). Δ is the Laplacian operator. $\frac{\partial U}{\partial \eta}$, $\frac{\partial V}{\partial \eta}$ and $\frac{\partial W}{\partial \eta}$ are respectively the normal derivatives of U, V and W on $\partial\Omega$. The three species are assumed to diffuse at rates D_i $(i = 1,2,3)$. a_0, b_0, v_0, d_0, a_1, v_1, v_2, d_2, c_3, v_3 and d_3 are assumed to be positives and are defined as follows: a_0 is the rate of growth of the prey U, b_0 measure mortality due to competition between individuals of the species U, v_0 is the maximum extent that the rate of reduction by individual U can reach, d_0 measures protection that prey U and intermediate predator V benefit through the environment, a_1 represents the mortality rate V in the absence of U, v_1 is the maximum value that the rate of reduction by the individual U can reach, v_2 is the maximum value that the rate of reduction by the individual V can reach, v_3 is the maximum value that the rate of reduction by the individual W can reach, d_2 is the value of V for which the rate of elimination by individual V becomes $\frac{v_2}{2}$, c_3 describes the growth rate of W, assuming that there are the same number of males and females and d_3 represents the residual loss caused by high scarcity of prey V of the species W.

The initial data $U_0(x,y)$, $V_0(x,y)$ and $W_0(x,y)$ are non-negative continuous functions on Ω. The vector η is an outward unit normal vector to the smooth boundary $\partial\Omega$. The homogeneous Neumann boundary condition signifies that the system is self-contained and there is no population flux across the boundary $\partial\Omega$.

The first model proposed in this topic is given by ordinary differential equations [Aziz-Alaoui, 2002] and read as:

$$\begin{cases} \dfrac{\partial U}{\partial T} = a_0 U - b_0 U^2 - \dfrac{v_0 V U}{U+d_0}, \\ \dfrac{\partial V}{\partial T} = -a_1 V + \dfrac{v_1 U V}{U+d_1} - \dfrac{v_2 W V}{V+d_2}, \\ \dfrac{\partial W}{\partial T} = c_3 W - \dfrac{v_3 W^2}{V+d_3}, \end{cases} \tag{3}$$

where U, V and W represent the population densities at time T. a_0, b_0, v_0, d_0, a_1, v_1, d_1, v_2, d_2, c_3, v_3 and d_3 are assumed to be positive. d_0 measures the extent to which the environment provides protection to prey U, d_1 has a similar meaning as d_0, d_2 is the value of V at which the per capita removal rate of V becomes $\frac{v_2}{2}$, d_3 represents the residual loss in species W due to severe scarcity of its favorite food V; the second term on the right-hand side in the third equation of (3) depicts the loss in the top-predator population. Other parameters are defined as parameters of system (2). The boundedness, existence of an attracting set, local and global stability of equilibria are proved.

The delayed model of (3) (see [Nindjin & Aziz-Alaoui, 2008]) is given by a system of three-delayed differential equations:

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = u(t)\left(a_1 - b_1 u(t-r_1) - \dfrac{v_0 v(t)}{u(t)+d_0}\right) \\ \dfrac{\partial v(t)}{\partial t} = v(t)\left(-a_2 + \dfrac{v_1 u(t-r_{12})}{u(t-r_{12})+d_0}\right. \\ \qquad\qquad \left. - b_2 v(t-r_2) - \dfrac{v_2 w(t)}{v(t)+d_2}\right) \\ \dfrac{\partial w(t)}{\partial t} = w(t)\left(a_3 - \dfrac{v_3 w(t-r_3)}{v(t-r_{23})+d_3}\right). \end{cases} \tag{4}$$

In system (4), r_1, r_2, r_3, r_{12} and r_{23} are non-negative constant. r_1 denotes the delay in the negative feedback. The authors assume that the prey growth rate response to resources limitations involves delay r_{12}, due to gestation of intermediate predator V, that is, delay in time for prey biomass increasing predator numbers. r_{23} can be regarded as a gestation period. Furthermore, they assume that the top predator growth rate response to resources limitations involves also a delay, so, r_3 has the same meaning as r_1. In addition, they have included the term $-b_2V(t-r_2)$ in the dynamics of predator V, to incorporate the negative feedback of intermediate predator crowding. The global stability and persistence properties are studied by using Lyapunov functional.

In [Zhou *et al.*, 2009], the authors considered a three-dimensional eco-epidemiological model with delay. The stability of the equilibria, existence of Hopf bifurcation and permanence are investigated.

Our goal, in this paper is to generalize the results presented in (see [Nindjin *et al.*, 2006; Nindjin & Aziz-Alaoui, 2008]) to a reaction–diffusion system defined on a circular domain and those presented in [Abid *et al.*, 2014] for two species. We study the local/global stability and the occurrence of Turing instability. In the end, we give some numerical simulations illustrating our results.

The current work is organized as follows. In Sec. 2, we give the spatiotemporal mathematical model. Section 3 presents the local/global stability analysis and Hopf bifurcation for the temporal system. In Sec. 4, we derive the analytical conditions for diffusion driven instability and we perform the conditions of pattern formation. In Sec. 5, we illustrate our results by numerical simulations. In the last section, a conclusion is given.

2. Mathematical Model and Preliminaries

In this section, we present some preliminary results on the boundedness of solution for system (2) on the disc domain.

Firstly, we reconsider system (2) defined on the circular domain Ω, then we can write x and y in polar coordinates as follows $x = r\cos\theta$ and $y = r\sin\theta$. By applying the polar coordinate transformation, we find $\Gamma = \{(r,\theta) : 0 < r < R, 0 \le \theta < 2\pi\}$. R is the radius of the disk Γ, with $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$. Without loss of generalities we denote also

$$u(t,x,y) = u(t, r\cos(\theta), r\sin(\theta)) = u(t,r,\theta),$$
$$v(t,x,y) = v(t, r\cos(\theta), r\sin(\theta)) = v(t,r,\theta) \quad \text{and}$$
$$w(t,x,y) = w(t, r\cos(\theta), r\sin(\theta)) = w(t,r,\theta)$$

are the densities of prey, predators and top predators respectively in polar coordinates.

Therefore, the Laplacian operator in polar coordinates is given by:

$$\Delta_{r\theta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{5}$$

To simplify system (2) we introduce some transformations of variables:

$$U = \frac{a_0}{b_0}u, \quad V = \frac{a_0^2}{b_0 v_0}v, \quad W = \frac{a_0^3}{b_0 v_0 v_2}w,$$
$$T = \frac{t}{a_0}, \quad r = \frac{r'}{a_0}, \quad \theta = \theta'$$

and

$$a = \frac{b_0 d_0}{a_0}, \quad b = \frac{a_1}{a_0}, \quad c = \frac{v_1}{a_0}, \quad d = \frac{d_2 v_0 b_0}{a_0^2},$$
$$p = \frac{c_3 a_0^2}{v_0 b_0 v_2}, \quad q = \frac{v_3}{v_2}, \quad s = \frac{d_3 v_0 b_0}{a_0^2},$$
$$\delta_1 = \frac{D_1}{a_0}, \quad \delta_2 = \frac{D_2}{a_0}, \quad \delta_3 = \frac{D_3}{a_0}.$$

Then the spatiotemporal system (2) in polar coordinates is written as follows:

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = \delta_1 \Delta_{r\theta} u(t,r,\theta) + f(u(t,r,\theta), v(t,r,\theta), w(t,r,\theta)) & \forall (r,\theta) \in \Gamma, \quad t > 0 \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = \delta_2 \Delta_{r\theta} v(t,r,\theta) + g(u(t,r,\theta), v(t,r,\theta), w(t,r,\theta)) & \forall (r,\theta) \in \Gamma, \quad t > 0 \\ \dfrac{\partial w(t,r,\theta)}{\partial t} = \delta_3 \Delta_{r\theta} w(t,r,\theta) + h(u(t,r,\theta), v(t,r,\theta), w(t,r,\theta)) & \forall (r,\theta) \in \Gamma, \quad t > 0 \\ \dfrac{\partial u(t,r,\theta)}{\partial n} = \dfrac{\partial v(t,r,\theta)}{\partial n} = \dfrac{\partial w(t,r,\theta)}{\partial n} = 0, & \forall (r,\theta) \in \partial\Gamma \\ u(0,r,\theta) = u_0(r,\theta) \geq 0, \quad v(0,r,\theta) = v_0(r,\theta) \geq 0, \quad w(0,r,\theta) = w_0(r,\theta) \geq 0 \end{cases} \tag{6}$$

where

$$\begin{cases} f(u(t,r,\theta), v(t,r,\theta), w(t,r,\theta)) = \left(1 - u(t,r,\theta) - \dfrac{v(t,r,\theta)}{u(t,r,\theta) + a}\right) u(t,r,\theta), \\ g(u(t,r,\theta), v(t,r,\theta), w(t,r,\theta)) = \left(-b + \dfrac{cu(t,r,\theta)}{u(t,r,\theta) + a} - \dfrac{w(t,r,\theta)}{v(t,r,\theta) + d}\right) v(t,r,\theta), \\ h(u(t,r,\theta), v(t,r,\theta), w(t,r,\theta)) = \left(p - \dfrac{qw(t,r,\theta)}{v(t,r,\theta) + s}\right) w(t,r,\theta). \end{cases} \tag{7}$$

Without diffusion, system (6) becomes

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = \left(1 - u(t,r,\theta) - \dfrac{v(t,r,\theta)}{u(t,r,\theta) + a}\right) u(t,r,\theta), \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = \left(-b + \dfrac{cu(t,r,\theta)}{u(t,r,\theta) + a} - \dfrac{w(t,r,\theta)}{v(t,r,\theta) + d}\right) v(t,r,\theta), \\ \dfrac{\partial w(t,r,\theta)}{\partial t} = \left(p - \dfrac{qw(t,r,\theta)}{v(t,r,\theta) + s}\right) w(t,r,\theta). \end{cases} \tag{8}$$

A steady state (u_e, v_e, w_e) of (8) is an equilibrium point of (6) which is a solution of the following system

$$\begin{cases} \delta_1 \Delta_{r\theta} u_e(t,r,\theta) + f(u_e(t,r,\theta), v_e(t,r,\theta), w_e(t,r,\theta)) = 0, \\ \delta_2 \Delta_{r\theta} v_e(t,r,\theta) + g(u_e(t,r,\theta), v_e(t,r,\theta), w_e(t,r,\theta)) = 0, \\ \delta_3 \Delta_{r\theta} w_e(t,r,\theta) + h(u_e(t,r,\theta), v_e(t,r,\theta), w_e(t,r,\theta)) = 0. \end{cases} \tag{9}$$

Then, we denote $E = (u, v, w)^T$ and

$$L(E) = \begin{pmatrix} f(u,v,w) \\ g(u,v,w) \\ h(u,v,w) \end{pmatrix} = \begin{pmatrix} \left(1 - u - \dfrac{v}{u+a}\right) u \\ \left(-b + \dfrac{cu}{u+a} - \dfrac{w}{v+d}\right) v \\ \left(p - \dfrac{qw}{v+s}\right) w \end{pmatrix}.$$

Then, problem (8) can be written as:

$$\frac{dE}{dt} = L(E). \quad (10)$$

It is obvious that, problem (10) has a positive steady state if and only

$$qc > bq + p \quad \text{and} \quad qc - bq - p > a(bq + p). \quad (11)$$

The positive steady state is uniquely given by

$$u^* = \frac{a(bq + p)}{qc - bq - p}, \quad v^* = (1 - u^*)(u^* + a) \quad \text{and} \quad w^* = \frac{p(v^* + s)}{q}. \quad (12)$$

The conditions (11) ensure that the system (6) has a positive equilibrium point corresponding to constant coexistence of the three species. We need to know the flow on the boundaries of $\mathbb{R}^3_+$, system (10) has four trivial boundary equilibria $E_0 = (0, 0, 0)$, $E_1 = (1, 0, 0)$, $E_2 = (0, 0, \frac{sp}{q})$ and $E_3 = (1, 0, \frac{sp}{q})$ and the nontrivial one is $E^* = (u^*, v^*, w^*)$.

Remark 2.1. Biologically, if $u = 0$ or $v = 0$ (i.e. when one of the populations of prey or predator dies), the introduction of top predators is not necessary. Using only the first two equations of system (10) and removing the last term of the second member of the second equation of (10), this system is written on the uv-plane as follows:

$$\begin{cases} \dfrac{du}{dt} = \left(1 - u - \dfrac{v}{u + a}\right) u, \\ \dfrac{dv}{dt} = \left(-b + \dfrac{cu}{u + a}\right) v. \end{cases} \quad (13)$$

The behaviors of solutions of system (13) are studied in [Aziz-Alaoui & Daher, 2003]. The system has two equilibria on the boundaries of $\mathbb{R}^2_+$, $E_{00} = (0, 0)$, $E_{11} = (1, 0)$. Obviously, these points are restrictions of E_0, E_1 in the uv-plane. The existence of an interior equilibrium of (13) in the positive first quadrant Int R^+_{uv} is proved and the local stability of equilibria of (13) is determined by computing the eigenvalues of the Jacobian matrix about each equilibrium (see [Aziz-Alaoui & Daher, 2003; Camara & Aziz-Alaoui, 2008])

$$J(E_{00}) = \begin{pmatrix} 1 & 0 \\ 0 & -b \end{pmatrix}, \quad J(E_{11}) = \begin{pmatrix} -1 & \dfrac{-1}{1 + a} \\ 0 & \dfrac{c - b - ab}{1 + a} \end{pmatrix}.$$

It is easy to verify that $E_{00} = (0, 0)$ is unstable if $b > 0$ and it is a hyperbolic saddle point which attracts in the v-direction and repels in the u-direction. For $E_{11} = (1, 0)$, the eigenvalues of the Jacobian matrix $J(E_{11})$ are

$$-1, \quad \frac{c - b - ab}{1 + a}. \quad (14)$$

If $c - b > ab$, then $E_{11} = (1, 0)$ is also a hyperbolic saddle point. If $c - b < ab$, then both eigenvalues are negative and E_{11} is locally asymptotically stable and if $c < b$, E_{11} is globally asymptotically stable (see [Aziz-Alaoui, 2002]). If $c - b > ab$, the corresponding instantaneous system of (10) is uniformly persistent which implies that system (10) must have a positive equilibrium.

Using the second and the third equations of system (10) and removing the second term of the second equation of (10), this system becomes restricted to the vw-plane R^+_{vw}:

$$\begin{cases} \dfrac{dv}{dt} = \left(-b - \dfrac{w}{v + d}\right) v, \\ \dfrac{dw}{dt} = \left(p - \dfrac{qw}{v + s}\right) w. \end{cases} \quad (15)$$

We can also compute explicitly $E_{22} = (0, \frac{sp}{q})$ is the restriction of the boundary equilibrium $E_2 = (0, 0, \frac{sp}{q})$ of system (10) in the vw-plan and the Jacobian matrix is

$$J(E_{22}) = \begin{pmatrix} -b - \dfrac{sp}{qd} & 0 \\ \dfrac{s^2p^2}{qa} & -\dfrac{2sp}{d} \end{pmatrix}.$$

As $\frac{2sp}{d} > 0$ and $b + \frac{sp}{qd} > 0$, the two eigenvalues are negative and E_{22} has a stable manifold of at least two dimensions.

For the equilibrium E_{33}, using the first and the third equations of system (10) and removing the last term of the second member of the second equation of (10) and the term v in the second member of the third equation of (10), this system becomes restricted to the uw-plane R^+_{uw} as follows:

$$\begin{cases} \dfrac{du}{dt} = (1 - u)u, \\ \dfrac{dw}{dt} = \left(p - \dfrac{qw}{s}\right) w. \end{cases} \quad (16)$$

We can also see that $E_{33} = (1, \frac{sp}{q})$ is the restriction of the boundary equilibrium $E_3 = (1, 0, \frac{sp}{q})$ of system (10) in the uw-plan and the Jacobian matrix is

$$J(E_{33}) = \begin{pmatrix} -1 & 0 \\ 0 & -\dfrac{2sp}{d} \end{pmatrix}.$$

As $\frac{2sp}{d} > 0$, the eigenvalues are negative and E_{33} has a stable manifold of at least two dimensions.

Next from the standpoint of biology, we are only interested in the dynamics of model (10) in the closed first quadrant $\mathbb{R}^3_+$. We denote $\mathbb{R}^3_+ = \{(u, v, w) \in \mathbb{R}^3, u_0 \geq 0, v_0 \geq 0, w_0 \geq 0\}$. We will investigate the asymptotic behavior of orbits starting in the positive cone $\operatorname{int}\mathbb{R}^3_+ = \{(u, v, w) \in \mathbb{R}^3, u_0 > 0, v_0 > 0, w_0 > 0\}$.

Throughout this paper, by saying that $E = (u, v, w)$ is positive, we mean that $u > 0, v > 0$ and $w > 0$. Concerning the boundedness of the solution for the model system (10), we state the following theorem:

Theorem 1. *If the condition (11) is satisfied, the assembly defined by*

$$\Theta \equiv [0, 1] \times [0, 1 + a] \times \left[0, \frac{p}{q}(1 + a + s)\right] \qquad (17)$$

(i) *is positively invariant region.*
(ii) *All solutions of (10) initiating in Θ are ultimately bounded with respect to $\mathbb{R}^3_+$ and eventually enter the attracting set Θ.*

Proof. From Eq. $(6)_1$, we have

$$\begin{cases} \dfrac{\partial u(t, r, \theta)}{\partial t} = \delta_1 \Delta_{r\theta} u(t, r, \theta) \\ \qquad\qquad + (1 - u(t, r, \theta)u(t, r, \theta)), \\ \dfrac{\partial u(0, r, \theta)}{\partial n} = 0, \\ u(0, r, \theta) = u_0(r, \theta) \leq u_{01} = \max\limits_{(r,\theta)\in\overline{\Gamma}} u_0(r, \theta). \end{cases} \qquad (18)$$

By the comparison principle, we have $u(t, r, \theta) \leq u_1 \leq 1$ with $u_1(t) = \frac{u_{01}}{u_{01}+(1-u_{01})e^{-t}}$ being a solution of the following ODE:

$$\begin{cases} \dfrac{du(t, r, \theta)}{dt} = (1 - u_1)u_1, \\ u_1(0) = u_{01} \leq 1. \end{cases} \qquad (19)$$

Then

$$\limsup_{t\to+\infty} u_1(t) = 1, \qquad (20)$$

from Eq. $(6)_2$, we have

$$\begin{cases} \dfrac{\partial v(t, r, \theta)}{\partial t} = \delta_2 \Delta_{r\theta} v + \left(-b + \dfrac{cu}{u + a} - \dfrac{w}{v + d}\right) v \\ \qquad\qquad \leq \delta_2 \Delta_{r\theta} v + \left(\dfrac{c}{1 + a} - b\right) v, \\ \dfrac{\partial v(0, r, \theta)}{\partial n} = 0, \\ v(0, r, \theta) = v_0(r, \theta) \leq v_{01} = \max\limits_{(r,\theta)\in\overline{\Gamma}} v_0(r, \theta). \end{cases} \qquad (21)$$

By the comparison principle, we have $v(t, r, \theta) \leq v_1 \leq 1$ with $v_1(t) = \frac{(1+a)v_{01}}{v_{01}+e^{-(c-b(1+a))t}}$ being a solution of the following ODE:

$$\begin{cases} \dfrac{dv(t, r, \theta)}{dt} = \left(\dfrac{c}{1 + a} - b\right) v_1, \\ v_1(0) = v_{01} \leq 1. \end{cases} \qquad (22)$$

Then

$$\limsup_{t\to+\infty} v_1(t) = 1 + a. \qquad (23)$$

From Eq. $(6)_3$, we have

$$\begin{cases} \dfrac{\partial w(t, r, \theta)}{\partial t} = \delta_3 \Delta_{r\theta} w + \left(p - \dfrac{qw}{v + s}\right) w \\ \qquad\qquad \leq \delta_3 \Delta_{r\theta} w + p\left(1 - \dfrac{w}{\dfrac{p}{q}(1 + a + s)}\right) w, \\ \dfrac{\partial w(0, r, \theta)}{\partial n} = 0, \\ w(0, r, \theta) = w_0(r, \theta) \leq w_{01} = \max\limits_{(r,\theta)\in\overline{\Gamma}} w_0(r, \theta). \end{cases} \qquad (24)$$

By the comparison principle, we have $w(t, r, \theta) \leq w_1 \leq 1$ with $w_1(t) = \frac{\frac{p}{q}(1+a+s)w_{01}}{w_{01}+(\frac{p}{q}(1+a+s)-w_{01})e^{-pt}}$ being a solution of the ODE:

$$\begin{cases} \dfrac{dw(t,r,\theta)}{dt} = p\left(1 - \dfrac{w}{\dfrac{p}{q}(1+a+s)}\right)w, \\ w_1(0) = w_{01} \leq 1. \end{cases} \quad (25)$$

Then

$$\limsup_{t\to+\infty} w_1(t) = \frac{p}{q}(1+a+s). \quad (26)$$

■

3. Analysis of Temporal System on a Disk

In this section, we will study the dynamical behavior of system (6) in the absence of diffusion, (i.e. taking diffusion coefficients δ_1, δ_2 and δ_3 equal to zero), in terms of local/global stability of the positive steady state $E^* = (u^*, v^*, w^*)$ and the conditions in which the strictly positive interior equilibrium enters into Hopf bifurcation.

3.1. *Equilibria and local stability*

The steady states are determined analytically by setting $\frac{dE}{dt} = 0$, by a simple computation, the trivial steady states are

$$E_0 = (0,0,0), \quad E_1 = (1,0,0),$$
$$E_2 = \left(0,0,\frac{sp}{q}\right) \quad \text{and} \quad E_3 = \left(1,0,\frac{sp}{q}\right)$$

and a unique interior one $E^* = (u^*, v^*, w^*)$.

The dynamical behavior of equilibrium points can be studied by computing the eigenvalues of the Jacobian matrix J of system (10) where

$$J(E^*) = \begin{pmatrix} 1 - 2u^* - \dfrac{av^*}{(u^*+a)^2} & -\dfrac{u^*}{u^*+a} & 0 \\ \dfrac{acv^*}{(u^*+a)^2} & \dfrac{cu^*}{u^*+a} - b - \dfrac{dw^*}{(v^*+d)^2} & -\dfrac{v^*}{v^*+d} \\ 0 & \dfrac{q(w^*)^2}{u^*+a} - b - \dfrac{dw^*}{(v^*+d)^2} & -\dfrac{2qw^*}{v^*+d} \end{pmatrix}. \quad (27)$$

Then, the Jacobian matrix evaluated at $E_0 = (0, 0,0)$ and $E_1 = (1,0,0)$ are respectively

$$J(E_0) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{} & 0 \\ \multicolumn{2}{c|}{J(E_{00})} & 0 \\ \hline 0 & -b & 0 \end{array}\right),$$
$$J(E_1) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{} & 0 \\ \multicolumn{2}{c|}{J(E_{11})} & 0 \\ \hline 0 & -b & 0 \end{array}\right). \quad (28)$$

The eigenvalues of $J(E_0)$ are 1, $-b$ and 0, then E_0 is nonhyperbolic. Furthermore, as one eigenvalue is a positive real, and another one is a negative real, E_0 is always unstable. Thus, for each orbit starting in $\text{int}(\mathbb{R}^3_+)$, the number of prey u and specialist predator v will not tend to zero.

The eigenvalues of $J(E_1)$ are 1, $\frac{c-b-ab}{1+a}$ and 0. If $ab > c-b$, two of the eigenvalues are negative, so E_1 has a stable manifold of at least two dimensions and if $ab < c-b$, E_1 is nonhyperbolic and E_1 is unstable.

For the equilibrium E_2 the Jacobian matrix evaluated at this equilibrium is

$$J(E_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & & J(E_{22}) \end{array}\right) \quad (29)$$

The eigenvalues of $J(E_2)$ are 1, $-b - \frac{sp}{qd}$ and $-\frac{2sp}{d}$. Hence the equilibrium $E_2 = (0,0,\frac{sp}{q})$ is a saddle with $\dim W^s(E_2) = 2$ and $\dim W^u(E_2) = 1$. For the equilibrium E_3 the Jacobian matrix evaluated at this equilibrium is

$$J(E_3) = \begin{pmatrix} -1 & \dfrac{-1}{1+a} & 0 \\ 0 & \dfrac{c}{1+a} - b - \dfrac{sp}{qd} & 0 \\ 0 & \dfrac{s^2p^2}{q(1+a)} & -\dfrac{2sp}{d} \end{pmatrix}. \quad (30)$$

The eigenvalues of $J(E_3)$ are -1, $\frac{c}{1+a}-b-\frac{sp}{qd}$, $-\frac{2sp}{d}$ respectively. Hence if $b+\frac{sp}{qd}>\frac{c}{1+a}$, the equilibrium $E_3=(1,0,\frac{sp}{q})$ is locally asymptotically stable. If $b+\frac{sp}{qd}<\frac{c}{1+a}$ the equilibrium $E_3=(1,0,\frac{sp}{q})$ is a saddle with $\dim W^s(E_3)=2$ and $\dim W^u(E_3)=1$.

In the following, we shall discuss the local stability of the positive steady state $E^*=(u^*,v^*,w^*)$ for the ODE system (10).

Theorem 2. *If condition (11) is satisfied and*

$$\frac{a+1}{qc}>\frac{2a}{qc-bq-p}$$

and

$$b+\frac{dp((1-u^*)(u^*+a)+s)}{q((1-u^*)(u^*+a)+d)^2}>\frac{cu^*}{u^*+a} \tag{31}$$

and

$$\frac{p^2((1-u^*)(u^*+a)+s)^2}{q(u^*+a)} > b+\frac{dp((1-u^*)(u^*+a)+s)}{q((1-u^*)(u^*+a)+d)^2}.$$

Then, the equilibrium solution $E^*=(u^*,v^*,w^*)$ is locally asymptotically stable.

Proof. Define $L_E(E^*)$ by

$$L_E(E^*)=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

according to (12), a direct calculation yields

$$\begin{cases} a_{11}=1-2u^*-\dfrac{a(1-u^*)}{(u^*+a)}, \quad a_{12}=-\dfrac{u^*}{u^*+a}, \quad a_{13}=0,\\[2ex] a_{21}=\dfrac{ac(1-u^*)}{(u^*+a)}, \quad a_{22}=\dfrac{cu^*}{u^*+a}-b-\dfrac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}, \quad a_{23}=-\dfrac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d},\\[2ex] a_{31}=0, \quad a_{32}=\dfrac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a}-b-\dfrac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2},\\[2ex] a_{33}=-\dfrac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}, \end{cases} \tag{32}$$

where

$$u^*=\frac{a(bq+p)}{qc-bq-p}.$$

The characteristic polynomial of $L_E(E^*)$ can be written as

$$\varphi(\lambda)=\lambda^3+B_1\lambda^2+B_2\lambda+B_3, \tag{33}$$

where

$$(1-u^*)(u^*+a)=\frac{aqc(qc-(bq+p)(a+1))}{(qc-bq-p)^2}, \quad \frac{(1-u^*)}{(u^*+a)}=\frac{qc-(bq+p)(a+1)}{aqc}, \quad \frac{u^*}{(u^*+a)}=\frac{bq+p}{qc}.$$

Hence

$$\begin{aligned} B_1&=-\mathrm{tr}(L_E(E^*))=-(a_{11}+a_{22}+a_{33})\\ &=u^*\left(2-\frac{a}{u^*+a}\right)+b+\frac{dp(qc-bq-p)^2(aqc(qc-(bq+p)(a+1))+s(qc-bq-p)^2)}{q(aqc(qc-(bq+p)(a+1))+d(qc-bq-p)^2)}\\ &\quad+\frac{2paqc(qc-(bq+p)(a+1))}{aqc(qc-(bq+p)(a+1))+d(qc-bq-p)^2}>0, \end{aligned}$$

$$B_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21}$$

$$= \left(-\frac{cu^*}{u^*+a} + b + \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}\right)\left(2u^* - \frac{(bq+p)(a+1)}{qc} + \frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}\right)$$

$$+ \left(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}\right)\left(2u^* - \frac{(bq+p)(a+1)}{qc}\right) + \left(\frac{ac(1-u^*)u^*}{(u^*+a)^2}\right)$$

$$+ \left(\frac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a} - b - \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}\right)\left(\frac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d}\right)$$

$$> 0,$$

$$B_3 = -\det(L_E(E^*))$$

$$= a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33}$$

$$= \left(\frac{ac(1-u^*)u^*}{(u^*+a)^2}\right)\left(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}\right)$$

$$+ \left(\frac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a} - b - \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}\right)\left(2u^* - \frac{(bq+p)(a+1)}{qc}\right)$$

$$\times \left(\frac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d}\right) + \left(-\frac{cu^*}{u^*+a} + b + \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}\right)\left(2u^* - \frac{(bq+p)(a+1)}{qc}\right)$$

$$\times \left(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}\right)$$

$$> 0,$$

$$B_1B_2 - B_3$$

$$= a_{11}^2(-a_{22} - a_{33}) + a_{22}^2(-a_{11} - a_{33})$$

$$+ a_{33}^2(-a_{11} - a_{22}) + a_{12}a_{21}(a_{11} + a_{22}) + a_{32}a_{23}(a_{33} + a_{22}) - 2a_{11}a_{22}a_{33}$$

$$= [a_{11}^2 - a_{32}a_{23}](-a_{22} - a_{33}) + [a_{33}^2 - a_{12}a_{21}](-a_{11} - a_{22}) + a_{22}^2(-a_{11} - a_{33}) - 2a_{11}a_{22}a_{33}$$

$$= \left(-\frac{cu^*}{u^*+a} + b + \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2} + \frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}\right)$$

$$\times \left[a_{11}^2 + \left(\frac{(1-u^*)(u^*+a)}{(1-u^*)(u^*+a)+d}\right)\left(\frac{\frac{p^2}{q}((1-u^*)(u^*+a)+s)^2}{u^*+a} - b - \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2}\right)\right]$$

$$+ a_{22}^2\left(2u^* - \frac{(bq+p)(a+1)}{qc} + \frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d}\right) + \left[a_{33}^2 + \left(\frac{acu^*(1-u^*)}{(u^*+a)^2}\right)\right]$$

$$\times \left(2u^* - \frac{(bq+p)(a+1)}{qc} - \frac{cu^*}{u^*+a} + b + \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2} \right) + 2\left(2u^* - \frac{(bq+p)(a+1)}{qc} \right)$$

$$\times \left(-\frac{cu^*}{u^*+a} + b + \frac{\frac{dp}{q}((1-u^*)(u^*+a)+s)}{((1-u^*)(u^*+a)+d)^2} \right) \left(\frac{2p((1-u^*)(u^*+a)+s)}{(1-u^*)(u^*+a)+d} \right)$$

$$> 0.$$

Using the conditions (31) we see that $B_1B_2 - B_3 > 0$. From the Routh–Hurwitz criterion, $E^* = (u^*, v^*, w^*)$ is local asymptotically stable. ■

Next, we will study the conditions under which the positive interior equilibrium enters into Hopf bifurcation.

3.2. *Hopf bifurcation*

We consider p as a parameter of bifurcation and p_{cr} the critical value or the bifurcating value of the concerned parameters. Recall that the characteristic equation (33) of (10) at $E^* = (u^*, v^*, w^*)$ is given by

$$\varphi(\lambda) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3. \tag{34}$$

Theorem 3. *The necessary and sufficient conditions for occurrence of Hopf bifurcation at $p = p_{\text{cr}}$ are the following*

(i) $B_i(p_{\text{cr}}) > 0,\ i = 1, 2, 3,$
(ii) $B_1(p_{\text{cr}})B_2(p_{\text{cr}}) - B_3(p_{\text{cr}}) = 0,$
(iii) $\text{Re}[\frac{d\lambda_i}{dp}]_{p=p_{\text{cr}}} \neq 0,\ i = 1, 2, 3.$

Proof. From the condition $B_1B_2 - B_3$, we get

$$\begin{aligned} B_1B_2 - B_3 &= [a_{11}^2 - a_{32}a_{23}](-a_{22} - a_{33}) \\ &\quad + [a_{33}^2 - a_{12}a_{21}](-a_{11} - a_{22}) \\ &\quad + a_{22}^2(-a_{11} - a_{33}) - 2a_{11}a_{22}a_{33} \\ &= 0. \end{aligned}$$

Since $B_2 > 0$ at $p = p_{\text{cr}}$, there exists an interval containing p in $(p_{\text{cr}} - \epsilon, p_{\text{cr}} + \epsilon)$, for every $\epsilon > 0$. Therefore, for $p \in (p_{\text{cr}} - \epsilon, p_{\text{cr}} + \epsilon)$ the characteristic equation cannot have roots containing negative real parts. For $p = p_{\text{cr}}$, we have

$$(\lambda^2 + B_2)(\lambda + B_1) = 0 \tag{35}$$

which has three roots $\lambda_1 = i\sqrt{B_2}$, $\lambda_2 = -i\sqrt{B_2}$ and $\lambda_3 = -B_1$. For $p \in (p_{\text{cr}} - \epsilon, p_{\text{cr}} + \epsilon)$, the roots are in general of the following form

$$\begin{aligned} \lambda_1(p) &= \alpha_1(p) + i\alpha_2(p), \\ \lambda_2(p) &= \alpha_1(p) - i\alpha_2(p), \\ \lambda_3(p) &= -B_1(p). \end{aligned}$$

In what follows, we verify the transversality condition $\text{Re}[\frac{d\lambda_i}{dp}]_{p=p_{\text{cr}}} \neq 0,\ i = 1, 2, 3$. Substituting $\lambda_1(p) = \alpha_1(p) + i\alpha_2(p)$ in (35) and calculating the derivative, we get

$$\begin{aligned} \Omega_1(p)\alpha_1'(p) - \Omega_2(p)\alpha_2'(p) + M_1(p) &= 0, \\ \Omega_2(p)\alpha_1'(p) + \Omega_1(p)\alpha_2'(p) + M_2(p) &= 0, \end{aligned}$$

where

$$\begin{aligned} \Omega_1(p) &= 3\alpha_1^2(p) + 2B_1(p)\alpha_1(p) + \beta_2(p) - 3\alpha_2^2(p), \\ \Omega_2(p) &= 6\alpha_1(p)\alpha_2(p) + 2B_1(p)\alpha_2(p), \\ M_1(p) &= \alpha_1^2(p)B_1'(p) + \beta_2'(p)\alpha_1(p) \\ &\quad + B_3'(p) - \alpha_2^2(p)B_1'(p), \\ M_2(p) &= 2\alpha_1(p)\alpha_2(p)B_1'(p) + B_2'(p)\alpha_2(p). \end{aligned}$$

Since $\Omega_2(p_{\text{cr}})M_2(p_{\text{cr}}) + \Omega_1(p_{\text{cr}})M_1(p_{\text{cr}}) \neq 0$, we have $\text{Re}[\frac{d\lambda_i}{dp}]_{p=p_{\text{cr}}} = \frac{\Omega_2 M_2 \Omega_1 M_1}{\Omega_2^2 + \Omega_1^2}|_{p=p_{\text{cr}}} \neq 0,\ i = 1, 2, 3$ and $\lambda_3(p_{\text{cr}}) = -B_1(p_{\text{cr}})$.

Hence the proof of the Theorem. ■

In the next paragraph, we establish the global stability of $E^* = (u^*, v^*, w^*)$ for some reasonable conditions on the parameters.

3.3. *Global stability of the nontrivial steady state*

In this subsection, we study the global stability of the homogeneous nontrivial equilibrium $E^* = (u^*, v^*, w^*)$ by using a suitable Lyapunov function.

Theorem 4. *If the condition (11) and the following conditions are verified*

$$\frac{v^*}{a^2} < \frac{1}{2(u^*+a)} + \frac{u^*}{2a^2} + \frac{1}{2}$$

and

$$\frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2} < \frac{s}{c(1+a+s)}\frac{v^*}{c(1+a+s)}$$

and

$$\frac{w^*(u^*+a)}{cd^2} + \frac{1}{2} + \frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2} < \frac{1}{2(u^*+a)} + \frac{u^*}{2a^2}.$$

Then the homogeneous nontrivial steady state (u^*, v^*, w^*) of system (8) is globally asymptotically stable.

Proof. Let us consider the following Lyapunov function,

$$V(t) = \left(u - u^* - u^*\ln\left(\frac{u}{u^*}\right)\right) + \frac{(u^*+a)}{c}\left(v - v^* - v^*\ln\left(\frac{v}{v^*}\right)\right) + \frac{c(v^*+s)}{q}\left(w - w^* - w^*\ln\left(\frac{w}{w^*}\right)\right).$$

Calculating the derivative of V along the solution of system, we have

$$\frac{dV}{dt} = (u-u^*)\left(1 - u - \frac{v}{u+a}\right) + \frac{(u^*+a)}{c}(v-v^*)\left(-b + \frac{cu}{u+a} - \frac{w}{v+d}\right) + \frac{c(v^*+s)}{q}(w-w^*)\left(p - \frac{qw}{v+s}\right).$$

Using the following results

$$1 = u^* + \frac{v^*}{u^*+a}, \quad b = \frac{cu^*}{u^*+a} - \frac{w^*}{v^*+d}$$

$$\text{and} \quad p = \frac{qw^*}{v^*+s}$$

we have

$$\begin{aligned}
\frac{dV}{dt} &= (u-u^*)\left(u^* + \frac{v^*}{u^*+a} - u - \frac{v}{u+a}\right) + \frac{(u^*+a)}{c}(v-v^*)\left(-\frac{cu^*}{u^*+a} + \frac{w^*}{v^*+d} + \frac{cu}{u+a} - \frac{w}{v+d}\right) \\
&\quad + \frac{c(v^*+s)}{q}(w-w^*)\left(\frac{qw^*}{v^*+s} - \frac{qw}{v+s}\right) \\
&= -(u-u^*)^2 + (u-u^*)\left(\frac{v^*(u-u^*) - a(v-v^*) - u^*(v-v^*)}{(u^*+a)(u+a)}\right) + (v-v^*)\left(\frac{a(u-u^*)}{(u+a)}\right) \\
&\quad + \frac{(u^*+a)}{c}(v-v^*)\left(\frac{-d(w-w^*) - v^*(w-w^*) + w^*(v-v^*)}{(v^*+d)(v+d)}\right) \\
&\quad + (w-w^*)\left(\frac{-s(w-w^*) + w^*(v-v^*) - v^*(w-w^*)}{c(v+s)}\right) \\
&= (u-u^*)^2\left[-1 + \frac{v^*}{(u^*+a)(u+a)}\right] + (v-v^*)^2\frac{w^*(u^*+a)}{c(v^*+d)(v+d)} + (w-w^*)^2\left[-\frac{s}{c(v+s)} - \frac{v^*}{c(v+s)}\right] \\
&\quad + (v-v^*)(u-u^*)\left[-\frac{a}{(u^*+a)(u+a)} - \frac{u^*}{(u^*+a)(u+a)} + \frac{a}{u+a}\right] \\
&\quad + (v-v^*)(w-w^*)\left[-\frac{d(u^*+a)}{c(v^*+d)(v+d)} - \frac{v^*(u^*+a)}{c(v^*+d)(v+d)}\right].
\end{aligned}$$

Then

$$\frac{dV}{dt} \le (u-u^*)^2\left[-1 + \frac{v^*}{(u^*+a)(u+a)}\right] + (v-v^*)^2\frac{w^*(u^*+a)}{c(v^*+d)(v+d)}$$

$$
\begin{aligned}
&+ (w-w^*)^2\left[-\frac{s}{c(v+s)} - \frac{v^*}{c(v+s)}\right] + \frac{(v-v^*)^2+(u-u^*)^2}{2}\\
&\times\left[-\frac{u^*}{(u^*+a)(u+a)} + \frac{a}{u+a}\left(1-\frac{1}{(u^*+a)}\right) + \frac{d(u^*+a)}{c(v^*+d)(v+d)} + \frac{v^*(u^*+a)}{c(v^*+d)(v+d)}\right]\\
\le{}& (u-u^*)^2\left[-1+\frac{v^*}{a^2}\right] + (v-v^*)^2\frac{w^*(u^*+a)}{cd^2} + (w-w^*)^2\left[-\frac{s}{c(v+s)} - \frac{v^*}{c(v+s)}\right]\\
&+ \frac{(v-v^*)^2+(u-u^*)^2}{2}\left[-\frac{u^*}{a^2}+1-\frac{1}{(u^*+a)}\right] + \frac{(v-v^*)^2+(w-w^*)^2}{2}\left[\frac{(u^*+a)}{cd} + \frac{v^*(u^*+a)}{cd^2}\right]\\
={}& (u-u^*)^2\left[-1+\frac{v^*}{a^2} - \frac{1}{2(u^*+a)} - \frac{u^*}{2a^2} + \frac{1}{2}\right]\\
&+ (v-v^*)^2\left[\frac{w^*(u^*+a)}{cd^2} - \frac{1}{2(u^*+a)} - \frac{u^*}{2a^2} + \frac{1}{2} + \frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2}\right]\\
&+ (w-w^*)^2\left[-\frac{s}{c(v+s)} - \frac{v^*}{c(v+s)} + \frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2}\right].
\end{aligned}
$$

From Theorem 1, we observe that $v(t) \le 1+a$ for all time t. So, $-\frac{1}{c(v+s)} \le -\frac{1}{c(1+a+s)}$. Substituting we get

$$
\begin{aligned}
\frac{dV}{dt} \le{}& (u-u^*)^2\left[\frac{v^*}{a^2} - \frac{1}{2(u^*+a)} - \frac{u^*}{2a^2} - \frac{1}{2}\right]\\
&+ (v-v^*)^2\left[\frac{w^*(u^*+a)}{cd^2} - \frac{1}{2(u^*+a)} - \frac{u^*}{2a^2} + \frac{1}{2} + \frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2}\right]\\
&+ (w-w^*)^2\left[-\frac{s}{c(1+a+s)} - \frac{v^*}{c(1+a+s)} + \frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2}\right],
\end{aligned}
$$

if

$$
\frac{v^*}{a^2} < \frac{1}{2(u^*+a)} + \frac{u^*}{2a^2} + \frac{1}{2}, \quad \frac{w^*(u^*+a)}{cd^2} + \frac{1}{2} + \frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2} < \frac{1}{2(u^*+a)} + \frac{u^*}{2a^2} \quad \text{and}
$$

$$
\frac{(u^*+a)}{2cd} + \frac{v^*(u^*+a)}{2cd^2} < \frac{s}{c(1+a+s)}\frac{v^*}{c(1+a+s)}.
$$

Then we deduce that $\frac{dV}{dt} < 0$ and by LaSalle's theorem, $E^* = (u^*, v^*, w^*)$ is globally asymptotically stable in the uvw-space, which completes the proof. ■

In biological terms, the system without diffusion has been studied in the preceding subsection. In what follows, the model with diffusion on the disk domain which is relevant to the real world, will be investigated.

4. Analysis of the Spatiotemporal Model on a Disk Domain

Our result implies that the constant positive steady state is globally asymptotically stable in the absence of the diffusion but may be unstable in the presence of the diffusion. In the next section, we will discuss the conditions that enable the occurrence of Turing instability.

4.1. *Diffusion driven instability*

In the reaction–diffusion system, the Turing instability occurs from a finite number of wave vectors producing stable spatial patterns depending essentially on the initial condition.

By setting

$$
W = \begin{pmatrix} u-u^* \\ v-v^* \\ w-w^* \end{pmatrix} \varphi(r,\theta)e^{\lambda t + ikr} \tag{36}
$$

where k is the wave number and $\varphi(r,\theta)$ is an eigenfunction of the Laplacian operator on a disc domain with zero flux on the boundary, i.e.

$$\begin{cases} \Delta_{r\theta}\varphi = -k^2\varphi, \\ \varphi_r(R,\theta) = 0. \end{cases}$$

Then by linearizing around (u^*, v^*, w^*), we have the following equation:

$$\frac{dW}{dt} = D\Delta W + L_E(E^*)W. \tag{37}$$

Consider now the system with diffusion (6) and substituting W by $\varphi e^{\lambda t}$ in Eq. (37) and canceling $e^{\lambda t}$, we get:

$$\lambda\varphi = L_E(E^*) - Dk^2\varphi. \tag{38}$$

We obtain the characteristic equation for the growth rate λ as a determinant of

$$\det(\lambda I_3 - L_E(E^*) + K^2 D) = 0 \Leftrightarrow \det\begin{pmatrix} \lambda - a_{11} + \delta_1 k^2 & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} + \delta_2 k^2 & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} + \delta_3 k^2 \end{pmatrix} = 0. \tag{39}$$

The characteristic polynomial from (39) is as follows

$$H(k^2) = \lambda^3 + \Phi_1(k^2)\lambda^2 + \Phi_2(k^2)\lambda + \Phi_3(k^2) = 0, \tag{40}$$

with

$$\Phi_1(k^2) = k^2(\delta_1 + \delta_2 + \delta_3) + B_1,$$
$$\Phi_2(k^2) = k^4(\delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3) - k^2(\delta_1(a_{22} + a_{33}) + \delta_2(a_{11} + a_{33}) + \delta_3(a_{11} + a_{22})) + B_2,$$
$$\Phi_3(k^2) = k^6\delta_1\delta_2\delta_3 - k^4(\delta_1\delta_2 a_{33} + \delta_1\delta_3 a_{22} + \delta_2\delta_3 a_{11}) + k^2(\delta_3(a_{11}a_{22} - a_{12}a_{21}) + \delta_2 a_{11}a_{33}) + B_3.$$

For the stability of the equilibrium point, according to the Routh–Hurwitz criteria, $\mathrm{Re}(\lambda) < 0$ if

$$\Phi_1(k^2) > 0, \tag{41}$$
$$\Phi_2(k^2) > 0, \tag{42}$$
$$\Phi_1(k^2)\Phi_2(k^2) - \Phi_3(k^2) > 0. \tag{43}$$

The Turing instability requires that the stable homogeneous equilibrium becomes unstable due to the interaction and diffusion of species.

Under the following conditions (of Turing):

$$\begin{matrix} \mathrm{Re}(\lambda(k^2 = 0)) < 0, \\ \mathrm{Re}(\lambda(k^2 > 0)) > 0, \end{matrix} \quad \text{for a } k^2 > 0. \tag{44}$$

We will study the sign of (41)–(43). Therefore, we get the following theorem.

Theorem 5. *If one of the following conditions is true*:

$$\Phi_1(k^2) < 0,$$
$$\Phi_2(k^2) < 0,$$
$$\Phi_1(k^2)\Phi_2(k^2) - \Phi_3(k^2) < 0.$$

Then, the positive equilibrium $E^* = (u^*, v^*, w^*)$ of system (6) is driven to instability.

Remark 4.1

- If the parameter λ is real, the spatial patterns are stable over time and in the case where λ is complex, spatial patterns vary over time. In either case, the sign of the real part of λ determines the patterns' growth: if $\mathrm{Re}(\lambda) > 0$ the linearized system (36) increases, there is currently pattern formation. By cons, if $\mathrm{Re}(\lambda) < 0$, the linear perturbation decreases with time and the solution of the perturbed system tends to homogeneous initial equilibrium (u^*, v^*, w^*).
- If k^2 is complex, one can observe complex spatial structures.
- For $k^2 = 0$, the characteristic equation is written as

$$\lambda^3 + \Phi_1(0)\lambda^2 + \Phi_2(0)\lambda + \Phi_3(0),$$

where

$$\Phi_1(0) = B_1 = -a_{11} - a_{22} - a_{33} = -\mathrm{tr}(L_E(E^*)),$$

if at least a_{11}, a_{22} or a_{33} is negative and $\Phi_1(0) = -\mathrm{tr}(L_E(E^*)) > 0$ and $\Phi_2(0) = B_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21} > 0$, if $a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} > a_{23}a_{32} + a_{12}a_{21}$, and $\Phi_3(0) = B_3 = -\det(L_E(E^*)) = a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33}$, if $a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} > a_{11}a_{22}a_{33}$, and $\Phi_3(0) = -\det(L_E(E^*)) > 0$.

$$\begin{aligned}\Phi_1(0)\Phi_2(0) - \Phi_3(0) &= -(a_{11}^2a_{22} + a_{11}^2a_{33} + 2a_{11}a_{22}a_{33} \\ &\quad + a_{11}a_{33}^2 + a_{11}a_{22}^2 + a_{22}^2a_{33} + a_{22}a_{33}^2) \\ &\quad + a_{12}a_{12}a_{22} + a_{22}a_{23}a_{32}\end{aligned}$$

$\Phi_1(0)\Phi_2(0) - \Phi_3(0) > 0$ if

$$\begin{aligned}&a_{11}^2a_{22} + a_{11}^2a_{33} + 2a_{11}a_{22}a_{33} + a_{11}a_{33}^2 \\ &\quad + a_{11}a_{22}^2 + a_{22}^2a_{33} + a_{22}a_{33}^2 \\ &\quad < a_{12}a_{12}a_{22} + a_{22}a_{23}a_{32}.\end{aligned} \tag{45}$$

Proof. For $k^2 \neq 0$, we have $\Phi_1(k^2) = -(a_{11} + a_{22} + a_{33}) + k^2(\delta_1 + \delta_2 + \delta_3)$. If $a_{11} + a_{22} + a_{33} < 0$, then $\Phi_1(k^2) > 0$ and instability of Turing does not occur.

Thereafter, we suppose in Eq. (42) $\rho = k^2 > 0$ we get:

$$\Phi_2(\rho) = \rho^2 p_1 - \rho p_2 + p_3, \tag{46}$$

where

$$\begin{aligned}p_1 &= \delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3, \\ p_2 &= \delta_1a_{22} + \delta_1a_{33} + \delta_2a_{11} + \delta_2a_{33} + \delta_3a_{11} + \delta_3a_{22}, \\ p_3 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{11} - a_{23}a_{23},\end{aligned}$$

a necessary condition for $E^* = (u^*, v^*, w^*)$ of (6) becomes unstable in that

$$\Phi_2(\rho) = \rho^2 p_1 - \rho p_2 + p_3 < 0. \tag{47}$$

For the instability, we need that $p_2 > 0$ and $p_2^2 - 4p_1p_3 > 0$ for some ρ. The equation $p_1\rho^2 - p_2\rho + p_3$ has two positive roots given by:

$$\rho_1 = \frac{p_2 - \sqrt{p_2^2 - 4p_1p_3}}{2p_1} \quad \text{and} \quad \rho_2 = \frac{p_2 - \sqrt{p_2^2 + 4p_1p_3}}{2p_1}. \tag{48}$$

The constant positive steady state $E^* = (u^*, v^*, w^*)$ of (6) is unstable and so (6) experiences Turing instability provided that $\rho_1 < \rho < \rho_2$.

The expressions $\Phi_3(k^2)$ and $\Phi_1(k^2)\Phi_2(k^2) - \Phi_3(k^2)$ are a cubic function of k^2 of the form

$$\Phi_3(k^2) = q_1(k^2)^3 + q_2(k^2)^2 + q_3k^2 + q_4, \tag{49}$$

$$\begin{aligned}q_1 &= \delta_1\delta_2\delta_3, \\ q_2 &= -(\delta_1\delta_2a_{33} + \delta_1\delta_3a_{22} + \delta_2\delta_3a_{11}), \\ q_3 &= \delta_1a_{22}h_w + \delta_2a_{11}a_{33} + \delta_3a_{11}a_{22} \\ &\quad - \delta_1a_{23}a_{32} - \delta_3a_{22}a_{21} \\ &= \delta_1(a_{22}a_{33} - a_{23}a_{32}) + \delta_2a_{11}a_{33} \\ &\quad + \delta_3(a_{11}a_{22} - a_{12}a_{21}), \\ q_4 &= \Phi_3(0) \\ &= a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33},\end{aligned}$$

with $q_1 = \det(D) \geq 0$ and $q_4 = -\det(L_E(E^*)) > 0$.

If Φ_3 has a minimum, one finds by simple calculation that

$$\frac{d\Phi_3}{d(k^2)} = 3q_1(k^2)^2 + 2q_2(k^2) + q_3 = 0 \tag{50}$$

and $\frac{d^2\Phi_3}{d^2(k^2)} > 0$, this minimum is reached for the solution of (50) at

$$k_{\mathrm{inf}}^2 = \frac{-q_2 + \sqrt{q_2^2 - 3q_1q_3}}{3q_1} \tag{51}$$

for q_2 expression if $a_{11} > 0$, $a_{22} > 0$ and $a_{33} > 0$ and for expression q_3, if $a_{22}a_{33} < a_{23}a_{32}$, $a_{11}a_{33} < 0$ and $a_{11}a_{22} < a_{12}a_{21}$ or $a_{22}a_{33} < 0$, $a_{11}a_{33} < 0$ and $a_{11}a_{22} < 0$ (i.e. (a_{22}, a_{33}) has different signs, the same applies to (a_{22}, a_{33}) and (a_{11}, a_{33})). Then

$$q_2 < 0 \quad \text{and} \quad q_3 < 0. \tag{52}$$

For condition (43)

$$\begin{aligned}\Psi(k^2) &= \Phi_1(k^2)\Phi_2(k^2) - \Phi_3(k^2) \\ &= r_1(k^2)^3 + r_2(k^2)^2 + r_3k^2 + r_4,\end{aligned} \tag{53}$$

where

$$\begin{aligned}r_1 &= 2\delta_1\delta_2\delta_3 + \delta_1^2\delta_3 + \delta_1^2\delta_2 + \delta_1\delta_2^2 + \delta_1\delta_3^2 \\ &\quad + \delta_3\delta_2^2 + \delta_2\delta_3^2 \\ &= (\delta_2 + \delta_3)(\delta_1^2 + \delta_2\delta_3 + \delta_1\delta_2 + \delta_1\delta_3), \\ r_2 &= -(\delta_1^2a_{22} + \delta_1^2a_{33} + \delta_2^2a_{11} + \delta_2^2a_{33} + \delta_3^2a_{11} \\ &\quad + \delta_3^2a_{22} + 2\delta_1\delta_2a_{11} + 2\delta_1\delta_2a_{33} + 2\delta_1\delta_3a_{11} \\ &\quad + 2\delta_1\delta_3a_{22} + 2\delta_1\delta_2a_{22} + 2\delta_1\delta_3a_{33} \\ &\quad + 2\delta_2\delta_3a_{11} + 2\delta_2\delta_3a_{22} + 2\delta_2\delta_3a_{33})\end{aligned}$$

$$= -a_{11}(\delta_3+\delta_2)(2\delta_1+\delta_2+\delta_3) - a_{22}(\delta_3+\delta_1)(\delta_1+2\delta_2+\delta_3) - a_{33}(\delta_1+\delta_2)(\delta_1+\delta_2+2\delta_3),$$

$$r_3 = \delta_1 a_{22}^2 + \delta_1 a_{33}^2 + \delta_2 a_{11}^2 + \delta_2 a_{33}^2 + \delta_3 a_{11}^2 + \delta_3 a_{22}^2 + 2\delta_1 a_{11}a_{22} + 2\delta_1 a_{11}a_{33} + 2\delta_1 a_{22}a_{33} - \delta_1 f_v g_u - \delta_1 f_w h_u + 2\delta_2 f_u g_v + 2\delta_2 a_{11}a_{33} + 2\delta_2 a_{22}a_{33} - \delta_2 a_{12}a_{21} - \delta_2 a_{23}a_{32} + 2\delta_3 a_{11}a_{22} + 2\delta_3 a_{11}a_{33} + 2\delta_1 a_{22}a_{33} - \delta_3 a_{23}a_{32}$$

$$= \delta_1 a_{22}^2 + \delta_1 a_{33}^2 + \delta_2 a_{11}^2 + \delta_2 a_{33}^2 + \delta_3 a_{11}^2 + \delta_3 a_{22}^2 + 2(\delta_1+\delta_2+\delta_3)(a_{11}a_{22}+a_{11}a_{33} + 2a_{33}a_{22}) - \delta_1 a_{12}a_{21} - \delta_2(a_{12}a_{21}+a_{23}a_{32}) - \delta_3 a_{23}a_{32},$$

$$r_4 = \Psi(0) = -(a_{11}^2 a_{22} + a_{11}^2 a_{33} + 2a_{11}a_{22}a_{33} + a_{11}a_{33}^2 + a_{11}a_{22}^2 + a_{22}^2 a_{33} + a_{22}a_{33}^2) + a_{12}a_{21}a_{22} + a_{22}a_{23}a_{32}.$$

With $r_1 \geq 0$ and $r_4 > 0$, when

$$a_{11}^2 a_{22} + a_{11}^2 a_{33} + 2a_{11}a_{22}a_{33} + a_{11}a_{33}^2 + a_{11}a_{22}^2 + a_{22}^2 a_{33} + a_{22}a_{33}^2 < a_{12}a_{12}a_{22} + a_{22}a_{23}a_{32}.$$

If Ψ has a minimum, by simple calculation one finds

$$\frac{d\Psi}{d(k^2)} = 3r_1(k^2)^2 + 2r_2(k^2) + r_3 = 0 \tag{54}$$

and $\frac{d^2\Psi}{d^2(k^2)} > 0$, this minimum is reached for the solution of (54) at

$$k_{1\inf}^2 = k_{\inf}^2 = \frac{-r_2 + \sqrt{r_2^2 - 3r_1r_3}}{3r_1} \tag{55}$$

for r_2 expression if $a_{11} > 0$, $a_{22} > 0$ and $a_{33} > 0$ and for expression r_3 if $a_{12}a_{21} > 0$, $(a_{12}a_{21}+a_{23}a_{32}) > 0$, $a_{23}a_{32} > 0$ and $\delta_1 a_{22}^2 + \delta_1 a_{33}^2 + \delta_2 a_{11}^2 + \delta_2 a_{33}^2 + \delta_3 a_{11}^2 + \delta_3 a_{22}^2 + 2(\delta_1+\delta_2+\delta_3)(a_{11}a_{22}+a_{11}a_{33}+2a_{33}a_{22}) < \delta_1 a_{12}a_{21} + \delta_2(a_{12}a_{21}+a_{23}a_{32}) + \delta_3 a_{23}a_{32}$. Then

$$r_2 < 0 \quad \text{and} \quad r_3 < 0. \tag{56}$$

By using the conditions of the homogeneous stable equilibrium of the system without diffusion ($\Phi_1(0) > 0, \Phi_2(0) > 0, \Phi_3(0) > 0$, $\Phi_1(0)\Phi_2(0) - \Phi_3(0) > 0$) and the necessary condition of the system with diffusion. (That is to say, at least one of the following conditions, ($\Phi_1(k^2) < 0, \Phi_2(k^2) < 0$, $\Phi_3(k^2) < 0, \Phi_1(k^2)\Phi_2(k^2) - \Phi_3(k^2) < 0$) for a certain $k^2 \neq 0$.)

The following proposition gives us a necessary condition, not sufficient for instability to the reaction–diffusion system with three species. ■

Proposition 1. *If these following conditions (i) and (ii) are satisfied*:

(i) *If one condition of (52) is verified and* $q_2^2 - 3q_1q_3 > 0$, *then* $k_{\inf}^2$ *is a positive real. If each condition of (52) and (56) is verified and* $q_2^2 - 3q_1q_3 > 0$, (*resp.*, $r_2^2 - 3r_1r_3 > 0$), *then* $k_{\inf}^2$ *is a positive real* (*resp.*, $k_{1\inf}^2$ *is a positive real*).

(ii) *Let*

$$\Phi_3(k_{\inf}^2) = \frac{2q_2^3 - 9q_1q_2q_3 + 27q_1^2q_4 - 2(q_2^2 - 3q_1q_3)^{\frac{3}{2}}}{27q_1^3}$$

and

$$\Psi(k_{1\inf}^2) = \frac{2r_2^3 - 9r_1r_2r_3 + 27r_1^2r_4 - 2(r_2^2 - 3r_1r_3)^{\frac{3}{2}}}{27r_1^3},$$

if $2q_2^3 - 9q_1q_2q_3 + 27q_1^2q_4 - 2(q_2^2 - 3q_1q_3)^{\frac{3}{2}} < 0$, (*resp.*, $2r_2^3 - 9r_1r_2r_3 + 27r_1^2r_4 - 2(r_2^2 - 3r_1r_3)^{\frac{3}{2}} < 0$), *then* $\Phi_3(k_{\inf}^2) < 0$ (*resp.*, $\Psi(k_{1\inf}^2) < 0$).

Then we observe the emergence of Turing instability for system (6).

4.2. *Pattern formation*

In this section, we assume that λ is complex, which involves the formation of spatiotemporal patterns. We assume that $\lambda = x + iy$ by direct computation in Eq. (40), we obtained the following two equations

$$x^3 - 3xy^2 + \Phi_1(x^2 - y^2) + \Phi_2 x + \Phi_3 = 0 \tag{57}$$

and

$$-y^3 + 3x^2y + 2\Phi_1 xy + \Phi_2 y = 0. \tag{58}$$

For Eq. (58)

$$-y^3 + 3x^2y + 2\Phi_1 xy + \Phi_2 y = y(-y^2 + 3x^2 + 2\Phi_1 x + \Phi_2) = 0.$$

Therefore $y = 0$ or $y^2 = 3x^2 + 2\Phi_1 x + \Phi_2$, this implies that λ is real. By substituting y^2 in (57), we obtain

$$H(x) \equiv 8x^3 + 8\Phi_1 x^2 + 2x(\Phi_1^2 + \Phi_2) + \Phi_1\Phi_2 - \Phi_3 = 0. \quad (59)$$

If the condition (53) is negative, then $H(0) = (\Phi_1\Phi_2 - \Phi_3)(k^2) < 0$, this implies that Eq. (59) has a positive real solution and there is formation of spatiotemporal patterns.

Remark 4.2

- If $\Phi_3(k^2) < 0$ and $\Phi_2(k^2) > 0$, then $H(0) > 0$ and there are temporally stable patterns.
- If $\Phi_3(k^2) > 0$, $\Phi_2(k^2) > 0$ and $H(0) > 0$ this implies that there is no formation of spatiotemporal patterns.

Otherwise, we also show after the extremum method that Eq. (59) admits a real positive solution.

$$\frac{dH(x)}{dx} = 24x^2 + 16\Phi_1 x + 2(\Phi_1^2 + \Phi_2) = 0.$$

Then

$$x = x_1 = \frac{-2\Phi_1 + \sqrt{4\Phi_1^2 - 3(\Phi_1^2 + \Phi_2)}}{6}$$

If $(\Phi_1^2 + \Phi_2) < 0$, then $x_1 > 0$. Thus $H(x)$ admits an extremum at x_1 such that $H(x_1) < 0$.

If $\Phi_2(k^2) > 0$, then $x = x_1 > 0$ because y is a positive real value and we deduce the formation of spatiotemporal patterns. However, there are currently more restriction if $\Phi_2(k^2) < 0$, for which q is a positive real value.

$$x_2 \equiv \frac{-\Phi_1 + \sqrt{4\Phi_1^2 - 3(\Phi_1^2 + \Phi_2)}}{3} > x_1.$$

Then

- If $\Phi_2(k^2) < 0, \Phi_1\Phi_2 - \Phi_3(k^2) < 0$, $\Phi_3(k^2) < 0$ or $\Phi_3(k^2) > 0$ and if $x > x_2$, we deduce the formation of spatiotemporal patterns. By cons, if $x < x_2$ then the patterns are temporally stable.

Remark 4.3. Assume that

$$\begin{gathered}\Phi_2(k^2) < 0, \quad \Phi_3(k^2) < 0, \\ \Phi_1\Phi_2 - \Phi_3(k^2) > 0 \quad \text{and} \\ \Phi_1\Phi_2 - \Phi_3(k^2) > 0.\end{gathered} \quad (60)$$

- If the condition (60) is verified, then the patterns are temporally stable,
- If the condition (60) is verified and if $H(\bar{x}) > 0$, then the patterns are temporally stable,
- If the condition (60) is verified and if $H(\bar{x}) < 0$ and $\bar{x} \le x < x_2$, then the patterns are temporally stable,
- If the condition (60) is verified and if $H(\bar{x}) < 0$ and $x > x_2$, then the patterns are temporally stable,

where $\bar{x}$ is the on-solution for x.

5. Numerical Analysis

It is clear that the analytical solution of the coupled reaction–diffusion system of predator–prey type is not always possible. Thus, we have to perform numerical simulations to solve them. The spatiotemporal system (6) is solved numerically in the disk $\Gamma = \{(r, \theta) : 0 < r < R, 0 \le \theta < 2\pi\}$, by using a finite difference scheme for the spatial derivatives with the zero-flux boundary conditions. In order to avoid numerical artifacts, the values of time (Δt) and space steps (Δr and $\Delta\theta$) have been chosen sufficiently small. For the numerical simulations, the initial condition is a small perturbation in the vicinity of equilibrium point (u^*, v^*, w^*). These initial conditions have been chosen as,

$$\begin{aligned} u(0, r, \theta) &= u^*((r\cos\theta)^2 + (r\sin\theta)^2) < 50, \\ v(0, r, \theta) &= v^*((r\cos\theta)^2 + (r\sin\theta)^2) < 50, \\ w(0, r, \theta) &= w^*((r\cos\theta)^2 + (r\sin\theta)^2) < 50. \end{aligned} \quad (61)$$

The parametric values are

$$\begin{gathered} a_0 = 0.5, \quad a_1 = 0.4, \quad b_0 = 0.36, \quad c_3 = 0.2, \\ d_0 = 0.3, \quad d_2 = 0.4, \quad d_3 = 0.4, \quad v_0 = 0.4, \\ v_1 = 0.8, \quad v_2 = 0.4, \quad v_3 = 0.6. \end{gathered} \quad (62)$$

In Fig. 1, the left figures are the evolution of the prey spatial distribution, the right are the top predators and the center are the predators. Initially we observe two waves of burst center of the disk, then these spirals burst leading to an aperiodic spatial distribution of some domain and this aperiodicity spreads throughout the area and remains in time. We then obtain the spatiotemporal chaos.

Next, we study the evolution of top predator density into functions of predator and prey densities for different values of time t (see Fig. 2).

We study the properties of the oscillations of populations in the UV-plane when the control parameter varies. For better studying the properties

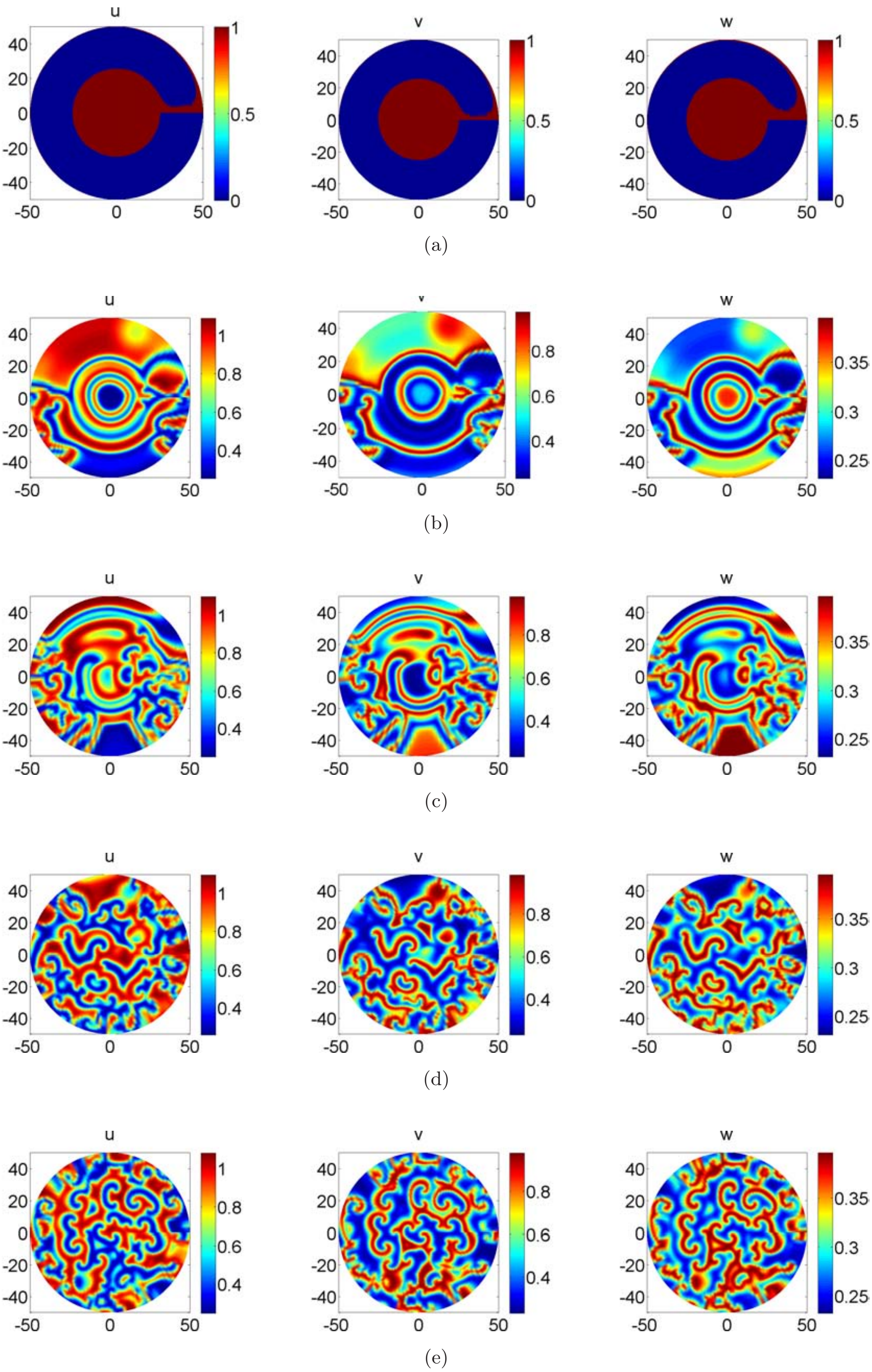


Fig. 1. Spatial distributions of prey (first column), predator (second column) and top predator (third column) are population densities of the spatial system (6). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$, at different time levels: for (a) $t = 0$, (b) $t = 1000$, (c) $t = 2000$, (d) $t = 4000$ and (e) $t = 20\,000$.

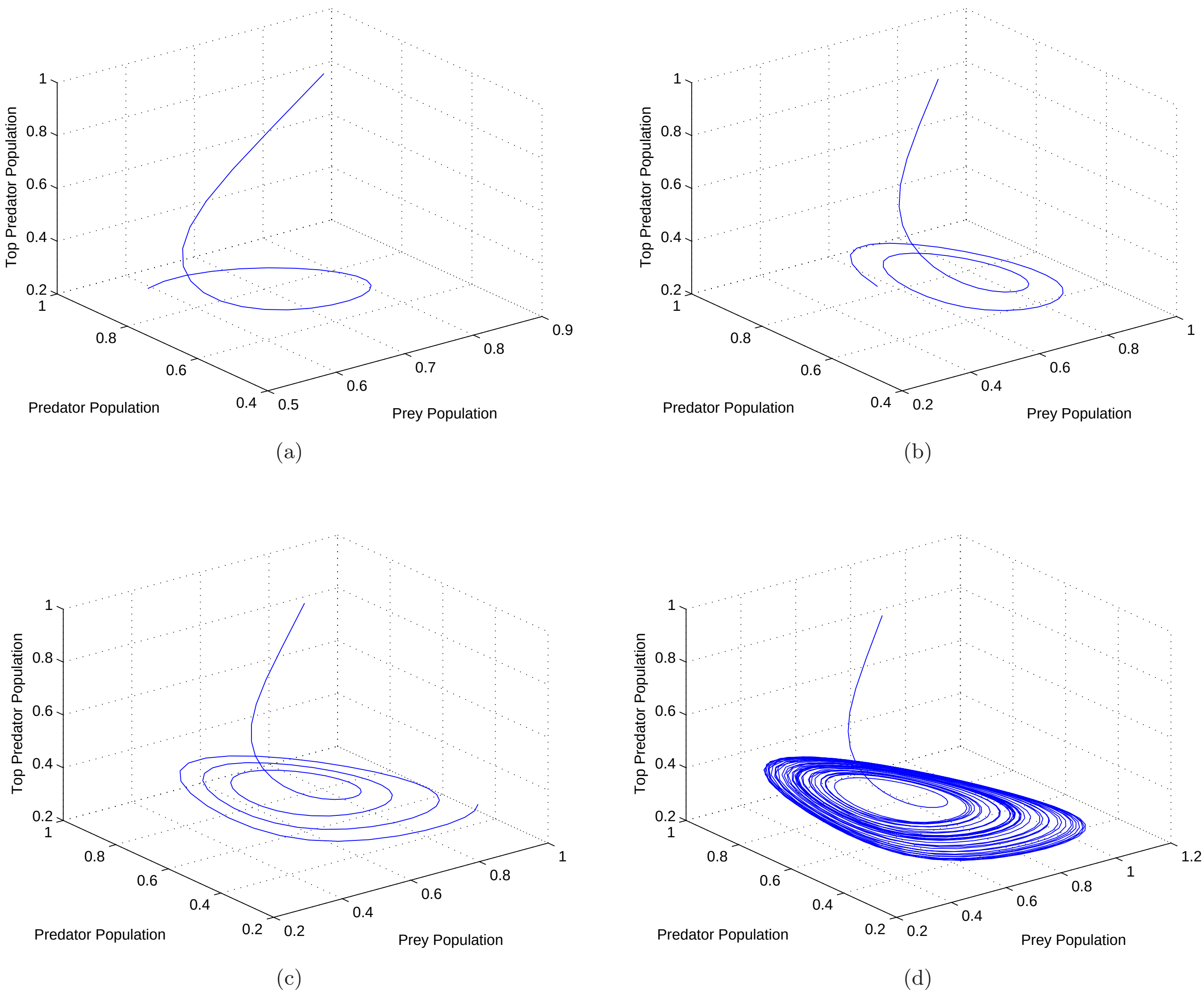


Fig. 2. Top predator density as a function of density of prey and predator with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$, at different time levels for (a) $t = 30$, (b) $t = 60$, (c) $t = 100$ and (d) $t = 1000$.

of the population dynamics as a whole, we estimate the species size of prey and predator by

$$U(t) = \int_0^R \int_0^{2\pi} u(t, r, \theta) dr d\theta \quad \text{and} \qquad (63)$$
$$V(t) = \int_0^R \int_0^{2\pi} v(t, r, \theta) dr d\theta.$$

We have considered p as a parameter of bifurcation with $p = \frac{c_3 a_0^2}{v_0 b_0 v_2}$ where c_3 describes the growth rate of W. We plot the phase portraits in the UV-plane, thus we vary c_3 between 0.18 and 0.26. For each value of c_3, the system (6) is solved with the initial condition (61), the other parameters are fixed in (62). We consider a transition time as fairly large so that the quantities V and U fall within the domain of attraction. First, we start by taking $c_3 = 0.255$ and then we decrease the values of c_3. For values of c_3 belonging to the interval $]0.245, 0.255[$ and for $c_3 = 0.255$, the solution of the system (6) is a hearth [Fig. 3(a)]. We observe for $c_3 = 0.245$ the first bifurcation [Fig. 3(b)], for the values of c_3 belonging to the interval $]0.2, 0.245[$ the system (6) exhibits quasiperiodic attractor solutions that are quasiperiodic in the phase plane (U, V) [Figs. 3(c)–3(e)]. Finally, for $c_3 < 0.2$, the solutions of (6) become chaotic [Fig. 3(f)].

Then, we show the effect of the bifurcation parameter on the formation of patterns for the different species for a fixed time, the other parameters

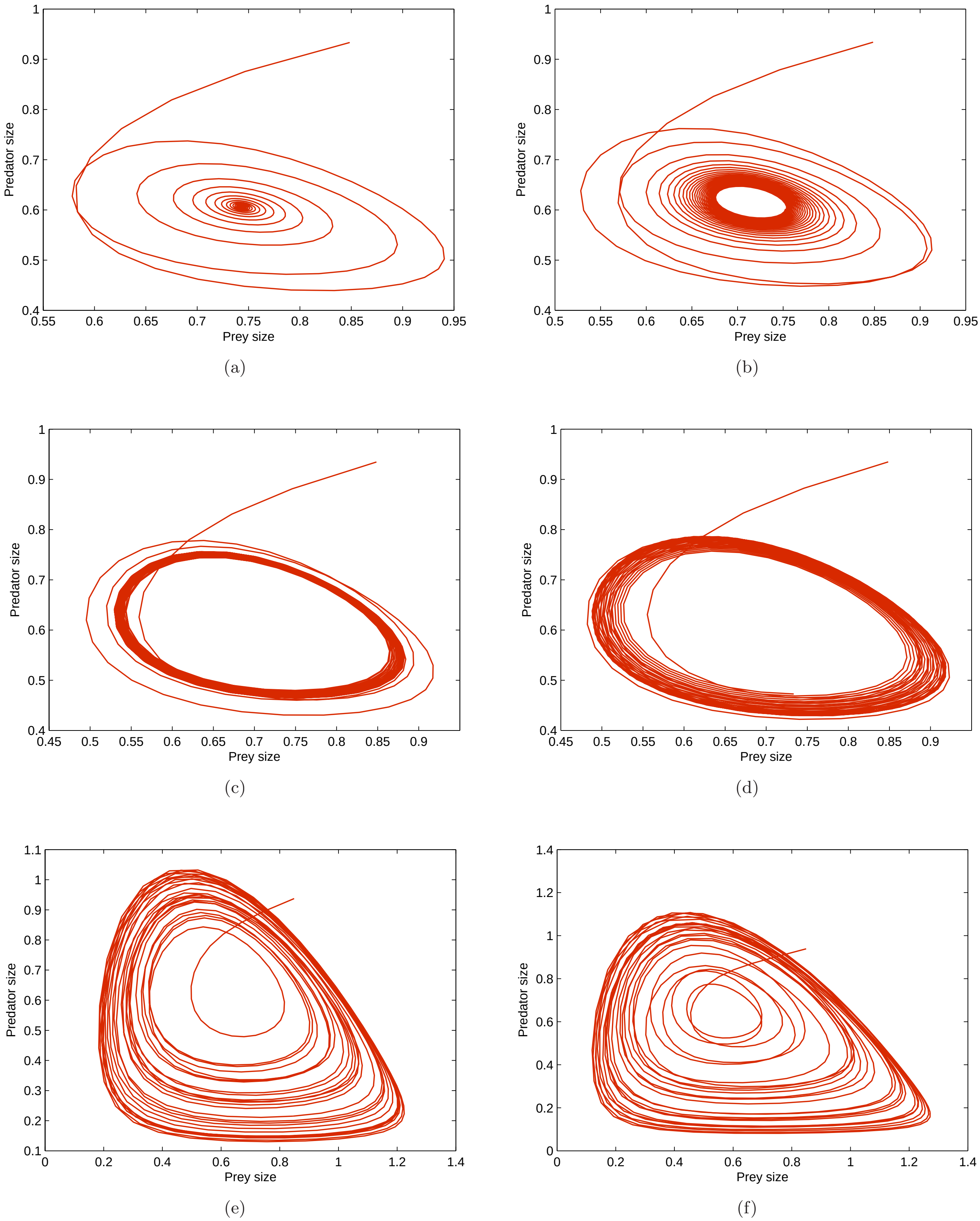


Fig. 3. Phase portraits in the uv-plane for system (6) for $t = 800$, showing the transition to chaos at different bifurcation parameters for (a) $c_3 = 0.255$, (b) $c_3 = 0.245$, (c) $c_3 = 0.238$, (d) $c_3 = 0.235$, (e) $c_3 = 0.2$ and (f) $c_3 = 0.18$. With diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$.

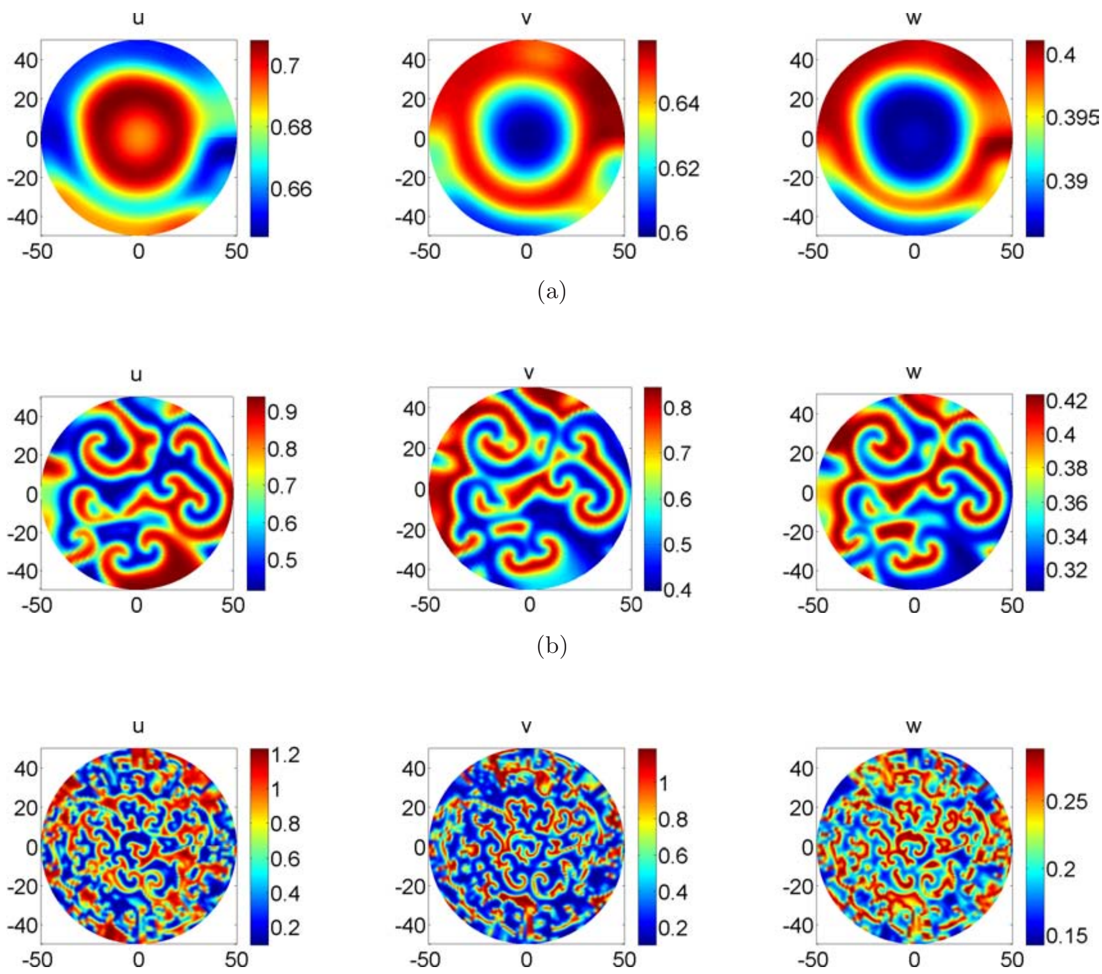


Fig. 4. Spatial distributions of prey (first column), predator (second column) and top predator (third column) are population densities of the spatial system (6). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$, for fixed time $t = 12\,000$ at different bifurcation parameters for (a) $c_3 = 0.23$, (b) $c_3 = 0.22$ and (c) $c_3 = 0.15$.

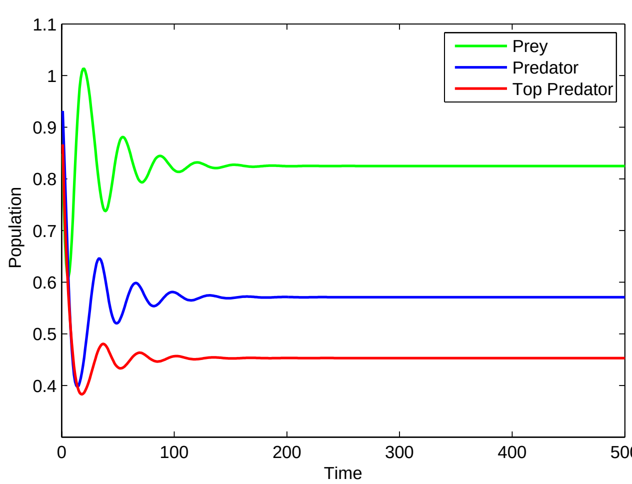


Fig. 5. Stable behavior of prey–predator populations and top predator for $c_3 = 0.26$ with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$ and $t = 500$, the other parameters are fixed in (62).

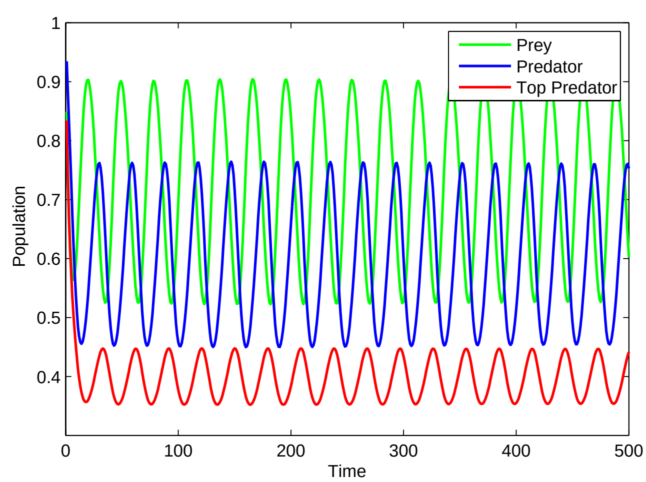


Fig. 6. Solution curves of prey–predator populations and top predator with time and $c_3 = 0.23$ with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$ and $t = 500$, the other parameters are fixed in (62).

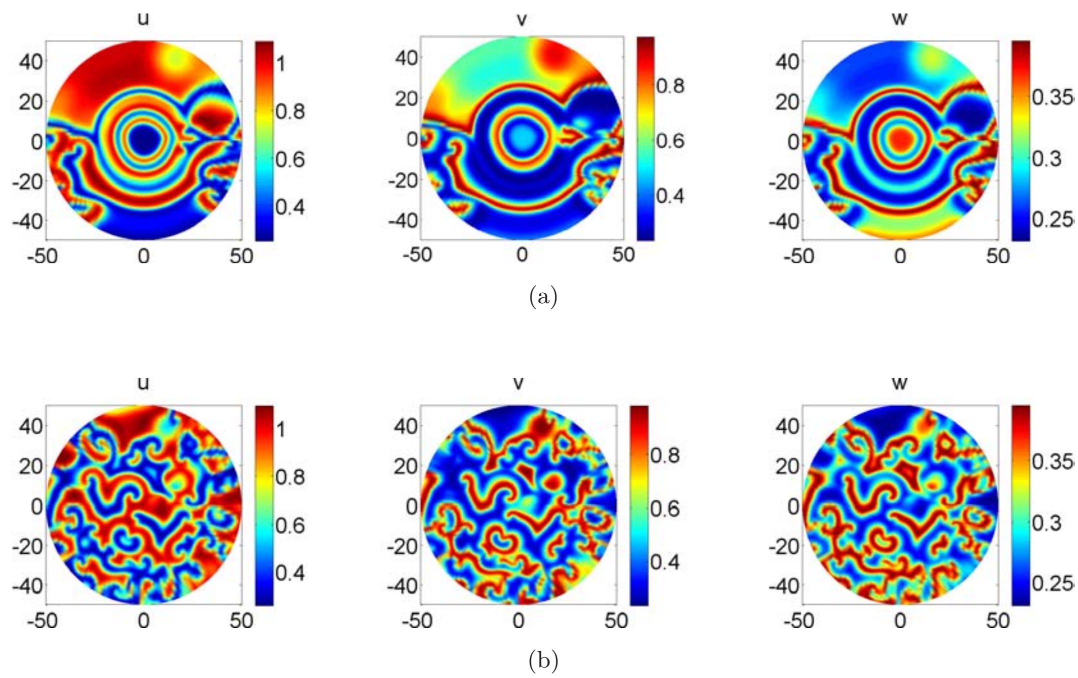


Fig. 7. Spatial distributions of prey (first column), predator (second column) and top predator (third column) are population densities of the spatial system 6. Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$, at different time levels: (a) $t = 1000$ and (b) $t = 4000$.

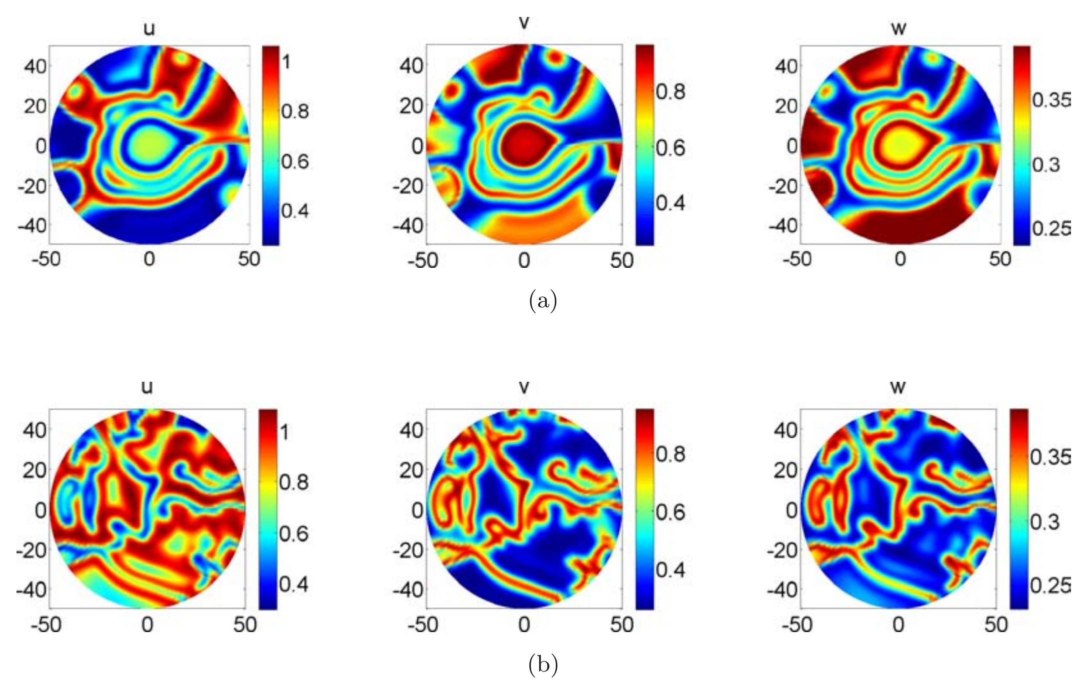
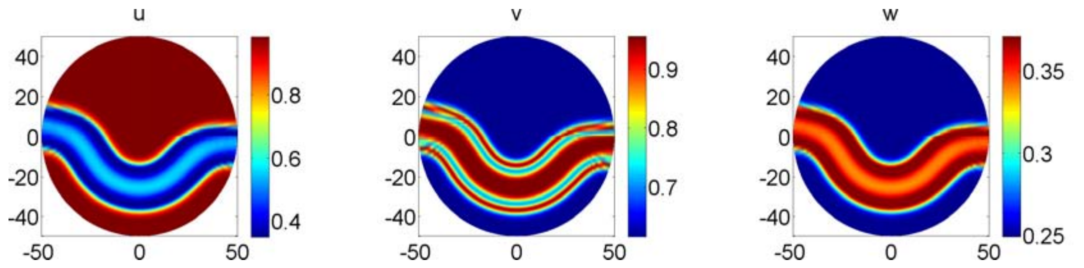
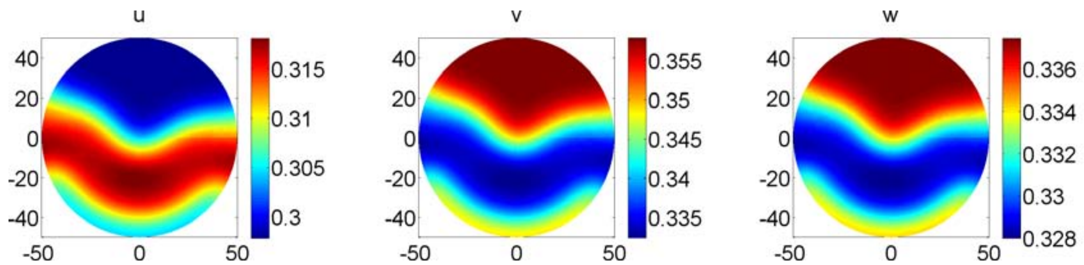


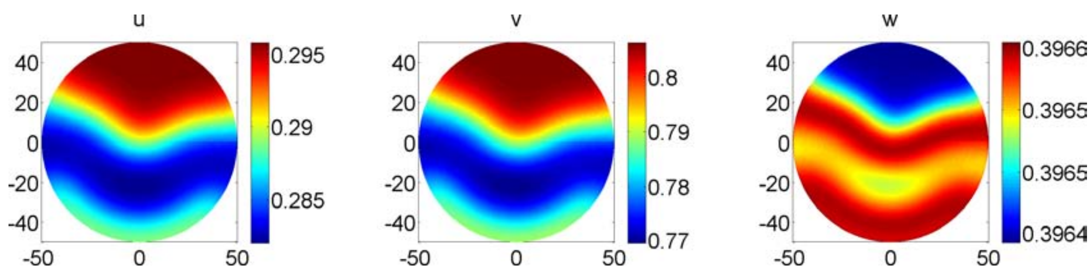
Fig. 8. Spatial distributions of prey (first column), predator (second column) and top predator (third column) are population densities of the spatial system (6). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.2$, $\delta_2 = 0.1$ and $\delta_3 = 0.5$, at different time levels: (a) $t = 1000$ and (b) $t = 4000$.



(a)



(b)



(c)

Fig. 9. Spatial distributions of prey (first column), predator (second column) and top predator (third column) are population densities of the spatial system (6). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 2.5$, $\delta_2 = 1.25$ and $\delta_3 = 6$, at different time levels: (a) $t = 1000$, (b) $t = 3000$ and (c) $t = 4000$.

are fixed in (62) and the initial conditions are given in (61).

In what follows, we plot for different values of c_3 the curves of densities of prey, predator and top predator with respect to time. The initial condition is given by (61). From Fig. 5, we observe that populations of prey, predator and top predator converge to their steady state and $E^* = (u^*, v^*, w^*)$ is locally asymptotically stable for system (3). If we decrease the value of the control parameter $c_3 = 0.03$, the equilibrium $E^* = (u^*, v^*, w^*)$ loses its stability and becomes unstable (Fig. 6). Now, we show the pattern formation with different time steps and diffusion coefficients. The initial condition given by (61) and the other parameters are fixed in (62). By varying the diffusivity coefficients parameters, we observed that the spatial structure changes over the times of the spatial system. In Fig. 7, we conclude that the labyrinth patterns prevail in the whole domain. If we increase the diffusion coefficients, we obtain the mixtures "labyrinth-spots" (see Fig. 8). In Fig. 9, with an increase of the diffusion coefficients, we conclude that the spot patterns of spatial are over the whole domain. From this figure, it is observed that the higher diffusivity coefficients stabilized the spatial system.

6. Conclusion

We have examined the dynamic behavior of a three-species food chain namely prey, predator and top predator. We have considered the response function as the modified Leslie–Gower Holling-type II schemes. We have obtained the boundedness conditions for the solutions, local stability by employing Routh–Hurwitz criteria, global analysis by constructing Lyapunov function. Furthermore, we have observed that the positive equilibrium enters a Hopf type bifurcation under the conditions of Theorem 3. We have proved the conditions that enable the occurrence of Hopf bifurcation and Turing instability in the circular spatial domain.

Finally, we carried out numerical simulations to substantiate the analytical findings. Our numerical analysis showed that the dynamics of a population may dramatically be affected by small changes in the value of the parameter c_3, at the same time we can see in Fig. 3 by plotting the phase portraits with different parameter sets and we have shown the transition to chaos. The nature of spatial patterns with respect to time (see Fig. 1) which leads to the formation of spatiotemporal chaos and the effect of the bifurcation parameter c_3 on the nature of the pattern (see Fig. 4) have been observed. Firstly, for $c_3 = 0.23$ the spot patterns of spatial over the whole domain [see Fig. 4(a)] are demonstrated, if we increase the parameter bifurcation for $c_3 = 0.15$, we find a mixture of stripe-spots patterns of spatial over the whole domain [see Fig. 4(b)]. Also, we have shown that if a diffusivity coefficient increases, then the population densities become uniform and spot pattern is observed (see Fig. 9).

Our result analytically and numerically show that the modeling by reaction–diffusion equation is a suitable tool for studying basic mechanisms of spatiotemporal dynamics in the real world food chain system.

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