

## OSCILLATIONS INDUCED BY QUIESCENT ADULT FEMALE IN A MODEL OF WILD Aedes Aegypti Mosquitoes

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**ABSTRACT.** *Aedes aegypti* (*Ae. aegypti*: mosquito) is a known vector of several viruses including yellow fever, dengue, chikungunya and zika. In the current paper, we present a delayed mathematical model describing the dynamics of *Ae. aegypti*. Our model is governed by a system of three delay differential equations modeling the interactions between three compartments of the *Ae. aegypti* life cycle (females, eggs and pupae). By using time delay as a parameter of bifurcation, we prove stability/switch stability of the possible equilibrium points and the existence of bifurcating branch of small amplitude periodic solutions when time delay crosses some critical value. We establish an algorithm determining the direction of bifurcation and stability of bifurcating periodic solutions. In the end, some numerical simulations are carried out to support theoretical results..

**1. Introduction and mathematical model.** Yellow fever, dengue, chikungunya and zika are dangerous diseases which still lack a cure, and they are spreading through a specific type of vector, the *Ae. aegypti* mosquito (called also yellow fever mosquito or *stegomyia aegypti*). The dynamics of *Ae. aegypti* are particularly impacted by climatic conditions like high temperature and frequent precipitation and the most affected areas are the ones with tropical climates which are favorable to its growth. However, if current studies and predictions about global climate change are now occurring, many new areas might start facing these diseases threat (see [15, 16, 35, 24, 3]). *Ae. aegypti* life cycle begins with eggs, which hatch into

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larvae under suitable conditions. The larvae develop into pupae that mature and emerge into adults (females and males). It's known that, males do not bite humans or animals of any species they live on fruit, unlike females they feed not only on fruit, but also on human or animal blood to provide protein for their eggs. After biting, female mosquitoes rest while their eggs develop. Once eggs are fully developed, the females oviposit and then proceed to find another blood meal thus completing the mosquito feeding cycle (see [13, 33, 28]).

According to the study given in [18]; in natural conditions females mosquitoes are fertilized very soon after their emergence by a single male during their lifetime, the seed being stored in three spermathecae. Once fully mature, females enter a phase of quiescence until a host is available for a blood meal. It's well known that quiescence is a type of dormancy which is defined by a physiological phenomenon which suspends development in some stage of life cycle and plays an important role in the maintenance of natural populations. Dormancy can take place in both plants and animals, in insects: it can occur in all different stages of life cycle (eggs, larvae, pupae and adult). This phenomenon can be influenced by climactic changes and was firstly described as a period of inactivity caused by low temperatures. According to definition of dormancy in insects, it is divided into two types: diapause and quiescence (see [23, 10]).

From Vacus [31], it's difficult to appreciate the longevity of females in the wild (see [31] page 17, paragraph 4). It seems to be usually in the range of 3 weeks to 3 months. However, in some species, a few individuals spend the unfavorable season (winter in temperate climate, dry season in tropical climate) in the state of quiescent adults that can survive thus 4 to 6 months. Therefore, we consider only the part which dies at the previous times  $t - \tau$  ( $\tau > 0$ ). Since in this work the mortality of females at the moment  $t$  is neglected, indeed, we assume it is very small compared to that of mosquitoes more old (those born at  $t - \tau$  and who have had a period of quiescence). We assume also that the delay for the female needs to produce new eggs is equal to time delay  $\tau$ . In [12], authors gave a review on the impact of dormancy (diapause and quiescence) on the life history of mosquitoes. In [26], authors studied the effect of quiescence of eggs population on the parameters and on the populations size.

Our goal in this work is to study how quiescent mature females affect the dynamics of *aedes aegypti* population? and how quiescent phase can induce oscillations in all stages of life cycle?

In [20], the times series of the estimates of mosquito density and corresponding meteorological conditions in the eight training Chinese cities (Guangzhou, Foshan, Santou, Shenzhen, Zhogshan, Zhuhai, Putian, Xishuangbanna) for SIR model are drawn from 2005 to 2015. These densities are characterized by oscillations.

Oscillations of all populations of *aedes aegypti* mosquitoes are also observed in Villa Carolina (VC), San Juan city, Puerto Rico from October 2007 to December 2008 [1] and in Bangkok from November 2009 to April 2010 (see [18]).

A study of diel landing periodicity of adult female of *aedes aegypti* mosquitoes at indoor and outdoor urban sites in Woodbrook and at indoor and outdoor rural sites in Tableland, Trinidad, West Indies (January-July, 1999) is given in [6].

In [5], authors consider a model in which taking into account the sterile insect technique to reduce or eradicate the wild mosquito population and the role of time delay of releasing sterile mosquitoes. The proposed model is given by a system of

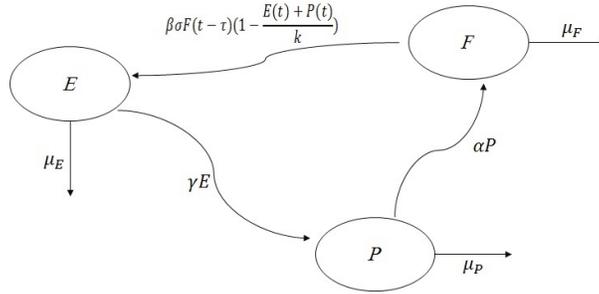


FIGURE 1. Schematic representation describing the interaction between females ( $F$ ), eggs ( $E$ ) and pupae ( $P$ ) of *Ae. aegypti* population.

delay differential equations and the existence of oscillations via Hopf bifurcation theory by using time delay as a parameter of bifurcation was studied.

To respond to our question, we propose a mathematical model describing the interactions between three stages of *aedes aegypti*: eggs, pupae and adult female in which we take into account the spent time (delay) of adult female in the quiescent phase.

Time delays of one type or another have been incorporated into biological models by many researchers, we refer to the monographs of Cushing [9], Gopalsamy [14], [19] and MacDonald [21] for general delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since time delay causes a stable equilibrium to become unstable and causes the populations to fluctuate.

Since only female mosquitoes are involved in the transmission of vector-borne diseases, the proposed model ignores males. Then, we consider three compartments of the *Ae. aegypti* life cycle: females ( $F$ ), eggs ( $E$ ) and pupal ( $P$ ) (see, Fig. 1). From the description above, we develop the following system of differential equations to describe *Ae. aegypti* dynamics without movement:

$$\begin{cases} \frac{dF(t)}{dt} = \alpha P(t) - \mu_F F(t - \tau), \\ \frac{dE(t)}{dt} = \beta \sigma F(t - \tau) \left(1 - \frac{E(t) + P(t)}{k}\right) - (\gamma + \mu_E) E(t), \\ \frac{dP(t)}{dt} = \gamma E(t) - (\alpha + \mu_P) P(t), \\ F(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad E(0) = E_0 \geq 0, \quad P(0) = P_0 \geq 0, \end{cases} \quad (1)$$

where  $\sigma$  ( $\text{day}^{-1}$ ) the oviposition rate,  $\gamma$  ( $\text{day}^{-1}$ ) the developing rate from egg to larva,  $\alpha$  ( $\text{day}^{-1}$ ) is the developing rate from pupa to adult stage,  $\mu_E$  ( $\text{day}^{-1}$ ) is the eggs unviability rate,  $\mu_F$  ( $\text{day}^{-1}$ ) the death rate of the female mosquitoes,  $\mu_P$  ( $\text{day}^{-1}$ ) the death rate of the pupae,  $\beta$  ( $\text{day}^{-1}$ ) the eggs fraction that turns into female mosquitoes,  $k$  charge capacity of the breeding places.

Our results demonstrate that in the case without quiescence (delay=0) of female mosquitoes, the stability of the equilibria depends on the population reproduction number. However small time of quiescence does affect this stability. But in the presence of large time of quiescence of female mosquitoes and for some parameters range, the size of each population exhibit oscillatory behavior.

The current work is organized as follows: In Section 2, we compute the population reproduction number and we prove stability/switch stability of the possible steady

states. In Sections 3 and 4, we show the existence of small amplitude periodic solutions when the delay crosses some critical value via Hopf bifurcation Theorem and stability of bifurcating branch of periodic solutions. In section 5, we prove how parameters can affect the populations sizes of mosquitoes. In sections 6 and 7, we give some numerical simulations supporting our theoretical results and conclusions.

**2. Stability analysis.** This section presents existence and stability results of system (1) of the steady states. An equilibrium point of a given system of equations ( $\dot{X}(t)$ ) (where  $X$  is a vector composed by state variables) is a steady-state solution, where  $X(t) = X^*$  for all  $t$ .

**Proposition 1.** *The model in (1) has exactly one equilibrium point on  $\partial\mathcal{D}$  given by  $E_0 = (0, 0, 0)$ . We label  $E_0$  the mosquito-free equilibrium point, where*

$$\mathcal{D} = \{(F, E, P) \in \mathbb{R}^3 / F \geq 0, E \geq 0, P \geq 0\}.$$

Next, we define the population reproduction number,  $R$ , as the expected number of female mosquitoes produced by a single female mosquito in her life time in the absence of density-dependence. In [32], a method for computing the reproduction number for epidemic models was developed. However, it can equivalently be used in ecological models where new births are treated as new infections. We determine the mosquito population reproduction number for model (1) using the next-generation technique.

Let  $X = (F, E, P) = (x_1, x_2, x_3)$ , the system (1) can be written as  $\dot{x}_i(t) = \mathcal{F}_i(X) - \mathcal{V}_i(X)$  for  $\tau = 0$  where  $\mathcal{F}_i$  is the rate of new recruitment (birth of eggs) in a compartment,  $\mathcal{V}_i = \mathcal{V}_i^- - \mathcal{V}_i^+$ , with  $\mathcal{V}_i^+$  being the rate of transfer of mosquitoes into a compartment and  $\mathcal{V}_i^-$  ( $i = 1, 2, 3$ ) is the rate of transfer of mosquitoes out

of the compartment. For this model  $\mathcal{F}$  and  $\mathcal{V}$  are given by:  $\mathcal{F} = \begin{pmatrix} \alpha P \\ 0 \\ 0 \end{pmatrix}$  and

$$\mathcal{V} = \begin{pmatrix} \mu_F F \\ (\gamma + \mu_E)E - \beta\sigma F \left(1 - \frac{E+P}{k}\right) \\ (\alpha + \mu_P)P - \gamma E \end{pmatrix}.$$

To obtain the next generation operator,  $\mathbf{FV}^{-1}$ , we calculate  $\mathbf{F}_{ij} = \frac{\partial \mathcal{F}_i}{\partial x_j}|_{E_0}$  and  $\mathbf{V}_{ij} = \frac{\partial \mathcal{V}_i}{\partial x_j}|_{E_0}$  to obtain

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{V} = \begin{pmatrix} \mu_F & 0 & 0 \\ \beta\sigma & \gamma + \mu_E & 0 \\ 0 & -\gamma & \alpha + \mu_P \end{pmatrix}$$

The population reproduction number,  $R$ , is the spectral radius of the next generation operator,  $\rho(\mathbf{FV}^{-1})$ . This value is given by

$$R = \frac{\alpha\gamma\sigma\beta}{\mu_F(\gamma + \mu_E)(\alpha + \mu_P)} \quad (2)$$

where  $\frac{\gamma}{\gamma + \mu_E}$  is the probability that an egg will succeed in becoming a (female) mosquito.  $\frac{\alpha}{\alpha + \mu_P}$  is the probability that a pupa will succeed in becoming an egg.

On the other hand,  $\frac{1}{\mu_F}$  is the average lifetime of a female mosquito and consequently  $\frac{\beta\sigma}{\mu_F}$  is the average number of eggs oviposited by each one.

The system of (1) has a persistent positive equilibrium solution  $E_1 = (F^*, E^*, P^*)$ , with its components given by

$$F^* = \frac{\alpha\gamma k(R-1)}{\mu_F R(\alpha + \mu_P + \gamma)}, \quad E^* = \frac{k(\alpha + \mu_P)(R-1)}{R(\alpha + \mu_P + \gamma)}, \quad P^* = \frac{\gamma k(R-1)}{R(\alpha + \mu_P + \gamma)},$$

with  $R$  given in equation (2) which exist in the interior of  $\mathcal{D}$  if  $R > 1$ .

**Theorem 2.1.** [29] *Suppose  $\tau = 0$ .*

*i) The mosquito-free equilibrium  $E_0$  is locally asymptotically stable when  $R < 1$  and unstable otherwise.*

*ii) If  $R > 1$ , the persistent positive equilibrium  $E_1$  is asymptotically stable.*

*iii) If  $R = 1$ ,  $E_0 = E_1$ .*

**Remark 1.** Note that for  $\tau = 0$  in Saulo et al. (2017) [29], Theorem 2.1 states that the equilibrium point  $E_0$  is asymptotically stable if  $\varphi(u) < 1$ . Then, the quantity  $\varphi(0)$  corresponds to  $R$  in our situation and we have the same results of stability.

**2.1. Local stability of mosquito-free equilibrium point.** The linearized system of system (1) around the mosquito-free equilibrium point  $E_0 = (0, 0, 0)$  is given by:

$$\frac{dX}{dt} = L_0 X(t) + L_\tau X(t - \tau)$$

$$\text{where } L_0 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & -(\gamma + \mu_E) & 0 \\ 0 & \gamma & -(\alpha + \mu_P) \end{pmatrix} \quad \text{and} \quad L_\tau = \begin{pmatrix} -\mu_F & 0 & 0 \\ \beta\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the associated characteristic equation is as follows

$$\mathcal{P}(\lambda) + \mathcal{Q}(\lambda)e^{-\lambda\tau} = 0$$

where

$$\begin{aligned} \mathcal{P}(\lambda) &= \lambda^3 + (\alpha + \mu_P + \gamma + \mu_E)\lambda^2 + (\alpha + \mu_P)(\gamma + \mu_E)\lambda, \\ \mathcal{Q}(\lambda) &= \mu_F\lambda^2 + \mu_F(\alpha + \mu_P + \gamma + \mu_E)\lambda + \mu_F(\alpha + \mu_P)(\gamma + \mu_E) - \beta\gamma\alpha\sigma. \end{aligned}$$

Define  $\mathfrak{F}$  by

$$\begin{aligned} \mathfrak{F}(y) &= |\mathcal{P}(iy)|^2 - |\mathcal{Q}(iy)|^2, \\ &= Y^3 + a_1 Y^2 + a_2 Y + a_3 \end{aligned}$$

where  $Y = y^2$  and

$$\begin{aligned} a_1 &= (\alpha + \mu_P)^2 + (\gamma + \mu_E)^2 - \mu_F^2, \\ a_2 &= (\alpha + \mu_P)^2(\gamma + \mu_E)^2 - \mu_F^2((\alpha + \mu_P)^2 + (\gamma + \mu_E)^2) - 2\mu_F\beta\gamma\alpha\sigma, \\ a_3 &= -(\mu_F(\alpha + \mu_P)(\gamma + \mu_E) - \beta\gamma\alpha\sigma)^2. \end{aligned}$$

Hence  $\mathfrak{F}$  is continue and  $\mathfrak{F}(0) = a_3 < 0$  and  $\lim_{Y \rightarrow +\infty} \mathfrak{F}(Y) = +\infty$ , then  $\mathfrak{F}$  crosses  $x$ -axis in some positive value. Then,  $E_0$  is asymptotically stable for delay smaller than a critical value of time delay and unstable otherwise (see [8, 4, 30]). As we are interested to the coexistence of populations of mosquitoes we will study the qualitative behavior and bifurcation at the persistent equilibrium point and not at the free equilibrium point. Next, we will study the local stability of the persistent equilibrium.

**2.2. Local stability of persistent positive equilibrium.** Linearizing system (1) around the persistent positive equilibrium  $E_1 = (F^*, E^*, P^*)$ , we obtain the following system

$$\frac{dX}{dt} = J_0 X(t) + J_\tau X(t - \tau)$$

where

$$J_0 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & -\left(\frac{\beta\sigma F^*}{k} + \gamma + \mu_E\right) & -\frac{\beta\sigma F^*}{k} \\ 0 & \gamma & -(\alpha + \mu_P) \end{pmatrix} \quad (3)$$

and

$$J_\tau = \begin{pmatrix} -\mu_F & 0 & 0 \\ \beta\sigma\left(1 - \frac{E^* + P^*}{k}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

The associated characteristic equation is given by

$$\Delta(\lambda) = \lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + (\mu_F \lambda^2 + \beta_1 \lambda + \beta_2) e^{-\lambda\tau} = 0 \quad (5)$$

where

$$\begin{aligned} \alpha_1 &= \frac{\beta\sigma F^*}{k} + \gamma + \mu_E + \alpha + \mu_P, \\ \alpha_2 &= \frac{\beta\sigma F^*}{k} (\alpha + \mu_P + \gamma) + (\gamma + \mu_E)(\alpha + \mu_P), \\ \beta_1 &= \mu_F \left( \frac{\beta\sigma F^*}{k} + \gamma + \mu_E + \alpha + \mu_P \right), \\ \beta_2 &= \mu_F (\alpha + \mu_P) \left( \gamma + \mu_E + \frac{\beta\sigma F^*}{k} \right) + \mu_F \frac{\beta\gamma\sigma F^*}{k} - \alpha\gamma(\gamma + \mu_E) \frac{E^*}{F^*}. \end{aligned}$$

Since the persistent positive equilibrium  $E_1$  is asymptotically stable for  $\tau = 0$  (see Theorem 2.1) and by the continuity property, it's still asymptotically stable for small  $\tau > 0$  (see [8, 4, 30]). To obtain the switch of stability, one needs to find a purely imaginary root for some critical value of  $\tau$ .

Let  $i\omega$  ( $\omega > 0$ ) be a root of Eq. (5), then we have

$$-i\omega^3 - \alpha_1 \omega^2 + i\alpha_2 \omega + (-\mu_F \omega^2 + i\beta_1 \omega + \beta_2)(\cos \omega\tau - i \sin \omega\tau) = 0. \quad (6)$$

Separating the real and imaginary parts, we find

$$\begin{cases} \alpha_1 \omega^2 = -\mu_F \omega^2 \cos \omega\tau + \beta_1 \omega \sin \omega\tau + \beta_2 \cos \omega\tau, \\ \omega^3 - \alpha_2 \omega = \mu_F \omega^2 \sin \omega\tau + \beta_1 \omega \cos \omega\tau - \beta_2 \sin \omega\tau. \end{cases} \quad (7)$$

By computation, we obtain

$$\omega^6 + (\alpha_1^2 - 2\alpha_2 - \mu_F^2) \omega^4 + (\alpha_2^2 + 2\mu_F \beta_2 + \beta_1^2) \omega^2 - \beta_2^2 = 0. \quad (8)$$

Let  $z = \omega^2$ , Eq. (8) becomes

$$h(z) = z^3 + (\alpha_1^2 - 2\alpha_2 - \mu_F^2) z^2 + (\alpha_2^2 + 2\mu_F \beta_2 + \beta_1^2) z - \beta_2^2 = 0. \quad (9)$$

As  $h(0) = -\beta_2^2 < 0$  and  $\lim_{z \rightarrow \infty} h(z) = +\infty$ , Eq. (9) has at least one positive root, denoted by  $z_0$ . Consequently, Eq. (8) has at least one positive root, denoted by  $\omega_0$ . This implies that the characteristic equation (5) has a pair of purely imaginary roots  $\pm i\omega_0$  at the critical values of time delay  $\tau = \tau_j$ , where

$$\tau_j = \frac{1}{\omega_0} \arccos \left( \frac{\beta_1 \omega_0^4 - \alpha_1 \mu_F \omega_0^3 + (\alpha_1 \beta_2 - \beta_1 \alpha_2) \omega_0^2}{\mu_F \omega_0^4 + (\beta_1^2 - 2\mu_F \beta_2) \omega_0^2 + \beta_2^2} \right) + \frac{2j\pi}{\omega_0}; \quad j = 0, 1, 2, \dots \quad (10)$$

**Lemma 2.2.** [27] Consider the transcendental equation:  $Q(\lambda) = \lambda^n + p_1^0 \lambda^{n-1} + \dots + p_{n-1}^0 \lambda + p_n^0 + e^{-\lambda \tau_1} (p_1^1 \lambda^{n-1} + \dots + p_{n-1}^1 \lambda + p_n^1) + \dots + e^{-\lambda \tau_s} (p_1^s \lambda^{n-1} + \dots + p_{n-1}^s \lambda + p_n^s)$ , where  $\tau_k > 0$  and  $p_j^k$ ,  $k = 1, \dots, s$ ;  $j = 1, \dots, n$  are constants. As  $(\tau_k)_{k=1, \dots, s}$  vary, the sum of the order of the zeros of the polynomial  $Q$  on the right half plane can change only if a zero appears on or crosses the imaginary axis.

Using Lemma 2.2, we obtain the following lemma.

**Lemma 2.3.** [30] As  $-\beta_2^2 < 0$ , all roots with positive real parts of Eq. (5) has the same sum to those of the same equation without delay for  $\tau \in [0, \tau_0)$ .

From Lemma 2.3, we have the following result.

**Corollary 1.** *i) The mosquito-free equilibrium point  $E_0$  is unstable for all  $\tau > 0$ .  
ii) If  $R > 1$ , the persistent positive equilibrium  $E_1$  is asymptotically stable for  $\tau < \tau_0$ .*

**3. Periodic solutions.** In this section, we study the existence of branch of periodic solutions bifurcating from the persistent positive equilibrium  $E_1$  by applying the Hopf bifurcation Theorem.

Let  $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$  be the eigenvalue of Eq. (5) such that  $\eta(\tau_0) = 0$  and  $\omega(\tau_0) = \omega_0$ .

Then, one needs to verify the transversality condition

$$\frac{d}{d\tau} \operatorname{Re} \lambda(\tau)|_{\tau=\tau_0} = \frac{d}{d\tau} \eta(\tau)|_{\tau=\tau_0} \neq 0.$$

**Lemma 3.1.** [36] Let  $z_0 = \omega_0^2$  and  $\beta_2 \neq \mu_F \omega_0^2$  and  $h'(z_0) \neq 0$ , where  $h$  is given by (9). Then

$$\left( \frac{d(\operatorname{Re} \lambda)}{d\tau} \right) \Big|_{\tau=\tau_j} \neq 0 \text{ and } \operatorname{sign} \left( \frac{d(\operatorname{Re} \lambda)}{d\tau} \right) \Big|_{\tau=\tau_j} = \operatorname{sign}(h'(z_0))$$

From Lemma 3.1 and Corollary 1, we deduce the following theorem.

**Theorem 3.2.** Suppose  $R > 1$  and  $h'(z_0) \neq 0$ , then, a Hopf bifurcation occurs at the persistent positive equilibrium  $E_1$  when  $\tau = \tau_j$ ,  $j = 0, 1, 2, \dots$ .

**4. Stability and direction of bifurcation.** In present section, we will determine some properties of Hopf bifurcation. The method we use is based on the normal form theory and the center manifold Theorem (see [17]). Let  $\tau = \tau_j + \mu$ ,  $\mu \in \mathbb{R}$  so that  $\mu = 0$  is Hopf bifurcation value for system (1). Let  $u_1(t) = F(t) - F^*(t)$ ,  $u_2(t) = E(t) - E^*(t)$ ,  $u_3(t) = P(t) - P^*(t)$  and  $x_i(t) = u_i(\tau t)$  for  $i = 1, 2, 3$ , system (1) will be written in  $C = C([-1, 0], \mathbb{R}^3)$  as follows

$$\frac{dx}{dt} = L_\mu x_t + f(\mu, x_t), \quad (11)$$

where  $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ ,  $x_t(\theta) = x(t + \theta)$ ,  $\theta \in [-1, 0]$ ,  $L_\mu : C \rightarrow \mathbb{R}^3$  and  $f : C \times \mathbb{R} \rightarrow \mathbb{R}^3$  are given by

$$L_\mu \phi = (\tau_j + \mu)[J_0 \phi(0) + J_\tau \phi(-1)], \quad (12)$$

and

$$f(\mu, \phi) = (\tau_j + \mu) \begin{pmatrix} 0 \\ \beta\sigma\phi_1(-1) \left( \frac{\phi_2(0) + \phi_3(0)}{k} \right) \\ 0 \end{pmatrix} \quad (13)$$

for  $\phi = (\phi_1, \phi_2, \phi_3)^T \in \mathbb{C}$ , where  $J_0$  and  $J_\tau$  are given in (3) and (4).

By Reisz Representation Theorem, there exists a function  $\eta(\theta, \mu)$  whose components are of bounded variation for  $\theta \in [-1, 0]$  such that

$$\mathbb{L}_\mu\phi = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta). \quad (14)$$

In view of equation (12), we have

$$d\eta(\theta, \mu) = (\tau_j + \mu)[J_0\delta(\theta) + J_\tau\delta(\theta + 1)], \quad (15)$$

where  $\delta(\theta)$  is Dirac delta function.

For  $\phi \in C^1([-1, 0], \mathbb{R}^3)$ , define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu)\phi(s) = \mathbb{L}_\mu\phi, & \theta = 0, \end{cases} \quad (16)$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\phi, \mu), & \theta = 0. \end{cases} \quad (17)$$

Then, system (11) is equivalent to

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t. \quad (18)$$

For  $\psi \in C^1([-1, 0], (\mathbb{R}^3)^*)$ , the adjoint operator  $A^*$  of  $A = A(0)$  is defined as

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1, 0), \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t) & s = 0. \end{cases} \quad (19)$$

Consider the following bilinear product

$$\langle \psi, \phi \rangle = \bar{\psi}(0) \cdot \phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \quad (20)$$

where  $\eta(\theta) = \eta(\theta, 0)$ .

If  $\mu = 0$ , system (18) undergoes a Hopf bifurcation and  $\pm i\omega_0\tau_j$  are eigenvalues of  $A$  and  $A^*$ . Let us now computing the eigenvectors of  $A$  and  $A^*$  corresponding to  $+i\omega_0\tau_j$  and  $-i\omega_0\tau_j$  respectively.

Let  $q(\theta) = (\alpha_1, \alpha_2, \alpha_3)^T e^{i\omega_0\tau_j\theta}$  be the eigenvector of  $A$  corresponding to eigenvalue  $i\omega_0\tau_j$  then

$$Aq(\theta) = i\omega_0\tau_j q(\theta). \quad (21)$$

For  $\theta = 0$ , we have

$$\tau_j \begin{pmatrix} i\omega_0 & \mu_F e^{-i\omega_0 \tau_j} & -\alpha \\ -\beta\sigma \left(1 - \frac{E^* + P^*}{k}\right) e^{-i\omega_0 \tau_j} & i\omega_0 + \left(\frac{\beta\sigma F^*}{k} + \gamma + \mu_E\right) & \frac{\beta\sigma F^*}{k} \\ 0 & -\gamma & i\omega_0 + (\alpha + \mu_P) \end{pmatrix} \times q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (22)$$

Solving system (21) and choosing  $\alpha_1 = 1$ , we get

$$\alpha_2 = \frac{i\omega_0 + \alpha + \mu_P}{\gamma} \alpha_3; \quad \alpha_3 = \frac{i\omega_0 \gamma}{\alpha \gamma - \mu_F e^{-i\omega_0 \tau_j} (i\omega_0 + \alpha + \mu_P)}.$$

Similarly assuming that  $q^*(s) = D(\alpha_1^*, \alpha_2^*, \alpha_3^*)^T e^{i\omega_0 \tau_j s}$  be the eigenvector of  $A^*$  corresponding to eigenvalue  $-i\omega_0 \tau_j$ , where the value of  $D$  is chosen to guarantee  $\langle q^*, q \rangle = 1$ . Then

$$A^* q^*(s) = -i\omega_0 \tau_j q^*(s), \quad (23)$$

where

$$\alpha_1^* = 1; \quad \alpha_2^* = -\frac{i\omega_0 e^{i\omega_0 \tau_j}}{\beta\sigma \left(1 - \frac{E^* + P^*}{k}\right)}; \quad \alpha_3^* = \alpha - \frac{F^* i\omega_0 e^{i\omega_0 \tau_j}}{(k - E^* - P^*)(i\omega_0 - \alpha - \mu_P)}.$$

To assure  $\langle q^*(s), q(\theta) \rangle = 1$ , one needs to determine the value of  $D$ .

From equation (20), we have

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}_2^*, \bar{\alpha}_3^*)^T e^{i\omega_0 \tau_j (\xi - \theta)} \\ &\quad \times d\eta(\theta)(1, \alpha_1, \alpha_2)^T e^{i\omega_0 \tau_j \xi} d\xi, \\ &= \bar{D}(1 + \alpha_2 \bar{\alpha}_2^* + \alpha_3 \bar{\alpha}_3^*) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}_2^*, \bar{\alpha}_3^*)^T \\ &\quad \times e^{i\omega_0 \tau_j (\xi - \theta)} d\eta(\theta)(1, \alpha_1, \alpha_2)^T e^{i\omega_0 \tau_j \xi} d\xi, \\ &= \bar{D} \left(1 + \alpha_2 \bar{\alpha}_2^* + \alpha_3 \bar{\alpha}_3^* \right. \\ &\quad \left. - \alpha_1 \bar{\alpha}_2^* \tau_j \beta\sigma \left(1 - \frac{E^* + P^*}{k}\right) e^{-i\omega_0 \tau_j} + \alpha_3 \bar{\alpha}_2^* \tau_j \frac{\beta\sigma F^*}{k} \right). \end{aligned}$$

By computation, we obtain

$$D = \left[1 + \bar{\alpha}_2 \alpha_2^* + \bar{\alpha}_3 \alpha_3^* - \bar{\alpha}_1 \alpha_2^* \tau_j \beta\sigma \left(1 - \frac{E^* + P^*}{k}\right) e^{i\omega_0 \tau_j} + \bar{\alpha}_3 \alpha_2^* \tau_j \frac{\beta\sigma F^*}{k}\right]^{-1}.$$

Next, we compute the coordinates on the center manifold  $\mathcal{C}_\mu$  at  $\mu = 0$  (see [17]). Let  $x_t$  be a solution of equation (18) when  $\mu = 0$  and define

$$z(t) = \langle q^*, x_t \rangle \quad \text{and} \quad W(t, \theta) = x_t(\theta) - 2\text{Re}z(t)q(\theta). \quad (24)$$

On the center manifold  $\mathcal{C}_0$ , we have

$$W(t, \theta) = W(z, \bar{z}, \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \dots, \quad (25)$$

where  $z$  and  $\bar{z}$  are the local coordinates on the center manifold  $\mathcal{C}_0$  in direction of  $q^*$  and  $\bar{q}^*$ . Noting that  $W$  is also real if  $x_t$  is real.

Now, we consider only the real solution  $x_t \in \mathcal{C}_0$  of equation (11), which gives

$$\dot{z} = i\omega_0\tau_j z + \bar{q}^*(0) \cdot f(0, W(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\}) = i\omega_0\tau_j z + g(z, \bar{z}), \quad (26)$$

where

$$g(z, \bar{z}) = \bar{q}^*(0) \cdot f_0(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2}. \quad (27)$$

From (25), we get

$$\begin{aligned} x_t(\theta) &= W(z, \bar{z}, \theta) + 2\text{Re}zq(\theta), \\ &= W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + z(1, \alpha_2, \alpha_3)^T e^{i\omega_0\tau_j\theta} \\ &\quad + \bar{z}(1, \bar{\alpha}_2, \bar{\alpha}_3)^T e^{-i\omega_0\tau_j\theta} + \dots \end{aligned}$$

So that

$$x_{jt} = W_{20}^j(\theta) \frac{z^2}{2} + W_{11}^j(\theta) z\bar{z} + W_{02}^j(\theta) \frac{\bar{z}^2}{2} + z\alpha_j e^{i\omega_0\tau_j\theta} + \bar{z}\bar{\alpha}_j e^{-i\omega_0\tau_j\theta}, \quad (28)$$

for  $j = 1, 2, 3$ .

From (27), we have

$$g(z, \bar{z}) = \tau_j \bar{D}(1, \bar{\alpha}_2^*, \bar{\alpha}_2^*) \cdot \begin{pmatrix} 0 \\ \beta\sigma x_{1t}(-1) \left( \frac{x_{2t}(0) + x_{3t}(0)}{k} \right) \\ 0 \end{pmatrix}. \quad (29)$$

Simplifying equation (29) and comparing with equation (27), we get

$$\begin{cases} g_{20} = 4\tau_j \bar{D} \frac{\alpha_2^* \alpha_2 \beta \sigma}{k} e^{-i\omega_0\tau_j}, \\ g_{11} = 4\tau_j \bar{D} \frac{\alpha_2^* \beta \sigma}{k} (\bar{\alpha}_2 e^{-i\omega_0\tau_j} + \alpha_2 e^{i\omega_0\tau_j}), \\ g_{02} = 4\tau_j \bar{D} \frac{\alpha_2^* \beta \sigma}{k} \bar{\alpha}_2 e^{i\omega_0\tau_j}, \\ g_{21} = \tau_j \bar{D} \frac{\alpha_2^* \beta \sigma}{k} \left( 2\bar{\alpha}_2 W_{20}^{(1)}(-1) + (W_{20}^{(2)}(0) + W_{20}^{(3)(0)}) e^{i\omega_0\tau_j} \right. \\ \quad \left. + 8\alpha_2 W_{11}^{(1)}(-1) + 2(W_{11}^{(2)}(0) + W_{11}^{(3)(0)}) e^{i\omega_0\tau_j} \right). \end{cases} \quad (30)$$

In order to compute  $g_{21}$ , we need to compute  $W_{20}(\theta)$  and  $W_{11}(\theta)$ .

From equations (24) and (26), we have

$$\begin{aligned} \dot{W} &= \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q}, \\ &= \begin{cases} AW - 2\text{Re}\bar{q}^* \cdot f_0q(\theta), & \theta \in [-1, 0), \\ AW - 2\text{Re}\bar{q}^* \cdot f_0q(\theta) + f_0, & \theta = 0, \end{cases} \\ &= AW + H(z, \bar{z}, \theta), \end{aligned} \quad (31)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \quad (32)$$

In view of (25), we get

$$AW = AW_{20}(\theta) \frac{z^2}{2} + AW_{11}(\theta) z\bar{z} + AW_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \quad (33)$$

Differentiating both sides of (25) with respect to  $t$ , we have

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}}. \quad (34)$$

According to (33) and (34) and comparing the coefficients, we obtain

$$(A - 2i\omega_0\tau_j)W_{20} = -H_{20}, \quad (35)$$

$$AW_{11} = -H_{11}. \quad (36)$$

For  $\theta \in [-1, 0)$  and from (34), we get

$$\begin{aligned} H(z, \bar{z}, \theta) &= -\bar{q}^*(0) \cdot f_0 q(\theta) - q^*(0) \cdot \bar{f}_0 \bar{q}(\theta), \\ &= -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta), \\ &= -(g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta))\frac{z^2}{2} - (g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta))z\bar{z} + \dots \end{aligned} \quad (37)$$

Comparing the coefficients with (32) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \quad (38)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (39)$$

From (35) and (38) and the definition of  $A$ , we have

$$W'_{20} = 2i\omega_0\tau_j W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \quad (40)$$

Note that  $q(\theta) = q(0)e^{i\omega_0\tau_j\theta}$ , then

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_j}q(\theta) + \frac{i\bar{g}_{02}}{3\omega_0\tau_j}\bar{q}(\theta) + E^1 e^{2i\omega_0\tau_j\theta}. \quad (41)$$

Similarly from (4) and (39) and the definition of  $A$ , we have

$$W'_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \quad (42)$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_j}q(\theta) + \frac{i\bar{g}_{11}}{\omega_0\tau_j}\bar{q}(\theta) + E^2, \quad (43)$$

where  $E^1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})$  and  $E^2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in \mathbb{R}^3$  are constant vectors to be determined.

From the definition of  $A$  and (35), we deduce

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_j W_{20}(0) - H_{20}(0), \quad (44)$$

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0). \quad (45)$$

From equation (4) and (32), we get

$$H_{20} = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \tau_j \begin{pmatrix} 0 \\ \beta\sigma \left( \frac{\alpha_2 + \alpha_3}{k} \right) \\ 0 \end{pmatrix}, \quad (46)$$

and

$$H_{11} = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_j \begin{pmatrix} 0 \\ \beta\sigma \left( \frac{Re\{\alpha_2\} + Re\{\alpha_3\}}{k} \right) \\ 0 \end{pmatrix}. \quad (47)$$

Using (41) and (46) in (44), we have

$$\left( 2i\omega_0\tau_j I - \int_{-1}^0 e^{2i\omega_0\tau_j\theta} d\eta(\theta) \right) E_1 = \tau_j \begin{pmatrix} 0 \\ \beta\sigma \left( \frac{\alpha_2 + \alpha_3}{k} \right) \\ 0 \end{pmatrix}, \quad (48)$$

that is,

$$\begin{pmatrix} 2i\omega_0 & \mu_F e^{-i\omega_0\tau_j} & -\alpha \\ -\beta\sigma \left( 1 - \frac{E^* + P^*}{k} \right) e^{-i\omega_0\tau_j} & 2i\omega_0 + \left( \frac{\beta\sigma F^*}{k} + \gamma + \mu_E \right) & \frac{\beta\sigma F^*}{k} \\ 0 & -\gamma & 2i\omega_0 + (\alpha + \mu_P) \end{pmatrix} \times \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \beta\sigma \left( \frac{\alpha_2 + \alpha_3}{k} \right) \\ 0 \end{pmatrix}. \quad (49)$$

Similarly using equation (43) and (47) in (45), we get

$$\begin{pmatrix} 0 & \mu_F & -\alpha \\ -\beta\sigma \left( 1 - \frac{E^* + P^*}{k} \right) e^{-i\omega_0\tau_j} & \frac{\beta\sigma F^*}{k} + \gamma + \mu_E & \frac{\beta\sigma F^*}{k} \\ 0 & -\gamma & (\alpha + \mu_P) \end{pmatrix} \times \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \beta\sigma \left( \frac{Re\{\alpha_2\} + Re\{\alpha_3\}}{k} \right) \\ 0 \end{pmatrix}. \quad (50)$$

Solving systems (49) and (50), we find  $E^1$  and  $E^2$ , using these values we determine  $W_{20}$  and  $W_{11}$  and hence  $g_{21}$ .

From the above analysis, we compute the elements of bifurcation (see Hassard et al. [17]).

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_j} \left( g_{11}g_{20} - 2\|g_{11}\|^2 - \frac{\|g_{02}\|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tau_j)\}}, \\ \beta_2 &= 2Re\{c_1(0)\}, \\ T_2 &= \frac{-Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_j)\}}{\omega_0\tau_j}. \end{aligned}$$

Then, we deduce the following result which allow us to determine the stability of bifurcating periodic solutions and the direction of bifurcation.

**Theorem 4.1.** *i) If  $\mu_2 > 0$  ( $\mu_2 < 0$ ), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for  $\tau > \tau_j$  ( $\tau < \tau_j$ ).  
 ii) The bifurcating solutions are orbitally stable (unstable) if  $\beta_2 < 0$  ( $\beta_2 > 0$ ).  
 iii) The period increases (decreases) if  $T_2 > 0$  ( $T_2 < 0$ ).*

Next, we use the following parameters values cited in Table 1

TABLE 1. Parameters estimation

parameter	value	reference
$\gamma$	0.90	Assumed
$\alpha$	0.6	Assumed
$\sigma$	4	[29]
$\beta$	0.4	[29]
$\mu_F$	0.16	Assumed
$\mu_E$	0.15	Assumed
$\mu_P$	0.01	Assumed
$k$	500	Assumed

**5. Sensitivity analysis.** The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter, it is used to discover parameters that have a high impact on  $R$ . If the variable is a differentiable function of the parameter, the sensitivity index is then defined using partial derivatives.

**Definition 5.1.** [25] The normalized forward sensitivity index of a variable  $f$  that depends differentiability on a parameter  $p$  is defined as:

$$\Upsilon_p^f = \frac{\partial f}{\partial p} \frac{p}{f}$$

To reduce the rate of the reproduction of mosquitoes, we need to know the importance of different factors involved in its production. So we investigate the sensitivity indices of the reproduction number  $R$  relative to the parameters involved. The sensitivity indices of the reproduction number  $R$  is given in Table 1. These indices allow us to measure the relative change in  $R$  with the change in a parameter. Using these indices, we find the parameters that highly effect  $R$ , and need to be targeted by intervention strategies. The effects of parameters are illustrated in Figs. 2-6 To identify a control strategies, we focus to control the parameters with highest sensitivity index. When a parameter with sensitivity index 1, increases by 10% would decrease (increase)  $R$  by 10%. From Table 2,  $\sigma$  and  $\beta$  have a highest sensitivity index +1 which means if we increase both of them by 10%, the quantity of  $R$  will increase by 10%. As the sensitivity index of  $\mu_F$  equal to  $-1$ , then if we increase it by 10%,  $R$  will decrease by 10%. If the sensitivity index of  $\gamma$  (resp.  $\mu_E$ ) that is +0,1428 (resp.  $-0,1428$ ) increase by 10%,  $R$  will increase (resp. decrease) by 1.428%. The sensitivity index of  $\alpha$  (resp.  $\mu_P$ ) that is +0,336 (resp.  $-0,336$ ) increase by 10%, then  $R$  will increase (resp. decrease) by 3.36%. Therefore, we deduce that the parameters  $\sigma$  and  $\beta$  and  $\mu_F$  have a highest sensitivity indices, then to reduce the reproduction of mosquitoes one needs to control both of them by decreasing  $\sigma$  and  $\beta$  and increasing  $\mu_F$ .

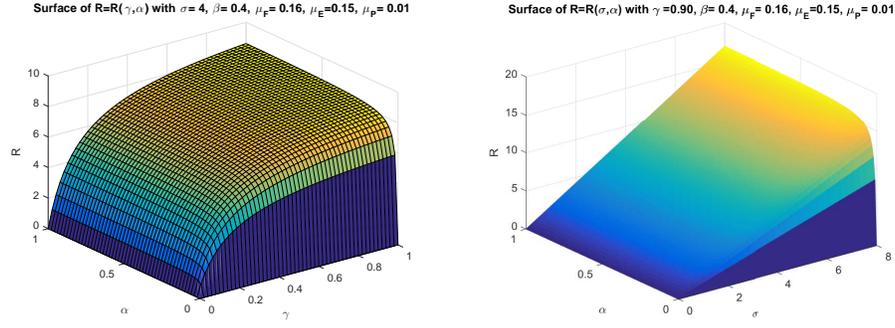


FIGURE 2. Surfaces representing the effect of parameters (left  $\gamma$  and  $\alpha$ ) and (right  $\sigma$  and  $\alpha$ ) on the variations of the population reproduction number  $R$

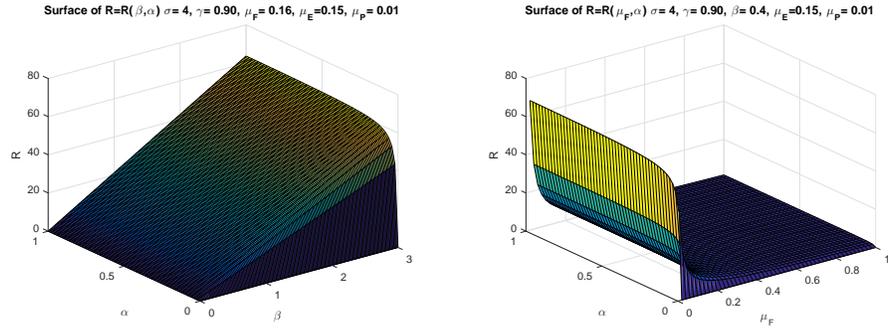


FIGURE 3. Surfaces representing the effect of parameters (left  $\beta$  and  $\alpha$ ) and (right  $\mu_F$  and  $\alpha$ ) on the variations of the population reproduction number  $R$

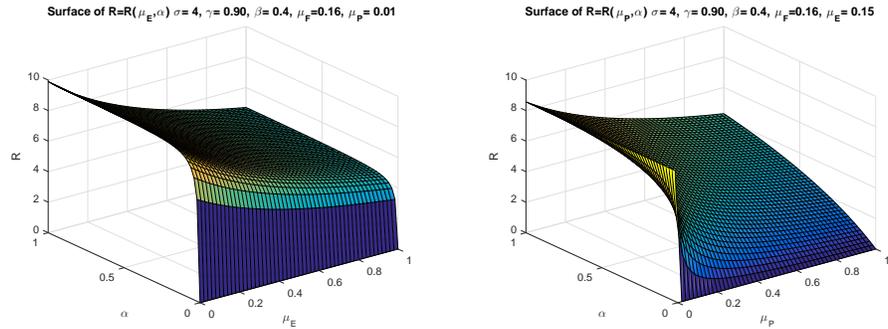


FIGURE 4. Surfaces representing the effect of parameters (left  $\mu_E$  and  $\alpha$ ) and (right  $\mu_P$  and  $\alpha$ ) on the variations of the population reproduction number  $R$

TABLE 2. The sensitivity indices of the population reproduction number  $R$

Parameter	Sensitivity index	Index at parameters value
$\gamma$	$\frac{\mu_E}{\gamma + \mu_E}$	+0.1428
$\alpha$	$\frac{\mu_P}{\alpha + \mu_P}$	+0.366
$\sigma$	+1	+1
$\beta$	+1	+1
$\mu_F$	-1	-1
$\mu_E$	$-\frac{\mu_E}{\gamma + \mu_E}$	-0.1428
$\mu_P$	$-\frac{\mu_P}{\alpha + \mu_P}$	-0.336

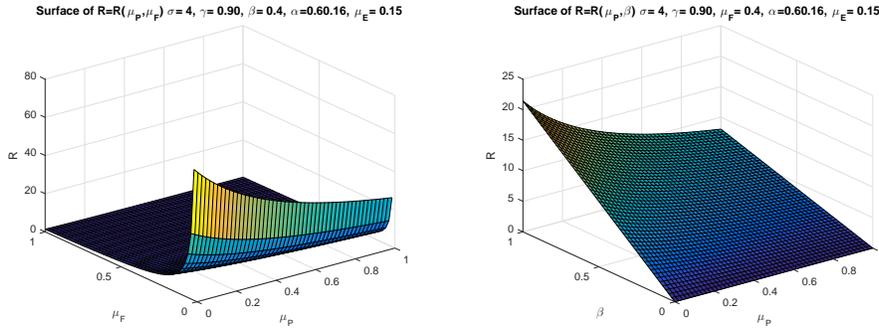


FIGURE 5. Surfaces representing the effect of parameters (left  $\mu_P$  and  $\mu_F$ ) and (right  $\mu_P$  and  $\beta$ ) on the variations of the population reproduction number  $R$

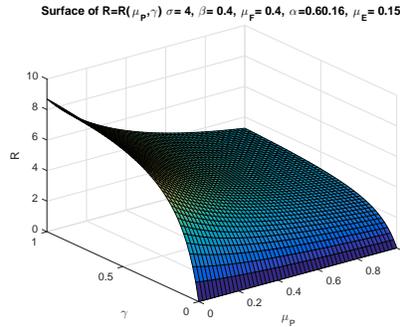


FIGURE 6. Surface representing the effect of parameters  $\mu_P$  and  $\gamma$  on the variations of the population reproduction number  $R$

**6. Numerical simulations.** In this section, we aim to provide a numerical simulation to substantiate the theoretical results established in the previous sections by using Matlab Software with the parameters given in Table (1).

i) For  $\tau = 0$  and  $\mu_F = 5$  and with ode45 package, we have  $R = 0.269$  and  $E_0$  is the only steady state which is asymptotically stable (see Figure 7).

ii) For  $\tau = 0$  and  $\mu_F = 0.16$ , smallwe have  $R = 8.43$  and  $E_0$  and  $E_1 = (984.99, 178.02, 262.66)$  are the steady states of the model. The trivial one is unstable and the non-trivial one is asymptotically stable (see Figure 8).

For  $\tau > 0$  and with the parameters cited in Table (1) and dde package we find:  $R = 8.43 > 1$ ,  $E_1 = (984.99, 178.02, 262.66)$ ,  $\omega_0 = 0.1386$ ,  $\tau_j = 10.35 + \frac{2j\pi}{\omega_0}$ . Then, the transversality condition is verified as follows  $sign\{\frac{d(Re\lambda)}{d\tau}\}_{\tau=\tau_j} = sign(0.0064)$ .

For  $\tau = 5 < \tau_0$  the equilibrium point  $E_1$  is asymptotically stable and unstable for  $\tau = 10.85 > \tau_0$  (see Figure 9). These results of stability and instability are confirmed by Mikhailov hodograph (Figure 10) and periodic solutions appear for  $\tau_0 = 10.35$  and  $\tau_0 + \varepsilon = 10.85$  (see Figures 11-12). If we increase the value of time delay, the periodic solutions disappear and chaotic solutions appear for  $\tau_0 + \varepsilon = 11.15$  (Figure 13).

By computation we find that  $c_1(0) = 2,8191.10^{-5} + i9,3145.10^{-5}$ ,  $\mu_2 = -4,4.10^{-3}$ ,  $\beta_2 = 5,6382.10^{-5}$  and  $T_2 = -3,7608.10^{-5}$  which imply that the Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for  $\tau < \tau_j$  and the bifurcating solutions are orbitally unstable and the period is decreasing.

With DDE-BIFTOOLS package, we plot the bifurcation diagram (the variation of the amplitude of periodic solutions with respect to time delay  $\tau$ ) (see Figure 14) which illustrate all branches of bifurcation.

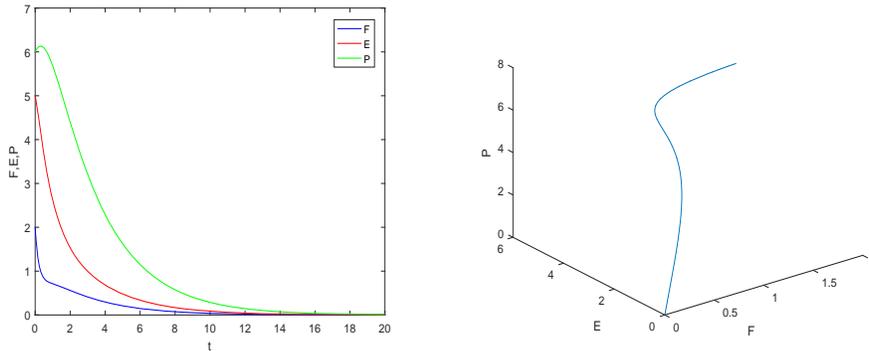


FIGURE 7. Stability of  $E_0$  for  $\tau = 0$  in  $(t, FEP)$  plane (left) and in  $(F, E, P)$  space (right) and non existence of  $E_1$  for  $\mu_F = 5$  and  $R = 0.269$ .

**7. Conclusions.** In this work, we studied (both analytically and numerically) the dynamics of the wild aedes aegypti mosquitoes with quiescent female phase by incorporating the time delay of quiescence. Our obtained results have shown that the growth time of quiescence of the female mosquito population has no destabilizing effect on the solution's behavior in the case when the population reproduction number  $R$  is small than 1 and in this case all solutions starting near the free equilibrium approach to this equilibrium. But if the population reproduction number  $R$  is greater than 1, the time of quiescence (delay) destabilize the solution's behavior and the free equilibrium becomes unstable and the persistent equilibrium is stable for some time of quiescence smaller than some critical value and solutions become to oscillate around this equilibrium. it can be understood that the wild mosquitoes coexist for a long or short term.

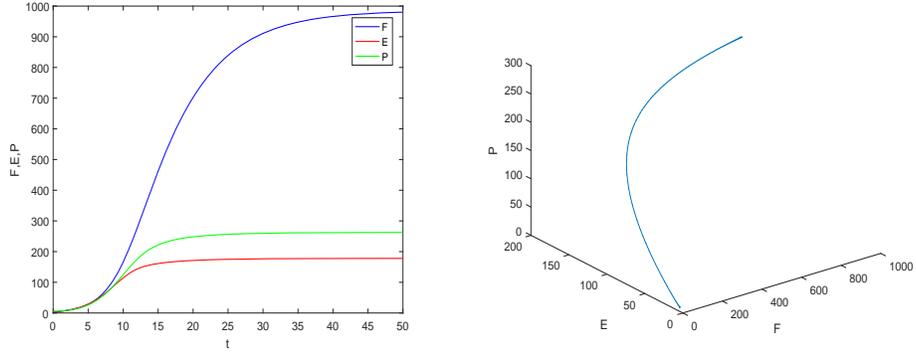


FIGURE 8. Instability of  $E_0$  and stability of  $E_1$  for  $\tau = 0$  in  $(t, FEP)$  plane (left) and in  $(F, E, P)$  space (right) with  $R = 8.43$ .

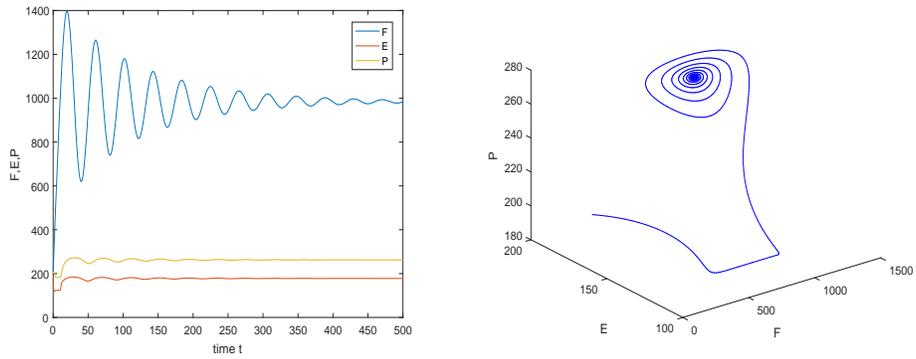


FIGURE 9. Instability of  $E_0$  and stability of  $E_1$  for  $\tau = 5$  in  $(t, FEP)$  plane (left) and in  $(F, E, P)$  space (right).

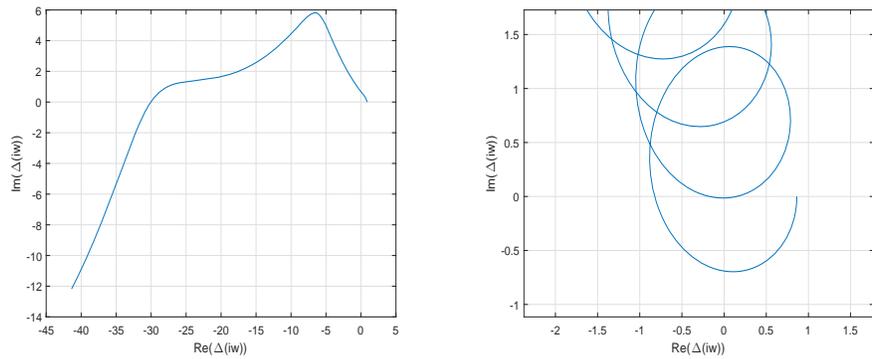


FIGURE 10. Mikhailov hodograph indicating the stability of  $E_1$  for  $\tau = 5$  (left) and instability of  $E_1$  for  $\tau = 50$  (right). Note that, The steady state  $E_1$  is stable if the curve crosses imaginary axis above zero (at all crossing points).

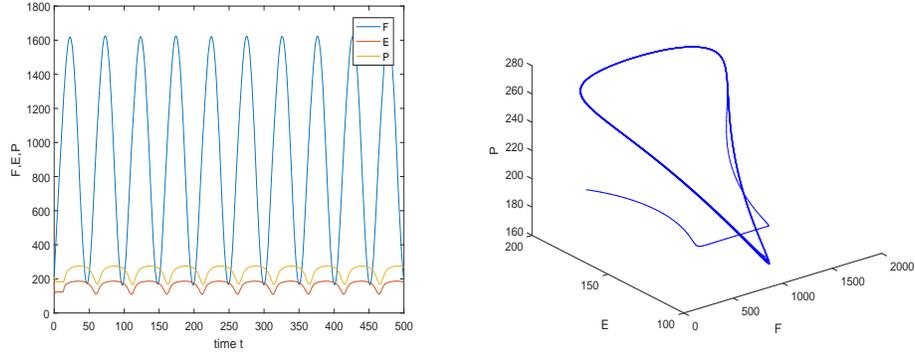


FIGURE 11. Periodic solution bifurcated from the steady state  $E_1$  for  $\tau = \tau_0 = 10.35$  in  $(t, FEP)$  plane (left) and in  $(F, E, P)$  space (right).

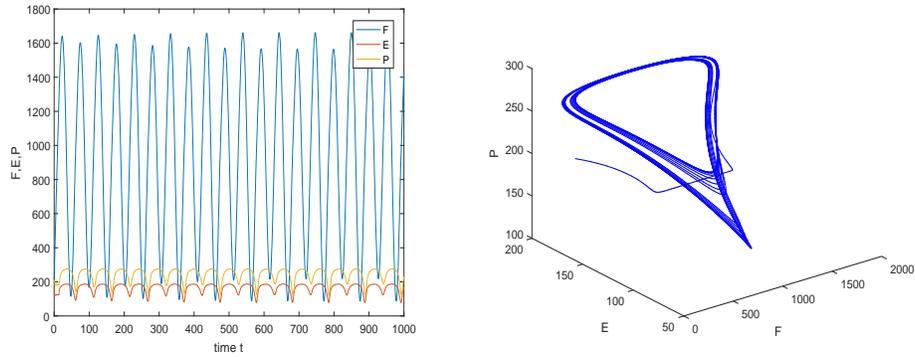


FIGURE 12. Periodic solution bifurcated from the steady state  $E_1$  for  $\tau = \tau_0 + \epsilon = 10.85$  with  $\epsilon = 0.5$  in  $(t, FEP)$  plane (left) and in  $(F, E, P)$  space (right).

As the time delay varies, we have established the existence and stability of periodic solutions. These obtained results describe the equilibrium of system process. In particular, when a stable periodic orbit exists, it can be understood that the three populations of female, eggs and pupae coexist for a long term and when an unstable periodic orbit exists and the three populations coexist for short term. The insertion of time of quiescence (delay) of female mosquitoes and the conditions on the parameters of the system are important in controlling the development and progression of mosquitoes.

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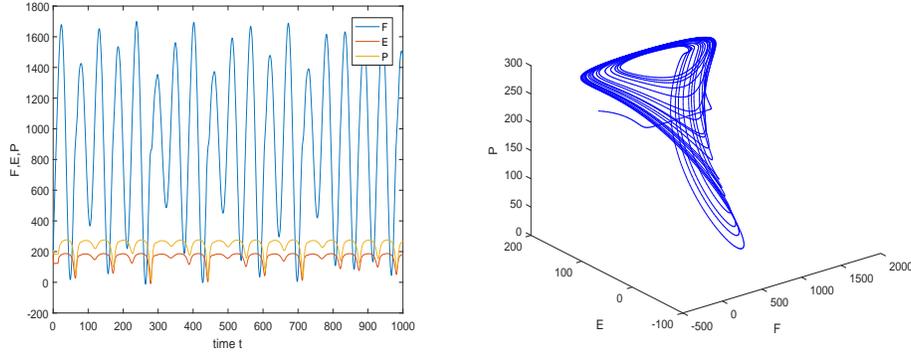


FIGURE 13. Chaotic solution bifurcated from the steady state  $E_1$  for  $\tau = \tau_0 + \epsilon = 11.15$  with  $\epsilon = 0.8$  in  $(t, FEP)$  plane (left) and in  $(F, E, P)$  space (right).

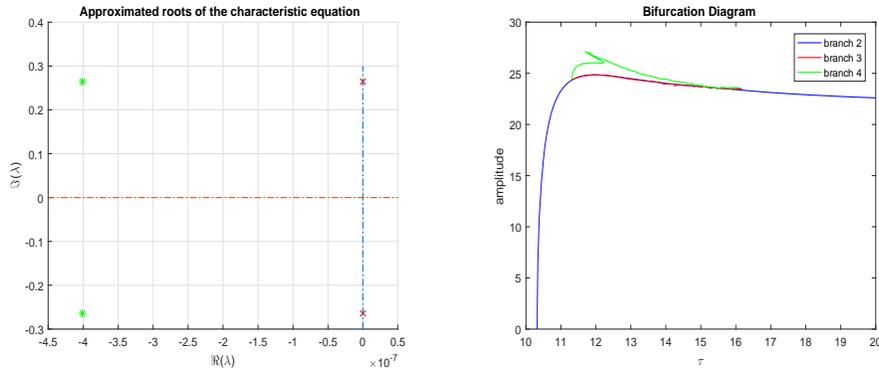


FIGURE 14. Existence of pair purely imaginary roots (left). The branches of bifurcation (diagram of bifurcation), branch 2 (blue line) is the branch of the periodic orbits that arise from the Hopf point, branch 3 (red line) is the branch of period doubling bifurcation point in branch 2 and branch 4 (green line) is the branch of period doubling bifurcation point in branch 3 (right). Note that, the branch 1 is the steady state  $E_1$  with amplitude 0.

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