Permanence and Extinction of a Diffusive Predator–Prey Model with Robin Boundary Conditions

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Abstract

The main concern of this paper is to study the dynamic of a predator–prey system with diffusion. It incorporates the Holling-type-II and a modified Leslie–Gower functional responses under Robin boundary conditions. More concretely, we study the dissipativeness of the system by using the comparison principle, and we derive a criteria for permanence and for predator extinction.

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1. Introduction, Mathematical Model and Preliminaries

1.1. Introduction and Mathematical Model

One of the dominant themes in both ecology and mathematical ecology is the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics. The investigations on predator–prey models have been mainly developed during the last decades, and more realistic models are derived in view of laboratory experiments and observations (see for example Chen and Shi 2012; Ko and Ryu 2006; Moussaoui and Bouguima 2016; Nindjin et al. 2006; Nindjin and Aziz-Alaoui 2008; Pao 1982; Saez and Gonzalez-Olivares 1999; Tanner 1975; Ye and Li 1990 and references cited therein). Aziz-Alaoui and Daher (2003) performed a global analysis of predator–prey system without diffusion. Concretely, they studied the following system:

$$\begin{cases} \frac{dU}{dT} = U(r_1 - b_1 U) - \frac{a_1 UV}{U+k_1}, \\ \frac{dV}{dT} = \left(r_2 - \frac{a_2 V}{U+k_2}\right) V, \\ U(0) = U_0 > 0, \quad V(0) = V_0 > 0, \end{cases} \tag{1}$$

where $U(T)$ and $V(T)$ are the population densities of the prey and predator species at time T, respectively, and $r_1, r_2, b_1, a_1, a_2, k_1, k_2$ are positive constants. These parameters are defined as follows: r_1 is the growth rate of prey U, b_1 measures the strength of intraspecific competition among individuals of species U, and it is related to the carrying capacity of the prey, a_1 is the maximum value of the per capita reduction rate of U due to V, k_1 (respectively k_2) measures the extent to which environment provides protection to prey U (respectively, to the predator V), r_2 describes the growth rate of V, and a_2 has a similar meaning to a_1. The model (1) is proposed based on the biological fact that the predator V is more capable of switching from its favorite food (the prey U) to other food options, thus it has better ability to survive when the prey population is low. The historical origin and applicability of this model is discussed in detail in Aziz-Alaoui (2002), Aziz-Alaoui and Daher (2003), Daher (2004), Nindjin et al. (2006), see also Nindjin and Aziz-Alaoui (2008), Letellier and Aziz-Alaoui (2002), Letellier et al. (2002).

For simplicity, we nondimensionalize system (1) with the following scaling,

$$t = r_1 T, u(t) = \frac{b_1}{r_1} U(T) \quad \text{and} \quad v(t) = \frac{a_2 b_1}{r_1 r_2} V(t),$$

to obtain the form,

$$\begin{cases} \frac{du}{dt} = u(1-u) - \frac{auv}{u+e_1}, \\ \frac{dv}{dt} = b\left(1 - \frac{v}{u+e_2}\right)v, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{cases} \tag{2}$$

where

$$a = \frac{a_1 r_2}{a_2 r_1}, \; b = \frac{r_2}{r_1}, \; e_1 = \frac{b_1 k_1}{r_1} \quad \text{and} \quad e_2 = \frac{b_1 k_2}{r_1}.$$

In Aziz-Alaoui and Daher (2003), the boundedness of solutions, the existence of positive invariance attracting set and the global stability of the coexisting interior equilibrium of system (2) are studied. See also Daher Okiye and Aziz-Alaoui (2003). Later, Nindjin et al. (2006) studied the global stability and persistence of the corresponding delayed system by using Lyapunov functional. The existence of periodic solutions and their stability are studied in Yafia et al. (2008, 2007), by considering the delay as a parameter of bifurcation.

Recently, Camara and Aziz-Alaoui (2008a, b) and Abid et al. (2015a, b) studied the diffusive version with Newman boundary conditions, and analyzed the stability, traveling wave, Turing and Hopf bifurcations.

It is much suitable to consider a reaction diffusion system, subject to general boundary conditions, we use here the Robin Boundary conditions and study the permanence condition of the following strongly-coupled PDE system:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = u\Big(1-u\Big) - \frac{auv}{u+e_1} & x \in \Omega,\ \ t>0 \\ \frac{\partial v}{\partial t} - d_2 \Delta v = bv\Big(1-\frac{v}{u+e_2}\Big) & x \in \Omega,\ \ t>0 \\ \beta u + (1-\beta)\frac{\partial u}{\partial \eta} = 0 & x \in \partial\Omega,\ \ t>0, \\ \gamma v + (1-\gamma)\frac{\partial v}{\partial \eta} = 0 & x \in \partial\Omega,\ \ t>0, \end{cases}$$

The spatial population densities of the prey and the predator species are respectively denoted by $u(x, t)$ and $v(x, t)$, $x \in \Omega$, where Ω is a bounded domain in $\mathbb{R}$ with smooth boundary, and $t > 0$. Δ denotes the Laplacian operator, the parameters $d_1,\ d_2 > 0$ are the diffusion coefficients of the corresponding species. $\beta, \gamma \in [0, 1]$ describes how likely it is for an individual reaching $\partial\Omega$ to leave Ω, $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on the boundary $\partial\Omega$.

Remark on the boundary conditions (BC)

– *Dirichlet boundary conditions*

The first observation is that the case $\beta = \gamma = 1$, corresponds to Dirichlet, used when the region Ω is rounded by a hostile environment for species.

– *Neumann boundary conditions*

The second observation is that in the case $\beta = \gamma = 0$ corresponding to no-flux (i.e., Neumann or reflecting) boundary conditions, species do not leave their domain (because of security or isolation) and do not cross the frontier, there is no diffusion along the boundaries, and any solution of the system without diffusion (1) is also solution of the system with diffusion (3).

– *Robin boundary conditions*, which we use in the present paper.

In general, in the case of Robin BCs, see Eq. (3), the value of β (resp. γ) represent the balance between the tendency of the prey (resp. the predator) to remain in its domain when it is close to the boundary, and its tendency to disappear beyond the boundary.

The system we study in the present paper may, for example, be considered as a representation of a snake-peacock food chain, nature abounds in systems which exemplify this model, see Aziz-Alaoui (2002), Letellier and Aziz-Alaoui (2002), Singh and Gakkhar (2014). Besides, as the model (1) is proposed based on the biological fact that the predator V is more capable of switching from its favorite food (the prey U) to other food options, and thus it has better ability to survive when the prey population is low, see Aziz-Alaoui (2002), Daher Okiye and Aziz-Alaoui (2003), it seems more reasonable to use Robin BCs that allow this balance at the edge of the domain. But, in doing so, we also consider, through the use of Robin's BCs, that the prey can also escape from its domain. Therefore, the model describes a situation where favorable/unfavorable conditions exist, for each specie, on a boundary of the domain. For motivation,[1] and for those specific questions, see especially the work of Bassett et al. (2017) and Kurowski et al. (2017), where movement toward a subsidy or toward/away from favorable/unfavorable regions are considered, assuming those features are on the interior of the domain. For more information about Robin BCs, see also Dai et al. (2015), Wang et al. (2012) for example.

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Now, in order to study the solution of (3) we consider initial conditions of the form,

$$u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 \;\; x \in \Omega, \tag{4}$$

where $u_0(x)$ and $v_0(x)$ are non-negative continuous functions. The existence and nonexistence of positive solutions of the corresponding stationary problem have been analyzed in some papers, see for example Abid et al. (2015a), Chen and Wang (2008) in the case of homogeneous Neumann boundary conditions, or Abid et al. (2015b), Camara and Aziz-Alaoui (2008a), Yafia et al. (2008) for the case of homogeneous Dirichlet boundary conditions. Other works exist, see for example Upadhyay et al. (2009, 2008) and references therin cited.

In the present article, we discuss the dynamics of (3) in terms of permanence, dissipativity and extinction of the predator ; we prove that the permanence of this system is fully determined by the signs of generalized principal eigenvalues.

The paper is organized as follows: Sect. 1.2 2.1 is devoted to some preliminaries, which are needed in next section. In Sect. 2, some conditions for the ultimate boundedness of solutions and permanence of this system are established. A sufficient condition of extinction of predator population is also given. We end with a brief section on conclusion.

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2. Permanence and Extinction

2.1. Preliminaires

The analysis of (3) uses a number of results mainly due to Cantrell and Cosner (1999, 2001, 2003, 1991, 1996) about single reaction–diffusion equations and related eigenvalue problems which we shall state below, and which for the sake of readability, we recall here.

Lemma 1 (Cantrell and Cosner 2003) *If* $m(x)$ *is continuous on* Ω *and positive on an open subset of* Ω, *the eigenvalue problem,*

$$\begin{cases} \Delta\Phi + \lambda m(x)\Phi = 0 \quad \text{in } \Omega \quad \beta\Phi + (1-\beta)\frac{\partial\Phi}{\partial n} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5}$$

has a unique positive principal eigenvalue $\lambda_1^+(m,\beta)$ *which admits a positive eigenfunction. The eigenvalue problem,*

$$\begin{cases} d\Delta\Psi + m(x)\Psi = \sigma\Psi \quad \text{in } \Omega \quad \beta\Psi + (1-\beta)\frac{\partial\Psi}{\partial n} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{6}$$

has a unique principal eigenvalue $\sigma_1(d,m,\beta)$ *which admits a positive eigenfunction. We have* $\sigma_1(d,m,\beta) > 0$ *if and only if* $d\lambda_1^+(m,\beta) < 1$.

These results follow from general results given in Cantrell and Cosner (2003), Hess (1991).

Lemma 2 (Cantrell and Cosner 2003) *Suppose that* $f(x, w)$ *is a smooth and decreasing function in* w, *with* $f(x,0) > 0$ *on an open subset of* Ω. *Suppose further that there exists a constant* K *so that* $f(x,w) < 0$ *for* $w > K$. *Then the equation,*

$$\begin{cases} \frac{\partial w}{\partial t} = d\Delta w + f(x,w)w \quad \text{in } \Omega \times (0,\infty) \\ \beta w + (1-\beta)\frac{\partial w}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{7}$$

has a unique positive equilibrium $\overline{w}$, *which is globally attracting for all solutions, if and only if* $d\lambda_1^+(f(x,0),\beta) < 1$ *(which is equivalent to* $\sigma_1(d, f(x,0),\beta) > 0$*). If* $d\lambda_1^+(f(x,0),\beta) \geq 1$ *then all positive solutions of* (7) *approach zero as* $t \to \infty$.

The case of Dirichlet boundary conditions is treated in Cantrell and Cosner (2003, 1991, 1998). A version of this result for the time-periodic case is given in Hess

(1991).

Remark 1 The condition $d\lambda_1^+(f(x,0),\beta) < 1$ is equivalent to $\sigma_1(d, f(x,0), \beta) > 0$ by Lemma 1.

2.2. Dissipativity

In this subsection, we will show that any nonnegative solution $(u(x, t), v(x, t))$ of (3) lies in a certain bounded region as $t \to \infty$ for all $x \in \Omega$.

Theorem 1 *All the solutions of* (3) *are nonnegative and defined for all* $t > 0$. *Furthermore, the nonnegative solution* $(u(x, t), v(x, t))$ *of* (3) *satisfies*

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x, t) \le 1,$$
$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x, t) \le 1 + e_2.$$

Proof The nonnegativity of the solutions of (3) is obvious since the initial value is nonnegative. We only consider the second part of the theorem. From the first equation of the system (3) it follows that,

$$\frac{\partial u}{\partial t} - d_1 \Delta u \le u\Big(1 - u\Big) \qquad x \in \Omega, \quad t > 0, \tag{8}$$

as long as u is defined as a function of t. Let z be the solution of the equation,

$$z'(t) = z(1 - z), \quad z(0) = \max_{x\in\bar{\Omega}} u(x, 0). \tag{9}$$

From the comparison principle Friedman (1964), we obtain $u(x, t) \le z(t)$. Now, taking into account that for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $z(t) < 1 + \epsilon$ for any $t \ge T_\epsilon$, which in turn implies that $u(x, t)$ is defined for all $t \ge 0$, and

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x, t) \le 1.$$

Having in mind that for a given $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $u(x, t) < 1 + \epsilon$ for any $x \in \Omega$ and $t \ge T_\epsilon$, and by using the second equation of (3), we get

$$\frac{\partial v}{\partial t} - d_2 \Delta v = bv\Big(1 - \frac{v}{u + e_2}\Big) \le bv\Big(1 - \frac{v}{1 + \epsilon + e_2}\Big) \tag{10}$$

for any $x \in \Omega$ and $t \geq T_\epsilon$, let z be the solution of the following initial value problem,

$$z'(t) = bz\Big(1 - \frac{z}{1 + \epsilon + e_2}\Big), \quad z(T_\epsilon) = \max_{x \in \bar{\Omega}} v(x, 0). \tag{11}$$

Then,

$$\lim_{t \to \infty} z(t) = 1 + \epsilon + e_2$$

Hence, by using the comparison principle, we obtain that $v(x, t) \leq z(t)$; which implies that

$$\limsup_{t \to \infty} \max_{x \in \bar{\Omega}} v(x, t) \le 1 + e_2 .$$

This completes the proof. $\square$

Remark 2 An immediate consequence of the proof of the former result is that for any $\epsilon > 0$, the rectangle $[0, 1 + \epsilon) \times [0, 1 + e_2 + \epsilon)$ is an absorbing set for the system (3) in $\mathbb{R}^2_+$.

2.3. Permanence

In this subsection, we investigate the permanence of system (3), for this aim we use Hale and Waltman Theorem stated in the "Appendix" (Hale and Waltman 1989; Hale 1988). To formulate such a result, we need to interpret (3) as a semi-flow on an appropriate space. Let use denote,

$$Y_\beta = \left\{ u \in C^1(\overline{\Omega}) : \beta u + (1 - \beta)\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\},$$

and

$$Y_\beta^+ = \begin{cases} \left\{ u \in Y_\beta; u > 0 \text{ in } \overline{\Omega} \right\}, & \text{if } \beta < 1 \\ \left\{ u \in Y_\beta; u > 0 \text{ in } \overline{\Omega} \text{ and } \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega \right\}, & \text{if } \beta = 1. \end{cases}$$

Then let $Y_0 = Y_\beta^+ \times Y_\gamma^+, \quad Y = Y_0 \cup \partial Y_0 \subseteq [C^1(\overline{\Omega})]^2.$

Theorem 2

a) *The system* (3) *generates a dissipative semi-flow on Y for which* Y_0 *and* ∂Y_0 *are forward invariant.*

b) *If* $d_1 \lambda_1^+(1, \beta) \geq 1$ *(equivalently* $\sigma_1(d_1, 1, \beta) \leq 0$*)* *then the system* (3) *is not permanent.*

c) *If* $d_1 \lambda_1^+(1, \beta) < 1$ *(equivalently* $\sigma_1(d_1, 1, \beta) > 0$*)* *the system is permanent if and only if* $\sigma_1(d_2, b, \gamma) > 0$.

Proof a) The proof that (3) generates a semi-flow $\{S(t)\}_{t \geq 0}$ on Y, which is defined by $[S(t)\phi](x) = U(x, t; u_0, v_0)$ where $U(x, t; u_0, v_0)$ is the classical solution of system (3) with initial condition $(u_0(x), v_0(x))$, follows from the results given in Cantrell and Cosner (1991), thus it is omitted here.

A direct application of the maximum principle shows that $S(t)$ is positively invariant on ∂Y_0, which in turn implies that $S(t)$ is positively invariant on Y_0. Moreover, from Theorem 1, it follows that $S(t)$ is pointwise dissipative.

b) Now, to show that (3) cannot be permanent if $d_1 \lambda_1^+(1, \beta) \geq 1$, let's remark that if (u, v) is a nonnegative solution to (3) then u is a subsolution to

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) & \text{in } \Omega \times (0, \infty) \\ \beta u + (1-\beta)\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \tag{12}$$

It follows that if $\widetilde{u}$ is a solution of (12) with $\widetilde{u}(x, 0) = u_0(x)$ then $0 \leq u \leq \widetilde{u}$ for all $t > 0$. However, by Lemma 2, $\widetilde{u} \to 0$ as $t \to +\infty$ if $d_1 \lambda_1^+(1, \beta) \geq 1$, hence $u \to 0$ as $t \to +\infty$. Therefore, (3) is not permanent.

c) If $d_1 \lambda_1^+(1, \beta) < 1$ and $\sigma_1(d_2, b, \gamma) > 0$, then the strong maximum principle implies that any solution of (3) which lies in ∂Y_0 must be of the form $(u, 0)$ or $(0, v)$. For solutions of the form $(u, 0)$ approaching $E_1 = (\overline{u}, 0)$ as $t \to +\infty$ and solutions of the form $(0, v)$ approaching $E_2 = (0, \overline{v})$ as $t \to +\infty$ by Lemma 2. Henceforth, we conclude that $\omega(\partial Y_0) = \{E_1, E_2\}$. It follows from these structural features that $\omega(\partial Y_0)$ is isolated and acyclic. By choosing $M_1 = E_1$ and $M_2 = E_2$, then $M = M_1 \cup M_2$ is the covering required by Theorem 3 in "Appendix". Taking into account that $\omega(\partial Y_0)$ is positively invariant and the fact that the stable manifold of E_1 is the u-axis and the stable manifold of E_2 is the v-axis, we obtain $W^s(E_1) \cap Y_0 = \emptyset$ and $W^s(E_2) \cap Y_0 = \emptyset$.

Since all hypotheses of Theorem 3 are fulfilled, we may conclude that system (3) is permanent.

Inversely, suppose that system (3) is permanent, it remains to prove that $\sigma_1(d_2, b, \gamma) > 0$. Since the system (3) is permanent, it has a positive equilibrium (u^*, v^*). The equation for v^* is,

$$\begin{cases} d_2 \Delta v^* + v^* \left(b - \frac{bv^*}{u^*+e_2}\right) = 0 & \text{in } \Omega \\ \gamma v^* + (1-\gamma)\frac{\partial v^*}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \quad 13$$

Since $v^* > 0$ in Ω we have,

$$\sigma_1\left(d_2, b - \frac{bv^*}{u^* + e_2}, \gamma\right) = 0. \quad 14$$

However, $u^* > 0$ and $v^* > 0$, so that, on Ω we have,

$$b - \frac{bv^*}{u^* + e_2} < b,$$

and by standard monotonicity results for eigenvalues we get,

$$0 = \sigma_1\left(d_2, b - \frac{bv^*}{u^* + e_2}, \gamma\right) < \sigma_1(d_2, b, \gamma). \quad 15$$

Therefore so permanence implies that $\sigma_1(d_2, b, \gamma) > 0$. $\square$

2.4. Predator Extinction

In this subsection, we investigate sufficient conditions for predator extinction.

Proposition 1 *If* $\sigma_1(d_2, b, \gamma) \leq 0$, *(i.e.,* $d_2\lambda_1^+(b, \beta) \geq 1$*) then* $v \to 0$ *as* $t \to +\infty$, *that is, the predator goes to extinction.*

Proof Suppose $(u(x,t), v(x,t))$ is a solution of (3) with initial condition $u_0(x) \geq 0 \ (\neq 0)$, $v_0(x) \geq 0 \ (\neq 0)$. Let $\widetilde{v}(t, x)$ be a solution of

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$$\frac{\partial \widetilde{v}}{\partial t} - d_2 \Delta \widetilde{v} - b\left(1 - \frac{\widetilde{v}}{1+e_2}\right)\widetilde{v} = 0, \quad \widetilde{v}(x,0) = v_0(x)$$

Using the inequality

$$0 = \frac{\partial v}{\partial t} - d_2 \Delta v - b\Big(1 - \frac{v}{u+e_2}\Big)v \geq \frac{\partial v}{\partial t} - d_2 \Delta v - b\Big(1 - \frac{v}{1+e_2}\Big)v$$

we get

$$0 = \frac{\partial \widetilde{v}}{\partial t} - d_2 \Delta \widetilde{v} - b\Big(1 - \frac{\widetilde{v}}{1+e_2}\Big)\widetilde{v} \geq \frac{\partial v}{\partial t} - d_2 \Delta v - b\Big(1 - \frac{v}{1+e_2}\Big)v.$$

The comparison principle gives $v \leq \widetilde{v}$ on Ω. Or, by Lemma 2, if $\sigma_1(d_2, b, \gamma) \leq 0$, (or equivalently $d_2 \lambda_1^+(b,\gamma) \geq 1$), then any solution of (16) approach 0 as $t \to +\infty$. This implies that $v \to 0$ as $t \to +\infty$. $\square$

3. Conclusion

In this paper, we have analysed a spatio-temporal system modelling predator–prey population with modified Leslie–Gower and Holling type-II functional response under Robin boundary condition, the dissipativity of nonnegative solutions of the system is established by the comparison principle for parabolic equations. We have obtained conditions for the permanence and the predator extinction of the system by using techniques in Cantrell and Cosner (2003), Hess (1991). Such results are determined by the sign of a generalized principal eigenvalue.

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4. Appendix: Permanence

For the convenience of the reader, we will summarize some facts contained in Hale and Waltman, see Hale and Waltman (1989), about the permanence for abstract dynamical systems. Suppose that Ω is a complete metric space with $\Omega = \Omega_0 \cup \partial\Omega_0$ for an open set Ω_0, where $\partial\Omega_0$ is the boundary of the set Ω_0. We will typically choose Ω_0 to be the positive cone in an ordered Banach space. A flow or semiflow on Ω under which Ω_0 and $\partial\Omega_0$ are forward invariant is said to be permanent if it is dissipative and if there is a number $\eta > 0$ such that any trajectory starting in Ω_0

will be at least a distance η from $\partial\Omega_0$ for all sufficiently large t. To state a theorem implying permanence we need a few definitions. An invariant set M for the flow or semiflow is said to be isolated if it has a neighborhood U such that M is the maximal invariant subset of U. Let $\omega(\partial\Omega_0) \subset \partial\Omega_0$ denote the union of the sets $\omega(u)$ over $u \in \Omega_0$ (This differs from the standard definition of the ω-limit set of a set but it is more convenient for our purposes; see Hutson and Schmitt (1992) for a discussion). The set $\omega(\Omega_0)$ is said to be isolated if it has a covering $M = \cup_{k=1}^{N} M_k$ of pairwise disjoint, both sets M_k which are isolated and invariant with respect to the flow or semiflow both on $\partial\Omega_0$ and on $\Omega = \Omega_0 \cup \partial\Omega_0$. The covering M is then called an isolated covering. Suppose N_1 and N_2 are isolated invariant sets (not necessarily distinct). The set N_1 is said to be chained to N_2 (denoted $N_1 \rightarrow N_2$) if there exists $u \in N_1 \cup N_2$ with $u \in W^u(N_1) \cap W^s(N_2)$. (As usual, W^u and W^s denote the unstable and stable manifolds, respectively). A finite sequence $N_1, N_2, \cdots, N_k$ of isolated invariant sets is a chain if $N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow \cdots \rightarrow N_k$. (This is possible for $k = 1$ if $N_1 \rightarrow N_1$.) The chain is called a cycle if $N_k = N_1$.The set $\omega(\partial\Omega_0)$ is said to be acyclic if there exists an isolated covering $\cup_{k=1}^{N} M_k$ such that no subset of $\{M_k\}$ is a cycle. We now state a theorem that can be used to establish permanence.

Theorem 3 (Hale and Waltman 1989) *Suppose that Ω is a complete metric space with $\Omega = \Omega_0 \cup \partial\Omega_0$ where Ω_0 is open. Suppose that a semiflow on Ω leaves both Ω_0 and $\partial\Omega_0$ forward invariant, maps bounded sets in Ω to precompact set for $t > 0$, and is dissipative. If in addition:*

(i) $\omega(\partial\Omega_0)$ *is isolated and acyclic,*

(ii) $W^s(M_k) \cap \Omega_0 = \emptyset$ *for all k, where $\cup_{k=1}^{N} M_k$ is the isolated covering used in the definition of acyclicity of $\partial\Omega_0$, then the semiflow is permanent; i.e., there exist $\eta > 0$ such that any trajectory with initial data in Ω_0 will be bounded away from $\partial\Omega_0$ by a distance greater than η for t sufficiently large.*

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