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Synchronization of Chaos

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Introduction: Chaotic Systems Can Synchronize

Synchronization is a ubiquitous phenomenon characteristic of many processes in natural systems and (nonlinear) science. It has permanently remained an objective of intensive research and is today considered as one of the basic nonlinear phenomena studied in mathematics, physics, engineering, or life science. This word has a Greek root, *syn* = common and *chronos* = time, which means to share the common time or to occur at the same time, that is, correlation or agreement in time of different processes (Boccaletti *et al.* 2002). Thus, synchronization of two dynamical systems generally means that one system somehow traces the motion of another. Indeed, it is well known that many coupled oscillators have the ability to adjust some common relation that they have between them due to weak interaction, which yields to a situation in which a synchronization-like phenomenon takes place.

The original work on synchronization involved periodic oscillators. Indeed, observations of (periodic) synchronization phenomena in physics go back at least as far as C Huygens (1673), who, during his experiments on the development of improved pendulum clocks, discovered that two very weakly coupled pendulum clocks become synchronized in phase: two clocks hanging from a common support (on the same beam of his room) were found to oscillate with exactly the same frequency and opposite phase due to the (weak) coupling in terms of the almost imperceptible oscillations of the beam generated by the clocks.

Since this discovery, periodic synchronization has found numerous applications in various domains, for instance, in biological systems and living nature where synchronization is encountered on different levels. Examples range from the modeling of the heart to the investigation of the circadian rhythm, phase locking of respiration with a mechanical ventilator, synchronization of oscillations of human insulin secretion and glucose infusion, neuronal information processing within a brain area and communication between different brain areas. Also, synchronization plays an important role in several neurological diseases such as epilepsies and pathological tremors, or in different forms of cooperative behavior of insects, animals, or humans (Pikovsky *et al.* 2001).

This process may also be encountered in celestial mechanics, where it explains the locking of revolution period of planets and satellites.

Its view was strongly broadened with the developments in radio engineering and acoustics, due to the work of Eccles and Vincent, 1920, who found synchronization of a triode generator. Appleton, Van der Pol, and Van der Mark, 1922–27, have, experimentally and theoretically, extended it and worked on radio tube oscillators, where they observed entrainment when driving such oscillators sinusoidally, that is, the frequency of a generator can be synchronized by a weak external signal of a slightly different frequency.

But, even though original notion and theory of synchronization implies periodicity of oscillators, during the last decades, the notion of synchronization has been generalized to the case of interacting chaotic oscillators. Indeed, the discovery of deterministic chaos introduced new types of oscillating systems, namely the chaotic generators.

Chaotic oscillators are found in many dynamical systems of various origins; the behavior of such systems is characterized by instability and, as a result, limited predictability in time.

Roughly speaking, a system is chaotic if it is deterministic, has a long-term aperiodic behavior, and exhibits sensitive dependence on initial conditions on a closed invariant set (the chaos theory is discussed in more detail elsewhere in this encyclopedia) (*see* Chaos and Attractors).

Consequently, for a chaotic system, trajectories starting arbitrarily close to each other diverge exponentially with time, and quickly become uncorrelated. It follows that two identical chaotic systems cannot synchronize. This means that they cannot produce identical chaotic signals, unless they are initialized at exactly the same point, which is in general physically impossible. Thus, at first sight, synchronization of chaotic systems seems to be rather surprising because one may intuitively (and naively) expect that the sensitive dependence on initial conditions would lead to an immediate breakdown of any synchronization of coupled chaotic systems. This scenario in fact led to the belief that chaos is uncontrollable and thus unusable. Despite this, in the last decades, the search for synchronization has moved to chaotic systems. Significant research has been done and, as a result, Yamada and Fujisaka (1983), Afraimovich *et al.* (1986), and Pecora and Carroll (1990) showed that two chaotic systems could be synchronized by coupling them: synchronization of chaos is actual and chaos could then be exploitable. Ever since, many researchers have discussed the theory and the design or applications of synchronized motion in coupled chaotic systems. A broad variety of applications has emerged, for example, to increase the power of lasers, to synchronize the output of electronic circuits, to control oscillations in chemical reactions, or to encode electronic messages for secure communications.

The publication of the seminal paper of Pecora and Caroll (1990) had a very strong impact in the domain of chaos theory and chaos synchronization, and their applications. It had stimulated very intense research activities and the related studies continue to attract great attention. Many authors have contributed to developing this domain, theoretically or experimentally (Boccaletti *et al.* 2002, Pecorra *et al.* 1997, references therein).

However, the special features of chaotic systems make it impossible to directly apply the methods developed for synchronization of periodic oscillators. Moreover, in the topics of coupled chaotic systems, many different phenomena, which are usually referred to as synchronization, exist and have been studied now for over a decade. Thus, more precise descriptions of such systems are indeed desirable.

Several different regimes of synchronization have been investigated. In the following, the focus will be on explaining the essentials on this large topic, subdivided into four basic types of synchronization of coupled or forced chaotic systems which have been found and have received much attention, while emphasizing on the first three:

- identical (or complete) synchronization (IS), which is defined as the coincidence of states of interacting systems;
- generalized synchronization (GS), which extends the IS phenomenon and implies the presence of some functional relation between two coupled systems; if this relationship is the identity, we recover the IS;
- phase synchronization (PS), which means entrainment of phases of chaotic oscillators, whereas their amplitudes remain uncorrelated; and
- lag synchronization (LS), which appears as a coincidence of time-shifted states of two systems.

Other regimes exist, some of them will be briefly pointed out at the end of this article; we also will briefly discuss the very relevant issue of the stability of synchronous motions.

Our discussion and examples given here are based on unidirectionally continuous systems, most of the exposed ideas can be easily extended to discrete systems.

Let us also emphasize that the same year, 1990, saw the publication of another seminal paper, by Ott, Grebogi, and Yorke (OGY) on the control of chaos (Ott *et al.* 1990). Recently, it has been realized that synchronization and control of chaos share a common root in nonlinear control theory. Both topics were presented by many authors in a unified framework. However, synchronization of chaos has evolved in its own right, even if it is nowadays known as a part of the nonlinear control theory.

Synchronization and Stability

For the basic master–slave configuration, where an autonomous chaotic system (the master)

$$\frac{dX}{dt} = F(X), \quad X \in \mathbb{R}^n \qquad [1]$$

drives another system (the slave),

$$\frac{dY}{dt} = G(X, Y), \quad Y \in \mathbb{R}^m \qquad [2]$$

synchronization takes place when Y asymptotically copies, in a certain manner, a subset X_p of X. That is, there exists a relation between the two coupled systems, which could be a smooth invertible function ψ, which transforms the trajectories on the attractor of a first system into those on the attractor of a second system. In other words, if we know, after a transient regime, the state of the first system, it allows us to predict the state of the second: $Y(t) = \psi(X(t))$. Generally, it is assumed that $n \geq m$; however, for the sake of easy readability (even if this is not a necessary restriction) the case $n = m$ will only be considered; thus, $X_p = X$. Henceforth, if we denote the difference $Y - \psi(X)$ by $X_\perp$, in order to arrive at a synchronized motion, it is expected that

$$||X_\perp|| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty \qquad [3]$$

If ψ is the identity function, the process is called IS.

Definition of IS System [2] synchronizes with system [1], if the set $M = \{(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n, Y = X\}$ is an attracting set with a basin of attraction $B(M \subset B)$ such that $\lim_{t\to\infty} \|X(t) - Y(t)\| = 0$, for all $(X(0), Y(0)) \in B$.

Thus, this regime corresponds to the situation where all the variables of two (or more) coupled chaotic systems converge.

If ψ is not the identity function, the phenomenon is more general and is referred to as GS.

Definition of GS System [2] synchronizes with system [1], in the generalized sense, if there exists a transformation $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, a manifold $M = \{(X, Y) \in \mathbb{R}^{n+m}, Y = \psi(X)\}$ and a subset B ($M \subset B$), such that for all $(X_0, Y_0) \in B$, the trajectory based on the initial conditions (X_0, Y_0) approaches M as time goes to infinity. This is explained further in the following.

Henceforth, in the case of IS, eqn [3] above means that a certain hyperplane M, called synchronization manifold, within $\mathbb{R}^{2n}$, is asymptotically stable. Consequently, for the sake of synchrony motion, we have to prove that the origin of the transverse system $X_\perp = Y - X$ is asymptotically stable. That is, to prove that the motion transversal to the synchronization manifold dies out.

However, significant progress has been made by mathematicians and physicists in studying the stability of synchronous motions. Two main tools are used in the literature for this aim: conditional Lyapunov exponents and asymptotic stability. In the examples given below, we will essentially formulate conditions for synchronization in terms of Lyapunov exponents, which play a central role in chaos theory. These quantities measure the sensitive dependence on initial conditions for a dynamical system and also quantify synchronization of chaos.

The Lyapunov exponents associated with the variational equation corresponding to the transverse system $X_\perp$:

$$\frac{\mathrm{d}X_\perp}{\mathrm{d}t} = DF(X)X_\perp \qquad [4]$$

where $DF(X)$ is the Jacobian of the vector field evaluated onto the driving trajectory X, are referred to as transverse or conditional Lyapunov exponents (CLEs).

In the case of IS, it appears that the condition $L^\perp_{\max} < 0$ is sufficient to insure synchronization, where $L^\perp_{\max}$ is the largest CLE. Indeed, eqn [4] gives the dynamics of the motion transverse to the synchronization manifold; therefore, CLEs indicate if this motion dies out or not, and hence, whether the synchronization state is stable or not. Consequently, if $L^\perp_{\max}$ is negative, it insures the stability of the synchronized state. This will be best explained using two examples below.

Even if there exist other approaches for studying synchronization, one may ask if this condition on $L^\perp_{\max}$ is true in general. To answer this question, mathematicians have recently formulated it in terms of properties of manifolds (or synchronization hyperplanes). Some rigorous results on (generalized) synchronization, when the system is smooth, are given by Josic (2000). This approach relies on the Fenichel theory of normally hyperbolic invariant manifolds and quantities that resemble Lyapunov exponents, and is referred to as differentiable GS. However, many situations correspond to the case where, in some region of values of parameters coupling, the function ψ is only continuous but not smooth, that is, the graph of ψ is a complicated geometrical object. This kind of synchronization is called nonsmooth GS (Afraimovich *et al.* 2001).

Furthermore, the mathematical theory of IS often assumes the coupled oscillators to be identical, even if, in practice, no two oscillators are exact copies of each other. This leads to small differences in system parameters and then to synchronization errors. These errors have been studied by many authors (see, e.g., Illing (2002), and references therein).

Identical Synchronization

Perhaps the best way to explain synchronization of chaos is through IS, also referred to as conventional or complete synchronization (Boccaletti *et al.* 2002). It is the simplest form of chaos synchronization and generalizes to the complete replacement which is explained below. It is also the most typical form of chaotic synchronization often observable in two identical systems.

There are various processes leading to synchronization; depending on the particular coupling configuration used these processes could be very different. So, one has to distinguish between the following two main situations, even if they are, in some sense, similar: the unidirectional and the bidirectional coupling. Indeed, synchronization of chaotic systems is often studied for schemes of the form

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= F(X) + kN(X - Y) \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= G(Y) + kM(X - Y) \end{aligned} \qquad [5]$$

where F and G act in $\mathbb{R}^n$, $(X, Y) \in (\mathbb{R}^n)^2$, is a scalar, and M and N are coupling matrices belonging to $\mathbb{R}^{n\times n}$. If $F = G$ the two subsystems X and Y are identical. Moreover, when both matrices are nonzero then the coupling is called bidirectional, while it is referred to as unidirectional if one is the zero matrix, and the other nonzero.

Constructing Pairs of Synchronized Systems: Complete Replacement

Pecora and Carroll (1990) proposed the use of stable subsystems of given chaotic systems to construct pairs of unidirectionally coupled synchronizing systems. Since then generalizations of this approach have been developed and various methods now exist to synchronize systems (Wu 2002, Hasler 1998).

One way to build a couple of synchronized systems is then to use the basic construction method introduced by Pecora and Carroll, who made an important observation. They found that, when they make a replica of part of a chaotic system and send a system variable from the original system (transmitter) to drive this replica (receiver), sometimes the replica subsystem and the original chaotic one lock in their steps and evolve together chaotically in synchrony. This method can be described as follows. Consider the autonomous n-dimensional dynamical system,

$$\frac{du}{dt} = F(u) \qquad [6]$$

divide this system into two subsystems ($u = (v, w)$),

$$\begin{aligned} \frac{dv}{dt} &= G(v, w) \\ \frac{dw}{dt} &= H(v, w) \end{aligned} \qquad [7]$$

where $v = (u_1, \ldots, u_m)$, $w = (u_{m+1}, \ldots, u_n)$, $G = (F_1, \ldots, F_m)$, and $H = (F_{m+1}, \ldots, F_n)$. Next, create a new subsystem w' identical to the w-subsystem. This yields a $(2n - m)$-dimensional system:

$$\begin{aligned} \frac{dv}{dt} &= G(v, w) \\ \frac{dw}{dt} &= H(v, w) \\ \frac{dw'}{dt} &= H(v, w') \end{aligned} \qquad [8]$$

The first state-variable component $v(t)$ of the (v, w) system is then used as the input to the w'-system. The coupling is unidirectional and the (v, w) subsystem is referred to as the driving (or master) system, the w'-subsystem as the response (or slave) system. In this context, the following notions and results are useful.

Definition If $\lim_{t \to +\infty} \|w'(t) - w(t)\| = 0$ and $w'(t)$ continues to remain in step with $w(t)$ in the course of the time, the two subsystems are said to be synchronized.

Definition The Lyapunov exponents of the response subsystem (w') for a particular driven trajectory $v(t)$ are called CLEs.

Let $w(t)$ be a chaotic trajectory with initial condition $w(0)$, and $w'(t)$ be a trajectory started at a nearly point $w'(0)$. The basic idea of the Pecora–Carroll approach is to establish the asymptotic stability of the solutions of w'-subsystem by means of CLEs. They have shown the following result (Pecora and Carroll 1990):

Theorem *A necessary and sufficient condition for the two subsystems, w and w', to be synchronized is that all of the CLEs be negative.*

Note that only a finite number of possible decompositions (or couplings) v–w exist; this is bounded by the number of different possible subsystems, namely $N(N-1)/2$. (For a description and mathematical analysis of various coupling schemes see Wu (2002).) Furthermore, by splitting the main system [6] in a different way, (complete) synchronization could not exist. Indeed, in general, only a few of the possible response subsystems possess negative CLEs, and may thus be used to implement synchronizing systems using the Pecora–Caroll method. In fact, it has been pointed out in the literature that in some cases, the CLE criterion is not as practical as some other criteria.

For simplicity, the idea will now be developed on the following three-dimensional simple autonomous system, which belongs to the class of dynamical systems called generalized Lorenz systems (see Derivière and Aziz-Alaoui (2003), and references therein):

$$\begin{aligned} \dot{x} &= -9x - 9y \\ \dot{y} &= -17x - y - xz \\ \dot{z} &= -z + xy \end{aligned} \qquad [9]$$

(This should be compared with the well-known Lorenz system:

$$\begin{aligned} \dot{x} &= -10x + 10y \\ \dot{y} &= 28x - y - xz \\ \dot{z} &= -\tfrac{8}{3}z + xy \end{aligned}$$

which differs in the signs of various terms and the values of coefficients.) From previous observations, it was shown that system [9] oscillates chaotically; its Lyapunov exponents are +0.601, 0.000, and –16.470; it exhibits the chaotic attractor of Figure 1, with a three-dimensional feature very similar to that of Lorenz attractor (in fact, it satisfies the condition $z < 0$, but in our context it does not matter).

Let us divide system [9] into two subsystems $v = x_1$ and $w = (y_1, z_1)$. By creating a copy

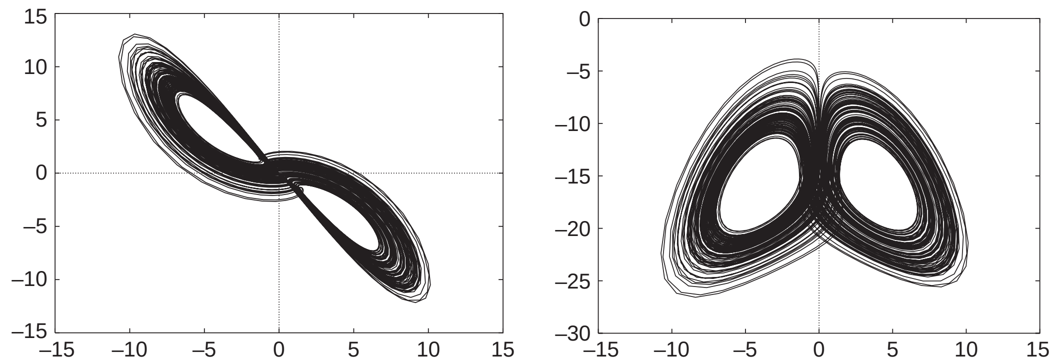

Figure 1 The chaotic attractor of system [9]: *x*–*y* and *x*–*z* plane projections.

$w' = (y_2, z_2)$ of the w-subsystem, we obtain the following five-dimensional dynamical system:

$$\begin{aligned} \dot{x}_1 &= -9x_1 - 9y_1 \\ \dot{y}_1 &= -17x_1 - y_1 - x_1 z_1 \\ \dot{z}_1 &= -z_1 + x_1 y_1 \\ \dot{y}_2 &= -17x_1 - y_2 - x_1 z_2 \\ \dot{z}_2 &= -z_2 + x_1 y_2 \end{aligned} \qquad [10]$$

In numerical experiments, it was observed that the motion quickly results in the two equalities, $\lim_{t\to+\infty} |y_2 - y_1| = 0$ and $\lim_{t\to+\infty} |z_2 - z_1| = 0$, to be satisfied, that is, $\lim_{t\to+\infty} \|w' - w\| = 0$. These equalities persist as the system evolves. Hence, the two subsystems w and w' are synchronized. Figure 2 illustrates this phenomenon.

It is also easy to verify that the synchronization persists even if a slight change in the parameters of the system is made. The CLEs of the linearization of the system around the synchronous state, the negativity of which determines the stability of the synchronized solution, are also computed easily.

Pecora–Carroll similarly built the system [10] by using the following steps. Starting with two copies of system [9], a signal $x(t)$ is transmitted from the first to the second: in the second system all x-components are replaced with the signal from the first system, that is, x_2 is replaced by x_1 in the second system. Finally, the $\mathrm{d}x_2/\mathrm{d}t$ equation is eliminated, since it is exactly the same as $\mathrm{d}x_1/\mathrm{d}t$ equation, and is superfluous. This then results in system [10]. For this reason, Pecora–Carroll called this construction a complete replacement. Thus, it is natural to think of the x_1 variable as driving the second system, but also to label the first system the drive and the second system the response. In fact, this method is a particular case of the unidirectional coupling method explained below. Note also that this method could be modified by using a partial substitution approach, in which a response variable is replaced with the drive counterpart only in certain locations (Pecora *et al.* 1997).

Unidirectional IS

The IS synchronization has also been called as one-way diffusive coupling, drive–response coupling, master–slave coupling, or negative feedback control.

System [5], $F = G$ and $N = 0$, becomes unidirectionally coupled, and reads

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= F(X) \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= F(Y) + kM(X - Y) \end{aligned} \qquad [11]$$

M is then a matrix that determines the linear combination of X components that will be used in the difference, and k determines the strength of the coupling (see, for an interesting review on this subject, Pecora *et al.* (1997)). In unidirectional synchronization, the evolution of the first system (the drive) is unaltered by the coupling, the second system (the response) is then constrained to copy the dynamics of the first. Let us consider an example with two copies of system [9], and for

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [12]$$

that is, by adding a damping term to the first equation of the response system, we get a following unidirectionally coupled system, coupled through a linear term $k > 0$ according to variables $x_{1,2}$:

$$\begin{aligned} \dot{x}_1 &= -9x_1 - 9y_1 \\ \dot{y}_1 &= -17x_1 - y_1 - x_1 z_1 \\ \dot{z}_1 &= -z_1 + x_1 y_1 \\ \dot{x}_2 &= -9x_2 - 9y_2 - k(x_2 - x_1) \\ \dot{y}_2 &= -17x_2 - y_2 - x_2 z_2 \\ \dot{z}_2 &= -z_2 + x_2 y_2 \end{aligned} \qquad [13]$$

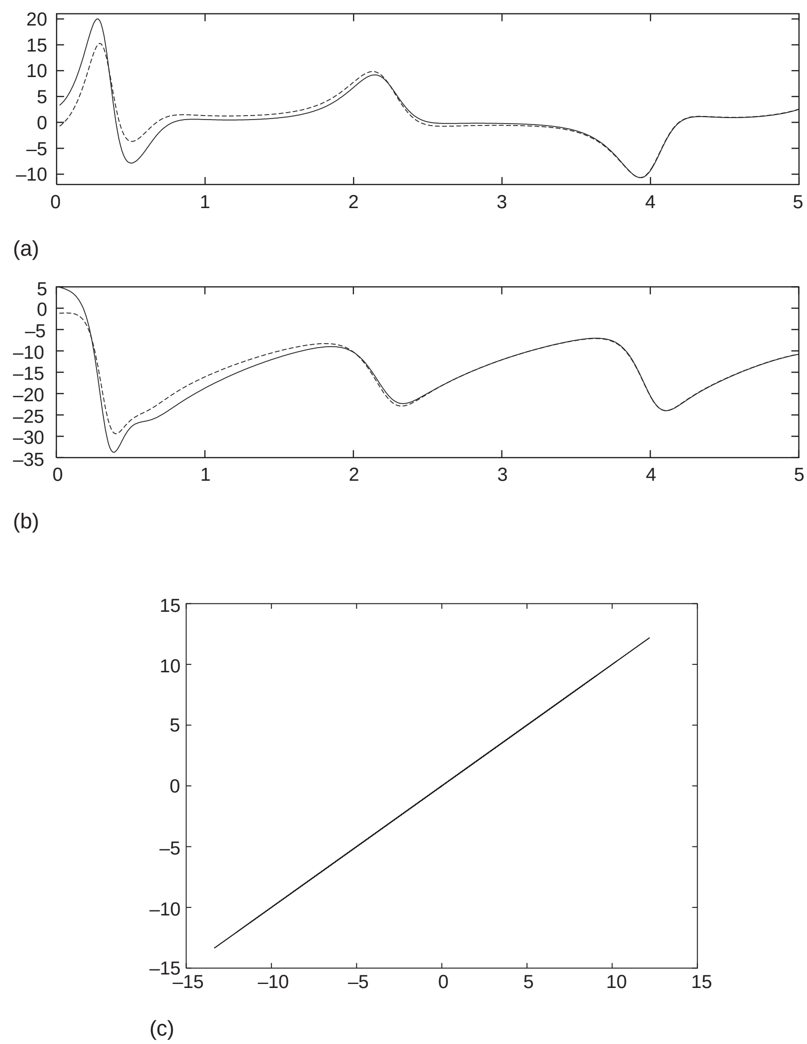

Figure 2 Complete replacement synchronization. Time series for (a) $y_i(t)$ and (b) $z_i(t), i = 1, 2$, in system [10]. The difference between the variable of the transmitter and the variable of the receiver asymptotes tends to zero as time progresses, that is, synchronization occurs after transients die down. (c) The plot of amplitudes y_1 against y_2, after transients die down, shows a diagonal line, which also indicates that the receiver and the transmitter are maintaining synchronization. The plot of z_1 against z_2 shows a similar behavior.

For $k = 0$, the two subsystems are uncoupled; for $k > 0$ both subsystems are unidirectionally coupled; and for $k \longrightarrow +\infty$, we recover the complete replacement coupling scheme explained above. Our numerical computations yield the optimal value $\tilde{k}$ for the synchronization; we found that for $k \geq \tilde{k} = 4.999$, both subsystems of [13] synchronize. That is, starting from random initial conditions, and after some transient time, system [13] generates the same attractor as for system [9] (see **Figure 1**). Consequently, all the variables of the coupled chaotic subsystems converge: x_2 converges to x_1, y_2 to y_1, and z_2 to z_1 (see **Figure 3**). Thus, the second system (the response) is locked to the first one (the drive).

Alternatively, observation of diagonal lines in correlation diagrams, which plot the amplitudes x_1 against x_2, y_1 against y_2, and z_1 against z_2, can also indicate the occurrence of system synchronization.

IS was the first for which examples of unidirectionally coupled chaotic systems were presented. It is important for potential applications of chaos synchronization in communication systems, or for time-series analysis, where the information flow is also unidirectional.

Bidirectional IS

A second brief example uses a bidirectional (also called mutual or two-way) coupling. In this situation, in contrast to the unidirectional coupling, both drive and response systems are connected in such a way that they influence each other's behavior. Many

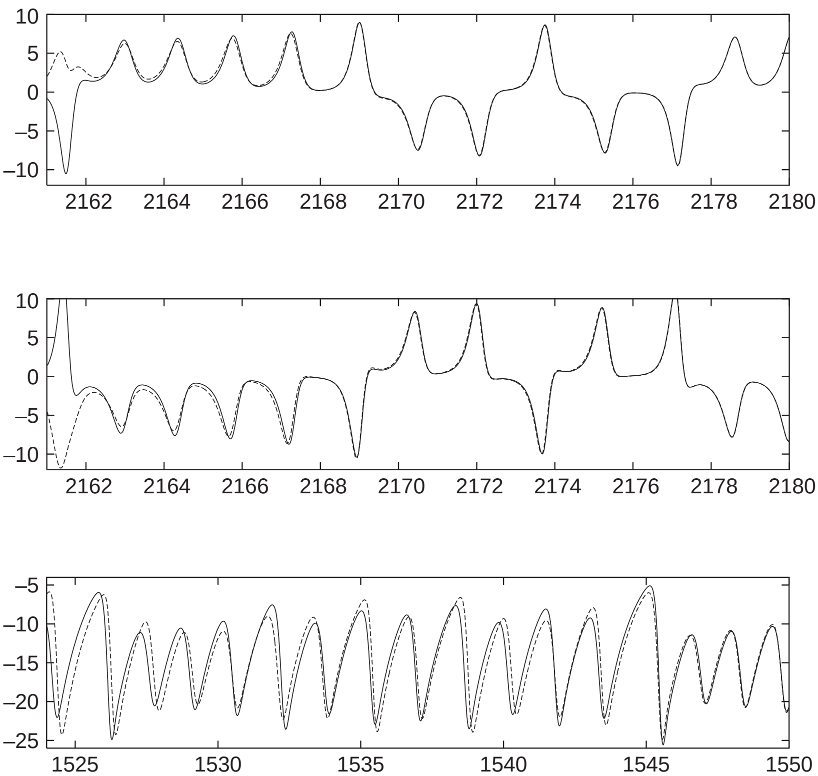

Figure 3 Time series for $x_i(t), y_i(t)$, and $z_i(t)(i=1,2)$ in system [13] for the coupling constant $k=5.0$, that is, beyond the threshold necessary for synchronization. After transients die down, the two subsystems synchronize perfectly.

biological or physical systems consist of bidirectionally interacting elements or components; examples range from cardiac and respiratory systems to coupled lasers with feedback. Let us then take two copies of the same system [9] as given above, but two-way coupled through a linear constant term $k > 0$ according to variables $x_{1,2}$:

$$\begin{aligned} \dot{x}_1 &= -9x_1 - 9y_1 - k(x_1 - x_2) \\ \dot{y}_1 &= -17x_1 - y_1 - x_1 z_1 \\ \dot{z}_1 &= -z_1 + x_1 y_1 \\ \dot{x}_2 &= -9x_2 - 9y_2 - k(x_2 - x_1) \\ \dot{y}_2 &= -17x_2 - y_2 - x_2 z_2 \\ \dot{z}_2 &= -z_2 + x_2 y_2 \end{aligned} \qquad [14]$$

We can get an idea of the onset of synchronization by plotting, for example, x_1 against x_2 for various values of the coupling-strength parameter k. Our numerical computations yield the optimal value $\tilde{k}$ for the synchronization: $\tilde{k} \simeq 2.50$ (**Figure 4**), both (x_i, y_i, z_i) subsystems synchronize and system [14] also generates the attractor of **Figure 1**.

Synchronization manifold and stability Geometri-Geometrically, the fact that systems [13] and [14], beyond synchronization, generate the same attractor as system [9], implies that the attractors of these combined drive–response six-dimensional systems are confined to a three-dimensional hyperplane (the synchronization manifold) defined by $Y = X$. After the synchronization is reached, this manifold is a stable submanifold in the full phase space $\mathbb{R}^6$. **Figure 5** gives an idea of what the geometry of the synchronous attractor of system [13] or [14] looks like, by exhibiting the projection of the phase space $\mathbb{R}^6$ onto (x_1, y_1, y_2) subspace. But, one can similarly plot any combination of variable x_i, y_i, and z_i $(i = 1, 2)$, and get the same result, since the motion, in case of synchronization, is confined to the hyperplane defined in $\mathbb{R}^6$ by the equalities $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.

This hyperplane is stable since small perturbations which take the trajectory off the synchronization manifold decay in time. Indeed, as stated earlier, CLEs of the linearization of the system around the synchronous state could determine the stability of the synchronized solution. This leads to requiring that the origin of the transverse system, $X_\perp$, is asymptotically stable. To see this, for both systems [13] and [14], we then switch to the new set of coordinates, $X_\perp = Y - X$, that is, $x_\perp = x_2 - x_1$, $y_\perp = y_2 - y_1$, and $z_\perp = z_2 - z_1$. The origin $(0, 0, 0)$ is obviously a fixed point for this transverse system,

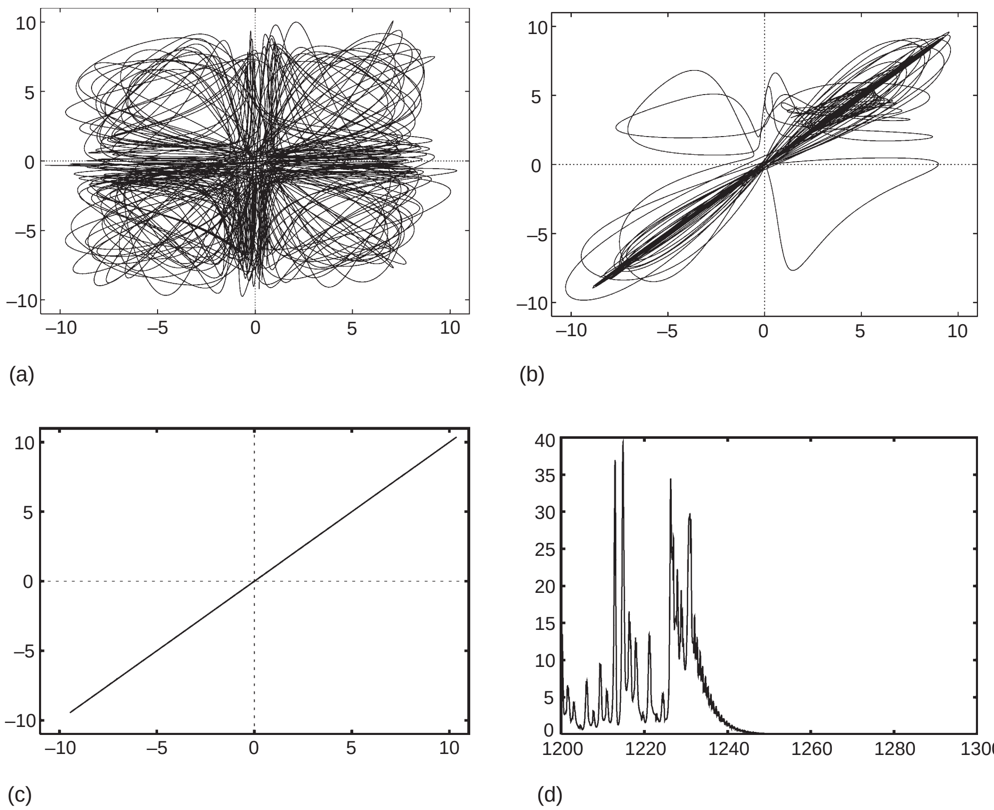

Figure 4 Illustration of the onset of synchronization of system [14]. (a)–(c) Plots of amplitudes x_1 against x_2 for values of the coupling parameter $k = 0.5, 1.5, 2.8$, respectively. The system synchronizes for $k \geq 2.5$. (d) Plot, for $k = 2.8$, of the norm $N(X) = \|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|$ versus t, which shows that the system synchronizes very quickly.

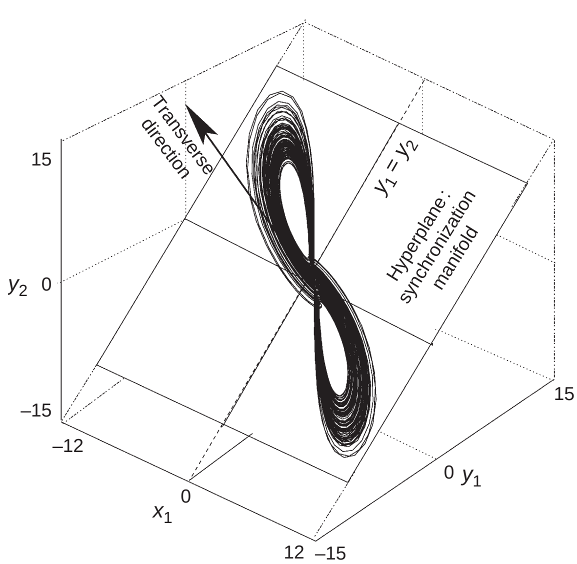

Figure 5 The motion of synchronized system [13] or [14] takes place on a chaotic attractor which is embedded in the synchronization manifold, that is, the hyperplane defined by $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.

within the synchronization manifold. Therefore, for small deviations from the synchronization manifold, this system reduces to a typical variational equation:

$$\frac{dX_\perp}{dt} = DF(X)X_\perp \qquad [15]$$

where $DF(X)$ is the Jacobian of the vector field evaluated onto the driving trajectory X, that is,

$$\begin{pmatrix} \frac{dx_\perp}{dt} \\ \frac{dy_\perp}{dt} \\ \frac{dz_\perp}{dt} \end{pmatrix} = V \begin{pmatrix} x_\perp \\ y_\perp \\ z_\perp \end{pmatrix} \qquad [16]$$

For systems [13] and [14], we obtain

$$V = V_i = \begin{pmatrix} -9 - k_i & -9 & 0 \\ -17 - z & -1 & -x \\ y & x & -1 \end{pmatrix} \qquad [17]$$

with $k_i = k$ for system [13] and $k_i = 2k$ for system [14]. Let us remark that the only difference between both matrices V_i is the coupling k which has a factor

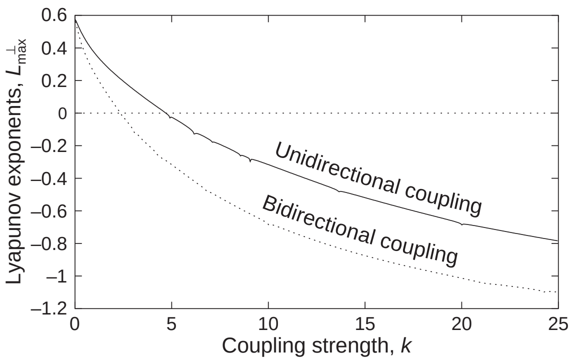

Figure 6 The largest transverse Lyapunov exponents $L^{\perp}_{\max}$ as a function of coupling strength k, in the unidirectional system [13] (solid) and the bidirectional system [14] (dotted).

2 in the bidirectional case. **Figure 6** shows the dependence of $L^{\perp}_{\max}$ on k, for both examples of unidirectionally and bidirectionally coupling systems. $L^{\perp}_{\max}$ becomes negative as k increases, which insures the stability of the synchronized state for systems [13] and [14].

Let us note that this can also be proved analytically as done by Derivière and Aziz-Alaoui (2003) by using a suitable Lyapunov function, and using some new extended version of LaSalle invariance principle.

Desynchronization motion Synchronization depends not only on the coupling strength, but also on the vector field and the coupling function. For some choice of these quantities, synchronization may occur only within a finite range $[k_1, k_2]$ of coupling strength; in such a case a desynchronization phenomenon occurs. Thus, increasing k beyond the critical value k_2 yields loss of the synchronized motion ($L^{\perp}_{\max}$ becomes positive).

Generalized Synchronization

Identical chaotic systems synchronize by following the same chaotic trajectory. However, real systems are in general not identical. For instance, when the parameters of two coupled identical systems do not match, or when these coupled systems belong to different classes, complete IS may not be expected, because there does not exist such an invariant manifold $Y = X$, as for IS. For non-identical systems, the possibility of some type of synchronization has been investigated (Afraimovich *et al.* 1986). It was shown that when two different systems are coupled with sufficiently strong coupling strength, a general synchronous relation between their states could exist and it could be expressed by a smooth invertible function, $Y(t) = \psi(X(t))$. This phenomenon, called GS, is thus a relaxed and extended form of IS in non-identical systems.

However, it may also occur for pairs of identical systems, for example, for systems having reflection symmetry, $F(-X) = -F(X)$. Besides these examples of GS, others also exist that exploit symmetries of the underlying systems (Parlitz and Kocarev 1999).

GS was introduced for unidirectionally coupled systems by Rulkov *et al.* (1995). For simplicity, we also focus on unidirectionally coupled continuous time systems:

$$\frac{dX}{dt} = F(X) \qquad \frac{dY}{dt} = G(Y, u(t)) \tag{18}$$

where $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$, $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $G: \mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}^m$, and $u(t) = (u_1(t), \ldots, u_k(t))$ with $u_i(t) = h_i(X(t, X_o))$. Two (non-identical) dynamical systems are said to be synchronized in a generalized sense if there is a continuous function ψ from the phase space of the first to the phase space of the second, taking orbits of the first system to orbits of the second.

The main problem is to know when and under what conditions system [18] undergoes GS. Many authors have addressed this question, and it has been shown that asymptotic stability is equally significant for this more universal concept (for some theoretical results, see Rulkov *et al.* (1995) and Parlitz and Kocarev (1999)). For unidirectionally coupled continuous time systems, the following results hold:

Theorem *A necessary and sufficient condition for system [18] to be synchronized in the generalized sense is that for each $u(t) = u(X(t, X_o))$ the system-is asymptotically stable.*

When it is not possible to find a Lyapunov function in order to use this theorem, one can numerically compute the CLEs of the response system, and use the following result:

Theorem *The drive and response subsystems of system [18] synchronize in the generalized sense iff all of the CLEs of the response subsystem are negative.*

The definition of ψ has the advantage that it allows the discussion of synchronization of non-identical systems and, at the same time, to consider synchronization in terms of the property of synchronization manifold. Therefore, it is important to study the existence of the transformation ψ and its nature (continuity, smoothness, ...). Unfortunately, except in special cases (Afraimovich *et al.* 1986), rarely will one be able to produce formulas exhibiting the mapping ψ.

An example of two unidirectionally coupled chaotic systems which synchronize in the generalized sense is given below. Consider the following Rössler system driven by system [9]:

$$\begin{aligned}
\dot{x_1} &= -9x_1 - 9y_1 \\
\dot{y_1} &= -17x_1 - y_1 - x_1 z_1 \\
\dot{z_1} &= -z_1 + x_1 y_1 \\
\dot{x_2} &= -y_2 - z_2 - k(x_2 - (x_1^2 + y_1^2)) \\
\dot{y_2} &= x_2 + 0.2y_2 - k(y_2 - (y_1^2 + z_1^2)) \\
\dot{z_2} &= 0.2 + z_2(x_2 - 9.0) - k(z_2 - (x_1^2 + z_1^2))
\end{aligned} \qquad [19]$$

As shown in **Figure 7**, it appears impossible to tell what the relation is between the transmitter subsystem (x_1, y_1, z_1) in eqn [19] and the two Rössler response subsystems (x_2, y_2, z_2) at $k = 1$ and $k = 100$.

However, GS occurs for large values of the coupling-strength parameter k. Therefore, for such values we expect that orbits of [19] will lie in the vicinity of a certain synchronization manifold. Indeed, let us define the set

$$S = \{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 : x_2 = x_1^2 + y_1^2, \\ y_2 = y_1^2 + z_1^2,\ z_2 = x_1^2 + z_1^2\}$$

Since the projections of S onto the coordinates (x_1, y_1, x_2), (y_1, z_1, y_2), and (x_1, z_1, z_2) are paraboloids, we can see how the synchronization manifold is approached. This is illustrated in **Figure 8**, where the (x_1, y_1, x_2) projections of typical trajectories are shown at four different coupling values. (See Josic (2000) for other examples and further developments; see also Pecora *et al.* (1997), where the authors summarize a method in order to get an idea on the functional relation occurring in case of GS, between two coupled systems.)

Phase Synchronization

For coupled non-identical chaotic systems, other types of synchronizations exist. Recently, a rather weak degree of synchronization, the PS, of chaotic systems has been described (Pikovsky *et al.* 2001). The Greek meaning of the word synchronization, mentioned in the introduction, is closely related to this type of processes. The synchronous motion is actually not visible. Indeed, in PS the phases of chaotic systems with PS are locked, that is, there exists a certain relation between them, whereas the amplitudes vary chaotically and are practically uncorrelated. Thus, it is mostly close to synchronization of periodic oscillators.

Definition PS of two coupled chaotic oscillators occurs if, for arbitrary integers n and m, the phase locking condition between the corresponding phases, $|n\phi_1(t) - m\phi_2(t)| \leq$ constant, holds and the amplitudes of both systems remain uncorrelated.

Let us note that such a phenomenon occurs when a zero Lyapunov exponent of the response system becomes negative, while, as explained above, identical chaotic systems synchronize by following the same chaotic trajectory, when their largest transverse Lyapunov exponent of the synchronized manifold decreases from positive to negative values.

Moreover, following the definition above, this phenomenon is best observed when a well-defined phase variable can be identified in both coupled systems. This can be done for strange attractors that spiral around a "hole," or a particular (fixed) point in a two-dimensional projection of the attractor. The typical example is given by the Rössler system, which, for some range of parameters, exhibits a Möbius-

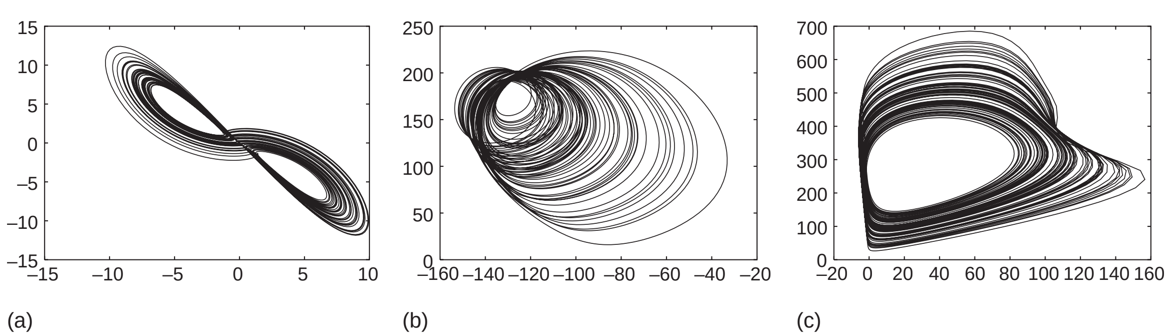

Figure 7 Projections onto the $(x–y)$ plane of typical trajectories of system [19]. (a) (x_1, y_1) projection, that is, a typical trajectory of system [9]; (b) and (c) (x_2, y_2) projections at, respectively, $k = 1$ and $k = 100$.

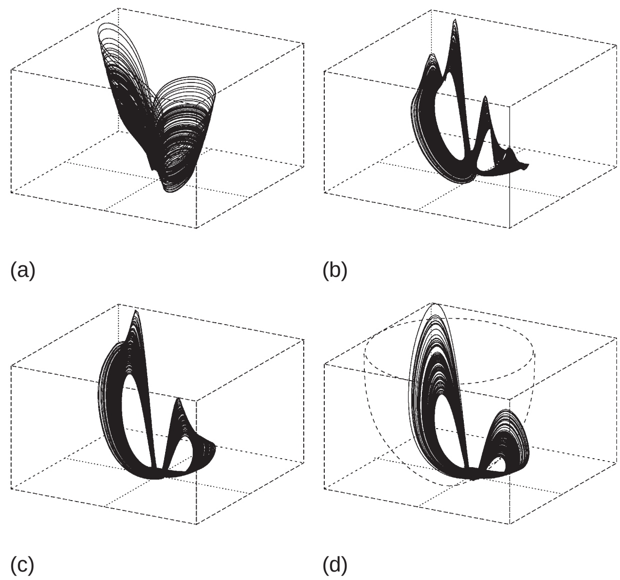

Figure 8 Generalized synchronization. (x_1, y_1, x_2) projections of typical trajectories of system [19] after transients die out, with (a) $k=1$, (b) $k=20$, (c) $k=100$, and (d) $k=200$. For the last value, the attractor lies in the set S, three-dimensional projections of which are paraboloïds.

strip-like chaotic attractor with a central hole. In such a case, a phase angle $\phi(t)$ can be defined that decreases or increases monotonically. For an illustration, we take the following two coupled Rössler oscillators:

$$\begin{aligned}\dot{x}_1 &= -\alpha_1 y_1 - z_1 + k(x_2 - x_1)\\ \dot{y}_1 &= \alpha_1 x_1 + 0.17 y_1\\ \dot{z}_1 &= 0.2 + z_1(x_1 - 9.0)\\ \dot{x}_2 &= -\alpha_2 y_2 - z_2 + k(x_1 - x_2)\\ \dot{y}_2 &= \alpha_2 x_2 + 0.17 y_2\\ \dot{z}_2 &= 0.2 + z_2(x_2 - 9.0)\end{aligned} \qquad [20]$$

with a small parameter mismatch $\alpha_{1,2} = 0.95 \pm 0.04, k$ governs the strength of coupling. If we can define a Poincaré section surface for the system, then, for each piece of a trajectory between two cross sections with this surface, we define the phase, as done in Pikovsky *et al.* (2001), as a piecewise linear function of time, so that the phase increment is 2π at each rotation:

$$\phi(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \quad t_n \le t \le t_{n+1}$$

where t_n is the time of the nth crossing of the secant surface.

In our example, the last has been chosen as the negative x-axis and represented by the wide segment in Figure 9a. This definition of phases is clearly ambiguous since it depends on the choice of the Poincaré section; nevertheless, defined in this way, the phase has a physically important property, it does correspond to the direction with the zero Lyapunov exponent in the phase space, its perturbations neither grow nor decay in time. Figure 9c shows that there is a transition from the nonsynchronous phase regime, where the phase difference increases almost linearly with time ($k=0.01$ and $k=0.05$), to a synchronous state, where the relation $|\phi_1(t) - \phi_2(t)| <$ constant holds ($k=0.1$), that is, the phase difference does not grow with time. However, the amplitudes are obviously uncorrelated as seen in Figure 9b. This example shows that PS could takes place for weaker degree of synchronization in chaotic systems. Readers can find more rigorous mathematical discussion on this subject, and on the definition of phases of chaotic oscillators, in Pikovsky *et al.* (2001), see also Boccaletti *et al.* (2002) and references therein.

Other Treatments and Types of Synchronization

Lag Synchronization

PS synchronization occurs when non-identical chaotic oscillators are weakly coupled: the phases are locked, while the amplitudes remain uncorrelated. When the coupling strength becomes larger, some relationships between amplitudes may be established. Indeed, it has been shown (Rosenblum *et al.* 1997), in symmetrically coupled non-identical oscillators and in time-delayed systems, that there exists

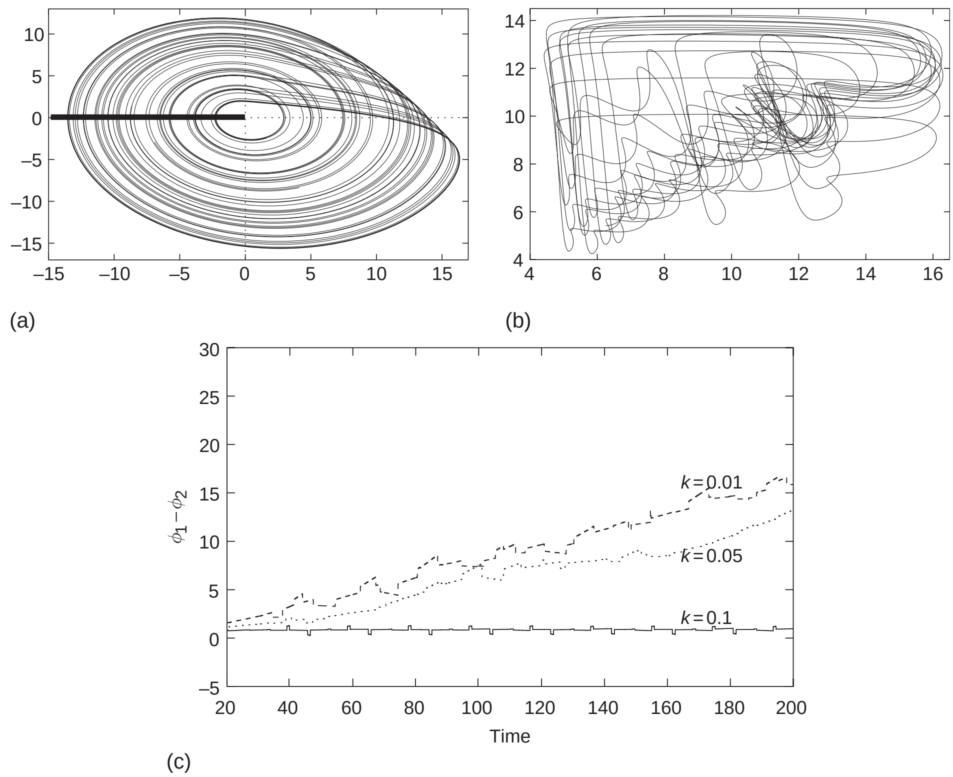

Figure 9 (a) Rössler chaotic attractor projection onto *x*–*y* plane. (b) Amplitudes A_1 versus A_2 for the phase synchronized case at $k = 0.1$. (c) Time serie of phase difference for different coupling strengths *k*; for $k = 0.01$ PS is not achieved, while for $k = 0.1$ PS takes place. Although the phases are locked, for $k = 0.1$, the amplitudes remain chaotic and uncorrelated.

a regime of LS. This process appears as a coincidence of time-shifted states of two systems:

$$\lim_{t\to+\infty} ||Y(t) - X(t-\tau)|| = 0$$

where τ is a positive delay.

Projective Synchronization

In coupled partially linear systems, it was reported by Mainieri and Rehacek (1999) that two identical systems could be synchronized up to a scaling factor. This type of chaotic synchronization is referred to as projective synchronization. Consider, for example, a three-dimensional chaotic system $\dot{X} = F(X)$, where $X = (x, y, z)$. Decompose X into a vector $v = (x, y)$ and a scalar z; the system can then be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g(v, z), \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = h(v, z)$$

In projective synchronization, two identical systems $X_1 = (x_1, y_1, z_1)$ (drive) and $X_2 = (x_2, y_2, z_2)$ (response) are coupled through the scalar variable z. It occurs if the state vectors v_1 and v_2 synchronize up to a constant ratio, that is, $\lim_{t\to+\infty} ||\alpha v_1(t) - v_2(t)|| = 0$, where α is called a scaling factor. For partially linear systems, it may automatically occur provided that the systems satisfy some stability conditions.

However, this process could not be classified as GS, even if there exists a linear relation between the coupled systems, because the response system of projective synchronization is not asymptotically stable. For more information about this subject, the reader is referred to Mainieri and Rehacek (1999).

Anticipating Synchronization

It is interesting to mention that a new form of synchronization has recently appeared, the so-called anticipating synchronization (Boccaletti *et al.* 2002). It shows that some coupled chaotic systems might synchronize such that their response anticipates the drivers by synchronizing with their future states.

It is also interesting to mention the nonlinear H_∞ synchronization method for nonautonomous schemes introduced by Suykens *et al.* (1997).

Spatio-Temporal Synchronization

Low-dimensional systems have rather limited usefulness in modeling real-world applications. This is why the synchronization of chaos has been carried out in high dimensions (see Kocarev *et al.* (1997) for a review). See also Chen and Dong (2001) for a discussion of special high-dimensional systems, namely large arrays of coupled chaotic systems.

Application to Transmission Systems and Secure Communication

Synchronization principles are useful in practical applications. Use of chaotic signals to transmit information has been a very active research topic in the last decade. Thus, it has been established that chaotic circuits may be used to transmit information by synchronization. As a result, several proposals for secure-communication schemes have been advanced (see, e.g., Cuomo *et al.* (1993), Hasler (1998), and Parlitz *et al.* (1999)). The first laboratory demonstration of a secure-communication system, which uses a chaotic signal for masking purposes, and which exploits the chaotic synchronization techniques to recover the signal, was reported by Kocarev *et al.* (1992).

It is difficult, within the scope of this article, to give a complete or detailed discussion, and it should be noted that there exist many competing and tested methods that are well established.

The main idea of the communication schemes is to encode a message by means of a chaotic dynamical system (the transmitter), and to decode it using a second dynamical system (the receiver) that synchronizes with the first. In general, secure-communication applications assume additionally that the coupled systems used are identical.

Different methods can be used to hide the useful information, for example, chaotic masking, chaotic switching, or direct chaotic modulation (Hasler 1998). For instance, in the chaotic masking method, an analog information carrying the signal $s(t)$ is added to the output $y(t)$ of the chaotic system in the transmitter. The receiver tries to synchronize with component $y(t)$ of the transmitted signal $s(t) + y(t)$. If synchronization takes place, the information signal can be retrieved by subtraction (Figure 10).

It is interesting to note that, in all proposed schemes for secure communications using the idea of synchronization (experimental realization or computer simulation), there is an inevitable noise degrading the fidelity of the original message.

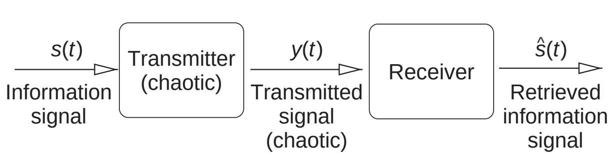

Figure 10 A typical communication setup.

Robustness to parameter mismatch was addressed by many authors (Illing *et al.* 2002). Lozi *et al.* (1993) showed that, by connecting two identical receivers in cascade, a significant amount of the noise can be reduced, thereby allowing the recovery of a much higher quality signal.

Furthermore, different implementations of chaotic secure communication have been proposed during the last decades, as well as methods for cracking this encoding. The methods used to crack such a chaotic encoding make use of the low dimensionality of the chaotic attractors. Indeed, since the properties of low-dimensional chaotic systems with one positive Lyapunov exponent can be reconstructed by analyzing the signal, such as through the delay-time reconstruction methods, it seems unlikely that these systems might provide a secure encryption method. The hidden message can often be retrieved easily by an eavesdropper without using the receiver. But, chaotic masking and encoding are difficult to break, using the state-of-the-art analysis tools, if sufficiently high dimensional chaos generators with multiple positive Lyapunov exponents (i.e., hyperchaotic systems) are used (see Pecora *et al.* (1997), and references therein).

Conclusion

In spite of the essential progress in theoretical and experimental studies, synchronization of chaotic systems continues to be a topic of active investigations and will certainly continue to have a broad impact in the future. Theory of synchronization remains a challenging problem of nonlinear science.

See also: Bifurcations of Periodic Orbits; Chaos and Attractors; Fractal Dimensions in Dynamics; Generic Properties of Dynamical Systems; Isochronous Systems; Lyapunov Exponents and Strange Attractors; Singularity and Bifurcation Theory; Stability Theory and KAM; Weakly Coupled Oscillators.

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