Bifurcations and synchronization in networks of unstable reaction-diffusion systems

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Abstract

This article is devoted to the analysis of the dynamics of a complex network of unstable reaction-diffusion systems. We demonstrate the existence of a non-empty parameter regime for which synchronization occurs in non-trivial attractors. We establish a lower bound of the dimension of the global attractor in an innovative manner, by proving a novel theorem of continuity of the unstable manifold, for which we invoke a principle of spectrum perturbation of non-bounded operators. Finally, we exhibit a co-dimension 2 bifurcation of the unstable manifold which shows that synchronization is compatible with instabilities.

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§1 Introduction

Christiaan Huygens was researching an accurate method of determining longitude for maritime navigation when he discovered, at the end of the seventeenth century, the principle of synchronization of two pendulum clocks [5]. Convinced that synchronization could be used to maintain pendulum clocks in agreement, Huygens observed that synchronization occurred when the pendulum clocks were interacting through small oscillations on a supporting vibrant frame. Nowadays, more than three centuries after his discovery, synchronization of dynamical systems has become an active topic in several domains of research. One pioneer work on the mathematical modeling of synchronized oscillators is probably the Kuramoto model [24]. Afterwards, a huge literature has emerged on this subject and it is now well understood how dynamical systems of finite dimension can be synchronized, even in the case of chaotic systems such as the Lorenz system [4]. In the mean time, the concept of complex network has been introduced and many applications to real-world systems have been considered, such as animal locomotion, coupled chemical reactions, neural networks, social networks, ecological meta-populations, epidemiological or behavioral networks (see for instance [6], [8], [11], [17], [19], [28], [31] and the references therein). However, synchronization of infinite dimensional dynamical systems remains an almost unexplored research field. In [23], the synchronization of two unidirectionally coupled reaction-diffusion systems is studied with a numerical approach, but the study of larger networks is not investigated. Recently, motivated by applications in neuroscience, Ambrosio & *al.* have proved that it is possible to synchronize FitzHugh-Nagumo reaction-diffusion systems [3]. Nevertheless, it is not known if synchronization is compatible with instabilities occurring in those complex

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dynamical systems: *does synchronization destroy instabilities of infinite dimensional systems, or are there different levels of instability for which synchronization can be maintained or not?* It is the purpose of this article to explore those open questions.

Since infinite dimensional dynamical systems can exhibit a complex asymptotic behavior, we place our study in the framework of attractors for continuous dynamical systems, which are invariant subsets of the phase space describing all the possible asymptotic states of the dynamical system. The concept of attractor has been developed for studying the asymptotic behavior of dissipative systems [25], and several sorts of attractors have been constructed: global attractors [26], [35], exponential attractors in a Hilbert setting [12] or in a Banach setting [13], pullback attractors for non-autonomous systems [15] or random attractors for stochastic systems [9]. One important class of systems for which the concept of attractor is well adapted is the class of reaction-diffusion systems. Those systems generate infinite dimensional dynamical systems and admit a great number of applications for chemical reactions, population dynamics, neuroscience or epidemiology (see the pioneer work of Turing [36] or [29], [34] and the references therein). For this class of systems, the nature of attractors has been widely studied, with estimations of their dimension [2], [14], [18], and investigations of their possible bifurcations [20], [27], [32].

In our work, we consider a complex network of infinite dimensional systems admitting unstable stationary solutions, determined by the Keener-Tyson reaction-diffusion system, which models unstable chemical reactions. This complex network is constructed in concordance with a finite graph, whose vertices are coupled with multiple instances of the Keener-Tyson model. We consider the coupling strength of the network as a control parameter for reaching synchronization. We show that it is possible to reach a synchronization state in a parameter regime for which the complex network admits non-trivial attractors (see Theorems 6 and 7). However, we show that synchronization is incompatible with attractors of arbitrarily large dimension. Roughly speaking, it seems possible to synchronize instabilities of low level, and impossible to synchronize instabilities of high level. Those results are established by estimating a lower bound of the dimension of attractors, applying a spectrum perturbation principle (exposed in [21]) to the unstable manifold of the unstable stationary solutions of the Keener-Tyson model. More precisely, we prove that the dimension of the local unstable manifold $\mathscr{W}_\xi^{loc}$ is constant for a sufficiently small coupling strength:

$$\dim \mathscr{W}_\xi^{loc}(\bar{U},\, \varepsilon) = \dim \mathscr{W}_0^{loc}(\bar{U},\, \varepsilon),$$

where ξ denotes the coupling strength of the complex network, $\bar{U}$ is an unstable stationary solution and ε is a parameter of the Keener-Tyson model, which also controls the dimension of the attractor (see Theorem 8). Furthermore, we observe that a variation of the coupling strength can provoke a series of bifurcations of the unstable manifold (see Proposition 1). Up to our knowledge, the results and methods presented in this work are new and overcome results established in previous works. Namely, the upper semi-continuity of attractors has been proved in [9] for instance, using the Hausdorff distance which is well suited for measuring the distances between two attractors, but this upper semi-continuity fails to detect changes of the dimension of the attractor; next, a principle of continuity of unstable manifolds has been proved in [10] but in a manner which does not control the dimension of the unstable manifold. Our methods are presented through the example of the Keener-Tyson model, but are sufficiently general to be applied to other parabolic problems, provided the existence of unstable stationary solutions.

Our article is organized as follows. In the next section, we recall important results of functional analysis in order to guaranty the self-sufficiency of the paper, and we present the setting of the problem, with the construction of a complex network of multiple instances of the Keener-Tyson model. In section 3, we establish intermediate results, with the existence of an invariant region which will be useful for proving a synchronization theorem, and dissipativity estimates which guaranty that the complex network generates a continuous dynamical system admitting the global attractor. Finally, we prove our main results in section 4, where the coexistence of synchronization and instabilities is investigated. We show that the unstable manifold contained in the global attractor undergoes a series of bifurcations which necessarily cross a synchronization threshold. We also explain why the continuity of exponential attractors is unable to detect bifurcations of the dimension of the unstable manifold.

§2 Problem statement

2.1. Preliminary results

In this section, we present several important results of functional analysis which shall be employed in our work, mainly about sectorial operators, semi-linear equations and attractors for infinite dimensional dynamical systems.

Notations. Throughout this paper, we will use classical notations for functional spaces: the space of continuous (respectively continuously differentiable) functions defined on an interval $I \subset \mathbb{R}$ with values in a Banach space X will be denoted $\mathscr{C}\big(I, X\big)$ (respectively $\mathscr{C}^1(I, X)$); Lebesgue spaces will be denoted $L^p(\Omega)$ and Sobolev spaces will be denoted $W^{k,p}(\Omega)$, where Ω denotes an open bounded domain in $\mathbb{R}^d$ with regular boundary $\partial\Omega$, $p \in [1, \infty]$ and $k \in \mathbb{N}$. Those functional spaces are Banach spaces whose norms will be denoted $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ respectively. For $p = 2$, we simply note $H^k(\Omega) = W^{k,2}(\Omega)$.

Sectorial operators. Let X be a Banach space and A a closed linear operator, densely defined in X. The domain of A will be denoted $\mathcal{D}(A)$. Assume that the spectrum of A satisfies $\sigma(A) \subset \{\lambda \in \mathbb{C}^*,\ |\arg(\lambda)| < \omega\}$, for $\omega \in]0, \pi]$ and furthermore that $\big\|(\lambda - A)^{-1}\big\|_{\mathscr{L}(X)} \leq \frac{M}{|\lambda|}$, for all $\lambda \in \mathbb{C}$ such that $|\arg(\lambda)| \geq \omega$, with $M \geq 1$, where $\mathscr{L}(X)$ denotes the Banach space of bounded linear operators defined in X. Then A is said to be *sectorial* in X. If A is a sectorial operator in X, it is seen that there exists a minimum coefficient ω satisfying the above properties; it is denoted ω_A and called *angle* of A. Sectorial operators admit fractional powers A^η with $0 < \eta < 1$, whose domains are denoted $\mathcal{D}(A^\eta)$ and can be described in terms of interpolation spaces (see for instance [37], Theorems 16.7 and 16.9).

Semi-linear equations. Let A be a sectorial operator in X of angle $\omega_A < \frac{\pi}{2}$, $0 < \eta < 1$, and F a non-linear operator defined in $\mathcal{D}(A^\eta)$ with values in X. We consider the Cauchy problem

$$\frac{dU}{dt} + AU = F(U), \quad t > 0, \qquad U(0) = U_0, \tag{1}$$

with $U_0 \in X$. We assume that F enjoys the property:

$$\|F(U) - F(V)\|_X \leq C_F\ (1 + \|A^\eta U\|_X + \|A^\eta V\|_X)\, \|U - V\|_X\,, \tag{2}$$

for all $U, V \in \mathcal{D}(A^\eta)$, with a positive constant C_F and a well-chosen $\eta \in (0, 1)$. The following theorem is proved in [37] (Theorem 4.4).

Theorem 1. *For all $U_0 \in X$, there exists $T_{U_0} > 0$ such that problem (1) admits a unique solution $U = U(t, U_0)$ in function space*

$$U \in \mathscr{C}\big((0, T_{U_0}]\,; \mathcal{D}(A)\big) \cap \mathscr{C}\big([0, T_{U_0}]\,; X\big) \cap \mathscr{C}^1\big((0, T_{U_0}]\,; X\big),$$

where T_{U_0} depends only on $\|U_0\|_X$. Furthermore, U satisfies

$$\|U(t)\|_X + t\,\|AU(t)\|_X \leq C_{U_0}, \quad 0 < t \leq T_{U_0},$$

where C_{U_0} is a positive constant which depends only on $\|U_0\|_X$.

Dynamical systems. If $\mathfrak{X}$ is a subset of X such that the solutions of problem (1) stemming from initial conditions in $\mathfrak{X}$ are global and remain in $\mathfrak{X}$, then $\mathfrak{X}$ is said to be *positively invariant*. In that case, one can define the mapping

$$\begin{array}{rlcl} \Theta : & (0, +\infty) \times \mathfrak{X} & \longrightarrow & \mathfrak{X} \\ & (t, U_0) & \longmapsto & S(t)U_0 \end{array} \tag{3}$$

where $S(t)$ denotes the semi-flow generated by problem (1), defined by $S(t)U_0 = U(t, U_0)$, for all $U_0 \in \mathfrak{X}$ and $t \geq 0$. Note that $S(t)$ satisfies $S(0) = Id$ (identity in X) and $S(t) \circ S(s) = S(t+s)$ for all non-negative t and s. The semi-flow $S(t)$ is called a *dynamical system*. X is called the *universal space* and $\mathfrak{X}$ is called the *phase space*.

Global attractor. Let X be a Banach space and $S(t)$ denote a dynamical system with phase space $\mathfrak{X}$. A subset $\mathfrak{B} \subset \mathfrak{X}$ is an *absorbing set* of the dynamical system $S(t)$ if for every bounded subset $B \subset \mathfrak{X}$, there exists a time t_B such that $S(t)B \subset \mathfrak{B}$ for all $t \geq t_B$. Now assume that the dynamical system $S(t)$ admits a compact absorbing set $\mathfrak{B} \subset X$. Then $S(t)$ possesses a global attractor $\mathscr{A}$, which is a compact set of X, invariant under the semi-flow $S(t)$, attracting each bounded set $B \subset \mathfrak{X}$ in the sense

$$\lim_{t\to\infty} \rho_H\big(S(t)B,\, \mathscr{A}\big) = 0,$$

where ρ_H denotes the Hausdorff semi-distance defined by

$$\rho_H(A,\, B) = \sup_{a\in A} \inf_{b\in B} d_X(a,\, b).$$

Note that the Hausdorff semi-distance ρ_H is not symmetric; thus the Hausdorff distance between A and B is defined by

$$\mathrm{dist}_H(A, B) = \max\big(\rho_H(A, B),\, \rho_H(B,\, A)\big). \tag{4}$$

It can be proved that the global attractor $\mathscr{A}$ is given by

$$\mathscr{A} = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)\mathfrak{B}},$$

where the closure is in X (see [35], Theorem I.1.1). Furthermore, if the absorbing set $\mathfrak{B}$ is a connected set, then $\mathscr{A}$ is also a connected set. If it exists, the global attractor $\mathscr{A}$ is necessarily unique.

Remark 1. *The global attractor is used in order to describe the asymptotic behavior of the dynamical system $S(t)$; it contains every equilibrium point of the dynamical system $S(t)$. One way to obtain lower bounds for the Hausdorff dimension $\dim_H \mathscr{A}$ of the global attractor is to estimate the dimension of the unstable manifold of an equilibrium of $S(t)$, which is always a subset of the global attractor.*

Unstable manifolds. Let X denote a Banach space and $S(t)$ a dynamical system defined in X, admitting a compact phase space $\mathfrak{X} \subset X$. Let $\bar{U}$ denote an equilibrium of $S(t)$. The unstable manifold $\mathscr{W}(\bar{U})$ is the set of elements U_0 in $\mathfrak{X}$ for which there exists a trajectory $\{U(t, U_0)\}_{t\leq 0}$ ending at U_0 such that $\lim_{t\to-\infty} U(t, U_0) = \bar{U}$. It is possible to represent local unstable manifolds by a localization method presented in [37] (Theorem 6.9). Furthermore, a variation of the dimension of the unstable manifold implies a change of the global attractor and constitutes a sufficient condition for a bifurcation to occur in the dynamical system $S(t)$. An illustrative example of a sequence of pitchfork bifurcations changing the dimension of the local

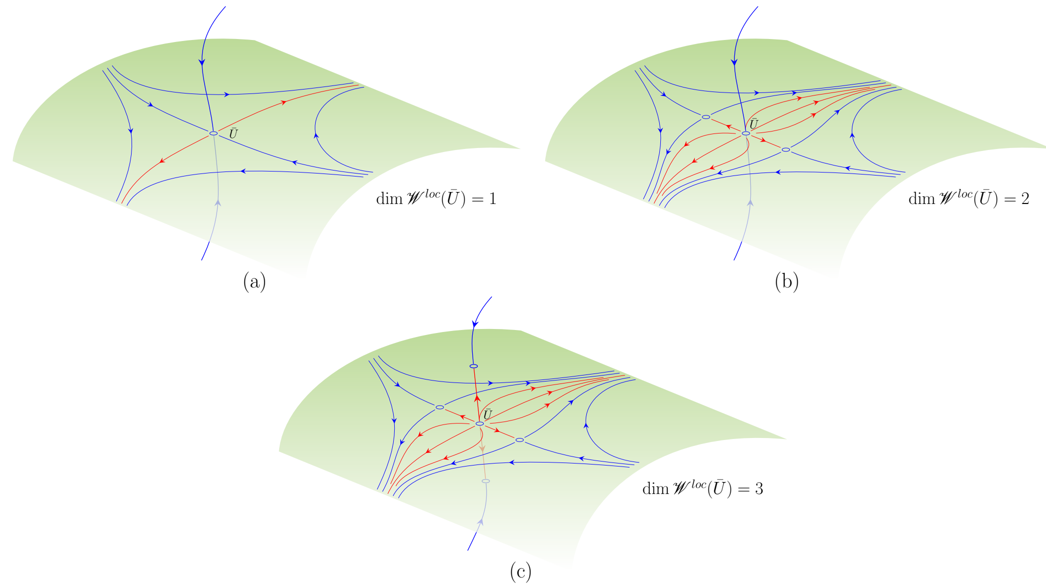


Figure 1: Illustrative example of a sequence of pitchfork bifurcations in a three-dimensional dynamical system, changing the dimension of the local unstable manifold (depicted in red). (a) The equilibrium point $\bar{U}$ is a saddle-node and $\dim \mathscr{W}^{loc}(\bar{U}) = 1$ (the Jacobian matrix of the system admits one positive eigenvalue and two negative eigenvalues). (b) The equilibrium point $\bar{U}$ is a saddle-node and $\dim \mathscr{W}^{loc}(\bar{U}) = 2$ (the Jacobian matrix admits two positive eigenvalues and one negative eigenvalue). (c) The equilibrium point $\bar{U}$ becomes a repulsive node and $\dim \mathscr{W}^{loc}(\bar{U}) = 3$ (the Jacobian matrix admits three positive eigenvalues).

unstable manifold is depicted in figure 1, for a three-dimensional dynamical system determined by a system of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^3,$$

where f is a function defined in $\mathbb{R}^3$.

Spectrum perturbation principle. Let X be a Banach space and A a densely defined closed operator in X. Then a part of $\sigma(A)$ consisting of a finite number of eigenvalues changes continuously with A: if B is a bounded operator such that $\|B\|_{\mathscr{L}(X)} \leq \delta$, then the finite part of eigenvalues of $A + B$ is unchanged for δ sufficiently small [21]. If A and B commute, it is possible to estimate the value of δ.

2.2. Keener-Tyson reaction-diffusion system

After those preliminary results, we aim to present the Keener-Tyson reaction-diffusion system, which models unstable chemical reactions. Let Ω denote an open bounded domain of $\mathbb{R}^d$ with $d \in \{1, 2, 3\}$, admitting a regular boundary $\partial\Omega$. The Keener-Tyson model is given by the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \dfrac{1}{\varepsilon^2}\left[u(1-u) - c\,v\,\dfrac{u-q}{u+q}\right] & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = b\Delta v + \dfrac{1}{\varepsilon}(u - v) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{5}$$

Here, u and v model chemical species in interaction, which are defined in $\Omega \times (0, \infty)$. The diffusion coefficients a and b are assumed to be positive. The parameters ε, c and q are positive real coefficients, and it is assumed that $q < 1$. The outward normal to $\partial\Omega$ is denoted by $\partial\nu$; the Neumann boundary condition models the impossibility for the chemical species to leave the domain Ω. The initial data u_0 and v_0 are defined in Ω and model the initial distributions of the chemical species within Ω. This reaction-diffusion system has been studied in [22] or [38] for instance. It is seen that system (5) admits two homogeneous stationary solutions $(0, 0)$ and $(\bar{u}, \bar{v})$, where $\bar{u}$ is the positive solution of the quadratic equation

$$(u + q)(1 - u) = c(u - q), \tag{6}$$

and $\bar{v} = \bar{u}$. Note that $q < \bar{u} < 1$. In [38], it is proved that one can find parameters values for which system (5) admits arbitrarily large exponential attractors. In this parameter regime, the stationary solution $(\bar{u}, \bar{v})$ admits a local unstable manifold $\mathscr{W}^{loc}\big((\bar{u}, \bar{v}), \varepsilon\big)$; in the case $\Omega \subset \mathbb{R}^3$, its dimension satisfies

$$\dim \mathscr{W}^{loc}\big((\bar{u}, \bar{v}), \varepsilon\big) \geq \frac{C}{\varepsilon^3},$$

where C denotes a positive constant. The example of an elementary bifurcation provoking an increase of the dimension of the unstable manifold is illustrated in figure 1.

2.3. Complex network of multiple instances of the Keener-Tyson model

Let us now describe the construction of a complex network of multiple instances of the Keener-Tyson model (5). Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ denote a graph composed with a finite set $\mathscr{V}$ of n vertices $(n \geq 2)$, and a finite set $\mathscr{E}$ of edges. We define a matrix of connectivity L in correspondence with the set of edges, by setting

$$L_{i,j} = +1 \text{ if } (j, i) \in \mathscr{E} (i \neq j), \quad L_{i,j} = 0 \text{ else}, \quad L_{j,j} = -\sum_{\substack{k=1 \\ k \neq j}}^{n} L_{k,j}. \tag{7}$$

We then construct a complex network of multiple instances of the Keener-Tyson model by coupling each vertex of the graph $\mathscr{G}$ with one instance of the Keener-Tyson model, which leads to the following equations:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = a_i \Delta u_i + \dfrac{1}{\varepsilon_i^2}\left[u_i(1 - u_i) - c_i v_i \dfrac{u_i - q_i}{u_i + q_i}\right] + \xi \displaystyle\sum_{k=1}^{n} L_{i,k} u_k & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v_i}{\partial t} = b_i \Delta v_i + \dfrac{1}{\varepsilon_i}(u_i - v_i) + \xi \displaystyle\sum_{k=1}^{n} L_{i,k} v_k & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u_i}{\partial \nu} = \dfrac{\partial v_i}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u_i(x, 0) = u_{i,0}(x), \quad v_i(x, 0) = v_{i,0}(x) & \text{in } \Omega, \end{cases} \tag{8}$$

for each $i \in \{1, \ldots, n\}$. As previously, the diffusion coefficients a_i and b_i $(1 \leq i \leq n)$ are assumed to be positive. The parameters ε_i, c_i, q_i $(1 \leq i \leq n)$ are positive real coefficients and we assume that

$$0 < q_i < 1. \tag{9}$$

The parameter ξ is a non-negative real coefficient which is called the *coupling strength* of the complex network. Here, the coupling terms $\xi \sum_{k=1}^{n} L_{i,k} u_k$ and $\xi \sum_{k=1}^{n} L_{i,k} v_k$ of system (8) admit a linear form, which is sufficiently general for the present work, since we will focus on the possible coexistence of synchronization and instabilities. However, it would be possible to consider coupling terms of a more general form. For instance, quadratic couplings are studied in [6].

Next, we consider three assumptions which will be useful in the sequel of the paper.

Assumption (A1). *The graph $\mathscr{G}$ is a complete bi-directional graph. In that case, the connectivity matrix of $\mathscr{G}$ reads*

$$L = \begin{bmatrix} -(n-1) & +1 & \dots & +1 \\ +1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & +1 \\ +1 & \dots & +1 & -(n-1) \end{bmatrix},$$

thus the complex network equations can be written

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = a_i \Delta u_i + \dfrac{1}{\varepsilon_i^2}\left[u_i(1-u_i) - c_i v_i \dfrac{u_i - q_i}{u_i + q_i}\right] + \xi \displaystyle\sum_{\substack{k=1 \\ k\neq i}}^{n} (u_k - u_i) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v_i}{\partial t} = b_i \Delta v_i + \dfrac{1}{\varepsilon_i}(u_i - v_i) + \xi \displaystyle\sum_{\substack{k=1 \\ k\neq i}}^{n} (v_k - v_i) & \text{in } \Omega \times (0, \infty). \end{cases} \tag{10}$$

Assumption (A2). *The graph $\mathscr{G}$ is bi-directional (or symmetric). In that case, the connectivity matrix of $\mathscr{G}$ satisfies $L_{i,j} = L_{j,i}$ for all i, j in $\{1, \dots, n\}$ such that $i \neq j$, and $\sum_{k=1}^{n} L_{i,k} = 0$ for all i in $\{1, \dots, n\}$.*

Obviously, assumption (A1) implies assumption (A2).

Assumption (A3). *The multiple instances of the Keener-Tyson model (5) composing the complex network (8) are identical. This means that there exists a common set of values a, b, c, q and ε such that the parameters a_i, b_i, c_i, q_i and ε_i $(1 \leq i \leq n)$ satisfy*

$$a_i = a, \quad b_i = b, \quad c_i = c, \quad q_i = q, \quad \varepsilon_i = \varepsilon,$$

for all i in $\{1, \dots, n\}$.

Note that if assumption (A3) is satisfied, then the diffusion coefficients of the chemical species u_i and v_i $(1 \leq i \leq n)$ can still be different, that is, $a \neq b$.

2.4. Abstract formulation of the complex network problem

In order to solve the complex network problem (8), we write it as an abstract Cauchy problem in an infinite dimensional space. To that aim, let us consider the Hilbert space $X = \left(L^2(\Omega)\right)^{2n}$ equipped with the product norm

$$\|U\|_X = \left(\sum_{i=1}^{n} \|u_i\|_{L^2(\Omega)}^2 + \sum_{i=1}^{n} \|v_i\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \tag{11}$$

for all $U = \left(u_i,\, v_i\right)_{1\leq i \leq n}^{\top} \in X$. We introduce the linear operators

$$A_{1,i} = -a_i \Lambda + id_{L^2(\Omega)}, \quad A_{2,i} = -b_i \Lambda + id_{L^2(\Omega)}, \quad A_i = \operatorname{diag}\left\{A_{1,i}, A_{2,i}\right\}, \quad 1 \leq i \leq n,$$

and $\mathbb{A} = \operatorname{diag}\{A_i,\ 1 \leq i \leq n\}$, where Λ denotes the realization of the Laplace operator Δ with Neumann boundary condition. It is known that $\mathbb{A}$ is a positive definite self-adjoint sectorial operator in X with angle strictly less than $\frac{\pi}{2}$. Furthermore, its domain is given by

$$\mathcal{D}(\mathbb{A}) = \left(H^2_N(\Omega)\right)^{2n}, \tag{12}$$

with

$$H^2_N(\Omega) = \left\{u \in H^2(\Omega)\ ;\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\},$$

where $H^2(\Omega)$ denotes the usual Sobolev space $W^{2,2}(\Omega)$ (the subscript N in the notation $H^2_N(\Omega)$ refers to the Neumann boundary condition). The operator $\mathbb{A}$ admits fractional powers $\mathbb{A}^\theta$, $0 \leq \theta \leq 1$, whose domains are given by

$$\mathcal{D}(\mathbb{A}^\theta) = \begin{cases} \left(H^{2\theta}(\Omega)\right)^{2n} & \text{if } 0 \leq \theta < \frac{3}{4}, \\ \left(H^{2\theta}_N(\Omega)\right)^{2n} & \text{if } \frac{3}{4} < \theta < 1, \end{cases} \tag{13}$$

where $H^s(\Omega)$ $(s \geq 0)$ denotes the usual Sobolev space $W^{s,p}(\Omega)$ with $p = 2$, and

$$H^s_N(\Omega) = \left\{u \in H^s(\Omega)\ ;\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}.$$

Now we fix an exponent $\eta \in \left(\frac{3}{4},\, 1\right)$. Since $d \in \{1,\, 2,\, 3\}$, the Sobolev injections guaranty that

$$H^{2\eta}(\Omega) \subset \mathscr{C}\left(\overline{\Omega}\right) \subset L^\infty(\Omega), \tag{14}$$

with continuous embeddings (see [1] for instance). For each $i \in \{1,\, \ldots,\, n\}$, we introduce the modified operator f_i defined in $\mathcal{D}\left(A_i^\eta\right)$ by

$$\begin{aligned} f_i(U_i) &= \left(\hat{f}_i(U_i),\ \check{f}_i(U_i)\right)^\top \\ &= \left(u_i + \frac{1}{\varepsilon_i^2}\left[u_i(1-u_i) - c_i\, v_i\, \frac{u_i - q_i}{|u_i| + q_i}\right],\ v_i + \frac{1}{\varepsilon_i}(u_i - v_i)\right)^\top, \end{aligned} \tag{15}$$

where $U_i = (u_i,\, v_i)^\top$. Note that the non-linearity of system (8) has been slightly modified, since the denominator $u_i + q_i$ has been replaced by $|u_i| + q_i$. However, it will be proved below that the solutions of the modified abstract problem (21) stemming from non-negative initial conditions remain non-negative (see Lemma 1), and thus fully coincide with the solutions of the non-modified problem (8).

It is proved in [37] that there exists a positive constant C_{ε_i} which depends on ε_i, such that

$$\left\|f_i(U_i) - f_i(\tilde{U}_i)\right\|_{(L^2(\Omega))^2} \leq C_{\varepsilon_i}\left(1 + \|U_i\|_{(L^\infty(\Omega))^2} + \left\|\tilde{U}_i\right\|_{(L^\infty(\Omega))^2}\right)\left\|U_i - \tilde{U}_i\right\|_{(L^2(\Omega))^2}, \tag{16}$$

for all U_i, $\tilde{U}_i$ in $\mathcal{D}\big(A_i^\eta\big)$. Moreover, it can be easily shown that the constant C_{ε_i} admits a quadratic expression

$$C_{\varepsilon_i} = \kappa_{i,0} + \frac{\kappa_{i,1}}{\varepsilon_i} + \frac{\kappa_{i,2}}{\varepsilon_i^2}, \tag{17}$$

with positive coefficients $\kappa_{i,0}$, $\kappa_{i,1}$ and $\kappa_{i,2}$ $(1 \leq i \leq n)$. Under assumption (A3), the operators A_i, f_i $(1 \leq i \leq n)$ will be written A and f respectively, and the constants C_{ε_i}, $\kappa_{i,j}$ $(0 \leq j \leq 2)$, will be written C_ε, κ_j respectively.

Next we introduce the non-linear operator F defined in $\mathcal{D}(\mathbb{A}^\eta)$ by

$$F(U) = \left(f_1(U_1),\, \ldots,\, f_n(U_n)\right)^\top, \tag{18}$$

with $U = (U_1, \ldots, U_n)^\top$, and the linear operator G_ξ defined in X by

$$G_\xi(U) = \big(g_1(U), \ldots, g_n(U)\big)^\top, \tag{19}$$

where g_i is given by

$$g_i(U) = \left(\xi \sum_{k=1}^{n} L_{i,k} u_k,\, \xi \sum_{k=1}^{n} L_{i,k} v_k\right)^\top, \quad 1 \leq i \leq n. \tag{20}$$

With these notations, the abstract formulation of problem (8) reads

$$\begin{cases} \dfrac{dU}{dt} + \mathbb{A}U = F(U) + G_\xi(U), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{21}$$

where U_0 denotes any initial condition in X.

Our first result states the existence and uniqueness of local solutions to the complex network problem (21).

Theorem 2. *For any initial condition $U_0 \in X$, there exists a positive time T_{U_0} depending only on $\|U_0\|_X$ such that problem (21) admits a unique solution U in function space*

$$\mathscr{C}\big((0, T_{U_0}],\ \mathcal{D}(\mathbb{A})\big) \cap \mathscr{C}\big([0, T_{U_0}],\ X\big) \cap \mathscr{C}^1\big((0, T_{U_0}],\ X\big). \tag{22}$$

Furthermore, U satisfies

$$\|U(t)\|_X + t\,\|\mathbb{A}U(t)\|_X \leq C_{U_0}, \quad 0 < t \leq T_{U_0}, \tag{23}$$

where $C_{U_0} > 0$ depends only on $\|U_0\|_X$.

Proof. By virtue of the embedding (14), estimation (16) implies that

$$\big\|f_i(U_i) - f_i(\tilde{U}_i)\big\|_{(L^2(\Omega))^2} \leq C_{\varepsilon_i}\big(1 + \|A_i^\eta U_i\|_{(L^2(\Omega))^2} + \big\|A_i^\eta \tilde{U}_i\big\|_{(L^2(\Omega))^2}\big)\, \big\|U_i - \tilde{U}_i\big\|_{(L^2(\Omega))^2},$$

for all $U_i,\ \tilde{U}_i$ in $\mathcal{D}\big(A_i^\eta\big)$. Thus we can conclude that there exists a positive constant $C_{\varepsilon,n}$ depending on $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and n, such that

$$\big\|F(U) - F(\tilde{U})\big\|_X \leq C_{\varepsilon,n}\left(1 + \|\mathbb{A}^\eta U\|_X + \big\|\mathbb{A}^\eta \tilde{U}\big\|_X\right) \big\|U - \tilde{U}\big\|_X,$$

for all U, $\tilde{U}$ in $\mathcal{D}(\mathbb{A}^\eta)$. In parallel, the linear operator G_ξ obviously satisfies

$$\big\|G_\xi(U) - G_\xi(\tilde{U})\big\|_X \leq C_{G_\xi} \big\|U - \tilde{U}\big\|_X,$$

for all U, $\tilde{U}$ in X, where C_{G_ξ} denotes a positive constant determined by G_ξ, and thus depending on the graph $\mathscr{G}$. Thus we have

$$\begin{aligned} &\big\|(F + G_\xi)(U) - (F + G_\xi)(\tilde{U})\big\|_X \\ &\quad\leq \big\|F(U) - F(\tilde{U})\big\|_X + \big\|G_\xi(U) - G_\xi(\tilde{U})\big\|_X \\ &\quad\leq C_{\varepsilon,n}\left(1 + \|\mathbb{A}^\eta U\|_X + \big\|\mathbb{A}^\eta \tilde{U}\big\|_X\right)\big\|U - \tilde{U}\big\|_X + C_{G_\xi}\big\|U - \tilde{U}\big\|_X \\ &\quad\leq (C_{\varepsilon,n} + C_{G_\xi})\left(1 + \|\mathbb{A}^\eta U\|_X + \big\|\mathbb{A}^\eta \tilde{U}\big\|_X\right)\big\|U - \tilde{U}\big\|_X. \end{aligned}$$

for all U, $\tilde{U}$ in $\mathcal{D}(\mathbb{A}^\eta)$. The conclusion follows from Theorem 4.4 in [37]. □

§3 Global solutions

In this section, we establish sufficient conditions which guarantee that the local solutions of the complex network problem (8) are global in time. This question is not trivial, since it is known that the solutions of reaction-diffusion systems can explode in finite time (see the survey given in [30] for instance). We first prove the existence of an invariant region which provides a uniform bound in $L^\infty(\Omega)^{2n}$; this uniform bound will in turn be useful for proving a sufficient condition of synchronization in the final section. Next, we establish energy estimates which assure the dissipativity of the complex network; from this dissipativity follows the existence of the global attractor.

3.1. Invariant region

First, we briefly demonstrate the non-negativity of the solutions stemming from non-negative initial data. We recall that a non-linear operator $\phi = (\phi_i)_{1\le i\le m}$ defined on $\mathbb{R}^m$ (with $m \in \mathbb{N}^*$) is said to be *quasi-positive* if it satisfies the property

$$\phi_i(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m) \ge 0, \tag{24}$$

for all $u = (u_1, \ldots, u_m) \in (\mathbb{R}^+)^m$ and for all $i \in \{1, \ldots, m\}$. The quasi-positivity can be used to prove the non-negativity of the solutions stemming from non-negative initial data (see e.g. [30]). Let us introduce the space of initial conditions

$$X_0 = \{u \in X \ ; \ u(x) \ge 0, \ \forall x \in \Omega\}, \tag{25}$$

where the inequality $u \ge 0$ has to be understood component-wise. It is easily seen that the operators f_i and g_i ($1 \le i \le n$), defined by equations (15) and (20) respectively, are quasi-positive. It follows that the operators F and G_ξ are also quasi-positive, and finally, that the sum operator $F + G_\xi$ is also quasi-positive. Thus we directly obtain the following lemma.

Lemma 1. *Let $U_0 \in X_0$ and U be the solution of problem (21) starting from U_0, defined on $[0, T_{U_0}]$. Then, its components are non-negative on $[0, T_{U_0}]$.*

Lemma 1 implies that the unique solution of the modified problem (21) is also the unique solution of the non-modified problem (8).

Our next result states the existence of an invariant region, under assumption (A1). Note that the existence of an invariant region is a sufficient condition for the local solutions of problem (8) to be global (see e.g. [34], Chapter 14).

Theorem 3. *If assumption (A2) holds ($\mathscr{G}$ is a bi-directional graph), then the flow induced by the complex network problem (8) admits a positively invariant region given by*

$$\mathscr{R} = \big\{(w_j)_{1\le j\le 2n} \in X \ ; \ w_j(x) \in [0, 1], \, x \in \Omega, \, 1 \le j \le 2n\big\}, \tag{26}$$

that is, if $U_0 \in \mathscr{R}$, then the solution $U(t, U_0)$ of the complex network problem (8) stemming from U_0, defined on $[0, T_{U_0}]$, satisfies $U(t, U_0) \in \mathscr{R}$ for all $t \in [0, T_{U_0}]$.

The proof of Theorem 3 uses the properties of a truncation function χ which are presented in the following lemma. Since its proof is elementary, we may skip it.

Lemma 2. *Let χ be the real-valued function defined on $\mathbb{R}$ by:*

$$\chi(s) = \begin{cases} \frac{1}{2}s^2 & \text{if } s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

The function χ is continuously differentiable on $\mathbb{R}$ and we have:

$$\chi'(s) = \begin{cases} s & \text{if } s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Furthermore, the function χ enjoys the following properties:

$$\begin{aligned} &\chi(s) \geq 0, \quad \chi'(s) \leq 0, \quad s\chi'(s) \geq 0, \quad s\chi'(s) = 2\chi(s), \quad \forall s \in \mathbb{R}, \\ &(s-r)\big(\chi'(s) - \chi'(r)\big) \geq 0, \quad \forall s, r \in \mathbb{R}. \end{aligned} \tag{27}$$

Proof of Theorem 3. Let consider an initial condition $U_0 \in \mathscr{R}$. We denote by $U = \big(u_i, v_i\big)_{1\leq i\leq n}$ the solution of the complex network problem (8) with a bi-directional graph, stemming from U_0 and defined on $[0, T_{U_0}]$ with $T_{U_0} > 0$. By virtue of Lemma 1, it is already known that $u_i(x, t) \geq 0$ and $v_i(x, t) \geq 0$, for all $i \in \{1, \ldots, n\}$, $x \in \Omega$ and $t \in [0, T_{U_0}]$. Next, let us fix $i \in \{1, \ldots, n\}$. We introduce the functions ω_i and ρ_i defined by

$$\begin{aligned} \omega_i(x, t) &= 1 - u_i(x, t), \quad (x, t) \in \Omega \times [0, T_{U_0}], \\ \rho_i(t) &= \int_\Omega \chi\big(\omega_i(x, t)\big) dx, \quad t \in [0, T_{U_0}]. \end{aligned}$$

We have $u_i(x, 0) \leq 1$ for all $x \in \Omega$, thus $\omega_i(x, 0) \geq 0$ for all $x \in \Omega$. Since $\chi(s) = 0$ for all non-negative s, it follows that $\rho_i(0) = 0$. Furthermore, since $\chi(s) \geq 0$ for all $s \in \mathbb{R}$, we have $\rho_i(t) \geq 0$ for all $t \in [0, T_{U_0}]$. Next, the function ρ_i is continuously differentiable on $[0, T_{U_0}]$, and we have:

$$\rho_i'(t) = \int_\Omega \frac{\partial \omega_i}{\partial t} \chi'(\omega_i) dx, \quad t \in [0, T_{U_0}],$$

where we omit the variables x and t in order to lighten our notations. By virtue of assumption (A2), we have

$$\sum_{k=1}^n L_{i,k} u_k = \sum_{\substack{k=1 \\ k\neq i}}^n L_{i,k}(u_k - u_i).$$

Now we compute:

$$\frac{\partial \omega_i}{\partial t} = -\frac{\partial u_i}{\partial t} = -a_i \Delta u_i - \frac{1}{\varepsilon_i^2}\left[u_i(1-u_i) - c_i v_i \frac{u_i - q_i}{u_i + q_i}\right] - \xi \sum_{\substack{k=1 \\ k\neq i}}^n L_{i,k}(u_k - u_i).$$

In parallel, we have $\Delta \omega_i = -\Delta u_i$, thus we obtain:

$$\begin{aligned} \rho_i'(t) &= \int_\Omega \left\{ a_i \Delta \omega_i - \frac{1}{\varepsilon_i^2}\left[u_i(1-u_i) - c_i v_i \frac{u_i - q_i}{u_i + q_i}\right] - \xi \sum_{\substack{k=1 \\ k\neq i}}^n L_{i,k}(u_k - u_i) \right\} \chi'(\omega_i) dx \\ &= a_i \int_\Omega (\Delta \omega_i) \chi'(\omega_i) dx - \frac{1}{\varepsilon_i^2} \int_\Omega u_i \omega_i \chi'(\omega_i) dx \\ &+ \frac{c_i}{\varepsilon_i^2} \int_\Omega v_i \frac{u_i - q_i}{u_i + q_i} \chi'(\omega_i) dx - \xi \sum_{\substack{k=1 \\ k\neq i}}^n L_{i,k} \int_\Omega (u_k - u_i) \chi'(\omega_i) dx. \end{aligned}$$

Since u_i satisfies the Neumann boundary condition on $\partial\Omega$, the same holds for ω_i and we have:

$$\int_\Omega (\Delta \omega_i) \chi'(\omega_i) dx = -\int_\Omega \nabla \omega_i \nabla \chi'(\omega_i) dx = -\int_\Omega |\nabla \chi'(\omega_i)|^2 dx \leq 0.$$

Next, we have $\omega_i \chi'(\omega_i) \geq 0$ by virtue of Lemma 2, thus:

$$-\frac{1}{\varepsilon_i^2} \int_\Omega u_i \omega_i \chi'(\omega_i) dx \leq 0.$$

Afterwards, we write:

$$\begin{aligned}\frac{c_i}{\varepsilon_i^2} \int_\Omega v_i \frac{u_i - q_i}{u_i + q_i} \chi'(\omega_i) dx &= \frac{c_i}{\varepsilon_i^2} \left[\int_\Omega v_i \frac{u_i - 1}{u_i + q_i} \chi'(\omega_i) dx + \int_\Omega v_i \frac{1 - q_i}{u_i + q_i} \chi'(\omega_i) dx \right] \\ &= -\frac{c_i}{\varepsilon_i^2} \int_\Omega \frac{v_i}{u_i + q_i} \omega_i \chi'(\omega_i) dx + \frac{c_i}{\varepsilon_i^2} \int_\Omega v_i \frac{1 - q_i}{u_i + q_i} \chi'(\omega_i) dx.\end{aligned}$$

We recall that $0 < q_i < 1$ (see equation (9)), thus we obtain:

$$\frac{c_i}{\varepsilon_i^2} \int_\Omega v_i \frac{u_i - q_i}{u_i + q_i} \chi'(\omega_i) dx \leq 0.$$

At this stage, we have proved that:

$$\rho_i'(t) \leq -\xi \sum_{\substack{k=1 \\ k \neq i}}^{n} L_{i,k} \int_\Omega (u_k - u_i) \chi'(\omega_i) dx.$$

Now we set

$$\rho = \sum_{i=1}^{n} \rho_i,$$

and we compute:

$$\rho'(t) \leq -\xi \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} L_{i,k} \int_\Omega (u_k - u_i) \chi'(\omega_i) dx.$$

We remark that

$$\begin{aligned}-\sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} L_{i,k} \int_\Omega (u_k - u_i) \chi'(\omega_i) dx &= -\sum_{\substack{1 \leq i,k \leq n \\ k < i}} L_{i,k} \int_\Omega (u_k - u_i) \big(\chi'(\omega_i) - \chi'(\omega_k)\big) dx \\ &= -\sum_{\substack{1 \leq i,k \leq n \\ k < i}} L_{i,k} \int_\Omega (\omega_i - \omega_k) \big(\chi'(\omega_i) - \chi'(\omega_k)\big) dx.\end{aligned}$$

The last property in Lemma 2 implies that:

$$-\sum_{\substack{1 \leq i,k \leq n \\ k < i}} L_{i,k} (\omega_i - \omega_k) \big(\chi'(\omega_i) - \chi'(\omega_k)\big) \leq 0, \tag{28}$$

thus we have:

$$\rho'(t) \leq 0, \quad t \in \big[0,\, T_{U_0}\big].$$

We conclude that ρ is a decreasing function on $\big[0,\, T_{U_0}\big]$. But we have $\rho(0) = 0$, thus $\rho(t) \leq 0$ on $\big[0,\, T_{U_0}\big]$. Since we also have $\rho(t) \geq 0$ on $\big[0,\, T_{U_0}\big]$, we obtain $\rho \equiv 0$. Finally, we have $\rho_i \geq 0$ for all $i \in \{1, \ldots, n\}$, from which we conclude that $\rho_i \equiv 0$, which means that $u_i(x, t) \leq 1$ for all $x \in \Omega$, $t \in \big[0,\, T_{U_0}\big]$ and $i \in \{1, \ldots, n\}$.

Similar computations show that $v_i(x, t) \leq 1$ for all $x \in \Omega$, $t \in \big[0,\, T_{U_0}\big]$ and $i \in \{1, \ldots, n\}$. The proof is complete. □

It is interesting to note that the property

$$-\sum_{\substack{1\le i,k\le n\\ k<i}} L_{i,k}(\omega_i-\omega_k)\big(\chi'(\omega_i)-\chi'(\omega_k)\big)\le 0$$

used at the end of the proof (see equation (28)) is similar to the property of the diffusion operator with Neumann boundary condition

$$\int_\Omega \big(\Delta\omega_i\big)\chi'(\omega_i)dx \le 0.$$

In other words, the coupling operator, in the case of a bi-directional graph, acts in a similar manner as the diffusion operator. For that reason, the coupling operator G_ξ defined by (19) is sometimes called *graph Laplacian*.

We emphasize that the symmetry of the coefficients $L_{i,k}$ is a key ingredient of the proof of Theorem 3; if the graph $\mathscr{G}$ is not symmetric, then the arguments used in the proof are not valid any longer. In that case, one should not hope to find an invariant region of rectangular shape.

Note also that recently, entropy methods have been employed in [16], for establishing, in all space dimensions, the existence of global solutions to reaction-diffusion systems similar to the complex network (8).

Finally, it is worth emphasizing that the region $\mathscr{R}$ given by (26) contains the non-trivial stationary solution

$$\bar U = \big(\bar u_i,\, \bar v_i\big)^\top_{1\le i\le n} \tag{29}$$

when $\xi = 0$, since we have $0 < q_i < 1$ for all $i \in \{1,\dots,n\}$ (see equation (9)). Under assumptions (A2) and (A3), elementary computations show that the region $\mathscr{R}$ even contains $\bar U$ for all positive ξ. As the invariant region provides a uniform bound which will be useful for proving a sufficient condition of synchronization in the final section, we shall investigate the possibility of reaching a synchronization state in the presence of the unstable stationary solution $\bar U$. In particular, it is a very interesting question to determine if synchronization perturbs the nature of the stationary solution $\bar U$.

3.2. Energy estimates

In this section, we aim to establish two energy estimates for the solutions of the complex network (8): the first one is obtained in the general case, but implies a restriction on the parameter regime, since the coefficients ε_i $(1\le i\le n)$, are required to be sufficiently small; the second energy estimate is obtained under assumption (A1), but it provides a uniform exponential decrease. From those energy estimates follows the existence of an absorbing set, as will be shown in the next section.

Theorem 4. *Let U_0 be any initial condition in the set of initial data X_0 defined by (25). We denote by $U(t, U_0)$ the solution of the complex network problem (8) stemming from U_0, defined on $[0, T_{U_0}]$. Then, for ε_i sufficiently small ($1\le i\le n$), there exist positive constants $\tilde\gamma$ and C such that*

$$\|U(t,U_0)\|_X \le \|U_0\|_X\, e^{-\tilde\gamma t} + C, \quad t\in[0, T_{U_0}]. \tag{30}$$

The following lemma is elementary, but useful for rearranging double finite sums.

Lemma 3. *Let n be a positive integer, $(a_i)_{1\le i\le n}$, $(b_{ki})_{1\le k,i\le n}$ and $(v_i)_{1\le i\le n}$ real coefficients. Then it holds that*

$$\sum_{i=1}^n \left(a_i v_i^2 + \sum_{\substack{k=1\\k\ne i}}^n b_{ki}v_k^2\right) = \sum_{i=1}^n c_i v_i^2,$$

with

$$c_i = a_i + \sum_{\substack{k=1\\k\neq i}}^{n} b_{ik}, \quad 1 \le i \le n.$$

Proof of Theorem 4. Let $U_0 \in X_0$ and $U(t, U_0)$ be the solution of problem (8) starting from U_0, defined on $[0, T_{U_0}]$. We introduce the energy functions E_1 and E_2 defined by

$$E_1(t) = \frac{1}{2}\sum_{i=1}^{n} \|u_i(t)\|^2_{L^2(\Omega)}, \quad E_2(t) = \frac{1}{2}\sum_{i=1}^{n} \|v_i(t)\|^2_{L^2(\Omega)}, \tag{31}$$

where $U = \big(u_i, v_i\big)^{\top}_{1\le i\le n}$. The energy functions E_1 and E_2 satisfy

$$E_1(t) + E_2(t) = \frac{1}{2}\|U(t, U_0)\|^2_X,$$

for all $t \in [0, T_{U_0}]$. Next, we compute

$$\begin{aligned}
\frac{dE_1}{dt}(t) &= \sum_{i=1}^{n}\int_\Omega u_i \frac{\partial u_i}{\partial t} dx \\
&= \sum_{i=1}^{n}\int_\Omega u_i \left\{ a_i \Delta u_i + \frac{1}{\varepsilon_i^2}\left[u_i(1-u_i) - c_i v_i \frac{u_i - q_i}{u_i + q_i}\right] + \xi \sum_{k=1}^{n} L_{i,k} u_k \right\} dx \\
&\le \sum_{i=1}^{n}\int_\Omega \left(\frac{1}{\varepsilon_i^2}u_i^2 - \frac{1}{\varepsilon_i^2}u_i^3 - \frac{1}{\varepsilon_i^2}c_i u_i v_i \frac{u_i - q_i}{u_i + q_i} + \xi L_{i,i}u_i^2 + \xi \sum_{\substack{k=1\\k\neq i}}^{n} L_{i,k}u_k u_i \right) dx,
\end{aligned}$$

where we have used the Neumann boundary condition and the maximum principle. Next, it is easy to see that

$$-\frac{1}{\varepsilon_i^2}c_i u_i v_i \frac{u_i - q_i}{u_i + q_i} \le \frac{1}{\varepsilon_i^2}c_i q_i v_i.$$

Meanwhile, we use Young inequality $u_i u_k \le \frac{u_i^2}{2} + \frac{u_k^2}{2}$, which leads to

$$\begin{aligned}
\frac{dE_1}{dt}(t) &\le \sum_{i=1}^{n}\int_\Omega \left(\frac{1}{\varepsilon_i^2}u_i^2 - \frac{1}{\varepsilon_i^2}u_i^3 + \frac{1}{\varepsilon_i^2}c_i q_i v_i + \xi L_{i,i}u_i^2 + \frac{\xi}{2}\sum_{\substack{k=1\\k\neq i}}^{n} L_{i,k}u_k^2 + \frac{\xi}{2}\sum_{\substack{k=1\\k\neq i}}^{n} L_{i,k}u_i^2 \right) dx \\
&\le \sum_{i=1}^{n}\int_\Omega \left(\zeta_{1,i}u_i^2 - \frac{1}{\varepsilon_i^2}u_i^3 \right) dx + \sum_{i=1}^{n}\int_\Omega \frac{1}{\varepsilon_i^2}c_i q_i v_i dx,
\end{aligned}$$

where $\zeta_{1,i}$ is a real coefficient obtained after rearranging the finite sums over k and i, whose expression is determined by Lemma 3 as

$$\zeta_{1,i} = \frac{1}{\varepsilon_i^2} + \frac{\xi}{2}\sum_{k=1}^{n} L_{i,k}, \quad 1 \le i \le n.$$

Similar computations show that

$$\frac{dE_2}{dt}(t) \le \sum_{i=1}^{n}\int_\Omega \frac{u_i^2}{2\varepsilon_i^2}dx + \sum_{i=1}^{n}\int_\Omega \left(\zeta_{2,i} - \frac{1}{\varepsilon_i} \right) v_i^2 dx,$$

where $\zeta_{2,i}$ is given by

$$\zeta_{2,i} = \frac{1}{2} + \frac{\xi}{2} \sum_{k=1}^{n} L_{i,k}, \quad 1 \le i \le n.$$

It follows that

$$\frac{dE_1}{dt}(t) + \frac{dE_2}{dt}(t) \le \sum_{i=1}^{n} \int_\Omega \left[\left(\zeta_{1,i} + \frac{1}{2\varepsilon_i^2} \right) u_i^2 - \frac{1}{\varepsilon_i^2} u_i^3 \right] dx + \sum_{i=1}^{n} \int_\Omega \left[\frac{1}{\varepsilon_i^2} c_i q_i v_i + \left(\zeta_{2,i} - \frac{1}{\varepsilon_i} \right) v_i^2 \right] dx.$$

We set

$$\gamma_i = - \left(\zeta_{2,i} - \frac{1}{\varepsilon_i} \right),$$

and we choose ε_i sufficiently small so that $\gamma_i > 0$. Now we use the elementary polynomial inequalities

$$\begin{aligned} &\left(\zeta_{1,i} + \frac{1}{2\varepsilon_i^2} \right) u_i^2 - \frac{1}{\varepsilon_i^2} u_i^3 \le -\frac{1}{2} u_i^2 + C_{1,i}, \\ &\frac{1}{\varepsilon_i^2} c_i q_i v_i - \gamma_i v_i^2 \le -\tilde{\gamma}_i v_i^2 + C_{2,i}, \end{aligned}$$

with positive constants $C_{1,i}$, $C_{2,i}$ and $\tilde{\gamma}_i$, given by:

$$C_{1,i} = \frac{(1 + 2\zeta_{1,i} + \varepsilon_i^{-2})^3}{54\varepsilon_i^{-4}}, \quad C_{2,i} = \frac{c_i^2 q_i^2}{2\varepsilon_i^4 \gamma_i}, \quad \tilde{\gamma}_i = \frac{\gamma_i}{2},$$

for $1 \le i \le n$. We obtain

$$\frac{dE_1}{dt}(t) + \frac{dE_2}{dt}(t) + \tilde{\gamma}\big(E_1(t) + E_2(t)\big) \le C,$$

with positive constants C and $\tilde{\gamma}$. The conclusion follows from Gronwall lemma. $\square$

We emphasize that the energy estimate (30) is valid for a restrictive parameter regime, since ε_i has to be chosen sufficiently small in order to guaranty that

$$\zeta_{2,i} - \frac{1}{\varepsilon_i} < 0,$$

where $\zeta_{2,i}$ depends on the coupling strength ξ. In particular, the coefficients $\zeta_{2,i}$ may increase with ξ, thus requiring a smaller value of ε_i; this should be seen as a restriction, since it will appear with Theorem 6 that ξ should admit a large value to guaranty synchronization. However, we can obtain a uniform in ξ energy estimate by assuming that the graph $\mathscr{G}$ is a complete bi-directional graph. Note that it is not necessary to assume that the instances of the complex network (8) are identical. This is stated in the following theorem.

Theorem 5. *Suppose that assumption (A1) holds (the graph $\mathscr{G}$ underlying the complex network problem (8) is a complete bi-directional graph). Let U_0 be any initial condition in the set of initial data X_0 defined by (25). We denote again by $U(t, U_0)$ the solution of the complex network problem (8) stemming from U_0, defined on $[0, T_{U_0}]$. Then there exist positive constants $\tilde{\gamma}$ and $\tilde{K}$ such that*

$$\|U(t, U_0)\|_X \le \|U_0\|_X \, e^{-\tilde{\gamma} t} + \tilde{K}, \quad t > 0, \tag{32}$$

where $\tilde{\gamma}$ and $\tilde{K}$ do not depend on ξ.

Proof. As before, let $U_0 \in X_0$ and $U(t, U_0)$ be the solution of problem (8) starting from U_0, defined on $[0, T_{U_0}]$. We examine again the energy functions E_1 and E_2 defined by (31). Now we have

$$\frac{dE_1}{dt}(t) = \sum_{i=1}^{n} \int_\Omega u_i \left\{ a_i \Delta u_i + \frac{1}{\varepsilon_i^2} \left[u_i(1-u_i) - c_i v_i \frac{u_i - q_i}{u_i + q_i} \right] + \xi \sum_{\substack{k=1 \\ k\neq i}}^{n} (u_k - u_i) \right\} dx.$$

By virtue of Young inequality and using Lemma 3, we have

$$\begin{aligned}
\sum_{i=1}^{n} u_i \sum_{\substack{k=1 \\ k\neq i}}^{n} (u_k - u_i) &= \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k\neq i}}^{n} u_i u_k - \sum_{i=1}^{n} (n-1) u_i^2 \\
&\leq \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k\neq i}}^{n} \left(\frac{u_i^2}{2} + \frac{u_k^2}{2} \right) - \sum_{i=1}^{n} (n-1) u_i^2 \\
&\leq \sum_{i=1}^{n} \left[\frac{-(n-1)}{2} u_i^2 + \sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{1}{2} u_k^2 \right] \\
&\leq \sum_{i=1}^{n} \left[\frac{-(n-1)}{2} + \sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{1}{2} \right] u_i^2 \\
&\leq 0.
\end{aligned}$$

It follows that

$$\frac{dE_1}{dt} \leq \sum_{i=1}^{n} \int_\Omega \left(\frac{1}{\varepsilon_i^2} u_i^2 - \frac{1}{\varepsilon_i^2} u_i^3 \right) dx + \sum_{i=1}^{n} \int_\Omega \frac{1}{\varepsilon_i^2} c_i q_i v_i dx.$$

Similarly, we have

$$\frac{dE_2}{dt} \leq \sum_{i=1}^{n} \int_\Omega \frac{1}{2\varepsilon_i} u_i^2 dx + \sum_{i=1}^{n} \int_\Omega \frac{-1}{2\varepsilon_i} v_i^2 dx.$$

We obtain

$$\frac{d(E_1+E_2)}{dt} \leq \sum_{i=1}^{n} \int_\Omega \left(\frac{2+\varepsilon_i}{2\varepsilon_i^2} u_i^2 - \frac{1}{\varepsilon_i^2} u_i^3 \right) dx + \sum_{i=1}^{n} \int_\Omega \left(\frac{1}{\varepsilon_i} c_i q_i v_i - \frac{1}{2\varepsilon_i} v_i^2 \right) dx.$$

Now we use the elementary polynomial inequalities

$$\begin{aligned}
\frac{2+\varepsilon_i}{2\varepsilon_i^2} u_i^2 - \frac{1}{\varepsilon_i^2} u_i^3 &\leq -\frac{1}{2} u_i^2 + C_{1,i}, \\
\frac{1}{\varepsilon_i} c_i q_i v_i - \frac{1}{2\varepsilon_i} v_i^2 &\leq -\frac{1}{4\varepsilon_i} v_i^2 + C_{2,i},
\end{aligned}$$

where $C_{1,i}$ and $C_{2,i}$ are positive constants, which are given for $1 \leq i \leq n$ by

$$C_{1,i} = \frac{(2+\varepsilon_i+\varepsilon_i^2)^3}{54\varepsilon_i^2}, \quad C_{2,i} = \frac{c_i^2 q_i^2}{\varepsilon_i}.$$

Note that $C_{1,i}$ and $C_{2,i}$ do not depend on ξ. We obtain

$$\frac{d(E_1+E_2)}{dt} \leq -\gamma(E_1+E_2) + K,$$

where γ and K are given by

$$\gamma = \min\left(1,\, \frac{1}{2\varepsilon_{\min}}\right), \quad K = |\Omega| \sum_{i=1}^{n} (C_{1,i} + C_{2,i}),$$

with $\varepsilon_{\min} = \min\limits_{1\le i\le n} \{\varepsilon_i\}$. This leads to the desired estimate. $\square$

§4 Large time behavior

In this final section, we prove our main results, investigating the possibility to reach a synchronization state within non-trivial attractors. We first prove a theorem which establishes a sufficient condition of synchronization in the complex network (8), under assumptions (A1) and (A3); this sufficient condition highlights the effect of the coupling strength ξ involved in the coupling terms of the complex network (see equation (19)), as well as the effect of the parameter ε of the initial Keener-Tyson model (5) which controls the dimension of the attractors (see equation (33) below). Next, we prove that the complex network (8) generates a dynamical system which admits the global attractor. Finally, we prove the continuity of the local unstable manifold of the stationary solution $\bar{U}$ of the complex network, employing an innovative method which provides a lower bound of the dimension of the global attractor and also proves that the unstable manifold undergoes a series of bifurcations which change its dimension.

4.1. Sufficient condition of synchronization

Let us first give a precise definition of identical synchronization for the complex network of multiple instances of the Keener-Tyson model.

Definition 1. *For every U_0 in X, let us denote by $U(t, U_0) = \big(U_i(t, U_0)\big)_{1\le i\le n}$ the solution of the complex network problem (8) stemming from U_0. Let $\mathscr{K}$ be a subset of X such that each solution of (8) stemming from $U_0 \in \mathscr{K}$ is global. We say that the complex network (8) synchronizes in $\mathscr{K} \subset X$ if for every initial condition U_0 in $\mathscr{K}$, we have:*

$$\lim_{t\to\infty} \|U_i(t, U_0) - U_j(t, U_0)\|_{L^2(\Omega)^2} = 0.$$

In the following theorem, we establish that the complex network (8) synchronizes in the invariant region $\mathscr{R}$ given by Theorem 3, under assumptions (A1) and (A3). The uniform bound provided by the invariant region plays a decisive role in the proof. Furthermore, we emphasize that the sufficient condition of synchronization (33) highlights the effect of the parameter ε of the Keener-Tyson model, thus is more precise than the sufficient conditions of synchronization established in [3] or in [7].

Theorem 6. *Let assumptions (A1) and (A3) hold. Then the complex network (8) synchronizes in $\mathscr{R}$, at an exponential rate, if*

$$n\,\xi > \tilde{\kappa}_0 + \frac{\tilde{\kappa}_1}{\varepsilon} + \frac{\tilde{\kappa}_2}{\varepsilon^2}, \tag{33}$$

where $\tilde{\kappa}_0$, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are positive constants determined by (17), which depend only on the parameters c and q of the Keener-Tyson model (5).

Proof. Let us consider an initial condition $U_0 \in \mathscr{R}$. We denote by $U = \big(u_i,\, v_i\big)^{\top}_{1\le i\le n}$ the solution of the complex network problem (8) with a complete bi-directional graph, stemming from U_0 and defined on $[0, \infty)$. By virtue of Theorem 3, we have

$$0 \le u_i(x, t) \le 1, \quad 0 \le v_i(x, t) \le 1, \quad t \ge 0, \quad x \in \Omega, \quad 1 \le i \le n.$$

We conclude that

$$\|u_i(t)\|_{L^\infty(\Omega)} \le 1, \quad \|v_i(t)\|_{L^\infty(\Omega)} \le 1, \quad t \ge 0, \quad 1 \le i \le n.$$

We introduce the energy functions:

$$E_{ik} = E_{1ik} + E_{2ik},$$

where

$$E_{1ik} = \frac{1}{2}\|u_i - u_k\|^2_{L^2(\Omega)}, \quad E_{2ik} = \frac{1}{2}\|v_i - v_k\|^2_{L^2(\Omega)},$$

for all i, k in $\{1, \ldots, n\}$ such that $i \neq k$. We compute:

$$\frac{d}{dt}E_{1ik} = \int_\Omega (u_i - u_k)\frac{\partial(u_i - u_k)}{\partial t}dx.$$

Using the fact the instances are identical, we have:

$$\begin{aligned}
\frac{\partial(u_i - u_k)}{\partial t} &= \frac{\partial u_i}{\partial t} - \frac{\partial u_k}{\partial t} \\
&= a\Delta u_i - a\Delta u_k + \hat{f}(U_i) - u_i - \hat{f}(U_k) + u_k + \xi\left[\sum_{\substack{j=1\\ j\neq i}}^{n}(u_j - u_i) - \sum_{\substack{j=1\\ j\neq k}}^{n}(u_j - u_k)\right].
\end{aligned}$$

Now we remark that

$$\begin{aligned}
\sum_{\substack{j=1\\ j\neq i}}^{n}(u_j - u_i) - \sum_{\substack{j=1\\ j\neq k}}^{n}(u_j - u_k) &= \sum_{\substack{j=1\\ j\neq i}}^{n} u_j - (n-1)u_i - \sum_{\substack{j=1\\ j\neq k}}^{n} u_j + (n-1)u_k \\
&= u_k - (n-1)u_i - u_i + (n-1)u_k \\
&= n(u_k - u_i).
\end{aligned}$$

We obtain

$$\begin{aligned}
\frac{d}{dt}E_{1ik} &= a\int_\Omega (u_i - u_k)\Delta(u_i - u_k)dx \\
&\quad + \int_\Omega (u_i - u_k)\big(\hat{f}(U_i) - u_i - \hat{f}(U_k) + u_k\big)dx \\
&\quad - n\xi\int_\Omega (u_k - u_i)^2 dx \\
&\le \int_\Omega (u_i - u_k)\big(\hat{f}(U_i) - u_i - \hat{f}(U_k) + u_k\big)dx - 2n\xi E_{1ik}.
\end{aligned}$$

Next we have, by virtue of Hölder's inequality:

$$\begin{aligned}
&\int_\Omega (u_i - u_k)\big(\hat{f}(U_i) - u_i - \hat{f}(U_k) + u_k\big)dx \\
&\qquad \le \left\|(u_i - u_k)\big(\hat{f}(U_i) - u_i - \hat{f}(U_k) + u_k\big)\right\|_{L^1(\Omega)} \\
&\qquad \le \|u_i - u_k\|_{L^2(\Omega)} \times \left\|\hat{f}(U_i) - u_i - \hat{f}(U_k) + u_k\right\|_{L^2(\Omega)} \\
&\qquad \le \|U_i - U_k\|_{L^2(\Omega)^2} \times \|f(U_i) - U_i - f(U_k) + U_k\|_{L^2(\Omega)^2}.
\end{aligned}$$

We recall that f satisfies the estimation (16). It follows that:

$$\begin{aligned}\|f(U_i) - U_i - f(U_k) + U_k\|_{L^2(\Omega)^2} &\le \|f(U_i) - f(U_k)\|_{L^2(\Omega)^2} + \|U_i - U_k\|_{L^2(\Omega)^2}\\ &\le \Big[C_\varepsilon\big(1 + \|U_i\|_{L^\infty(\Omega)^2} + \|U_k\|_{L^\infty(\Omega)^2}\big) + 1\Big]\,\|U_i - U_k\|_{L^2(\Omega)^2}\\ &\le (1 + 3C_\varepsilon)\,\|U_i - U_k\|_{L^2(\Omega)^2}\,.\end{aligned}$$

We conclude that

$$\begin{aligned}\frac{d}{dt}E_{1ik} &\le -2n\xi E_{1ik} + (1 + 3C_\varepsilon)\,\|U_i - U_k\|^2_{L^2(\Omega)^2}\\ &\le -2n\xi E_{1ik} + 2(1 + 3C_\varepsilon)E_{ik}.\end{aligned}$$

Similar computations show that we also have

$$\frac{d}{dt}E_{2ik} \le -2n\xi E_{2ik} + 2(1 + 3C_\varepsilon)E_{ik},$$

from which we deduce

$$\frac{d}{dt}E_{ik} + 2(n\xi - \tilde{C}_\varepsilon)E_{ik} \le 0,$$

where $\tilde{C}_\varepsilon$ is a positive quantity. By virtue of equation (17), $\tilde{C}_\varepsilon$ is given by a quadratic expression on ε:

$$\tilde{C}_\varepsilon = 1 + 3C_\varepsilon = \tilde{\kappa}_0 + \frac{\tilde{\kappa}_1}{\varepsilon} + \frac{\tilde{\kappa}_2}{\varepsilon^2},$$

where $\tilde{\kappa}_0$, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are positive constants which depend only on the parameters c and q of the Keener-Tyson model (5). Applying Gronwall lemma leads to

$$E_{ik}(t) \le E_{ik}(0)e^{-2(n\xi - \tilde{C}_\varepsilon)t}, \quad t \ge 0,$$

which completes the proof. □

It is possible to relax assumption (A1) and (A3) in several ways (see [3] or [7]). However, the symmetry of the graph $\mathscr{G}$ represents, here, a key ingredient for reaching synchronization. Indeed, it is possible to construct a simple two nodes network with a unique directed edge, which does not synchronize.

Note that the sufficient condition of synchronization (33) can be fulfilled for any $\varepsilon > 0$. When the number n of vertices in the graph $\mathscr{G}$ is fixed, this sufficient condition determines a synchronization threshold in the (ε, ξ) plane which is depicted in figure 3(b) below. In the sequel, we examine the possibility to fulfill the sufficient synchronization (33) while conserving the instability of the stationary solution $\bar{U}$.

4.2. Global attractor of the complex network

The energy estimates proved in Theorems 4 and 5 guarantee that the solutions of the complex network problem (8) starting in the set X_0 given by (25) are global in time. Therefore, we can construct a family of semi-flows $S_\xi(t)$ parametrized by ξ as follows: for any $\xi \ge 0$ and any initial condition $U_0 \in X_0$, we denote by $U_\xi(t, U_0)$ the solution of the complex network problem (8), defined for $t \ge 0$ and we set

$$S_\xi(t)U_0 = U_\xi(t, U_0). \tag{34}$$

The energy estimates (30) and (32) are also useful for proving the existence of an absorbing set, which in turn guarantees the existence of the global attractor for each dynamical system $S_\xi(t)$.

Theorem 7. *Suppose that the assumptions of Theorem 4 or 5 hold. Then, for each $\xi \geq 0$, the complex network problem (8) generates a dynamical system $S_\xi(t)$ which admits the global attractor $\mathscr{A}_\xi$. Furthermore, $\mathscr{A}_\xi$ is connected and contains the trivial stationary solution 0 for all $\xi \geq 0$. If assumptions (A2) and (A3) hold, then the non-trivial stationary solution $\bar{U}$ defined by (29) also belongs to $\mathscr{A}_\xi$ for all $\xi \geq 0$ and we have*

$$\dim_H \mathscr{A}_\xi \geq 1, \quad \xi \geq 0.$$

Proof. For each $\xi \geq 0$, we have, by virtue of energy estimate (30):

$$\|S_\xi(t)U\|_X \leq \|U\|_X \, e^{-\tilde{\gamma}_\xi t} + C_\xi,$$

for all $U \in X_0$, $t \geq 0$, with positive constants $\tilde{\gamma}_\xi$ and C_ξ. Now let us consider a bounded subset B of X. There exists a positive constant C_B such that $\|U\|_X \leq C_B$ for all U in X. We set $t_B = \frac{\log C_B}{\tilde{\gamma}_\xi}$, which leads to

$$\|S_\xi(t)U\|_X \leq 1 + C_\xi,$$

for all $t \geq t_B$ and $U \in B \cap X_0$. Following the reasoning of [37] (Chapter 6, Section 5.2), we solve the complex network problem (21)

$$\frac{dU}{dt} + \mathbb{A}U = F(U) + G_\xi(U), \quad t > 0,$$

with an initial condition $S_\xi(s)U$ such that $s \geq 0$ and $U \in B \cap X_0$. By virtue of Theorem 2 and estimation (23), there exist a positive time τ which depends only on $\|S_\xi(s)U\|_X$, and an increasing function μ, such that

$$(t-s)\,\|\mathbb{A}\, S_\xi(t)U\|_X \leq \mu\left(\|S_\xi(s)U\|_X\right), \quad s < t \leq s + \tau.$$

We apply this inequality at $s = t - (1 + t_B + \tau)$, which leads to

$$\|\mathbb{A}\, S_\xi(t)U\|_X \leq \frac{1}{1 + t_B + \tau} \mu\left(\left\|S_\xi\big(t - (1 + t_B + \tau)\big)U\right\|_X\right), \quad 1 + t_B + \tau < t.$$

We introduce $\tilde{t}_B = 1 + \tau + 2t_B$ and obtain

$$\sup_{U \in B \cap X_0} \sup_{t \geq \tilde{t}_B} \|\mathbb{A}\, S_\xi(t)U\|_X \leq \tilde{C}_\xi,$$

with $\tilde{C}_\xi = \mu(1 + C_\xi) > 0$. Now we set $\mathfrak{B}_\xi = \overline{B}^{\mathcal{D}(\mathbb{A})}(0, \tilde{C}_\xi)$ (closed ball in X, with closure in $\mathcal{D}(\mathbb{A})$, centered at 0, of radius $\tilde{C}_\xi$). As such, $\mathfrak{B}_\xi$ is a compact and connected absorbing set, whose existence implies the existence of the global attractor $\mathscr{A}_\xi$, which is also connected.

Next, it is easily seen that the trivial stationary solution 0 belongs to $\mathscr{A}_\xi$ for all $\xi \geq 0$ (see Remark 1), and that the non-trivial stationary solution $\bar{U}$ also does if assumptions (A2) and (A3) hold. Indeed, elementary computations show that the stationary solution $(\bar{u}, \bar{v})$ of the Keener-Tyson model (5) satisfies in this case

$$\frac{1}{\varepsilon_i^2}\left[\bar{u}(1-\bar{u}) - c_i \bar{v} \frac{\bar{u} - q_i}{\bar{u} + q_i}\right] + \xi \sum_{k=1}^{n} L_{i,k} \bar{u} = \frac{1}{\varepsilon_i}(\bar{u} - \bar{v}) + \xi \sum_{k=1}^{n} L_{i,k} \bar{v} = 0,$$

for all $i \in \{1, \ldots, n\}$ and all $\xi \geq 0$. Therefore, the non-trivial stationary solution $\bar{U}$ defined by (29) is a stationary homogeneous solution of the complex network problem (8) for all $\xi \geq 0$; hence, $\bar{U}$ belongs to $\mathscr{A}_\xi$ for all $\xi \geq 0$.

Now we consider the function λ defined by

$$\lambda(x) = d_X(x, 0), \quad x \in \mathscr{A}_\xi,$$

where d_X denotes the distance induced by the norm of the Banach space X (see equation (11)). The function λ is continuous and $\mathscr{A}_\xi$ is connected, thus $\lambda(\mathscr{A}_\xi)$ is connected in $\mathbb{R}$. Moreover, $\lambda(\mathscr{A}_\xi)$ is non-empty since $d_X(0, \bar{U}) > 0$, and the function λ is also Lipschitz, thus we have

$$\dim_H \lambda(\mathscr{A}_\xi) \leq \dim_H \mathscr{A}_\xi.$$

Finally, $\lambda(\mathscr{A}_\xi)$ is a non-empty connected subset of $\mathbb{R}$, thus we have $\dim_H \lambda(\mathscr{A}_\xi) = 1$, which completes the proof. □

It is worth emphasizing that the assumptions of the latter Theorem are compatible with the sufficient condition of synchronization (33). Thus we have proved an important pattern: **it is possible to reach a synchronization state in the complex network** (8) **while preserving instabilities within a non-trivial attractor**. With slight modifications, this pattern shall be recovered for a large class of infinite dimensional dynamical systems, provided the existence of at least two stationary solutions.

Note that it can also be shown that the complex network problem (8) generates a continuous dynamical system $S_\xi(t)$ defined in X, admitting exponential attractors $\mathscr{M}_\xi$ of finite fractal dimension (see [37], Chapter 6). Under assumption (A1), the uniform in ξ energy estimate (32) could even be used to construct a uniform in ξ absorbing set $\mathfrak{B}$ for the family of dynamical systems $S_\xi(t)$ (see [38]). Therefore, we would prove that the family of exponential attractors $\mathscr{M}_\xi$ parametrized by ξ enjoys a continuity property with respect to a variation of ξ, in the sense of the Hausdorff distance defined by (4):

$$\mathrm{dist}_H(\mathscr{M}_\xi, \mathscr{M}_0) \to 0 \text{ as } \xi \to 0. \tag{35}$$

However, this continuity property would not imply[4] that the Hausdorff dimension of $\mathscr{M}_\xi$ is close to the Hausdorff dimension of $\mathscr{M}_0$, mainly because the Hausdorff dimension is sensitive to "*holes*", whereas the Hausdorff distance is not [33].

Now we intend to deepen our study of the global attractor $\mathscr{A}_\xi$, in order to determine the level of instability which would be compatible with a synchronization state of the complex network (8). As illustrated with the example given in figure 1, a detailed analysis of the unstable manifold can provide refined estimates of the dimension of the global attractor.

Let us introduce the notations

$$\varphi(u, v) = u(1-u) - cv\frac{u-q}{u+q}, \quad \psi(u, v) = u - v,$$

for $u \geq 0$ and $v \geq 0$. We consider the stationary solution $(\bar{u}, \bar{v})$ of the Keener-Tyson model (5), given by (6). We also assume that the parameters c, q satisfy the condition

$$0 < \frac{\partial \varphi}{\partial u}(\bar{u}, \bar{v}) < 1. \tag{36}$$

Under the latter assumption, and in the case $\Omega \subset \mathbb{R}^3$, it has been proved in [38] that the dynamical system induced by the Keener-Tyson model (5) admits a local unstable manifold $\mathscr{W}^{loc}\big((\bar{u}, \bar{v}), \varepsilon\big)$ whose dimension satisfies, for ε sufficiently small,

$$\dim \mathscr{W}^{loc}\big((\bar{u}, \bar{v}), \varepsilon\big) \geq \frac{C}{\varepsilon^3}, \tag{37}$$

where C denotes a positive constant. At the opposite, if ε is sufficiently large, it is seen that $(\bar{u}, \bar{v})$ is exponentially stable and that

$$\dim \mathscr{W}^{loc}\big((\bar{u}, \bar{v}), \varepsilon\big) = 0. \tag{38}$$

[4]Consider $X = \mathbb{R}$, $A = [0, 1]$ and $B_m = \{\frac{k}{m}\ ;\ 0 \leq k \leq m\}$ for each integer $m > 0$. Then it holds that $\dim_H A = 1$, $\dim_H B_m = 0$ for all $m > 0$, whereas $\mathrm{dist}_H(A, B_m) \to 0$ as $m \to \infty$.

Since the unstable manifold of a stationary solution is always a subset of the global attractor (see Remark 1 and Figure 1), we can deduce that the global attractor $\mathscr{A}_0$ of the dynamical system $S_0(t)$ induced by the complex network (8) with $\xi = 0$ also satisfies for ε sufficiently small

$$\dim_H \mathscr{A}_0 \geq \frac{C_n}{\varepsilon^3}, \tag{39}$$

for a positive quantity C_n which may depend on the number n of vertices in the graph $\mathscr{G}$. We are interested in establishing a similar property for $\mathscr{A}_\xi$ with $\xi > 0$.

4.3. Unstable manifold of the complex network

Let us suppose that assumptions (A2) and (A3) hold. In this case, $\bar{U}$ belongs to $\mathscr{A}_\xi$ for all $\xi \geq 0$. Since the operator $F_\xi = F + G_\xi$ defined by equations (18) and (19) is Fréchet differentiable in a neighborhood of $\bar{U}$, it follows from Theorem 6.9 in [37], that the complex network problem (8) can be localized and complexified, in such a way that the stationary solution $\bar{U}$ is hyperbolic and admits a local unstable manifold $\mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon)$ (note that the phase space can be slightly modified in this complexification process). Furthermore, the dimension of the local unstable manifold $\mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon)$ is given by the number of eigenvalues of the operator $\bar{\mathbb{A}}_\xi = \mathbb{A} - F'_\xi(\bar{U})$ having a negative real part, where $F'_\xi(\bar{U})$ denotes the Fréchet derivative of the operator F_ξ, evaluated at $\bar{U}$. The following theorem states that $\mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon)$ and $\mathscr{W}_0^{loc}(\bar{U}, \varepsilon)$ have the same dimension for ξ sufficiently small.

Theorem 8. *Suppose that assumptions (A2) and (A3) hold and that the property* (36) *is satisfied. Then for any $\varepsilon > 0$, there exists a positive ξ_ε such that*

$$\dim \mathscr{W}_\xi^{loc}(\bar{U},\, \varepsilon) = \dim \mathscr{W}_0^{loc}(\bar{U},\, \varepsilon),$$

for $0 \leq \xi \leq \xi_\varepsilon$.

Proof. Let us introduce the matrices Q and R defined by

$$Q = \begin{bmatrix} 1 + \frac{1}{\varepsilon^2}\left(1 - 2\bar{u} - \frac{2cq\bar{v}}{(\bar{u}+q)^2}\right) & -\frac{c(\bar{u}-q)}{\varepsilon^2(\bar{u}+q)} \\ \frac{1}{\varepsilon} & 1 - \frac{1}{\varepsilon} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the n blocks matrices Q_n and R_n given by

$$Q_n = \begin{bmatrix} Q & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & Q \end{bmatrix}, \quad R_n = \begin{bmatrix} L_{1,1}R & L_{1,1}R & \dots & L_{1,n}R \\ L_{2,1}R & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & L_{n-1,n}R \\ L_{n,1}R & \dots & L_{n,n-1}R & L_{n,n}R \end{bmatrix}.$$

The Fréchet derivative of the non-linear operator $F_\xi = F + G_\xi$, where and F and G_ξ are defined by equations (18) and (19) respectively, evaluated at $\bar{U}$, is given by

$$F'_\xi(\bar{U}) = Q_n + \xi R_n.$$

Next we introduce the operator

$$\bar{\mathbb{A}}_\xi = \mathbb{A} - F'_\xi(\bar{U}).$$

We write

$$\bar{\mathbb{A}}_\xi = \bar{\mathbb{A}}_0 + T(\xi), \tag{40}$$

with $\bar{\mathbb{A}}_0 = \mathbb{A} - Q_n$ and $T(\xi) = -\xi R_n$. In this way, the operator $\bar{\mathbb{A}}_\xi$ is seen as a perturbation of $\bar{\mathbb{A}}_0$. Let $\sigma(\bar{\mathbb{A}}_\xi)$ denote the spectrum of the operator $\mathbb{A}_\xi$. Now we examine the separation condition

$$\sigma(\bar{\mathbb{A}}_\xi) \cap \{\lambda \in \mathbb{C}\ ;\ \Re e\lambda = 0\} = \varnothing. \tag{41}$$

It is already known that $\bar{\mathbb{A}}_0$ satisfies this spectrum separation. Indeed, the intersection of $\sigma(\bar{\mathbb{A}}_0)$ with the half-plane $\{\lambda \in \mathbb{C}\ ;\ \Re e\lambda < 0\}$ is a finite set of negative eigenvalues, which proves that $\bar{U}$ admits a local unstable manifold $\mathscr{W}_0^{loc}(\bar{U}, \varepsilon)$ which is tangent to the eigenspace corresponding to these negative eigenvalues; the dimension of $\mathscr{W}_0^{loc}(\bar{U}, \varepsilon)$ is the dimension of this eigenspace (see [37], Chapter 6, section 6.3). Finally, since $T(\xi)$ is a bounded operator, by virtue of Kato's theorem for the spectrum of perturbed operators (see [21], Chapter 4, section §3), it follows that

$$\dim \mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon) = \dim \mathscr{W}_0^{loc}(\bar{U}, \varepsilon),$$

for ξ sufficiently small. □

The association of Theorem 8 and of the lower bound (37) directly provides the promised refined lower bound of the global attractor $\mathscr{A}_\xi$.

Corollary 1. *Suppose that assumptions (A2) and (A3) hold and that the property* (36) *is satisfied, with $\Omega \subset \mathbb{R}^3$. Then for any $\varepsilon > 0$ sufficiently small, there exists a positive ξ_ε such that*

$$\dim_H \mathscr{A}_\xi \geq \dim \mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon) \geq \frac{C_n}{\varepsilon^3},$$

for $0 \leq \xi \leq \xi_\varepsilon$, where C_n denotes a positive quantity which depends on the number n of vertices in the graph $\mathscr{G}$ and on the parameters c and q of the Keener-Tyson model (5)*.*

4.4. Bifurcations of the stationary solution

The spectrum perturbation principle is the main ingredient of the proof of Theorem 8. However, this principle shows more than the local continuity of the unstable manifold with respect to the parameter ξ, which corresponds to the coupling strength of the complex network (8). First, we have for ε sufficiently small and $0 \leq \xi \leq \xi_\varepsilon$:

$$\dim \mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon) \geq \frac{C_n}{\varepsilon^3}.$$

At the opposite, for ε sufficiently large and $0 \leq \xi \leq \xi_\varepsilon$, we have, by virtue of Theorem 8 and equation (38):

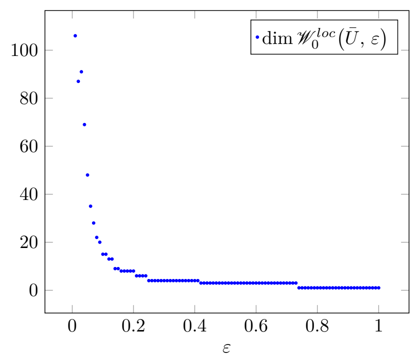
$$\dim \mathscr{W}_\xi^{loc}(\bar{U}, \varepsilon) = 0.$$

We obtain the following proposition.

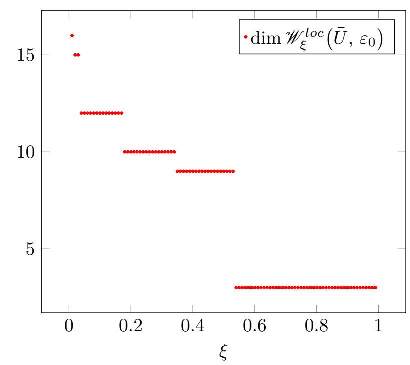
Proposition 1. *Suppose that assumptions (A2) and (A3) hold and that the property* (36) *is satisfied. Then the complex network problem* (8) *undergoes a sequence of bifurcations which provoke a variation of the dimension of the unstable manifold, under a variation of both parameters ξ and ε.*

Note that the nature of the bifurcations, whose existence is established by the latter proposition, is not investigated here. We remark that the continuity property of exponential attractors (see equation (35)) fails to detect those bifurcations. In other words, exponential attractors seem to damp the structural changes of the dynamical system. The same holds for the continuity property of the unstable manifold, established in [10] by using the Hausdorff distance, which similarly fails to detect the bifurcations. Thus the spectrum perturbation principle invoked in the proof of Theorem 8 and Proposition 1 represents an innovative method which leads to refined results.

We have illustrated in figures 2 and 3 the sequence of bifurcations of the unstable manifold $\mathscr{W}_{\xi}^{loc}(\bar{U},\, \varepsilon)$, with respect to both parameters ξ and ε. Those numerical results have been obtained for the complex network (8) with 4 nodes, $\Omega = (0,\, \pi)^2$, $a = b = 1$, $q = 0.03$ and $c = 0.7$. The computations are been performed in a Debian/GNU-Linux environment with the free software FreeFem++. We observe that the dimension of the unstable manifold increases as ε tends to 0, or as ξ does. However, it is not known, at this stage, if the critical value ξ_ε given in Theorem 8 changes continuously with ε. In a separate paper, we aim to present a deepened numerical approach in order to visualize the shape of those bifurcation lines. Of great interest is the intersection of the synchronization threshold defined by the inequality (33) with the bifurcation lines in the $(\varepsilon,\, \xi)$ plane, which is illustrated in figure 3(b). It is easy to approximate the coefficients $\tilde{\kappa}_0$, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ involved in this synchronization threshold, but it seems more delicate to estimate the value of the border of each bifurcation step. Note that the critical value ξ_ε can be roughly estimated when the perturbation $T(\xi)$ in the operator $\bar{\mathbb{A}}_\xi$ (see equation (40)) is seen to commute with $\bar{\mathbb{A}}_0$ (see [21], Chapter 4, Theorem 3.6).



(a)



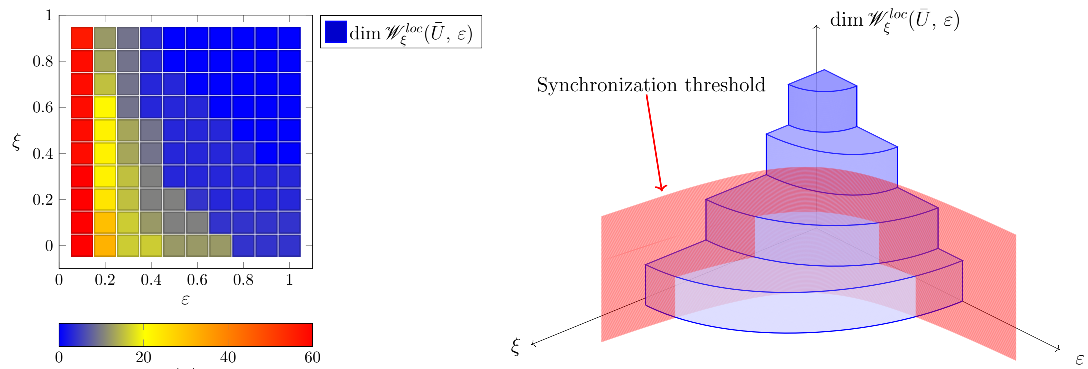
(b)

Figure 2: (a) Variation of the dimension of the local unstable manifold $\mathscr{W}_0^{loc}(\bar{U},\, \varepsilon)$ with respect to ε. (b) Variation of the dimension of the local unstable manifold $\mathscr{W}_\xi^{loc}(\bar{U},\, \varepsilon_0)$ with respect to ξ for a given $\varepsilon_0 > 0$.

Let us finally discuss the possible coexistence of synchronization with instabilities. The coupling strength ξ of the complex network (8) has to overcome the threshold defined by (33) in order to guaranty synchronization; but if ξ overcomes the critical value ξ_ε, then the dimension of the local unstable manifold is likely to decrease. Analogously, we can consider the effect of a variation of the number n of vertices in the graph $\mathscr{G}$. An increase of n obviously favors synchronization, since the coefficients $\tilde{\kappa}_0$, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ appearing in the inequality (33) do not depend on n. However, the dimension of the local unstable manifold $\mathscr{W}_\xi^{loc}(\bar{U},\, \varepsilon)$ is bounded from below by a quantity which is likely to increase with n (see Corollary 1). Overall, synchronization appears to be compatible with a low level of instability, but incompatible with a high level of instability.

Conclusion

In this article, we have studied the possibility to synchronize unstable dynamical systems of infinite dimension, while preserving the nature of the instabilities occurring in those systems. Considering a complex network of multiple instances of the Keener-Tyson model, we have proved a sufficient condition of synchronization, which highlights the effect of the main parameters of the model. In parallel, we have analyzed the asymptotic behavior of the solutions, by establishing a lower bound of the dimension of the global attractor. This lower bound has been obtained by



(a) (b)

Figure 3: (a) Variation of the dimension of the local unstable manifold $\mathscr{W}_0^{loc}(\bar{U},\, \varepsilon)$ with respect to ε and ξ. (b) Symbolic figure illustrating the intersection of the synchronization threshold given by equation (33) with the bifurcations lines changing the dimension of the local unstable manifold $\mathscr{W}_0^{loc}(\bar{U},\, \varepsilon)$.

examining the effect of the coupling strength of the network on the dimension of the unstable manifold of a non-trivial stationary solution. A spectrum perturbation principle for non-bounded operators has been applied, which, up to our knowledge, represents a novelty in the study of the asymptotic behavior of infinite dimensional dissipative systems. This method also proves its efficiency, since it allows to exhibit a sequence of bifurcations which affect the dimension of the global attractor. Our results show that there exists a non-empty parameter regime for which synchronization occurs in non-trivial attractors. However, many questions are still unresolved. Indeed, it seems very delicate to estimate the optimal level of instability which is compatible with synchronization. Furthermore, the topology of the graph underlying the network has been subject to some technical assumptions of symmetry, thus the case of non-symmetric networks, which are known to provoke original dynamics, is undoubtedly a promise for original discoveries.

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