Recovering trajectories of chaotic piecewise linear dynamical systems

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Abstract

The purpose of this paper is to present some results on recovering trajectories of chaotic dynamical systems from their ε-perturbed trajectories. Such a mathematical problem appears to be equivalent to the extraction of an information bearing signal from an input signal, to the reduction of a noise in an input signal, etc.

1 Introduction

When almost each trajectory of a dynamical system is Lyapunov stable, the problem of extraction of efficient signal from input signal in receiving system is well known and detailed methods of solutions are available. For a chaotic dynamical system, where there is a sensitive dependence on initial data and every trajectory on the strange attractor is unstable, the situation is absolutely different.

The elaboration of methods of extraction of efficient signal from input signal (in order to reduce the noise in an input signal, to encode/decode electronic messages for secure communication, etc.) for chaotic dynamical system is still a very attractive problem. There are different approaches to investigate such a type of task (for instance, [AL95],[CP92],[H94],[KHECP92],[LC93],[LA94],[PC90],[S95]).

Let us point out that this problem is equivalent to the problem of recovering a trajectory of the system from its ε-perturbed trajectory. This is the purpose of our paper in which we present some results on recovering the trajectories of chaotic dynamical systems (in particular, systems given by piecewise linear maps) from their ε-perturbed trajectories.

2 Definitions and arguments

In this section we will define the notion of the recovering trajectories property in dynamical systems and will give some relevant results.

Let $\{X, f\}$ be a dynamical system defined by the map $f : X \to X$, where $X \subset R^k \quad (k \geq 1)$, and let $\|\cdot\|$ be metric in R^k.

Definition 1 *A sequence $\{x_i\}_{i=0}^n, n \leq \infty$, is called a true trajectory of the dynamical system if $x_{i+1} = f(x_i)$. A sequence $\{x_i\}_{i=0}^n, n \leq \infty$, is called an ε-trajectory of the dynamical system if $\|x_{i+1} - f(x_i)\| < \varepsilon$.*

The notion "ε-trajectory" (or "pseudotrajectory", which is the same) is an object of modern theory of dynamical systems [A67],[KH95]. Because numerical trajectories are always only ε-trajectories (where ε is calculation precision), the question "Do ε-trajectories represent true trajectories?" [HYG87], [HYG88] is pertinent.

Definition 2 *A sequence $\{\overline{x}_i\}_{i=0}^n$ is called an ε-perturbed trajectory of the dynamical system, if there exists a true trajectory $\{x_i\}_{i=0}^n$ such that $\|\overline{x}_i - x_i\| < \varepsilon$ for all i.*

According to these definitions every ε-perturbed trajectory is an ε-trajectory too. But the converse is not true (an ε-trajectory is not always an ε'-perturbed trajectory of some true trajectory with $\varepsilon' \leq \varepsilon$).

Nevertheless, there are classes of dynamical systems (for instance, structurally stable systems), for which every ε-trajectory with $\varepsilon < \varepsilon_0$ (which depends on dynamical system) is an ε'-perturbed trajectory and, moreover, it is possible to take ε' as small as desired. Such a property of ε-trajectories of dynamical system is called shadowing property.

Our purpose now is to recover trajectories of a chaotic dynamical system from its ε-perturbed trajectories.

At once the question arises about the univalent solvability of this task: whether it always has a unique (for given accuracy) solution?

Definition 3 *We will say that ε-perturbed trajectory $\{\overline{x}_i\}_{i=0}^n$ of the dynamical system determines a true trajectory of this system on some time interval $[m_-, n - m_+] \subset [0, n]$ with the accuracy δ, if*

a) there is a trajectory $\{x_i\}_{i=0}^n$ such that $\|\overline{x}_i - x_i\| < \delta$ for $m_- \le i \le n - m_+$ and $\|\overline{x}_i - x_i\| < \varepsilon$ for the others i;

b) if $\{x_i'\}_{i=m_-}^{m_+}$ and $\{x_i''\}_{i=m_-}^{m_+}$ are two such trajectories, then $\|x_i' - x_i''\| < \delta$ for all $m_- \le i \le n - m_+$.

Of course, it is desirable to deal with dynamical systems, for which every ε-perturbed trajectory determines a true trajectory with an accuracy given in advance.

Definition 4 *We will say that a dynamical system has the recovering trajectories property, if for this dynamical system, there exists $\varepsilon_0 > 0$, such that for every $\varepsilon < \varepsilon_0$ and any δ there are integers $m_-, m_+ \ge 0$ such that every ε-perturbed trajectory, given on the time interval $[0, n]$ with $n > m_- + m_+$, defines a true trajectory with accuracy δ on the time interval $[m_-, n - m_+]$.*

On one hand, structurally stable systems are included in the class of dynamical systems, possessing this property. On the other hand, it is not difficult to give examples of dynamical systems which do not possess such a property. For example, for ε-trajectory with the shadowing property, the a) from Definition 3 is fulfilled but the b) does not always hold. We won't discuss this question here in detail. Let us only note that this task is not soluble for trajectory on a chaotic strange attractor of any one-dimensional continuous map because this map on the attractor is not a one-to-one map. Moreover, there are "pasting points", which one can take as close as desired from another. This is a reason why it is impossible to restore identically a long enough piece of trajectory, which is dense on the attractor (one is a "typical" trajectory: such trajectories form a residual set on the attractor).

For one-dimensional piecewise discontinuous maps, which have uniformly separated preimages of any point, the problem of recovering the trajectories is solved identically (the preimages are separated uniformly, if there is δ such that, if $f(x') = f(x'')$, then either $x' = x''$, or $\|x' - x''\| > \delta$).

Both integers m_- and m_+ determine the speed of recovering the true trajectory; m_- corresponds to "the synchronization time" for one part of (relative) coordinates, m_+ corresponds to "the antisynchronization time" for the other part of coordinates. It is expected naturally, that they depend on Lyapunov exponents of the "strange" attractor. It is not difficult to see, that for structurally stable systems, m_-, m_+ can be estimated from the expressions

$$\varepsilon\lambda_-^{m_-} \approx \delta, \quad \varepsilon\lambda_+^{-m_+} \approx \delta$$

where

$$\lambda_- = \max_{|\lambda_j|<1} |\lambda_j|, \quad \lambda_+ = \min_{|\lambda_j|>1} |\lambda_j|,$$

and $\lambda_j, j = 1, 2, \ldots, k$, are the multipliers of the attractor of the dynamical system considered.

In this paper we consider dynamical systems with only discrete time given by maps. Of course, recovering trajectories also has a profit for dynamical systems with continuous time (for example, given by differential equations). In the case of continuous time, such a task is essentially more complicated but it is simplified, if we deal with differential equations, the right part of which is defined by piecewise linear functions. Using Poincaré section (as secants it is natural to choose hyperplanes which define the partition of phase space on sets where the equations are linear), we can reduce the task of recovering trajectories for differential equations to the same task for the maps.

3 Algorithm of recovering the trajectories

3.1 Description of the algorithm

Let a dynamical system given by the map $f : X \to X,\ X \subset R^k, k \ge 1$, having the recovering trajectories property with some $\varepsilon_0 > 0$. As mentioned, the map f must either be a one-to-one map, or possess the property of δ_0-separation of preimages with some $\delta_0 \ge \varepsilon_0$ (so that the map f is not a continuous 1-D map).

Let us suppose that the dynamical system has a hyperbolic chaotic attractor, so that both trajectories on the attractor and these attracted by the attractor have a sensitive dependence on initial conditions. Moreover, the absolute value of the multipliers $\lambda_1, \ldots, \lambda_k$ of this attractor (and, consequently, the absolute value of the multipliers for almost all of such trajectories) is not equal to 1 .

For a given δ-accuracy, which is necessary for the reconstruction of the "true" trajectory, if we know

$\lambda_- = \max_{|\lambda_s|<1} |\lambda_s|$, and $\lambda_+ = \min_{|\lambda_s|>1} |\lambda_s|$, we can evaluate the "recovering times" m_-, m_+.

Let $\{\overline{x}_i\}_{i=0}^{n}, n \leq \infty$, be an ε-perturbed trajectory of this dynamical system with some $\varepsilon \leq \varepsilon_0$. How to compute a "true" trajectory $\{x_i\}_{i=m_-}^{n-m_+}$?

Let $B_\varepsilon(\overline{x}) = \{x \in X : \|x - \overline{x}\| \leq \varepsilon\}$. As far the point $\overline{x}_i$ of the ε-perturbed trajectory is distant from a point x_i we have to find (not farther then of ε), then $x_i \in B_\varepsilon(\overline{x})$. Due to the hyperbolicity of the dynamical system, for ε small enough the intersection of the ball $B_\varepsilon(\overline{x})$ with the hyperplane corresponding to the "stable manifold" of the trajectory through the point x, is contracted under the map f on the average in λ_- times and, simultaneously, the intersection with the hyperplane corresponding to the "unstable manifold" of the trajectory through the x, is expanded on average in λ_+ times, at least.

That is why, the recovering trajectory algorithm can be the following:

1) We construct the sets

$$B_\varepsilon(\overline{x}_0), B_\varepsilon(\overline{x}_1), \ldots, B_\varepsilon(\overline{x}_i), B_\varepsilon(\overline{x}_{i+1}), \ldots,$$

each of the balls $B_\varepsilon(\overline{x}_i)$, as we know, contains an unknown point x_i.

2) If both "times" m_- and m_+ differ a little, using the sets $S_i^0 = B_\varepsilon(\overline{x}_i), i = 0, 1, 2, \ldots$, of "level" 0, we sequently construct sets of "levels" $j = 1, 2, \ldots, m$, where $m = \max\{m_-, m_+\}$, by the formula

$$S_i^j = f(S_{i-1}^{j-1}) \cap S_i^{j-1} \cap f^{-1}(S_{i+1}^{j-1}), \quad i = j, j+1, j+2, \ldots,$$

where f^{-1} is the inverse function of f (f^{-1} may be multivalued, it is always multivalued when $k = 1$).

3) At the typical situation we get that diam $S_i^m < \delta$, $i = m, m+1, \ldots$, and we can take any point from the set S_i^m as x_i.

If both "times" m_- and m_+ differ drastically it is better to modify a few the algorithm and to define the sets S_i^j starting with the "level" $j = m_* = \min\{m_-, m_+\} + 1$ by the formula

$$S_i^j = S_i^{j-1} \cap f^{\mp 1}(S_{i\pm 1}^{j-1}), \quad i = m_*, m_* + 1, \ldots$$

with f^{-1} if $m_- < m_+$ and $f^{+1} (= f)$ if $m_- > m_+$.

In the case, where all multipliers are larger than 1 in absolute value (this is always the case when $k = 1$), the algorithm is somewhat simplified and the formula for S_i^j is transformed into the form

$$S_i^j = S_i^{j-1} \cap f^{-1}(S_{i+1}^{j-1}), \quad i = 0, 1, 2, \ldots.$$

In this case it is possible to reconstruct the trajectory, starting from the first point.

3.2 Algorithm for one-dimensional dynamical system

Since the inversed map f^{-1} is multivalued, then computing the true trajectory $\{x_i\}_{i=0}^{n-m_+}$, it is necessary to walk on "nonregular" (with variable branching) tree defined by intersections of preimages of the interval with the intervals themselves. For carrying out efficient computations (cutting of the empty intersections and doing a rational passage along the tree) it is reasonable to use the recursion for the construction of the algorithm.

We will explain our algorithm using the operators of the pseudo-C language.

Input:

– the map (for simplification we will restrict ourselves by using a prime model)

$$x_{i+1} = f(x_i) = \begin{cases} f_l(x_i) = ax + b, & \text{if } 0 \leq x_i \leq p; \\ f_r(x_i) = cx + d, & \text{if } p < x_i \leq 1; \end{cases}$$

– ε-perturbed trajectory $\{\overline{x}_i\}_{i=0}^{n}$;

– ε is an evaluation of perturbations;

– δ is the accuracy used in recovering the true trajectory.

Output:

the true trajectory $\{\tilde{x}_i\}_{i=0}^{n-m_+}, m_+ = m(\delta)$, recovered with given accuracy, where $m_+ = m(\delta)$ is computated in the algorithm.

Algorithm:

proc recovering$(f, \{\overline{x}_i\}_{i=0}^{n}, \varepsilon, \delta)$
  $\lambda = Lyap(f)$; // *estimate the multiplier* $\lambda = \lambda_+$ *for the ma*
  $m_+ = m(\delta, \lambda)$; // *define an evaluation* m_+ *– a number of*
  // *steps, which are necessary for recovering the true traject*
  for $j = 0$ to $n - m_+$ do
    $B(\overline{x}_j) = [\overline{x}_j - \varepsilon; \overline{x} +_j +\varepsilon]$;
    inv_recover$(f, \{\overline{x}_i\}_{i=j}^{j+m_+}, B(\overline{x}_j), m_+)$;
  end // *for*
end // *proc recovering*
proc inv_recover$(f, \{\overline{x}_i\}_{i=0}^{m_+}, B(\overline{x}_i), m)$
  if $m = 0$ then
    output $\tilde{x} \in B(\overline{x}_i)$;
  $B^l(\overline{x}_{i-1}) = f_l^{-1}(B(\overline{x}_i)) \cup B(\overline{x}_{i-1})$;
  $B^r(\overline{x}_{i-1}) = f_r^{-1}(B(\overline{x}_i)) \cup B(\overline{x}_{i-1})$;
  $m = m - 1$;
  if $B^l(\overline{x}_{i-1}) \neq \emptyset$ then
    inv_recover$(f, \{\overline{x}_i\}_{i=0}^{m_+}, B^l(\overline{x}_{i-1}))$
  if $B^r(\overline{x}_{i-1}) \neq \emptyset$ then
    inv_recover$(f, \{\overline{x}_i\}_{i=0}^{m_+}, B^r(\overline{x}_{i-1}))$
end // *proc inv_recover*

3.3 Algorithm for dynamical system with 2-D piecewise map

In contrast to this one-dimensional case, due to the complexity of solving the problem, we will use here a piecewise linearity of a nonlinear map as maximum, as is possible and as a basic model let us take the dynamical system defined by the Lozi map

$$\begin{cases} x_{i+1} = -a|x_i| + y_i + 1; \\ y_{i+1} = bx_i. \end{cases}$$

In the dynamical system with the Lozi map for any convex polygon V_r, which completely lies in the right (V_l – in the left) half-plane R^2, its image/preimage is also a convex polygon. It is necessary to preserve such a property during the computations in order to use efficient algorithms of determination of the polygon [PS85]. Therefore, it is necessary to divide S into two polygons S_r and S_l, belonging to right and left half-planes R^2 respectively, if image/preimage S of the convex polygon $\overline{S}_i, m_- \leq i \leq n$ (or $\tilde{S}_i, 0 \leq i \leq n - m_+$) intersects the line $x = 0$.

Now the other concept of application of the recursion to create a recovering algorithm, precisely, computations, by preservation convexity of all polygons $\overline{S}_i, \tilde{S}_i$, is fulfilled on some "nonregular" trees and, therefore, it is necessary to simplify the circuit along the tree as much, as possible, at the expense of cutting-off "empty" branches, which will appear on the intersection of polygons.

The structure of the algorithm of recovering trajectory of the dynamical system with Lozi map is developped using the same method, which has been illustrated in the construction of our algorithm for one-dimension case and with the use of all needed methods of computing geometry.

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