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DIMENSION ESTIMATE OF ATTRACTORS FOR COMPLEX NETWORKS OF REACTION-DIFFUSION SYSTEMS APPLIED TO AN ECOLOGICAL MODEL

Guillaume Cantin*

Laboratoire de Mathématiques Appliquées du Havre, Normandie University
FR CNRS 3335, ISCN, 76600 Le Havre, France

M. A. Aziz-Alaoui

Laboratoire de Mathématiques Appliquées du Havre, Normandie University
FR CNRS 3335, ISCN, 76600 Le Havre, France

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Abstract. The asymptotic behavior of dissipative evolution problems, determined by complex networks of reaction-diffusion systems, is investigated with an original approach. We establish a novel estimation of the fractal dimension of exponential attractors for a wide class of continuous dynamical systems, clarifying the effect of the topology of the network on the large time dynamics of the generated semi-flow. We explore various remarkable topologies (chains, cycles, star and complete graphs) and discover that the size of the network does not necessarily enlarge the dimension of attractors. Additionally, we prove a synchronization theorem in the case of symmetric topologies. We apply our method to a complex network of competing species systems modeling an heterogeneous biological ecosystem and propose a series of numerical simulations which underpin our theoretical statements.

1. **Introduction.** In this article, we aim to bring a novel contribution to the study of the asymptotic behavior of dissipative evolution problems, by establishing an innovative estimation of the dimension of their possible attractors. We focus on evolution problems determined by complex networks of reaction-diffusion systems. Those complex networks can be constructed in concordance with a finite graph, whose vertices are associated with non-identical instances of a given reaction-diffusion system, as will be shown below. Under reasonable assumptions which cover a wide class of systems, we show that such complex networks generate continuous dynamical systems whose asymptotic behavior can be described by a family of exponential attractors of finite fractal dimension. Furthermore, we establish an estimate of this dimension in terms of the topology of the graph associated to the complex network. Up to our knowledge, this estimate has never been proved before.

A huge number of studies have been devoted to complex networks of dynamical systems given by ordinary differential equations (ODE), but only a few works are studying complex networks of dynamical systems given by partial differential

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* Corresponding author.

equations (PDE). Those studies are motivated by numerous applications of great interest, including neural networks, epidemiological networks or geographical networks (see for instance [2, 5, 7, 8, 40]). Complex networks can also be applied to ecological models, since they can reproduce the heterogeneity of biological environments fragmented by urban and industrial expansion, which threats the natural equilibrium of biodiversity [19]. Emergent properties and self-organization, such as synchronization, which is a form of control of the asymptotic behavior, are some of the topics which are commonly analyzed (see [3, 4, 10, 17] or [35]); but the question to determine the dynamics of the network, assuming that the dynamics of each vertex is known and that the topology of the subsequent graph is given, remains open in the general case. Furthermore, complex networks of reaction-diffusion systems have also been studied in [6], in order to approximate a fourth order parabolic problem, which shows again the wide potential of application of complex networks. In the case of the finite dimension, that is, when the complex network is determined by a system of ODE, it is sometimes possible to describe partially the asymptotic behavior of the resulting dynamical system. Nevertheless, in the case of the infinite dimension, that is, when the complex network is given by a system of PDE, only sporadic results have been proved. Recently, the asymptotic behavior of solutions of Keller–Segel equations in network shaped domains has been studied in [23], where the convergence towards stationary solutions is investigated. In another recent paper, conditions of synchronization have been obtained in [2] for a neural network built with the FitzHugh–Nagumo reaction-diffusion system. Thus it appears essential to develop a novel approach in order to analyze the asymptotic behavior of complex networks in the case of infinite dimension and to generalize what has been proved in particular cases.

Here, we establish an upper bound on the fractal dimension of exponential attractors for complex networks of reaction-diffusion systems. Our main result is stated in Theorem 4.1, in which we prove the following asymptotic estimate

$$d_F(\mathfrak{M}) \leq 1 + C\,|\Omega|\,\left(C_g\right)^{d/2}$$

where $d_F(\mathfrak{M})$ denotes the fractal dimension of an exponential attractor $\mathfrak{M}$ of the complex network, C_g is the Lipschitz constant of the coupling operator of the network, Ω is a bounded domain in $\mathbb{R}^d$ with $d \in \{1,\, 2,\, 3\}$ and C is a positive constant. Additionally, the Lipschitz constant C_g is estimated in terms of the number of vertices in the network and the maximal coupling strength (see Proposition 3). In order to derive the estimate of $d_F(\mathfrak{M})$, we revisit the method proposed in [14]. This technique has been used several times for the analysis of various parabolic problems in a Hilbert setting (see [16] for instance) and has been extended to Banach spaces and non-autonomous systems (for instance in [15, 25]). Furthermore, this method overcomes the well-known defects of robustness of the global attractor (see [26, 29, 30] or [38]). Here, the main difficulty we face consists in isolating the effect of the topology of the graph underlying the complex network, with respect to other parameters of the system. Estimating the dimension of attractors for complex networks can be of great interest in regard to the synchronization topic, which also makes our contribution original; roughly speaking, complex networks admitting small attractors are susceptible to exhibit synchronization. However, we emphasize that synchronization is likely to occur even in the case of complex networks admitting large attractors; the most relevant example of that situation is analyzed in [3], where it is shown that chaotic dynamics can be synchronized. Here, we prove

that synchronization can be reached in the case of symmetric topologies (*i.e.* with bi-directed couplings) , in particular for the complete bi-directed graph topology; this result is stated in Theorem 4.2 below. Furthermore, we analyze the effect of remarkable asymmetric topologies corresponding to oriented chains, cycles and star graphs, and discover that the number of vertices does not necessarily enlarge the size of the attractors (see Proposition 4). Our framework is concentrated around reaction-diffusion systems which admit a rich variety of solutions, and can model a great number of real-world applications (see for instance [13, 24, 31, 41]). In particular, we apply our theoretical results to a complex network built with multiple instances of a competing species model (presented in [22] or [27]) for which the coupling is relevant and adapted to the *metapopulation* approach [20]. However, our method can easily be applied to other parabolic problems.

This paper is organized as follows. In the next section, for the self-sufficiency of the paper, we recall some important results of functional analysis concerning interpolation spaces, sectorial operators and semi-linear equations. We also present the concept of exponential attractor of finite fractal dimension and recall the main assumptions required for using the technique given by [14]. In section 3, we show how to construct a complex network of non-identical systems, stemming from a reaction-diffusion system and a finite graph. We briefly demonstrate the existence and uniqueness of local solutions, and set a minimal number of hypotheses under which the complex network generate a continuous dynamical system admitting a family of exponential attractors. Our main results are presented in section 4, where we establish an estimate of the fractal dimension of those attractors in terms of the coupling parameters. Additionally, we establish sufficient conditions on the topology for synchronization in the network. We complete our results with the analysis of several remarkable asymmetric topologies. In the final section, we apply our theoretical statements to a competing species model embedded in a complex network structure which reproduces an heterogeneous biological environment, and we illustrate our approach with numerical simulations.

2. **Preliminaries.**

In this section, we present some important results of functional analysis that will be used in the present work, so as to guaranty the self-sufficiency of the article and the comfort of the reading.

2.1. **Functional spaces and interpolation theory.**

Throughout this paper, the symbol C will denote an absolute positive constant, whereas the symbol C_α will design a positive constant depending on a given object α.

We will use the classical notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, where Ω denotes an open bounded domain in $\mathbb{R}^d$ with regular boundary $\partial\Omega$, $p \in [1, \infty]$ and $k \in \mathbb{N}$. Those functional spaces are Banach spaces whose norms will be denoted $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ respectively. For $p = 2$, we simply note $H^k(\Omega) = W^{k,2}(\Omega)$; $H^k(\Omega)$ is a Hilbert space whose inner product will be denoted $\big(\cdot\,,\,\cdot\big)_{H^k(\Omega)}$.

Let X_0 and X_1 denote two Banach spaces, with dense and continuous embedding $X_1 \subset X_0$. Several methods have been proposed (see for instance [1, 39] or [32]) in order to construct a family of Banach spaces which are called *interpolation spaces* and denoted $\big([X_0,\, X_1]_\alpha\big)_{0\le\alpha\le 1}$. The interpolation spaces satisfy the following properties:

(i) $[X_0,\, X_1]_0 = X_0$ and $[X_0,\, X_1]_1 = X_1$ with isometries;

(ii) $X_1 \subset [X_0,\, X_1]_\alpha \subset X_0$ with dense and continuous embeddings, for all $\alpha \in]0,\, 1[$;
(iii) for $\alpha \in]0,\, 1[$, it holds that $\|x\|_{[X_0,\, X_1]_\alpha} \leq \|x\|_{X_0}^{1-\alpha} \|x\|_{X_1}^{\alpha}, \forall x \in X_1$;
(iv) $[X_0,\, X_1]_\beta \subset [X_0,\, X_1]_\alpha$ with dense and continuous embeddings, for all coefficients α, β such that $0 \leq \alpha < \beta \leq 1$.

Those interpolation spaces can be used in order to define Sobolev of fractional order $H^s(\Omega)$ where exponent $s \geq 0$ is not necessarily an integer. In order to avoid any misunderstanding, it is worth noting that some authors prefer to write $(X_1,\, X_0)_\alpha$ instead of $[X_0,\, X_1]_\alpha$.

2.2. **Continuous dynamical systems generated by semi-linear parabolic equations.** Here, we recall the definition of a *sectorial operator* and present an existence theorem for semi-linear parabolic equations. We refer to [42] or [18] for details concerning this class of equations. Then we show how semi-linear parabolic equations can generate a continuous dynamical system.

Let X be a Banach space and A a closed linear operator, densely defined in X. Assume that the spectrum of A satisfies

$$\sigma(A) \subset \{\lambda \in \mathbb{C}^*,\ |\arg(\lambda)| < \omega\},$$

for $\omega \in]0,\, \pi]$ and furthermore that

$$\left\|(\lambda - A)^{-1}\right\|_{\mathscr{L}(X)} \leq \frac{M}{|\lambda|},$$

for all $\lambda \in \mathbb{C}$ such that $|\arg(\lambda)| \geq \omega$, with $M \geq 1$. Then A is said to be *sectorial* in X. If A is a sectorial operator in X, it is seen that there exists a minimum coefficient ω satisfying the above properties; it is denoted ω_A and called *angle* of A. Sectorial operators admit fractional powers whose domains can be described in terms of interpolation spaces (see for instance [42], Theorems 16.7 and 16.9). Let A be a sectorial operator in X of angle $\omega_A < \frac{\pi}{2}$ and F a non-linear operator defined in $\mathcal{D}(A^\eta)$ (where η is an exponent such that $0 < \eta < 1$) with values in X. We consider the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + Au = F(u), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

with $u_0 \in X$. We assume that F enjoys the property:

$$\|F(u) - F(v)\|_X \leq C_F\ (1 + \|A^\eta u\|_X + \|A^\eta v\|_X)\, \|u - v\|_X\,, \tag{2.2}$$

for all $u,\, v \in \mathcal{D}(A^\eta)$, for a positive constant C_F. The following theorem is proved in [42].

Theorem 2.1. *For all $u_0 \in X$, there exists $T_{u_0} > 0$ such that problem (2.1) admits a unique solution $u = u(t,\, u_0)$ in function space*

$$u \in \mathscr{C}\big((0,\, T_{u_0}]\,;\, \mathcal{D}(A)\big) \cap \mathscr{C}\big([0,\, T_{u_0}]\,;\, X\big) \cap \mathscr{C}^1\big((0,\, T_{u_0}]\,;\, X\big),$$

where T_{u_0} depends only on $\|u_0\|_X$. Furthermore, u satisfies

$$\|u(t)\|_X + t\, \|Au(t)\|_X \leq C_{u_0}, \quad 0 < t \leq T_{u_0},$$

where $C_{u_0} > 0$ depends only on $\|u_0\|_X$.

Note that many other existence results have been established for semi-linear equations using semi-groups methods (see for instance [28] or [33]).

Now we continue with the presentation of the concept of *continuous dynamical system* generated by a semi-linear parabolic equation. If Φ is a compact subset of

X such that the solutions of problem (2.1) stemming from initial conditions in Φ are global and remain in Φ, then Φ is said to be *positively invariant*. In that case, one can define the mapping

$$\begin{array}{rccc} G: & (0,\,+\infty)\times\Phi & \longrightarrow & \Phi \\ & (t,\,u_0) & \longmapsto & S(t)u_0 \end{array} \tag{2.3}$$

where $S(t)$ denotes the semi-flow generated by problem (2.1), defined by

$$S(t)u_0 = u(t,\,u_0),$$

for all $u_0 \in \Phi$ and $t \geq 0$. Note that $S(t)$ satisfies $S(0) = Id$ (identity in X) and $S(t)\circ S(s) = S(t+s)$ for all non-negative t and s. Furthermore, Φ admits a metric structure since it can be equipped by the distance induced by X. If the mapping G is continuous in $(0,\,+\infty)\times\Phi$, then the triplet $\big(S(t),\,\Phi,\,X\big)$ is called a *continuous dynamical system*. X is called the *universal space* and Φ is called the *phase space*.

2.3. **Exponential attractors of finite fractal dimension.** Let X be a Banach space and $\big(S(t),\,\Phi,\,X\big)$ denote a continuous dynamical system with compact phase space $\Phi\subset X$. It is well-known (see for instance [38]) that $\big(S(t),\,\Phi,\,X\big)$ possesses a global attractor $\mathfrak{A} = \cap_{t\geq 0} S(t)\Phi$, which is used in order to describe the asymptotic behavior of the considered dynamical system. However, it is seen that the global attractor $\mathfrak{A}$ may present some defects. Indeed, the rate of convergence of the solutions towards the global attractor $\mathfrak{A}$ is not always known; furthermore, $\mathfrak{A}$ can react discontinuously to a small perturbation of the dynamical system. For those reasons, the concept of *exponential attractor* has been proposed in [14]. Namely, a subset $\mathfrak{M}\subset\Phi$ is said to be an *exponential attractor* of $\big(S(t),\,\Phi,\,X\big)$ if it is a positively invariant, compact subset of Φ containing the global attractor $\mathfrak{A}$, which attracts bounded subsets of Φ at an exponential rate for the Haussdorff pseudo-distance ρ_H defined by

$$\rho_H(A,\,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|_X\,.$$

Note that $\mathfrak{M}$ is not unique, since its image by $S(t)$ is another exponential attractor.

Since $\mathfrak{M}$ is compact, for any $\varepsilon>0$, it can be covered by a finite number of closed balls of radius ε. Let $N(\varepsilon)$ denote the minimal number of balls of radius ε which cover $\mathfrak{M}$. Then the fractal dimension of $\mathfrak{M}$ is defined by

$$d_F(\mathfrak{M}) = \limsup_{\varepsilon\to 0}\frac{\log N(\varepsilon)}{\log\frac{1}{\varepsilon}}.$$

The existence of exponential attractors can be established by virtue of the following theorem, which is proved in [14].

Theorem 2.2. *Assume that the mapping G defined by (2.3) satisfies the Lipschitz condition*

$$\|G(t,\,u)-G(s,\,v)\|_X \leq C\big(\,|t-s| + \|u-v\|_X\,\big), \quad t>0, s>0, u,v\in\Phi,$$

for a given positive constant C. Assume furthermore that there exists a positive time t^*, a real coefficient $\delta^*\in\big(0,\,\frac{1}{8}\big)$ and an orthogonal projection P^* of rank N^* such that either

$$\|S(t^*)u - S(t^*)v\|_X \leq \delta^*\,\|u-v\|_X \tag{2.4}$$

or

$$\big\|(Id-P^*)\big(S(t^*)u - S(t^*)v\big)\big\|_X \leq \big\|P\big(S(t^*)u - S(t^*)v\big)\big\|_X \tag{2.5}$$

holds for each pair $u, v \in \Phi$. Then the dynamical system $\big(S(t), \Phi, X\big)$ admits an exponential attractor $\mathfrak{M}$ of finite fractal dimension $d_F(\mathfrak{M})$. Moreover, the following estimate holds

$$d_F(\mathfrak{M}) \leq 1 + N^* \max\left[1, \frac{\log\left(1 + \frac{2L^*}{\delta^*}\right)}{\log\left(\frac{1}{4\delta^*}\right)}\right], \tag{2.6}$$

where L^* denotes the Lipschitz constant of $S(t^*)$ on Φ.

The dichotomy principle (2.4)-(2.5) is usually called the *squeezing property*. We shall apply the latter theorem in Section 4 in order to derive an estimate of the fractal dimension of exponential attractors for complex networks of reaction-diffusion systems. Note that the squeezing property is proved in [42] for a wide class of systems, but the fractal dimension is not estimated.

3. **Semi-flow generated by the complex network.** In this section, we first show how to construct a complex network of dynamical systems, stemming from a reaction-diffusion system and a finite graph. Then we prove that the complex network problem admits local in time solutions and present reasonable assumptions under which the solutions are global.

3.1. **Construction of the complex network problem.** Let Ω denote an open domain in $\mathbb{R}^d$ with $d \in \{1, 2, 3\}$. We assume that Ω admits a regular boundary $\partial\Omega$ and we consider a reaction-diffusion system of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + \varphi(u) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Here, $u = (u_1, \ldots, u_m)^T$ is defined in $\Omega \times (0, \infty)$ with values in $\mathbb{R}^m$; D is a diagonal matrix of order m with positive entries; φ is a non-linear operator whose form will be detailed below and u_0 is a given initial condition.

Additionally, we consider a graph $\mathscr{G} = \big(\mathscr{N}, \mathscr{E}\big)$ made with a finite set $\mathscr{N}$ of n vertices ($n \in \mathbb{N}^*$) and a finite set $\mathscr{E}$ of edges. We associate each vertex j of $\mathscr{G}$ with an instance of the latter reaction-diffusion system (3.1), and we define a coupling operator g as follows. We define the matrix of connectivity $L = \big(L_{i,j}\big)_{1 \leq i,j \leq n}$, in concordance with the set $\mathscr{E}$ of edges, by setting

$$L_{j,k} = +1 \text{ if } (k, j) \in \mathscr{E} \text{ with } k \neq j, \quad L_{k,k} = -\sum_{j=1, j\neq k}^{n} L_{j,k}, \tag{3.2}$$

thus L is a matrix of order n whose sum of coefficients of each column is null. We assume that the set of edges $\mathscr{E}$ does not possess any loop (we recall that a loop is an edge that connects a vertex to itself). We also introduce a matrix of coupling strengths $\mathcal{H}$ of order m defined by

$$H = \operatorname{diag}\big(\sigma_1, \ldots, \sigma_m\big), \tag{3.3}$$

with $\sigma_i \geq 0$, $1 \leq i \leq m$. We introduce the notation

$$Hu_j = \big(\sigma_1 u_{1,j}, \ldots, \sigma_m u_{m,j}\big)^T, \quad 1 \leq j \leq n,$$

and finally define the coupling operator g by setting

$$g_j(u) = \sum_{k=1}^{n} L_{j,k} H u_k, \quad 1 \le j \le n. \tag{3.4}$$

In this way, the equations of the complex network can be written

$$\begin{cases} \dfrac{\partial u_j}{\partial t} = D_j \Delta u_j + \varphi_j(u_j) + g_j(u_1, \dots, u_n) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u_j}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u_j(x, 0) = u_{j,0}(x) & \text{in } \Omega, \end{cases} \tag{3.5}$$

for $1 \le j \le n$, where $u_j = (u_{1,j}, \dots, u_{m,j})^T$ is defined in $\Omega \times (0, +\infty)$ with values in $\mathbb{R}^m$. In our notation $u_{i,j}$, the first subscript i ($1 \le i \le m$) refers to the i-th component of u_j, whereas the second subscript j refers to the vertex j ($1 \le j \le n$) of $\mathscr{G}$ associated with one instance of system (3.1). The subscript j in D_j and φ_j means that the values of the parameters involved in D and φ can be different from one instance of system (3.1) to another, which justifies the expression *complex network of non-identical systems.*

Remark 3.1. *We emphasize that the n instances of the reaction-diffusion system (3.1), which compose the complex network problem (3.5), are all set in the same domain Ω, thus problem (3.5) can be seen as a multi-layer model, in which the couplings are understood to be point-wise. The case of a complex network of systems defined in distinct domains $\Omega_1, \dots, \Omega_n$ with non point-wise couplings is delicate and should be handled in $\left(L^2(\Omega_1 \times \cdots \times \Omega_n)\right)^{m \times n}$; this case will be treated in a separate paper.*

Remark 3.2. *Complex networks of the form (3.5) have been considered for a great number of applications. For instance, in neural networks of FitzHugh–Nagumo type, we have $u = (v, w)$, where v is the membrane voltage and w is a recovery variable of a given neuron; in that case, the couplings correspond to chemical exchanges which occur for instance in the synapses. Otherwise, in epidemiological networks, u stores the states of several subgroups of a population affected by a disease; in that second example, the couplings correspond to point-wise physical displacements of individuals from one vertex in the network to another. In the final section of this paper, we will apply our method to a network of competing species models, where the couplings will similarly correspond to migrations of biological individuals.*

3.2. **Abstract formulation of the complex network problem.** We handle the complex network problem (3.11) in Hilbert space

$$X = \left(L^2(\Omega)\right)^{n \times m},$$

equipped with the product norm defined by

$$\|u\|_X = \left(\sum_{j=1}^{n} \sum_{i=1}^{m} \|u_{i,j}\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

for all $u \in X$. For each $j \in \{1, \dots, n\}$, we consider the diagonal operator $A_j = \operatorname{diag}\{A_{1,j}, \dots, A_{m,j}\}$, where $A_{i,j}$, $1 \le i \le m$, is the realization of $-D_{i,j}\Delta u_{i,j} + u_{i,j}$ in $L^2(\Omega)$, under Neumann boundary condition on $\partial\Omega$. The operators $A_{i,j}$, $1 \le i \le m$, are sectorial and positive definite self-adjoint operators of $L^2(\Omega)$, with angles

strictly lesser than $\frac{\pi}{2}$ (see for instance [42], Theorem 2.6). They admit a common domain given by

$$H_N^2(\Omega) = \left\{ u \in H^2(\Omega) \; ; \; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}. \tag{3.6}$$

Hence, $A = \operatorname{diag}\{A_i,\, 1 \leq i \leq n\}$ is a sectorial and positive definite self-adjoint operator of the product space X, with angle strictly lesser than $\frac{\pi}{2}$.

We fix $\eta \in \left(\frac{3}{4},\, 1\right)$, and consider the fractional power operator A^η, whose domain is given by the interpolation space (see [42], Theorem 16.7):

$$\mathcal{D}(A^\eta) = \left(\left[L^2(\Omega),\; H_N^2(\Omega) \right]_\eta \right)^{nm} = \left(H_N^{2\eta}(\Omega) \right)^{nm}, \tag{3.7}$$

with the norm equivalence:

$$\frac{1}{C} \left\| u \right\|_{H^{2\eta}(\Omega)} \leq \left\| (A_{i,j})^\eta u \right\|_{L^2(\Omega)} \leq C \left\| u \right\|_{H^{2\eta}(\Omega)}, \tag{3.8}$$

for all $u \in \mathcal{D}\left((A_{i,j})^\eta\right)$, $1 \leq i \leq m$, $1 \leq j \leq n$, for a given constant $C > 0$. Since $2\eta > \frac{3}{2}$ and Ω is bounded, the embedding theorems for Sobolev spaces [1] guarantee that:

$$H^{2\eta}(\Omega) \subset \mathscr{C}\left(\overline{\Omega}\right), \tag{3.9}$$

with continuous embedding. Additionally, due to the boundedness of Ω, it is clear that

$$\mathscr{C}\left(\overline{\Omega}\right) \subset L^\infty(\Omega) \subset L^2(\Omega), \tag{3.10}$$

with continuous embeddings.

The complex network problem (3.5) can be written

$$\begin{cases} \dfrac{du}{dt} + Au = f(u) + g(u), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{3.11}$$

where u, f and g are given by

$$\begin{aligned} u &= \left((u_{i,1})_{1 \leq i \leq m},\, \ldots,\, (u_{i,n})_{1 \leq i \leq m} \right)^T, \\ f(u) &= \left(u_1 + \varphi_1(u_1),\, \ldots,\, u_n + \varphi_n(u_n) \right)^T, \\ g(u) &= \left(g_1(u),\, \ldots,\, g_n(u) \right)^T. \end{aligned}$$

Next, we assume that the non-linear operator f satisfies the estimation

$$\left\| f(u) - f(v) \right\|_X \leq C_f \left(1 + \left\| A^\eta u \right\|_X + \left\| A^\eta v \right\|_X \right) \left\| u - v \right\|_X, \tag{3.12}$$

for all u, v in $\mathcal{D}(A^\eta)$. In parallel, we assume that the coupling operator g satisfies

$$\left\| g(u) - g(v) \right\|_X \leq C_g \left\| u - v \right\|_X, \tag{3.13}$$

for all u, v in X. Obviously, assumption (3.13) is always fulfilled by virtue of the definition (3.4) of g; we will see below that C_g can be estimated in terms of the number of vertices in the network and the maximal coupling strength.

The following theorem guarantees the existence and uniqueness of local solutions for the complex network problem (3.11).

Theorem 3.1. *For any $u_0 \in X$, there exists $T_{u_0} > 0$ such that the abstract problem (3.11) admits a unique solution u in the function space*

$$\mathscr{C}\big((0,\, T_{u_0}],\; \mathcal{D}(A)\big) \cap \mathscr{C}\big([0,\, T_{u_0}],\; X\big) \cap \mathscr{C}^1\big((0,\, T_{u_0}],\; X\big). \tag{3.14}$$

Furthermore, u satisfies the estimate

$$\|u(t)\|_X + t\,\|Au(t)\|_X \leq C_{u_0}, \quad \forall t \in]0,\, T_{u_0}], \tag{3.15}$$

where C_{u_0} is a positive constant depending only on $\|u_0\|_X$.

Proof. We have already noticed that A is a sectorial operator of X with angle lesser than $\frac{\pi}{2}$. Now, let $u,\, v \in \mathcal{D}(A^\eta)$. We have

$$\begin{aligned}\|(f+g)(u) - (f+g)(v)\|_X \leq & \|f(u) - f(v)\|_X + \|g(u) - g(v)\|_X \\ \leq & C_f\,(1 + \|A^\eta u\|_X + \|A^\eta v\|_X)\,\|u - v\|_X + C_g\,\|u - v\|_X \\ \leq & (C_f + C_g)\,(1 + \|A^\eta u\|_X + \|A^\eta v\|_X)\,\|u - v\|_X\,.\end{aligned}$$

for all $u,\, v$ in $\mathcal{D}(A^\eta)$. The conclusion directly follows from Theorem 2.1. □

3.3. **Energy estimates and global existence.** Here, we investigate sufficient conditions for proving that the local solutions of the complex network problem (3.11) are global in time. It is well-known that the solutions of reaction-diffusion systems can explode in finite time. However, it can be proved that the solutions are global if the non-linearities satisfy an under-polynomial growth property (see for instance the survey given in [34]). Entropy methods have also been used for studying this feature (see e.g. [12]). Recently, global existence of weak solutions has been established for reaction-diffusion systems under the assumption that the non-linearities enjoy a quadratic growth property [36]. Nevertheless, the analysis of the asymptotic behavior of the solutions requires stronger hypotheses in order to guaranty the existence of exponential attractors. For instance, weak solutions can be global in time, while blowing up in L^∞ infinitely many times. Thus we assume in the present work an a priori L^2-type estimation of the local solutions, and we list the consequences of that estimation: global existence of the local solutions, generation of a continuous dynamical system, existence of a family of exponential attractors.

Proposition 1. *Let $u(t,\, u_0)$ denote the solution of the complex network problem (3.11) stemming from $u_0 \in X$. Assume that there exist positive constants C_1, C_2 and δ such that*

$$\|u(t,\, u_0)\|_X \leq C_1 e^{-\delta t}\,\|u_0\|_X + C_2, \quad 0 < t \leq T_{u_0}. \tag{3.16}$$

Then the solution $u(t,\, u_0)$ is global in X, that is $T_{u_0} = +\infty$. Furthermore, the mapping

$$\begin{array}{rccc} G: & (0,\, +\infty) \times X & \longrightarrow & X \\ & (t,\, u_0) & \longmapsto & u(t,\, u_0) \end{array}$$

generates a continuous dynamical system $\big(S(t),\, \Phi,\, X\big)$ defined in X, whose phase space Φ is a compact subset of X and a bounded subset of $\mathcal{D}(A)$. Finally, the continuous dynamical system $\big(S(t),\, \Phi,\, X\big)$ admits exponential attractors.

Remark 3.3. *We emphasize that the dissipation assumption (3.16) is fulfilled for a wide class of models, provided that the non-linear operator φ involved in the initial problem (3.1) admits an under-polynomial growth and that the coupling operator g satisfies a conservation law. This is the case of the competing species model which shall be presented in section 5.*

Proof. First, it is clear that the a priori estimate (3.16) implies that the solution $u(t,\, u_0)$ stemming from $u_0 \in X$ is global in X. Next, the continuity of the mapping G is a consequence of Proposition 6.2 in [42]. Afterwards, let us consider a bounded subset $B \subset X$. One can find a positive constant C_B such that $\|u\|_X \leq C_B$ for all u

in B. Since $\delta > 0$, there exists $t_B > 0$ such that $e^{-\delta t} C_B < 1$. Indeed, it suffices to set $t_B = \frac{1+\log C_B}{\delta}$. We obtain

$$\sup_{u_0 \in B} \sup_{t \geq t_B} \|u(t,\, u_0)\|_X \leq C_1 + C_2.$$

By virtue of Proposition 6.1 in [42], the latter inequality implies that the stronger dissipative condition holds:

$$\sup_{u_0 \in B} \sup_{t \geq t_B} \|u(t,\, u_0)\|_{\mathcal{D}(A)} \leq C_3, \tag{3.17}$$

where C_3 is a positive constant. Now we consider the closed ball

$$\mathscr{B} = \overline{B}^{\mathcal{D}(A)}(0,\, C_3),$$

where the closure is in $\mathcal{D}(A)$. By virtue of Proposition 6.4 in [42], it is seen that $\mathscr{B}$ is a compact set of X. Furthermore, inequality (3.17) implies that $\mathscr{B}$ is an absorbing set. Finally, we consider

$$\Phi = \overline{\bigcup_{t \geq t_{\mathscr{B}}} S(t)\mathscr{B}}^{X}, \tag{3.18}$$

where the closure is in X, and $t_{\mathscr{B}}$ denotes a positive time such that $S(t)\mathscr{B} \subset \mathscr{B}$ for all $t \geq t_{\mathscr{B}}$. We easily verify that Φ is an invariant set for the semi-flow $S(t)$ induced by the complex network problem (3.11). In this way, we have proved that $\big(S(t),\, \Phi,\, X\big)$ is a continuous dynamical system. Finally, since Φ is a compact set of X and a bounded set of $\mathcal{D}(A)$, we deduce from [42] (Section 5.3) that $\big(S(t),\, \Phi,\, X\big)$ admits exponential attractors. The proof is complete. □

Energy estimates of type (3.16) can sometimes be established after proving that the solutions of the complex network problem (3.11) satisfy the non-negativity property, that is, solutions stemming from non-negative initial data remain non-negative in the future. This preservation of the non-negativity can be demonstrated by assuming that the non-linear operator f is *quasi-positive*. To that aim, we recall that a non-linear operator $F = (F_i)_{1 \leq i \leq m}$ defined on $\mathbb{R}^m$ (with $m \in \mathbb{N}^*$) is said to be *quasi-positive* if it satisfies the property

$$F_i(u_1,\, \ldots,\, u_{i-1},\, 0,\, u_{i+1},\, \ldots,\, u_m) \geq 0, \tag{3.19}$$

for all $u = (u_1,\, \ldots,\, u_m) \in (\mathbb{R}^+)^m$ and for all $i \in \{1,\, \ldots,\, m\}$. Let us introduce the space of initial conditions

$$X_0 = \{u \in X \;;\; u(x) \geq 0,\; \forall x \in \Omega\}, \tag{3.20}$$

where the inequality $u \geq 0$ has to be understood component-wise. We easily show that the sum of two quasi-positive operators is quasi-positive. Thus we directly obtain the following proposition, which we shall invoke in the final section for analysing the solutions of a complex network of competing species models.

Proposition 2. *The coupling operator g defined by (3.4) is quasi-positive. Suppose moreover that f is quasi-positive. Let $u_0 \in X_0$ and u be the solution of problem (3.11) starting from u_0, defined on $[0,\, T_{u_0}]$. Then, its components are non-negative on $[0,\, T_{u_0}]$.*

4. **Fractal dimension of exponential attractors.** In this section, we explore the influence of the topology of the graph underlying the complex network problem (3.11), and of the coupling strengths $\sigma_1,\, \ldots,\, \sigma_m$, on the dimension of the exponential attractors whose existence is guaranteed by proposition 1.

4.1. **Estimate of the fractal dimension of exponential attractors for the complex network problem.** Let $\big(S(t),\, \Phi,\, X\big)$ denote the continuous dynamical system generated by the complex network problem (3.11). Under the assumptions (3.12), (3.13) and (3.16), it has been proved in Proposition 1 that $\big(S(t),\, \Phi,\, X\big)$ admits a family of exponential attractors. In the sequel, we denote by $\mathfrak{M}$ one of those exponential attractors. For estimating the fractal dimension of $\mathfrak{M}$, we shall apply the method given in [14]. This method requires that the non-linearity $f+g$ involved in (3.11) satisfies an estimation of the type

$$\|(f+g)(u)-(f+g)(v)\|_X \leq C\left\|A^{\beta}(u-v)\right\|_X,$$

for all u, v in Φ, with an exponent β lesser than $\frac{1}{2}$. Obviously, the coupling operator g defined by (3.4) satisfies this requirement with $\beta=0$. However, the non-linear operator f satisfies a weaker estimate given by (3.12). We may obtain a stronger estimate of f, remarking that the phase space Φ is a bounded subset of $\mathcal{D}(A)$, which guarantees that there exists a positive constant C_Φ such that

$$\|Au\|_X \leq C_\Phi, \quad u\in\Phi. \tag{4.1}$$

Using the above estimates, we obtain the following theorem.

Theorem 4.1. *There exists a positive constant $C^*>0$ such that the fractal dimension of the exponential attractor $\mathfrak{M}$ satisfies*

$$d_F(\mathfrak{M}) \leq 1 + C\,|\Omega|\,\big(C^*\big)^{-d/2}. \tag{4.2}$$

Furthermore, C^* is given by

$$C^* = \frac{1-\exp\Big\{-C_A^2\big[C_g + C_f\,(1+2C_{A,\eta}C_\Phi)\,\big]\Big\}}{C_A^2\big[C_g + C_f\,(1+2C_{A,\eta}C_\Phi)\,\big]^2}, \tag{4.3}$$

where the positive constants C_f, C_g, C_Φ are defined by (3.12), (3.13) and (4.1) respectively. The positive constant C_A depends only on the diffusion operator A and the constant $C_{A,\eta}$ depends only on A and η, where the exponent η is fixed in $(\frac{3}{4},\,1)$. $|\Omega|$ denotes the diameter of the open bounded domain $\Omega\subset\mathbb{R}^d$ with $d\in\{1,\,2,\,3\}$.

In particular, the following asymptotic estimate holds:

$$d_F(\mathfrak{M}) \leq 1 + C\,|\Omega|\,\big(C_g\big)^{d/2}, \tag{4.4}$$

as C_g tends to infinity, where C is a positive constant.

Proof. Our goal is to prove that there exists $t^*>0$ such that $S^*=S(t^*)$ satisfies the squeezing property (2.4)-(2.5). The proof is divided into three steps. First, we estimate the Lipschitz constant L^* of S^* on Φ, and we choose properly t^*. Next, we estimate the squeezing coefficient δ^* involved in (2.4)-(2.5) and finally, we choose the projection rank N^* defined in Theorem 2.2 sufficiently large in order to guaranty that $\delta^*<\frac{1}{8}$.

Step 1: estimation of the Lipschitz constant L^*. Let us consider a basis of X composed with eigenvectors of operator A:

$$Aw_k = \lambda_k w_k, \quad k\geq 1,$$

with $0\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_k\to+\infty$. We set $X_k=\mathrm{Span}(w_1,\,\ldots,\,w_k)$; we consider the orthogonal projection P_k on X_k, and $Q_k=I-P_k$. Let $t^*>0$. Our aim is to show that for all $\delta>0$, there exists N^* such that for all u, v in Φ:

$$\|Q_{N^*}(S^*u-S^*v)\|_X > \|P_{N^*}(S^*u-S^*v)\|_X,$$

implies

$$\|S^*u - S^*v\|_X < \delta \|u - v\|_X .$$

We consider u_0, v_0 in Φ, and we denote $u(t) = S(t)u_0$, $v(t) = S(t)v_0$. Since Φ is positively invariant, we have $u(t) \in \Phi$ and $v(t) \in \Phi$ for all $t > 0$. We introduce

$$w(t) = u(t) - v(t), \quad \lambda(t) = \frac{\left\|A^{1/2}w(t)\right\|_X^2}{\|w(t)\|_X^2}, \quad w^* = w(t^*), \quad \lambda^* = \lambda(t^*).$$

First, it is easily seen that $\lambda^* > \frac{1}{2}\lambda_{N^*+1}$, where λ_{N^*+1} denotes the smallest eigenvalue of A over $Q_{N^*+1}X$. Next, since u and v satisfy (3.11), the inner product of $\frac{dw}{dt}$ and w in X leads to

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 + \lambda \|w\|_X^2 = \big(f(u) - f(v),\, w\big)_X + \big(g(u) - g(v),\, w\big)_X,$$

where we omit the time dependence in order to lighten our notations. By virtue of assumption (3.13), we have

$$\big|\big(g(u) - g(v),\, w\big)_X\big| \le C_g \|w\|_X^2 .$$

Now our aim is to estimate the inner product $\big(f(u) - f(v),\, w\big)_X$. We have

$$\big|\big(f(u) - f(v),\, w\big)_X\big| \le \|f(u) - f(v)\|_X \|w\|_X \le C_f\big(1 + \|A^\eta u\|_X + \|A^\eta v\|_X\big) \|w\|_X^2 .$$

The continuous embedding $\mathcal{D}(A) \subset \mathcal{D}(A^\eta)$ guarantees that there exists a positive constant $C_{A,\eta}$ such that $\|A^\eta u\|_X \le C_{A,\eta} \|Au\|_X$ for all u in $\mathcal{D}(A)$. It follows that

$$\big|\big(f(u) - f(v), w\big)_X\big| \le C_f\big(1 + C_{A,\eta} \|Au\|_X + C_{A,\eta} \|Av\|_X\big) \|w\|_X^2 .$$

By virtue of (4.1), we finally obtain

$$\big|\big(f(u) - f(v),\, w\big)_X\big| \le C_f\,(1 + 2C_{A,\eta}C_\Phi) \|w\|_X^2 , \quad t > 0.$$

Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 + [\lambda - C_g - C_f\,(1 + 2C_{A,\eta}C_\Phi)] \|w\|_X^2 \le 0, \quad t > 0.$$

It follows from Gronwall lemma that

$$\|w(t)\|_X \le \delta(t) \|w(0)\|_X , \quad t > 0,$$

where

$$\delta(t) = \exp\left\{-\int_0^t \lambda(\tau)d\tau + \big[C_g + C_f\,(1 + 2C_{A,\eta}C_\Phi)\,\big]t\right\}.$$

Consequently, the Lipschitz constant of $S(t)$ on Φ can be estimated by

$$\mathrm{Lip}_\Phi\big(S(t)\big) \le \exp\big\{\big[C_g + C_f\,(1 + 2C_{A,\eta}C_\Phi)\,\big]t\big\} .$$

Now we introduce $t^* > 0$ given by

$$t^* = \frac{1}{C_g + C_f\,(1 + 2C_{A,\eta}C_\Phi)}. \tag{4.5}$$

The Lipschitz constant L^* of S^* on Φ is finally estimated by

$$L^* \le e. \tag{4.6}$$

Step 2: estimation of δ^*. Now we estimate the quantity $\lambda(\tau)$ for $\tau \le t^*$. Let us introduce $\xi(t) = w(t) \times \|w(t)\|_X^{-1}$. We have

$$\frac{1}{2}\frac{d}{dt}\lambda(t) = \frac{1}{\|w\|_X^2}\left[\left(\frac{\partial w}{\partial t},\, Aw\right)_X - \left(\frac{\partial w}{\partial t},\, w\right)_X \lambda(t)\right]$$

$$=\frac{1}{\|w\|_X}\left(-Aw+\left[f(u)-f(v)+g(u)-g(v)\right],\,(A-\lambda)\xi\right)_X.$$

Basic computations show that $\frac{1}{\|w\|_X}(Aw,\,(A-\lambda)\xi)_X=\|(A-\lambda)\xi\|_X^2$, from which we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\lambda(t)+\|(A-\lambda)\xi\|_X^2 \leq&\frac{1}{\|w\|_X}\|f(u)-f(v)+g(u)-g(v)\|_X\times\|(A-\lambda)\xi\|_X\\ \leq&\frac{1}{\|w\|_X}\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]\|w\|_X\times\|(A-\lambda)\xi\|_X.\end{aligned}$$

Now the continuous embedding $\mathcal{D}(A^{1/2})\subset X$ guarantees that a positive constant C_A can be found such that $\|w\|_X\leq C_A\left\|A^{1/2}w\right\|_X$ for all w in $\mathcal{D}(A^{1/2})$. It follows that

$$\frac{1}{2}\frac{d}{dt}\lambda(t)+\|(A-\lambda)\xi\|_X^2\leq C_A\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]\sqrt{\lambda}\times\|(A-\lambda)\xi\|_X.$$

By virtue of Young inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\lambda(t)+\|(A-\lambda)\xi\|_X^2\leq C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2\frac{\lambda}{2}+\frac{\|(A-\lambda)\xi\|_X^2}{2},$$

and consequently

$$\frac{d}{dt}\lambda(t)\leq\theta\,\lambda(t),\quad t>0,$$

where $\theta=C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2$. Applying once again Gronwall lemma leads to

$$\lambda(t)\leq\lambda(s)\exp\left\{\int_s^t\theta d\tau\right\},$$

for all s and t such that $0\leq s<t$. Setting $t=t^*$ and inverting the above inequality leads to

$$\lambda(s)\geq\lambda(t^*)\exp\left\{-\int_s^{t^*}\theta d\tau\right\},$$

for all $s\in[0,\,t^*)$. Since $\lambda(t^*)>\frac{1}{2}\lambda_{N^*+1}$, we obtain

$$\int_0^{t^*}\lambda(s)ds\geq\frac{1}{2}\lambda_{N^*+1}\int_0^{t^*}\exp\left\{-\int_s^{t^*}\theta d\tau\right\}ds.$$

Now we compute

$$\begin{aligned}\int_0^{t^*}\exp\left\{-\int_s^{t^*}\theta d\tau\right\}ds=&\int_0^{t^*}\exp\left\{C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2(s-t^*)\right\}\\ =&\left[\frac{\exp\left\{C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2(s-t^*)\right\}}{C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2}\right]_0^{t^*}\\ =&\frac{1-\exp\left\{-C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2t^*\right\}}{C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2}\\ =&\frac{1-\exp\left\{-C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]\right\}}{C_A^2\left[C_g+C_f\left(1+2C_{A,\eta}C_\Phi\right)\right]^2}.\end{aligned}$$

Let us introduce the positive constant C^* defined by

$$C^* = \frac{1 - \exp\Big\{ - C_A^2 \big[C_g + C_f \left(1 + 2C_{A,\eta} C_\Phi\right) \big] \Big\}}{C_A^2 \big[C_g + C_f \left(1 + 2C_{A,\eta} C_\Phi\right) \big]^2}. \tag{4.7}$$

Note that C^* depends on f, g, η, A and Φ. It follows that

$$\int_0^{t^*} \lambda(s) ds \geq \frac{C^*}{2} \lambda_{N^*+1},$$

and consequently we obtain the following estimation of δ^*:

$$\delta^* \leq \exp\left\{ -\frac{C^*}{2} \lambda_{N^*+1} + 1 \right\}.$$

Additionally, basic computations show that the following asymptotic estimate holds $C^* \sim C_g^{-1}$ as C_g tends to infinity.

Step 3: choice of N^*. Now, in order to guaranty $\delta^* < \frac{1}{8}$, it suffices to choose N^* so that

$$\lambda_{N^*+1} > \frac{2(1 + \log 8)}{C^*}.$$

It is well-known [39] that

$$\lambda_{N^*+1} = C \left(\frac{N^*}{|\Omega|} \right)^{2/d},$$

so we choose a constant $C > 0$ such that

$$N^* = C\, |\Omega| \left[\frac{2(1 + \log 8)}{C^*} \right]^{d/2}$$

implies $\delta^* < \frac{1}{8}$. It follows that there exists a positive constant $\tilde{C}$ such that

$$N^* \sim \tilde{C}\, |\Omega| \left(C_g \right)^{d/2}$$

as C_g tends to infinity, which leads to the expected estimate of $d_F(\mathfrak{M})$ and completes the proof. □

Remark 4.1. *Estimate (4.4) shows that the fractal dimension of $\mathfrak{M}$ is likely to grow with C_g. We shall investigate in the next paragraph the effect of the topology and of the coupling strengths on the Lipschitz constant C_g. The optimality of estimate (4.4) is not discussed in the present work. In a separate paper, we aim to establish a lower bound of $d_F(\mathfrak{M})$, by applying another method, which requires to construct the unstable manifold of an unstable equilibrium of the complex network problem (3.11). However, it is observed in numerous cases that the lower bounds of exponential attractors can have a commensurate order with upper bounds (see [16] for instance).*

4.2. **Influence of the topology and of the coupling strengths on the coupling operator.** Now our aim is to estimate the Lipschitz constant C_g of the coupling operator g defined by (3.4), in terms of the number of vertices of the graph underlying the complex network problem (3.11), and of the coupling strengths σ_i (with $1 \leq i \leq m$) stored in the matrix H defined by (3.3). The following proposition establishes a first bound which is valid for any topology.

Proposition 3. *Let $\sigma_{\max} = \max(\sigma_i,\, 1 \leq i \leq m)$. The Lipschitz constant C_g of the coupling operator g defined by (3.4) satisfies*

$$C_g \leq n(n-1)\sigma_{\max},$$

where n denotes the number of vertices of the graph $\mathscr{G}$ underlying the complex network problem (3.11).

Proof. Let $u, v \in X$. By virtue of (3.4), we have for each $j \in \{1, \dots, n\}$

$$g_j(u) - g_j(v) = \sum_{k=1}^{n} L_{j,k} H(u_k - v_k).$$

Using (3.3), we obtain for $1 \le j \le n$ and $1 \le i \le m$:

$$\begin{aligned}
&\left\| \left(g_j(u) - g_j(v)\right)_i \right\|^2_{L^2(\Omega)} \\
=&\left\| \sum_{k=1}^{n} L_{j,k} \sigma_i (u_{i,k} - v_{i,k}) \right\|^2_{L^2(\Omega)} \le \sigma^2_{\max}(n-1)^2 \left\| \sum_{k=1}^{n} (u_{i,k} - v_{i,k}) \right\|^2_{L^2(\Omega)} \\
\le&\sigma^2_{\max}(n-1)^2 \left(\sum_{k=1}^{n} \|u_{i,k} - v_{i,k}\|_{L^2(\Omega)} \right)^2 \le \sigma^2_{\max}(n-1)^2 n \sum_{k=1}^{n} \|u_{i,k} - v_{i,k}\|^2_{L^2(\Omega)},
\end{aligned}$$

since $(x_1 + x_2 + \cdots + x_n)^2 \le n(x_1^2 + x_2^2 + \cdots + x_n^2)$ for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Consequently, we have

$$\begin{aligned}
&\|g_j(u) - g_j(v)\|^2_{L^2(\Omega)^m} \\
=&\sum_{i=1}^{m} \left\| \left(g_j(u) - g_j(v)\right)_i \right\|^2_{L^2(\Omega)} \le \sigma^2_{\max}(n-1)^2 n \sum_{i=1}^{m} \sum_{k=1}^{n} \|u_{i,k} - v_{i,k}\|^2_{L^2(\Omega)} \\
\le&\sigma^2_{\max}(n-1)^2 n \sum_{k=1}^{n} \sum_{i=1}^{m} \|u_{i,k} - v_{i,k}\|^2_{L^2(\Omega)} \le \sigma^2_{\max}(n-1)^2 n \sum_{k=1}^{n} \|u_k - v_k\|^2_{L^2(\Omega)^m} \\
\le&\sigma^2_{\max}(n-1)^2 n \|u - v\|^2_X .
\end{aligned}$$

Finally, we obtain

$$\|g(u) - g(v)\|^2_X \le \sigma^2_{\max}(n-1)^2 n \sum_{j=1}^{n} \|u - v\|^2_X \le \sigma^2_{\max}(n-1)^2 n^2 \|u - v\|^2_X ,$$

which leads to the desired estimate of C_g. □

It is worth noting that in the latter proof, each diagonal coefficient of the matrix of connectivity L has been estimated by $(n-1)$. However, we can obtain better estimates of the constant C_g when the topology of the graph admits a particular structure. The following proposition establishes a bound for the cases of an oriented chain, an oriented cycle and an oriented star (see figure 1). We emphasize that in the case of an oriented chain, an oriented cycle or a star oriented from exterior toward interior, the upper bound of the Lipschitz constant does not depend on the number of vertices of the graph $\mathscr{G}$.

Proposition 4. *If the graph $\mathscr{G}$ is given by an oriented chain, an oriented cycle or a star oriented from exterior toward interior, we have*

$$C_g \le 2\sigma_{\max}.$$

If the graph $\mathscr{G}$ is given by a star of n vertices ($n \ge 3$) oriented from interior toward exterior, we have

$$C_g \le \sigma_{\max}\sqrt{n^2 - n - 2}.$$

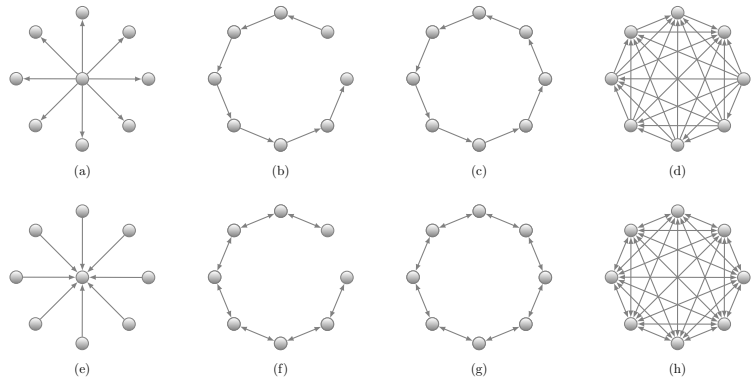

FIGURE 1. Several graph topologies. (a) Star oriented from interior toward exterior. (b) Oriented chain. (c) Oriented cycle. (d) Oriented complete topology. (e) Star oriented from exterior toward interior. (f) Bi-directed chain. (g) Bi-directed cycle. (h) Bi-directed complete topology.

Proof. Let $u, v \in X$. First, assume that the graph $\mathscr{G}$ is given by an oriented chain. After rearranging the labels of the vertices of $\mathscr{G}$, we may without loss of generality assume that the oriented chain has the form

$$(1) \to (2) \to \cdots \to (n).$$

Consequently, the matrix of connectivity L can be written

$$L = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ +1 & -1 & 0 & & 0 \\ 0 & +1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 \\ 0 & \dots & 0 & +1 & 0 \end{bmatrix}. \tag{4.8}$$

Hence we have for each $i \in \{1, \dots, m\}$

$$\left\| \left(g_1(u) - g_1(v)\right)_i \right\|^2_{L^2(\Omega)} = \sigma_i^2 \left\| \sum_{k=1}^{n} L_{1,k}(u_{i,k} - v_{i,k}) \right\|^2_{L^2(\Omega)} = \sigma_i^2 \left\| u_{i,1} - v_{i,1} \right\|^2_{L^2(\Omega)},$$

since $|L_{1,1}| = 1$ whereas $L_{1,k} = 0$ for $k > 1$. We can deduce from the latter inequality that

$$\|g_1(u) - g_1(v)\|^2_{L^2(\Omega)^m} \leq \sigma_{\max}^2 \|u_1 - v_1\|^2_{L^2(\Omega)^m}.$$

Similarly, we have for $1 < j < n$

$$\|g_j(u) - g_j(v)\|^2_{L^2(\Omega)^m} \leq 2\,\sigma_{\max} \left(\|u_{j-1} - v_{j-1}\|^2_{L^2(\Omega)^m} + \|u_j - v_j\|^2_{L^2(\Omega)^m} \right),$$

where we have used the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$. Finally, we have for $j = n$:

$$\|g_n(u) - g_n(v)\|^2_{L^2(\Omega)^m} \leq \sigma_{\max}^2 \|u_{n-1} - v_{n-1}\|^2_{L^2(\Omega)^m}.$$

We obtain

$$\|g(u) - g(v)\|^2_X$$

$$\leq \sigma_{\max}^2 \Big[\|u_1 - v_1\|^2_{L^2(\Omega)^m} + 2 \sum_{1<j<n} \Big(\|u_{j-1} - v_{j-1}\|^2_{L^2(\Omega)^m} + \|u_j - v_j\|^2_{L^2(\Omega)^m} \Big) + \|u_{n-1} - v_{n-1}\|^2_{L^2(\Omega)^m} \Big] \leq 4\, \sigma_{\max}^2 \, \|u - v\|^2_X \,,$$

which yields the desired estimate of C_g in the case of an oriented chain. For an oriented cycle, L can be written

$$L = \begin{bmatrix} -1 & 0 & \dots & 0 & +1 \\ +1 & -1 & 0 & \dots & 0 \\ 0 & +1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 \\ 0 & \dots & 0 & +1 & -1 \end{bmatrix}. \tag{4.9}$$

For an oriented star, L can be written

$$L = \begin{bmatrix} -(n-1) & 0 & \dots & 0 \\ +1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ +1 & 0 & \dots & 0 \end{bmatrix} \text{ or } L = \begin{bmatrix} 0 & +1 & +1 & \dots & +1 \\ 0 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -1 \end{bmatrix}, \tag{4.10}$$

for a star oriented from interior toward exterior and from exterior toward interior respectively. The other estimates are obtained using similar computations. □

4.3. **Symmetric topologies and synchronization.** Here we focus on symmetric topologies (*i.e.* topologies corresponding to bi-directed edges) and prove that the attractor $\mathfrak{M}$, even if large, is likely to contain synchronization states, provided the coupling strengths σ_i $(1 \leq i \leq m)$ are sufficiently strong. Those synchronization states correspond to the situation when all the vertices of the complex network, whose evolutions are determined by the variables $u_j \in L^2(\Omega)^m$, exhibit the same asymptotic dynamics, that is

$$\|u_j(t) - u_k(t)\|_{L^2(\Omega)^m} \to 0 \text{ as } t \to +\infty,$$

for $1 \leq j, k \leq n$ (the latter definition corresponds to *identical* synchronization [3]). We assume that the topology of the complex network is determined by a symmetric graph, such that the diagonal coefficients of the matrix L satisfy

$$L_{j,j} = -p, \quad 1 \leq j \leq n, \tag{4.11}$$

with $1 \leq p \leq n-1$, and such that

$$L_{j,l} = L_{k,l}, \quad 1 \leq l \leq n, \quad l \neq j, \quad l \neq k, \tag{4.12}$$

for all j, k in $\{1, \dots, n\}$. One example of a topology satisfying the above properties is that of a complete bi-directed graph (see figure 1 (h)); in that case, the matrix of connectivity can be written

$$L = \begin{bmatrix} -(n-1) & +1 & \dots & +1 \\ +1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & +1 \\ +1 & \dots & +1 & -(n-1) \end{bmatrix}.$$

We assume that the non-linear operator φ involved in the native reaction-diffusion system (3.1) satisfies

$$\|\varphi(u)-\varphi(v)\|_{L^2(\Omega)^m} \leq C_\varphi \|u-v\|_{L^2(\Omega)^m}, \tag{4.13}$$

for all u, v in $\mathcal{D}(\varphi)$, where C_φ denotes a positive constant; note that this requirement is straightly satisfied if the non-linear operator f enjoys property (3.12). Furthermore, we assume that the matrices of diffusion D_j, $1 \leq j \leq n$, are identical, that is

$$D_j = D, \quad \forall j \in \{1, \ldots, n\}. \tag{4.14}$$

Note that condition (4.14) does *not* imply that the diffusion coefficients of the components $u_{j,1}, \ldots, u_{j,m}$ should be identical ($1 \leq j \leq n$). More precisely, assumption (4.14) implies that $d_{i,j} = d_{i,k}$ for all $i \in \{1, \ldots, m\}$ and j, $k \in \{1, \ldots, n\}$, but we can still have $d_{i_1,j} \neq d_{i_2,j}$ for some i_1, $i_2 \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$; thus non trivial dynamics leading for example to the formation of Turing patterns are likely to occur. The following theorem generalizes the result presented in [2].

Theorem 4.2. *Assume that L is a symmetric matrix and that properties (4.11)-(4.12)-(4.13)-(4.14) are fulfilled. Then for any $u_0 \in \Phi$, the solution $u(t, u_0)$ synchronizes in the following sense*

$$\|u_j(t)-u_k(t)\|_{L^2(\Omega)^m} \to 0 \text{ as } t \to +\infty,$$

for all j, $k \in \{1, \ldots, n\}$, provided the coupling strengths satisfy

$$2(p-1)\sigma_i > 1 + C_\varphi^2, \tag{4.15}$$

for all $i \in \{1, \ldots, m\}$.

Remark 4.2. *In the case of a bi-directed complete topology, the sufficient condition (4.15) becomes*

$$2(n-2)\sigma_i > 1 + C_\varphi^2,$$

for all $i \in \{1, \ldots, m\}$. Roughly speaking, this condition is satisfied in small networks with a strong coupling strength (that is, n small and σ_i large), or in large networks with weak coupling (that is, n large and σ_i small). Hence the size of the network and the coupling strength are linked by an inverse power law which is characteristic of emergent properties of complex systems (see [2] for instance).

Proof. Let us introduce the energy functions defined by

$$E_{i,j,k} = \frac{1}{2}\int_\Omega (u_{i,j}-u_{i,k})^2 dx,$$

for $i \in \{1, \ldots, m\}$ and $j, k \in \{1, \ldots, n\}$. We compute the derivative of $E_{i,j,k}$ with respect to t:

$$\begin{aligned}
\frac{dE_{i,j,k}}{dt} &= \int_\Omega (u_{i,j}-u_{i,k})\frac{\partial(u_{i,j}-u_{i,k})}{\partial t}dx \\
&= \int_\Omega (u_{i,j}-u_{i,k})\big(d_{i,j}\Delta u_{i,j} - d_{i,k}\Delta u_{i,k}\big)dx \\
&\quad + \int_\Omega (u_{i,j}-u_{i,k})\big(\varphi_{i,j}(u_j) - \varphi_{i,k}(u_k)\big)dx \\
&\quad + \int_\Omega (u_{i,j}-u_{i,k})\big(g_{i,j}(u) - g_{i,k}(u)\big)dx.
\end{aligned}$$

By virtue of assumption (4.14), we have $d_{i,j} = d_{i,k}$, from which we can deduce

$$\int_\Omega (u_{i,j} - u_{i,k})\big(d_{i,j}\Delta u_{i,j} - d_{i,k}\Delta u_{i,k}\big)dx = -d_{i,j}\int_\Omega |\nabla(u_{i,j} - u_{i,k})|^2\, dx \le 0.$$

Next, using assumption (4.13) and Young inequality, we have

$$\begin{aligned}&\int_\Omega (u_{i,j} - u_{i,k})\big(\varphi_{i,j}(u_j) - \varphi_{i,k}(u_k)\big)dx\\ \le& E_{i,j,k} + \frac{1}{2}\left\|\varphi_{i,j}(u_j) - \varphi_{i,k}(u_k)\right\|^2_{L^2(\Omega)} \le (1 + C_\varphi^2)E_{i,j,k}.\end{aligned}$$

Finally, we examine the effect of the coupling operator:

$$\begin{aligned}g_{i,j}(u) - g_{i,k}(u) =& \sigma_i\left(\sum_{l=1}^n L_{j,l}u_{i,l} - \sum_{l=1}^n L_{k,l}u_{i,l}\right)\\ =&\sigma_i\left(\sum_{l\neq j} L_{j,l}u_{i,l} - pu_{i,j} - \sum_{l\neq k} L_{k,l}u_{i,l} + pu_{i,k}\right)\\ =&\sigma_i\Big(\sum_{l\neq j, l\neq k} L_{j,l}u_{i,l} + u_{i,k} - pu_{i,j} - \sum_{l\neq k, l\neq j} L_{k,l}u_{i,l} - u_{i,j} + pu_{i,k}\Big)\\ =& -\sigma_i(p-1)(u_{i,j} - u_{i,k}).\end{aligned}$$

We obtain

$$\frac{dE_{i,j,k}}{dt} + \big[2(p-1)\sigma_i - 1 - C_\varphi^2\big]E_{i,j,k} \le 0, t > 0.$$

Applying Gronwall lemma leads to the conclusion. □

5. **Application to a complex network of competing species models.** In this section, we handle a complex network problem built with non-identical instances of a competing species model, which takes into account the heterogeneity of the biological environment of the species. This model is set in a bounded domain $\Omega \subset \mathbb{R}^2$, whose boundary $\partial\Omega$ is assumed to be regular; it is given by the following system of two reaction-diffusion equations:

$$\begin{cases}\dfrac{\partial u}{\partial t} = d_1\Delta u + \alpha_1 u - \beta_1 u^2 - \gamma_1 uv,\\[2mm] \dfrac{\partial v}{\partial t} = d_2\Delta v + \alpha_2 v - \beta_2 v^2 - \gamma_2 uv.\end{cases} \tag{5.1}$$

Here, u and v correspond to the densities of some biological individuals. The coefficients d_i, α_i, β_i and γ_i ($i \in \{1, 2\}$) are assumed to be positive. The terms $\alpha_1 u - \beta_1 u^2$ and $\alpha_2 v - \beta_2 v^2$ correspond to the logistic growths of both species, whereas the terms $\gamma_1 u\, v$ and $\gamma_2 u\, v$ represent competition between those two species. One can find a detailed presentation of the dynamics of system (5.1) in [22] or [27] for instance. Recently, special attention has been paid to the role of heterogeneity in space or resource on the dynamics of such models (see for example [9] or [21]). Here, so as to take into account spatial heterogeneity of the biological environment in which the competing species evolve, we study a complex network of multiple instances of system (5.1), with Neumann boundary condition modeling the situation where biological individuals cannot leave the domain Ω. Such heterogeneous environments can be viewed through the *metapopulation* approach [20]; they can for instance correspond to natural habitats fragmented by urban and industrial expansion, which threats the equilibrium of natural ecosystems and thus the diversity of wildlife [19].

Here, we choose to model migrations of biological individuals from one region to another by point-wise couplings (see Remark 3.1). The heterogeneity of the biological environment also appears through the fact that the multiple instances of the competing species model are non-identical, which means that the biological parameters of each instance can differ from one region of the ecosystem to another. This heterogeneity in biological parameters is likely for example to favour one of the two species in a given region of the ecosystem, and the other species in another region of the ecosystem.

System (5.1) can be rewritten

$$\frac{\partial U}{\partial t} = D\Delta U + \varphi(U),$$

with $U = (u,\, v)^T$, $D = \operatorname{diag}\{d_1,\, d_2\}$ and

$$\varphi(U) = \left(\alpha_1 u - \beta_1 u^2 - \gamma_1 uv,\, \alpha_2 v - \beta_2 v^2 - \gamma_2 uv\right)^T.$$

Following the framework presented in section 2, we consider a graph $\mathscr{G}$ of n vertices and build a complex network problem by introducing

$$\begin{aligned}
&U = (U_1,\, \ldots,\, U_n)^T, \quad U_j = (u_j,\, v_j)^T, \quad 1 \le j \le n,\\
&f_j(U_j) = \begin{pmatrix} (\alpha_{1,j}+1)u_j - \beta_{1,j}u_j^2 - \gamma_{1,j}u_jv_j \\ (\alpha_{2,j}+1)v_j - \beta_{2,j}v_j^2 - \gamma_{2,j}u_jv_j \end{pmatrix},\\
&f(U) = \left(f_1(U_1),\, \ldots,\, f_n(U_n)\right)^T,\\
&\alpha_j = \max(\alpha_{1,j}\ ;\ \alpha_{2,j}),\\
&\beta_j = \max(\beta_{1,j}\ ;\ \beta_{2,j}),\\
&\gamma_j = \max(\gamma_{1,j}\ ;\ \gamma_{2,j}), \quad 1 \le j \le n.
\end{aligned} \tag{5.2}$$

Finally, we consider the Hilbert space $X = L^2(\Omega)^{2n}$, the diffusion operators $A_{i,j}$ defined in X by $A_{i,j} = -d_{i,j}\Delta u_j + u_j$ with Neumann boundary condition ($i \in \{1,\, 2\}$, $1 \le j \le n$) and the diagonal operator

$$A = \operatorname{diag}\{A_{i,j},\, 1 \le i \le 2,\, 1 \le j \le n\}.$$

5.1. **Existence and fractal dimension of exponential attractors.** First we show that the non-linear operator f defined in (5.2) satisfies property (3.12). We emphasize that the constant C_f in estimation (5.3) below depends on the number n of vertices in $\mathscr{G}$.

Proposition 5. *The non-linear operator f defined in (5.2) satisfies*

$$\left\|f(U) - f(\tilde{U})\right\|_X \le C_f\left(1 + \|A^\eta U\|_X + \left\|A^\eta \tilde{U}\right\|_X\right)\left\|U - \tilde{U}\right\|_X, \tag{5.3}$$

for all $U, \tilde{U} \in \mathcal{D}(A^\eta)$, with

$$C_f \le C\sqrt{n}\max_{1\le j\le n}(\alpha_j, \beta_j, \gamma_j),$$

where C denotes an absolute positive constant.

Proof. Let $U,\, \tilde{U} \in \mathcal{D}(A^\eta)$, with $U = (U_1,\, \ldots, U_n)^T$, $U_j = (u_j,\, v_j)^T$ and $\tilde{U} = (\tilde{U}_1, \ldots, \tilde{U}_n)^T$, $\tilde{U}_j = (\tilde{u}_j, \tilde{v}_j)^T$. Using the identity $a^2 - b^2 = (a-b)(a+b)$ for $a, b \in \mathbb{R}$ and the continuous embedding $H^{2\eta}(\Omega) \subset L^\infty(\Omega)$ (see (3.9) and (3.10)), we obtain

$$\left\|u_j^2 - \tilde{u}_j^2\right\|_{L^2(\Omega)} \le \left(\|u_j\|_{L^\infty(\Omega)} + \|\tilde{u}_j\|_{L^\infty(\Omega)}\right)\|u_j - \tilde{u}_j\|_{L^2(\Omega)}$$

$$\leq \left(\|(A_{1,j})^\eta u_j\|_{L^2(\Omega)} + \|(A_{1,j})^\eta \tilde{u}_j\|_{L^2(\Omega)} \right) \|u_j - \tilde{u}_j\|_{L^2(\Omega)},$$

for each $j \in \{1, \ldots, n\}$. Similarly, using the triangular inequality, we can write

$$\begin{aligned}
&\|u_j v_j - \tilde{u}_j \tilde{v}_j\|_{L^2(\Omega)} \\
&\leq \|u_j\|_{L^\infty(\Omega)} \|v_j - \tilde{v}_j\|_{L^2(\Omega)} + \|\tilde{v}_j\|_{L^\infty(\Omega)} \|u_j - \tilde{u}_j\|_{L^2(\Omega)} \\
&\leq \|(A_{1,j})^\eta u_j\|_{L^2(\Omega)} \|v_j - \tilde{v}_j\|_{L^2(\Omega)} + \|(A_{2,j})^\eta \tilde{v}_j\|_{L^2(\Omega)} \|u_j - \tilde{u}_j\|_{L^2(\Omega)},
\end{aligned}$$

for $1 \leq j \leq n$. Summing the latter inequalities for $1 \leq j \leq n$ leads to the desired estimate. □

Estimate (5.3) guarantees that Theorem 3.1 can be applied. Hence the complex network of non-identical systems (5.1) admits local solutions in function space (3.14). Afterwards, we easily verify that the non-linear operator f defined in (5.2) is quasi-positive. By virtue of proposition 2, we can deduce that the solutions of the complex network enjoy the non-negativity preservation property. We shall use this preservation property in order to derive an estimation in X of the solutions; this result is stated in the following proposition.

Proposition 6. *Let $U_0 \in X$ denote any initial condition with non-negative components. There exist positive constant δ, C_1 and C_2 which do not depend on U_0, such that the solution $U(t)$ of the complex network of multiple instances of system (5.1) stemming from U_0 satisfies*

$$\|U(t)\|_X \leq C_1 e^{-\delta t} \|U_0\|_X + C_2, \quad t > 0. \tag{5.4}$$

Proof. We introduce

$$E_1(t) = \sum_{j=1}^n \frac{1}{2} \int_\Omega u_j^2(x,t)dx = \frac{1}{2} \sum_{j=1}^n \|u_j\|^2_{L^2(\Omega)}, \quad t > 0.$$

We compute the derivative of E_1 with respect to t:

$$\begin{aligned}
\frac{dE_1}{dt}(t) &= \sum_{j=1}^n u_j \frac{\partial u_j}{\partial t} dx \\
&= \sum_{j=1}^n \int_\Omega u_j \left(d_{1,j} \Delta u_j + \alpha_{1,j} u_j - \beta_{1,j} u_j^2 - \gamma_{1,j} u_j v_j + \sigma_1 \sum_{k=1}^n L_{j,k} u_k \right) dx,
\end{aligned}$$

where we omit the variables x and t in order to lighten our notations. The maximum principle for diffusion operators with Neumann boundary condition implies that

$$\int_\Omega u_j \Delta u_j dx = -\int_\Omega |\nabla u_j|^2 dx \leq 0, \quad 1 \leq j \leq n.$$

Furthermore, the non-negativity of the solution guarantees that

$$\int_\Omega (-\gamma_{1,j} u_j v_j) dx \leq 0, \quad 1 \leq j \leq n.$$

We obtain

$$\frac{dE_1}{dt}(t) \leq \sum_{j=1}^n \int_\Omega \left(\alpha_{1,j} u_j^2 - \beta_{1,j} u_j^3 + \sigma_1 L_{j,j} u_j^2 + \sigma_1 \sum_{k \neq j} L_{j,k} u_k u_j \right) dx$$

$$\leq \sum_{j=1}^{n} \int_{\Omega} \left(\alpha_{1,j} u_j^2 - \beta_{1,j} u_j^3 + \sigma_1 L_{j,j} u_j^2 + \frac{\sigma_1}{2} \sum_{k \neq j} L_{j,k} u_j^2 + \frac{\sigma_1}{2} \sum_{k \neq j} L_{j,k} u_k^2 \right) dx,$$

where we have used the Young inequality $u_j u_k \leq \frac{u_j^2}{2} + \frac{u_k^2}{2}$, and the non-negativity of the off-diagonal terms $L_{j,k}$ for $j \neq k$. After rearranging the finite sums over k and j, we obtain

$$\frac{dE_1}{dt}(t) \leq \sum_{j=1}^{n} \int_{\Omega} (\zeta_{1,j} u_j^2 - \beta_{1,j} u_j^3) dx,$$

with $\zeta_{1,j} \in \mathbb{R}$, $1 \leq j \leq n$. Next, we use the polynomial inequality

$$\zeta_{1,j} u_j^2 - \beta_{1,j} u_j^3 \leq -\frac{1}{2} u_j^2 + \frac{(2\zeta_{1,j} + 1)^3}{27 \beta_{1,j}^2}, \quad 1 \leq j \leq n,$$

which leads to

$$\frac{dE_1}{dt}(t) + E_1(t) \leq |\Omega| \sum_{j=1}^{n} \frac{(2\zeta_{1,j} + 1)^3}{27 \beta_{1,j}^2},$$

and consequently, by using Gronwall lemma

$$\sum_{j=1}^{n} \|u_j(t)\|_{L^2(\Omega)}^2 \leq e^{-t} \sum_{j=1}^{n} \|u_j(0)\|_{L^2(\Omega)}^2 + 2\,|\Omega| \sum_{j=1}^{n} \frac{(2\zeta_{1,j} + 1)^3}{27 \beta_{1,j}^2}.$$

We can similarly obtain

$$\sum_{j=1}^{n} \|v_j(t)\|_{L^2(\Omega)}^2 \leq e^{-t} \sum_{j=1}^{n} \|v_j(0)\|_{L^2(\Omega)}^2 + 2\,|\Omega| \sum_{j=1}^{n} \frac{(2\zeta_{2,j} + 1)^3}{27 \beta_{2,j}^2},$$

with $\zeta_{2,j} \in \mathbb{R}$, $1 \leq j \leq n$. Summing the above inequalities leads to the desired estimate of $\|U(t)\|_X$, with

$$\delta = 1, \quad C_1 = 1, \quad C_2 = 2\,|\Omega| \sum_{j=1}^{n} \left(\frac{(2\zeta_{1,j} + 1)^3}{27 \beta_{1,j}^2} + \frac{(2\zeta_{2,j} + 1)^3}{27 \beta_{2,j}^2} \right). \qquad \square$$

Estimate (5.4) guarantees that proposition 1 and Theorem 4.1 can be applied. Consequently, the complex network of systems (5.1) generates a continuous dynamical system $\big(S(t),\, \Phi,\, X\big)$ which admits a family of exponential attractors of finite fractal dimension. Furthermore, the fractal dimension of those attractors can be estimated by (4.2). It is the purpose of the next section to illustrate the estimate of the fractal dimension by numerical simulations.

5.2. **Numerical simulations.** Here, we present a series of numerical simulations of the complex network of competing species models (5.1). Those numerical simulations have been obtained by the implementation of a splitting scheme with discretization of time and finite elements in space (see [11], [37]). The computations have been performed with the free software FreeFem++, on the calculation server of the Laboratory of Applied Mathematics of Le Havre Normandy, in a GNU/Linux environment. It is worth noting that attractors of infinite dimensional dynamical systems cannot be easily observed; thus our aim is not to completely visualize the attractors, but only to illustrate the estimate of the fractal dimension, by showing that the couplings are likely to create new equilibrium states. We also show a numerical simulation of a complete bi-directed topology with strong coupling strengths, in order to illustrate the synchronization theorem (Theorem 4.2).

We consider a circular domain Ω of radius $L = 250$ and a graph of four vertices. We experiment 4 topologies: the first topology corresponds to the absence of coupling, whereas the three other topologies are depicted in figure (2).

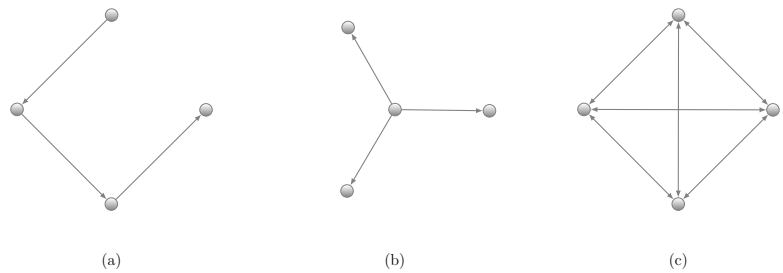

FIGURE 2. Three topologies for a complex network of competing species. (a) Oriented chain. (b) Star oriented from center toward periphery. (c) Bi-directed complete graph.

We set the diffusion coefficients to the non trivial case $d_1 \neq d_2$ with

$$d_1 = 15, \quad d_2 = 1, \tag{5.5}$$

for each vertex; the numerical values of other parameters of the competing species model (5.1) are indicated in table 1.

TABLE 1. Values of the parameters for a complex network of 4 non-identical competing species models.

Vertex 1		**Vertex 2**	
Parameter	Value	Parameter	Value
$\alpha_{1,1}$	1.0	$\alpha_{1,2}$	1.0
$\alpha_{2,1}$	1.0	$\alpha_{2,2}$	1.0
$\beta_{1,1}$	0.1	$\beta_{1,2}$	1.0
$\beta_{2,1}$	1.0	$\beta_{2,2}$	0.1
$\gamma_{1,1}$	0.1	$\gamma_{1,2}$	1.0
$\gamma_{2,1}$	1.0	$\gamma_{2,2}$	0.1

Vertex 3		**Vertex 4**	
Parameter	Value	Parameter	Value
$\alpha_{1,3}$	0.5	$\alpha_{1,4}$	10.0
$\alpha_{2,3}$	0.5	$\alpha_{2,4}$	10.0
$\beta_{1,3}$	0.1	$\beta_{1,4}$	5.0
$\beta_{2,3}$	0.1	$\beta_{2,4}$	5.0
$\gamma_{1,3}$	0.5	$\gamma_{1,4}$	4.0
$\gamma_{2,3}$	0.5	$\gamma_{2,4}$	4.0

Finally, we introduce various possible initial conditions by setting

$$\psi_1(x,\, y) = \frac{20}{1 + 0.1(x - 3L/4)^2 + 0.1(y - L/2)^2},$$
$$\psi_2(x,\, y) = \frac{20}{1 + 0.1(x - L/4)^2 + 0.1(y - L/2)^2},$$

$$\psi_3(x,\, y) = \frac{20}{1 + 0.1(x - L/2)^2 + 0.1(y - L/2)^2}.$$

The numerical results of the first scenario, corresponding to the absence of couplings, are presented in figure 3, in which we give the values of u_1, u_2, u_3 and u_4 for 3 different times.

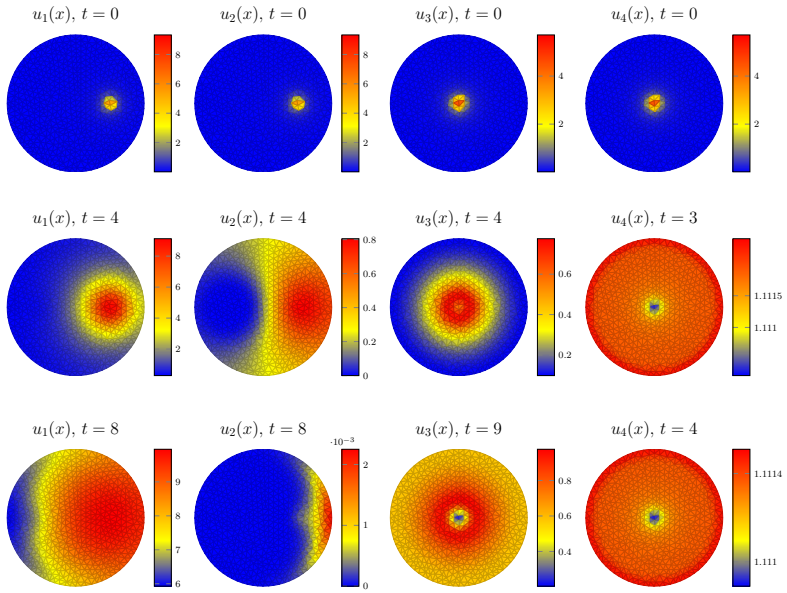

FIGURE 3. Numerical simulation of a complex network of competing species models in absence of coupling, showing the densities u_1, u_2, u_3 and u_4 for three different times (similar computations would show the densities v_1, v_2, v_3 and v_4): u_1 persists on vertex (1), whereas u_2 vanishes on vertex (2); in parallel, u_3 and v_3 coexist on vertex (3), and similarly, u_4 and v_4 coexist on vertex (4).

We observe that u_1 persists on vertex (1), whereas u_2 vanishes on vertex (2). In parallel, u_3 and v_3 coexist on vertex (3), and similarly, u_4 and v_4 coexist on vertex (4). Note that the asymptotic phase is rapidly reached on vertex (4). Those dynamics can obviously be predicted by examining the values of the parameters chosen for each vertex (see table 1).

The numerical results of the complex network built on an oriented chain (4) $\to$ (1) $\to$ (2) $\to$ (3) are presented in figure 4. The coupling strengths have been set to $\sigma_1 = 0.5$ and $\sigma_2 = 0.0$. First, we remark that the transitional phase is completely modified. Additionally, the asymptotic phase is perturbed: the domination of u_1 on vertex (1) is attenuated; u_2 seems to persist on vertex (2), whereas u_2 vanishes in absence of coupling; u_3 dominates on vertex (3), whereas u_3 and v_3 coexist in absence of coupling; u_4 and v_4 still coexist. Changing the order of the vertices in the oriented chain generates other dynamics. This example illustrates that the couplings of the complex network create new equilibrium states, which roughly speaking corresponds to an enlarged attractor.

Remark 5.1. *From the biological point of view, those new equilibrium states can be interpreted as perturbations of the natural equilibrium of wildlife caused by the fragmentation of the initial habitat of the species. In particular, the coexistence of two competing species can be affected, as shown in figures 3 and 4 for the third vertex, for which the species coexist in absence of couplings, whereas one of the species vanishes for non trivial couplings.*

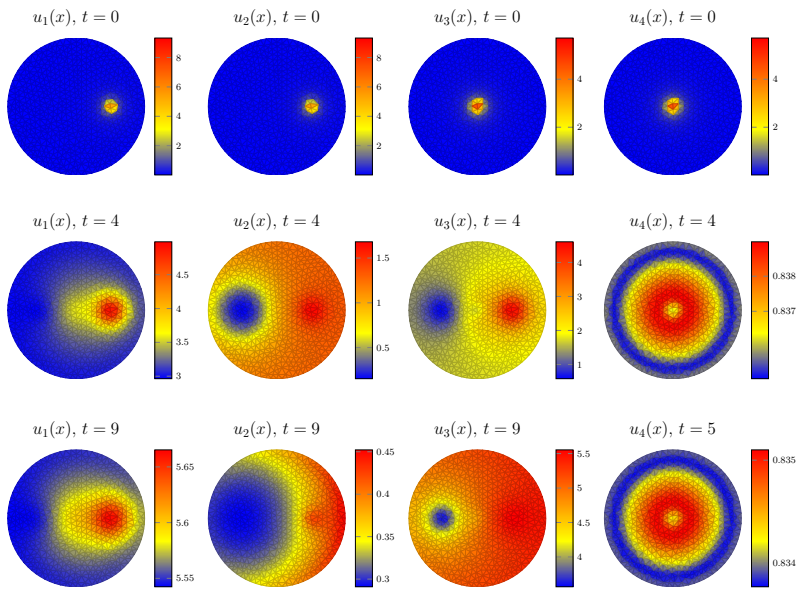

FIGURE 4. Numerical simulation of a complex network of competing species models, built on an oriented chain: the domination of u_1 on vertex (1) is attenuated; u_2 seems to persist on vertex (2), whereas u_2 vanishes in absence of coupling; u_3 dominates on vertex (3), whereas u_3 and v_3 coexist in absence of coupling; u_4 and v_4 still coexist.

Next we present in figure 5 the numerical results obtained with a star centered at vertex (4), oriented towards vertices (1), (2) and (3). The coupling strengths have been set to $\sigma_1 = 0.1$ and $\sigma_2 = 0.9$. Once again, we observe that the asymptotic dynamics are modified; in particular, u_2 persists on vertex (2), whereas u_2 vanishes in absence of coupling.

Finally, we present in figure 6 the numerical results obtained with a bi-directed complete graph topology. The coupling strengths have been set to $\sigma_1 = \sigma_2 = 2.0$. The synchronization of the four vertices is eloquent and occurs rapidly, which illustrates Theorem 4.2. As mentioned before, we recall that synchronization is not contradictory with enlarged attractors.

Remark 5.2. *From the biological point of view, the synchronization state of the complex network equipped with a bi-directed complete graph topology and strong couplings might signify that the homogeneity of an ecosystem is preserved, despite fragmentation, if the migrations of the species from one region to another are of sufficient intensity.*

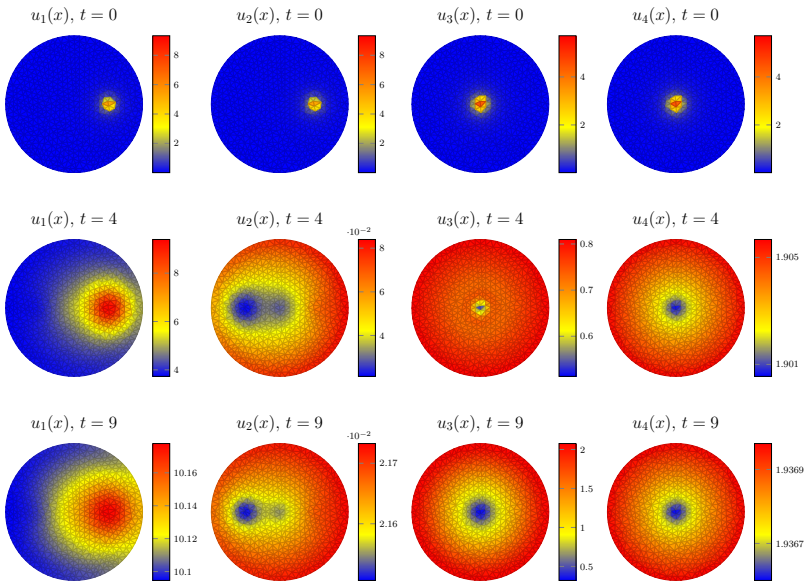

FIGURE 5. Numerical simulation of a complex network of competing species models, built on a star oriented from center toward periphery: the asymptotic dynamics are modified; in particular, u_2 persists on vertex (2), whereas u_2 vanishes in absence of coupling.

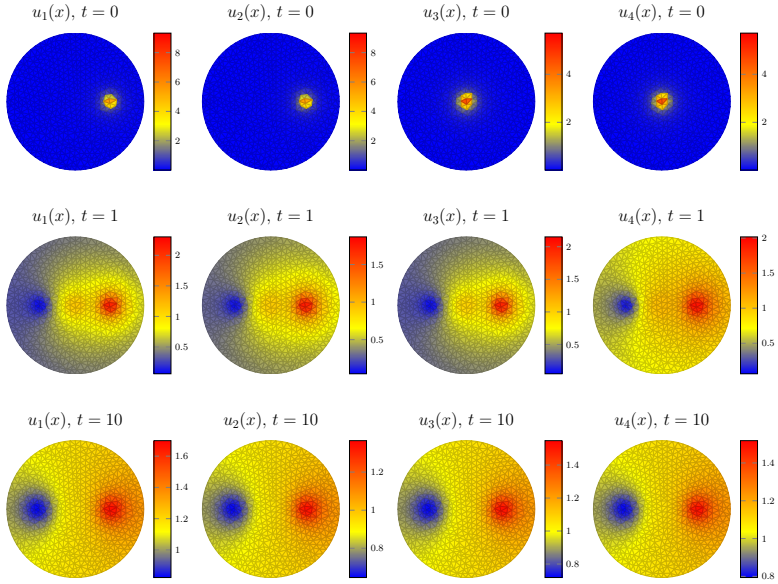

FIGURE 6. Numerical simulation of a complex network of competing species models, built on a bi-directed complete graph topology. After a brief transitional phase, synchronization of the four vertices occurs rapidly, which illustrates Theorem 4.2.

Conclusion. In this paper, we have studied the asymptotic behavior of complex networks of reaction-diffusion systems. Under reasonable assumptions which cover a wide class of systems, we have proved that those complex networks generate continuous dynamical systems which admit exponential attractors of finite fractal dimension. An innovative estimate of the fractal dimension of those attractors has been established, which clarifies the effect of the topology on the asymptotic dynamics of the complex network. Furthermore, we have proved that a symmetric topology with strong couplings leads to synchronization and we have investigated the effect of asymmetric topologies.

In a future work, we aim to complete the main result of the present work, by establishing a lower bound of the fractal dimension of exponential attractors for similar complex networks of reaction-diffusion systems. It is known that such an estimation can be obtained by approximating the dimension of the unstable manifold of a given equilibrium of the system. It is reasonable to expect that such a lower bound would reveal what kind of topology would enlarge the dimension of attractors.

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E-mail address: guillaumecantin@mail.com