Discrete Applied Mathematics 306 (2022) 133–137

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Independent vertex sets in the Zykov sum

Yunhua Liao [a,b,*], M.A. Aziz-Alaoui [b], Yaoping Hou [c]

[a] Department of Mathematics, Hunan University of Technology and Business, Changsha, Hunan, 410205, China
[b] Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, 76600 Le Havre, France
[c] Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China

ARTICLE INFO

Article history:
Received 5 August 2019
Received in revised form 15 July 2021
Accepted 27 September 2021
Available online 22 October 2021

Keywords:
Independence polynomial
Zykov sum
Generalized join
Fibonacci number
Real-world networks

ABSTRACT

Let $G = (V, E)$ be a connected graph. $\mathcal{H}$ denotes a family of pairwise disjoint graphs $\{H_v\}_{v\in V}$. The Zykov sum of G and $\mathcal{H}$, denoted by $G[\mathcal{H}]$, is the graph obtained from G by replacing every vertex v of G with graph H_v and all vertices of H_u, H_v are adjacent if $uv \in E$. In this paper, we first give a decomposition formula for the independence polynomial $I(G[\mathcal{H}]; x)$. Then, we derive a formula expressing the Fibonacci number of $G[\mathcal{H}]$ in terms of the independence polynomial of graph G and the Fibonacci number of H_v. Finally, as applications, we compute the independence polynomials and the Fibonacci numbers of several interesting graphs, such as the windmill graphs, the path network and the ring network.

1. Introduction

Throughout this paper $G = (V, E)$ is a connected and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $|V|$ denote the cardinality of V. For $S \subseteq V(G)$ we use $G - S$ for the subgraph induced by $V(G) - S$, and write $G - v$, whenever $S = \{v\}$. The *neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u : u \in V(G), uv \in E(G)\}$, and $N[v] = N(v) \cup \{v\}$. The *join* of two disjoint graphs G_1 and G_2 is the graph $G_1 + G_2$ such that $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1),\ v \in V(G_2)\}$. We use K_n, P_n, C_n and S_n for a complete graph, a path, a cycle and a star, all of order n, respectively.

An *independent set* in graph G is a set of pairwise non-adjacent vertices. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set of G. The set of all independent sets of G is denoted by $Ind(G)$. Let $f_k = f_k(G)$ be the number of independent sets of cardinality k in G, with the convention that $f_0 = 1$. The idea of counting independent sets in graphs seems to begin with a paper of Prodinger and Tichy [20] in which they defined, for a graph G, the *Fibonacci number* $f(G)$ to be the total number of independent sets of G, that is, $f(G) = |Ind(G)| = \sum_{k=0}^{\alpha(G)} f_k$. $f(G)$ is a parameter of interest to chemists and is the so-called *Merrifield–Simmons index* [3,10,13,16–19,24] of a graph which is related to stability in molecules. The polynomial

$$I(G; x) = \sum_{k=0}^{\alpha(G)} f_k x^k = \sum_{I\in Ind(G)} x^{|I|}$$

is called the *independence polynomial* of G [7], or the *Fibonacci polynomial* of G [9]; see also [6,15,21–23,25,26].

* Corresponding author at: Department of Mathematics, Hunan University of Technology and Business, Changsha, Hunan, 410205, China.
E-mail address: yunhua.liao@hutb.edu.cn (Y. Liao).

https://doi.org/10.1016/j.dam.2021.09.027

Let $\mathcal{H}_G$ be a family of pairwise disjoint graphs $\{H_v\}_{v\in G}$. The *Zykov sum* [1,4,27] (or *generalized join*, [2]) $G[\mathcal{H}_G]$ of G and $\mathcal{H}_G$ is the graph obtained from G by replacing every vertex v of G with graph H_v and all vertices of H_u and H_v are adjacent if $uv \in E(G)$, that is, $V(G[\mathcal{H}_G]) = \bigcup_{v\in V(G)} V(H_v)$ and

$$E(G[\mathcal{H}_G]) = \left(\bigcup_{v\in V(G)} E(H_v)\right) \cup \left(\bigcup_{uv\in E(G)} \{st : s \in V(H_u), t \in V(H_v)\}\right).$$

For example, the join $G_1 + G_2$ is the Zykov sum $P_2[G_1, G_2]$. The *composition* (or *lexicographic product*) of two disjoint graphs G and H, denoted by $G[H]$, as the Zykov sum, where $H_v = H$ for every $v \in V(G)$. The independence polynomial and the Fibonacci number of the composition graph $G[H]$ have been studied by Brown et al. [1] and Dosal-Trujillo and Galeana-Sanchez [4]. In this paper, motivated by the previous results, we consider the independence polynomial and the Fibonacci number of the Zykov sum $G[\mathcal{H}_G]$.

2. Preliminary results

In this section, we list some necessary results which are needed in this paper.

Lemma 2.1. *Let G, H be graphs.*
(i) [7] $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$ for each $v \in V(G)$.
(ii) [1,4] $I(G[H]; x) = I(G; I(H; x) - 1)$ and $f(G[H]) = I(G; f(H) - 1)$.
(iii) [8] $I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1$.

Lemma 2.2 ([9,22]). *(i) $I(K_n; x) = 1 + nx$;*
(ii) $I(S_n; x) = x + (1 + x)^{n-1}$;
(iii) $I(P_n; x) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} x^k$;
(vi) $I(C_n; x) = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k} \binom{n-1-k}{k-1} x^k$.

Dosal-Trujillo and Galeana-Sanchez [4] generalized Lemma 2.1(i) to vertex subset elimination.

Lemma 2.3 ([4]). *If G is a graph with U a subset of its vertices, such that for every $u, v \in U$, $N(u) \cap (V(G) - U) = S = N(v) \cap (V(G) - U)$ and H is the vertex induced subgraph $G\langle U\rangle$, then*

$$I(G; x) = I(G - U; x) + (I(H; x) - 1)\, I\left(G - (U \cup S); x\right).$$

3. The independence polynomial of $G[\mathcal{H}_G]$

In this section, we want to generalize Lemma 2.1(ii).

Lemma 3.1. *Let G be a connected and simple graph and let $\mathcal{H}_G = \{H_v\}_{v\in V(G)}$ be a family of pairwise disjoint graphs. v is a vertex of G. Then*

$$I\left(G[\mathcal{H}_G]; x\right) = I\left((G - v)[\mathcal{H}_{G-v}]; x\right) + (I(H_v; x) - 1)\, I\left((G - N[v])[\mathcal{H}_{G-N[v]}]; x\right).$$

Proof. Let $U = V(H_v)$. Then $G[\mathcal{H}_G] - U$ is isomorphic to $(G - v)[\mathcal{H}_{G-v}]$ and $G[\mathcal{H}_G] - (U \cup S)$ is isomorphic to $(G - N[v])[\mathcal{H}_{G-N[v]}]$, the result follows from Lemma 2.3. □

Theorem 3.2. *Let G be a connected and simple graph and let $\mathcal{H}_G = \{H_v\}_{v\in V(G)}$ be a family of pairwise disjoint graphs. v is a vertex of G. Then*

$$I\left(G[\mathcal{H}_G]; x\right) = 1 + \sum_{\emptyset \neq I \in Ind(G)} \prod_{v\in I} (I(H_v; x) - 1). \tag{1}$$

Particularly, if $I(H_v; x) = c(x)$ for every $v \in V(G)$, then

$$I(G[\mathcal{H}_G]; x) = I(G; c(x) - 1).$$

Proof. We will proceed by induction on $|V(G)|$.

When $|V(G)| = 1$, we have $I(G; x) = 1 + x$, and $I(G[H_v]; x) = I(H_v; x) = 1 + (I(H_v; x) - 1)$.

Suppose that Eq. (1) is true for every graph G' with $1 \leq |V(G')| < n$. Let G be a graph of order n and let u be a vertex of G. $Ind^{\bullet}(G)$ denotes the set of all nonempty independent sets of G, i.e. $Ind^{\bullet}(G) = Ind(G) - \{\emptyset\}$. Let $P = Ind^{\bullet}(G - u)$,

$Q = \{I \cup \{u\} : I \in Ind^{\bullet}(G - N[u])\}$, $R = \{u\}$. It is clear that P, Q, R are disjoint subsets of $Ind^{\bullet}(G)$ and $Ind^{\bullet}(G) = P \cup Q \cup R$. Moreover,

$$\sum_{I\in P}\prod_{v\in I}(I(H_v;x)-1) = \sum_{I\in Ind^{\bullet}(G-u)}\prod_{v\in I}(I(H_v;x)-1);$$

$$\sum_{I\in Q}\prod_{v\in I}(I(H_v;x)-1) = (I(H_u;x)-1)\left(\sum_{I\in Ind^{\bullet}(G-N[u])}\prod_{v\in I}(I(H_v;x)-1)\right);$$

$$\sum_{I\in R}\prod_{v\in I}(I(H_v;x)-1) = I(H_u;x)-1.$$

By Lemma 3.1, we have:

$$I\left(G[\mathcal{H}_G];x\right) = I\left((G-v)[\mathcal{H}_{G-v}];x\right) + (I(H_v;x)-1)\, I\left((G-N[v])[\mathcal{H}_{G-N[v]}];x\right).$$

Using the inductive hypothesis in $I\left((G-u)[\mathcal{H}_{G-u}];x\right)$ and $I\left((G-N[u])[\mathcal{H}_{G-N[u]}];x\right)$, we have:

$$\begin{aligned}
I(G[\mathcal{H}_G];x) &= 1 + \sum_{I\in Ind^{\bullet}(G-u)}\prod_{v\in I}(I(H_v;x)-1)\\
&\quad + (I(H_u;x)-1)\left(\sum_{I\in Ind^{\bullet}(G-N[u])}\prod_{v\in I}(I(H_v;x)-1)\right) + (I(H_u;x)-1)\\
&= 1 + \sum_{I\in P}\prod_{v\in I}(I(H_v;x)-1) + \sum_{I\in Q}\prod_{v\in I}(I(H_v;x)-1) + \sum_{I\in R}\prod_{v\in I}(I(H_v;x)-1)\\
&= 1 + \sum_{I\in Ind^{\bullet}(G)}\prod_{v\in I}(I(H_v;)-1).
\end{aligned}$$

Thus, Eq. (1) holds. If $I(H_v;x) = c(x)$ for every $v \in V(G)$, then

$$\begin{aligned}
I\left(G[\mathcal{H}_G];x\right) &= 1 + \sum_{\emptyset\neq I\in Ind(G)}\prod_{v\in I}(c(x)-1)\\
&= 1 + \sum_{\emptyset\neq I\in Ind(G)}(c(x)-1)^{|I|}\\
&= I(G;c(x)-1). \quad \square
\end{aligned}$$

Now, we study the Fibonacci number.

Corollary 3.3. *Let G be a connected and simple graph and let $\mathcal{H}_G = \{H_v\}_{v\in V(G)}$ be a family of pairwise disjoint graphs. Then*

$$f(G[\mathcal{H}_G]) = 1 + \sum_{\emptyset\neq I\in Ind(G)}\prod_{v\in I}(f(H_v)-1).$$

Particularly, if $f(H_v) = c$ for every $v \in V(G)$, then

$$f(G[\mathcal{H}_G]) = I(G;c-1).$$

4. Applications

In this section, we study the independence polynomials and the Fibonacci numbers of several kinds of graphs. Let $\overline{K_n}$ denote the complement of the complete graph. It is well known that $I(\overline{K_n};x) = (1+x)^n$.

4.1. Complete graph network

Let $H_1, H_2, \ldots, H_n$ be n disjoint graphs. The *complete graph network* is the Zykov sum $K_n[H_1, H_2, \ldots, H_n]$. By Theorem 3.2, we have

$$I(K_n[H_1,H_2,\ldots,H_n];x) = I(H_1;x) + I(H_2;x) + \cdots + I(H_n;x) + n - 1.$$

The complete multipartite graph $K_{n_1,n_2,\ldots,n_s}$ is a special case of the complete graph network. It can be viewed as $K_s[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_s}}]$. Thus,

$$I(K_{n_1,n_2,\ldots,n_s};x) = (1+x)^{n_1} + (1+x)^{n_2} + \cdots + (1+x)^{n_s} + s - 1.$$

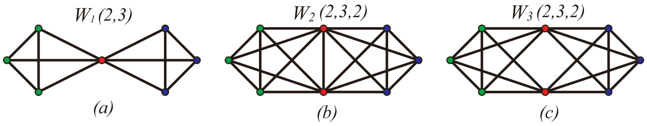
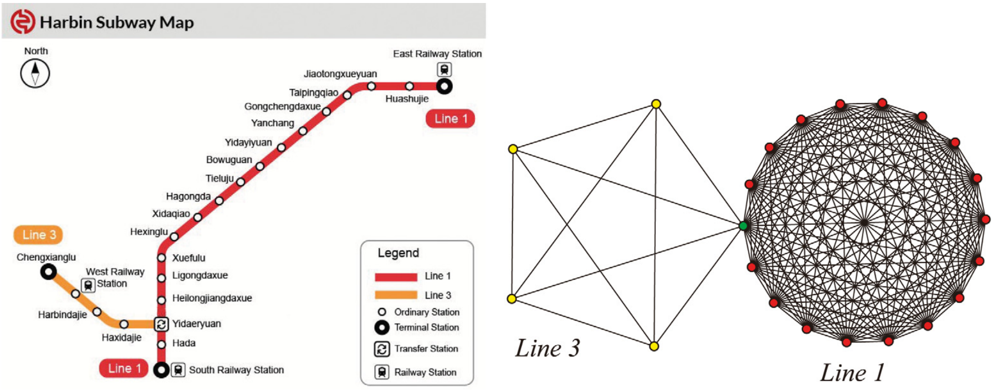


Fig. 1. Some windmill graphs.



(a) Topological representation   (b) Representation in P-space

Fig. 2. (Color online) Metro network of Harbin.

4.2. Generalized windmill graph

The windmill graphs were recently introduced by Estrada [5] and Kooij [14]. The windmill graph of the first type $W_1(n, s)$ consists of n copies of the complete graph K_s, with every vertex connected to a common vertex, see Fig. 1(a). The windmill graph of the second type $W_2(n, s, t)$ is the graph obtained from $W_1(n, s)$ by replacing the central vertex by t central vertices which form a complete graph K_t, see Fig. 1(b). The windmill graph of the third type $W_3(n, s, t)$ is the graph obtained from $W_1(n, s)$ by replacing the central vertex by t central vertices which form a empty graph $\overline{K_t}$, see Fig. 1(c). It was shown that windmill graphs arise naturally in certain real-world networks, such as citation networks [5], the public transport networks [14], see Fig. 2. Note that the network represented in Fig. 2(b) is not an actual windmill graph because the numbers of vertices in the two cliques representing Line 1 and Line 3 are not equal. In order to get a better model, we generalize the family of windmill graphs. Let $H_0, H_1, H_2, \ldots, H_n$ be $n + 1$ disjoint connected graphs. The generalized windmill graph $W_4(H_0, H_1, \ldots, H_n)$ is the Zykov sum $S_{n+1}[H_0, H_1, \ldots, H_n]$ where the central vertex of S_{n+1} was replaced by graph H_0. For example, the graph in Fig. 2(b) is the generalized windmill graph $W_4(K_1, K_{17}, K_4)$. Obviously, it holds that $W_1(n, s) = W_4(K_1, K_s, \ldots, K_s)$, $W_2(n, s, t) = W_4(K_t, K_s, \ldots, K_s)$, and $W_3(n, s, t) = W_4(\overline{K_t}, K_s, \ldots, K_s)$. By Lemma 3.1, we have

$$I(W_4(H_0, H_1, \ldots, H_n); x) = \prod_{k=1}^{n} I(H_k; x) + I(H_0; x) - 1. \tag{2}$$

Then,

$$\begin{aligned} I(W_1(n, s); x) &= (1 + sx)^n + x; \\ I(W_2(n, s, t); x) &= (1 + sx)^n + tx; \\ I(W_3(n, s, t); x) &= (1 + sx)^n + (1 + x)^t - 1. \end{aligned}$$

4.3. Path network and cycle network

The path network and ring network were introduced by Jiang and Yan [11,12] who determined the two-point resistances of these two networks. Given n positive integers $m_1, m_2, \ldots, m_n$. The path network $P[m_i]_1^n$ is the network with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i \cap V_j = \emptyset$ if $i \neq j$ and $|V_i| = m_i$, and with edge set $E = \{uv : u \in V_i,\ v \in V_{i+1},\ i = 1, 2, \ldots, n-1\}$. The ring network $C[m_i]_1^n$ is the network with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i \cap V_j = \emptyset$ if $i \neq j$ and $|V_i| = m_i$, and with edge set $E = \{uv : u \in V_i,\ v \in V_{i+1},\ i = 1, 2, \ldots, n\}$, where $V_{n+1} = V_1$. We denote by $P[m]$ and $C[m]$ the path network and the ring network when $m_i = m$ for $1 \leq i \leq n$, respectively. Clearly, $P[m]$ is the composition graph $P_n[\overline{K_m}]$ and $C[m]$ is the composition graph $C_n[\overline{K_m}]$. According to Lemma 2.2(iii,vi) and Corollary 3.3, we can derive their Fibonacci numbers,

$$f(P[m]) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} (2^m-1)^k;$$

$$f(C[m]) = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k} \binom{n-1-k}{k-1} (2^m-1)^k.$$

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 12001186), the Scientific Research Fund of Hunan Province Education Department (No. 20B158), the Normandie region France and the XTerm ERDF project (European Regional Development Fund) on Complex Networks and Applications.

References

[1] J.I. Brown, C.A. Hickman, R.J. Nowakowski, On the location of roots of independence polynomials, J. Algebraic Combin. 19 (2004) 273–282.
[2] Y. Chen, H.Y. Chen, The characteristic polynomial of a generalized join graph, Appl. Math. Comput. 348 (2019) 456–464.
[3] G. Chen, Z.X. Zhu, The number of independent sets of unicyclic graphs with given matching number, Discrete Appl. Math. 160 (2012) 108–115.
[4] L.A. Dosal-Trujillo, H. Galeana-Sanchez, On the Fibonacci number of the composition of graphs, Discrete Appl. Math. 266 (2019) 213–218.
[5] E. Estrada, When local and global clustering of networks diverge, Linear Algebra Appl. 488 (2016) 249–263.
[6] I. Gutman, Independence vertex sets in some compound graphs, Publ. Inst. Math. 52 (1992) 5–9.
[7] I. Gutman, F. Harary, Generalizations of the matching polynomial, Util. Math. 24 (1983) 97–106.
[8] C. Hoede, X.L. Li, Clique polynomials and independent set polynomials of graphs, Discrete Math. 125 (1994) 219–228.
[9] G. Hopkins, W. Staton, Some identities arising from the Fibonacci numbers of certain graphs, Fibonacci Quart. 22 (1984) 255–258.
[10] Y.F. Huang, L.S. Shi, X.Y. Xu, The Hosoya index and the Merrifield–Simmons index, J. Math. Chem. 56 (2018) 3136–3146.
[11] Z.Z. Jiang, W.G. Yan, Resistances between two nodes of a ring network, Physica A 484 (2017) 21–26.
[12] Z.Z. Jiang, W.G. Yan, Resistances between two nodes of a path network, Appl. Math. Comput. 361 (2019) 42–46.
[13] A. Knopfmacher, R.F. Tichy, S. Wagner, V. Ziegler, Graphs, partitions and Fibonacci numbers, Discrete Appl. Math. 155 (2007) 1175–1187.
[14] R. Kooij, On generalized windmill graphs, Linear Algebra Appl. 565 (2019) 25–46.
[15] V.E. Levit, E. Mandrescu, On the roots of independence polynomials of almost all very well-covered graphs, Discrete Appl. Math. 156 (2008) 478–491.
[16] S.C. Li, X.C. Li, W. Jing, On the extremal Merrifield–Simmons index and Hosoya index of quasi-tree graphs, Discrete Appl. Math. 157 (2009) 2877–2885.
[17] X.L. Li, H.X. Zhao, I. Gutman, On the Merrifield–Simmons index of trees, MATCH Commun. Math. Comput. Chem. 54 (2005) 389–402.
[18] A.S. Pedersen, P.D. Vestergaard, The number of independent sets in unicyclic graphs, Discrete Appl. Math. 152 (2005) 246–256.
[19] G. Perarnau, W. Perkins, Counting independent sets in cubic graphs of given girth, J. Combin. Theory Ser. B 133 (2018) 211–242.
[20] H. Prodinger, R.F. Tichy, Fibonacci numbers of graphs, Fibonacci Quart. 20 (1982) 16–21.
[21] V.R. Rosenfeld, The independence polynomial of rooted products of graphs, Discrete Appl. Math. 158 (2010) 551–558.
[22] L.Z. Song, W. Staton, B. Wei, Independence polynomials of k-tree related graphs, Discrete Appl. Math. 158 (2010) 943–950.
[23] L.Z. Song, W. Staton, B. Wei, Independence polynomials of some compound graphs, Discrete Appl. Math. 160 (2012) 657–663.
[24] Z.H. Zhang, Merrifield–Simmons index and its entropy of the 4-8-8 lattice, J. Stat. Phys. 154 (2014) 1113–1123.
[25] B.X. Zhu, Clique cover products and unimodality of independence polynomials, Discrete Appl. Math. 206 (2016) 172–180.
[26] B.X. Zhu, Y. Chen, Log-concavity of independence polynomials of some kinds of trees, Appl. Math. Comput. 342 (2019) 35–44.
[27] A.A. Zykov, On some properties of linear complexes, Math. Sb. 24 (1949) 163–188.