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# The number of spanning trees of a hybrid network created by inner-outer iteration

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**Abstract**

We consider the number of spanning trees in a novel hybrid network created by inner-outer iteration. The hybrid network is small-world but not self-similar. Firstly, we introduce two particular electrically equivalent transformations: delta-edge transformation and Sierpinski-delta transformation. By using these two particular transformations, we find the changes of the conductances of corresponding electrically equivalent networks. Secondly, based on the inner-outer iteration structure characteristics, we obtain a closed-form formula for the number of spanning trees of the hybrid network, as well as its spanning tree entropy. Finally, we compare our result with those of previously investigated networks with the same average degree, and give an explanation for their differences.

Keywords: spanning tree, electrically equivalent transformation, scale-free network

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Counting spanning trees in networks is a fascinating and central issue in statistical physics, because of its inherent relevance to diverse aspects in related fields. For example, the number of spanning trees is a crucial measure of the network's reliability [1]. Again for instance, it is exactly the number of recurrent configurations of the Abelian sandpile model [2, 3], which have been studied extensively in recent years as a paradigm of the self-organized criticality [4–6]. On the other hand, the number of spanning trees has numerous connections with other interesting problems associated with networks, such as transport [7], community structure detection [8], the Ising model partition function [9], loop-erased random walks [10], and many others [11].

In view of their relevance to diverse aspects of networks and a wide range of applications, spanning trees in networks have become a focus of some recent research [12–15]. Particularly, in the physics literature a lot of effort has been devoted to enumerating spanning trees in specific self-similar networks by using different techniques. Examples include the Farey graph [16, 17], $(x, y)$-flower [12], Apollonian network [18–20], Sierpinski gasket [21, 22], pseudofractal scale-free web [23], and so on. However, very little work appears to have been done on counting the spanning trees in a non-self-similar graph.

In this paper, inspired by the work in [13, 24], we investigate the number of spanning trees in a hybrid network proposed by Zhang *et al* [25], which follows an exponential degree distribution and has a small-world effect. Unlike many other deterministic networks, the hybrid network is not self-similar. We employ the electrically equivalent technique to determine the number of spanning trees of the hybrid network, and compute its spanning tree entropy.

## 2. Preliminaries

Throughout this paper, $G = (V, E, w)$ is a connected graph (network) with an positive edge-weight function $w$. When $w(e) = 1$ for every edge $e \in G$, $G$ is the usual unweighted graph. Let $|S|$ denote the cardinality of the set $S$, then the order (number of vertices) and the size (number of edges) of $G$ are $|V|$ and $|E|$, respectively. Let $T$ be a spanning

 

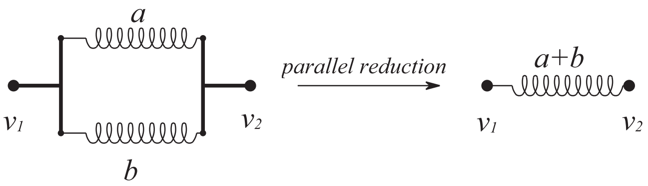


**Figure 1.** Parallel reduction.

tree of the weighted graph $G$, the *weight* $W(T)$ of $T$ is defined as the product of weights of edges in $T$, i.e.,

$$W(T) = \prod_{e \in E(T)} w(e).$$

We write $W(G)$ to denote the *weighted number of spanning trees* in $G$:

$$N_{WST}(G) = \sum_{T \in G} W(T).$$

Note that if $w(e) = 1$ for every edge $e \in G$, then $N_{WST}(G)$ is the usual number of spanning trees, and we write $N_{ST}(G) = N_{WST}(G)$ in this case.

### *2.1. Electrical Networks*

Following [20, 24, 26–28], we recall some concepts, notations, and results from electrical network theory. An edge-weighted graph can be regarded as an electrical network in which the weights are the conductances of the respective edges. Two weighted graphs $G$ and $H$ are called *electrically equivalent* with respect to $D \subseteq V(G) \cap V(H)$, if they cannot be distinguished by applying voltages to $D$ and measuring the resulting currents on $D$. The following theorem was introduced by Teufl and Wagner [26].

**Theorem 2.1.** *Let $G$ be a weighted graph that can be partitioned into two edge-disjoint subgraphs $S_1$ and $S_2$ (inheriting weights in an obvious way) such that $V(S_1) \cup V(S_2) = V(G)$ and $V(S_1) \cap V(S_2) == S^*$. Let $S_2'$ be a weighted graph with $E(S_1) \cap E(S_2') = \varnothing$ and $V(S_1) \cap V(S_2') = S^*$, such that $S_2$ and $S_2'$ are electrically equivalent with respect to $S^*$. Let $G' = S_1 \cup S_2'$, if $N_{WST}(G) \neq 0$ and $N_{WST}(S_2) \neq 0$, the following ratio holds:*

$$\frac{N_{WST}(G')}{N_{WST}(G)} = \frac{N_{WST}(S_2')}{N_{WST}(S_2)}. \tag{1}$$

Theorem 2.1 means that if electrical network $G'$ can be obtained from electrical network $G$ by an electrically equivalent transformation, then the weighted number of spanning trees in $G'$ and $G$ are related by equation (1). Here, we will consider the effect of the following four basic electrically equivalent transformations [13, 26]:

- Parallel reduction: Two parallel edges $e_1$ and $e_2$ with conductances $a$ and $b$ can be replaced by a single edge with conductance $a + b$, see figure 1. The weighted number of spanning trees remains unchanged (the corresponding factor is equal to 1).
- Series reduction: Two serial edges with conductances $a$ and $b$ can be merged into a single edge with conductance $\frac{ab}{a+b}$, see figure 2. The weighted number of spanning trees will vary as follows:

$$N_{WST}(G') = \frac{1}{a+b} N_{NST}(G).$$

- Delta-wye transform: A triangle with conductances $a$, $b$, $c$ can be transformed into a star with conductances $x = \frac{ab+bc+ca}{a}$, $y = \frac{ab+bc+ca}{b}$, $z = \frac{ab+bc+ca}{c}$, see figure 3. The weighted number of spanning trees will vary as follows:

$$N_{WST}(G') = \frac{(ab+bc+ca)^2}{abc} N_{NST}(G).$$

- Wye-Delta transform: A star graph with conductances $x$, $y$, $z$ can be transformed into a triangle with conductances $a = \frac{yz}{x+y+z}$, $b = \frac{xz}{x+y+z}$, $c = \frac{xy}{x+y+z}$, see figure 3. The weighted number of spanning trees will vary as follows:

$$N_{WST}(G') = \frac{1}{x+y+z} N_{NST}(G).$$

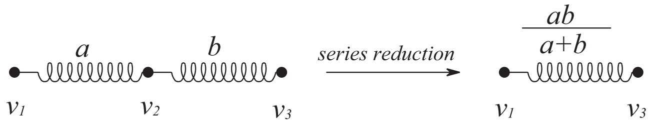


**Figure 2.** Series reduction.

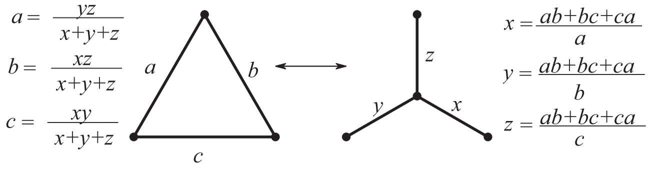


**Figure 3.** Delta-wye and wye-delta transformations.

For example, we consider the network that is shown in figure 4. A few applications of the four basic transformations suffice to count spanning trees of this network. The evolution of three electrically equivalent topologies and the corresponding calculations of the conductances are as follows: (1) Since the conductance of each edge of the original network $G_1$ is 1, the corresponding conductance of the resulting edges in graph $G_2$ for delta-wye transformation is 3, and $N_{WST}(G_2) = 9N_{WST}(G_1)$. (2) When two serial edges with conductances 1 and 3 are merged into a new edges, the conductance the new edge is $\frac{3}{4}$. Since series reduction has been applied three times, $N_{WST}(G_3) = \left(\frac{1}{4}\right)^3 N_{WST}(G_2) = \frac{9}{64} N_{WST}(G_1)$. (3) The conductance of the edge in $G_4$ is $\frac{9}{4}$ which is the sum of the three original conductances, and $N_{WST}(G_4) = N_{WST}(G_3) = \frac{9}{64} N_{WST}(G_1)$. It is clear that $N_{WST}(G_4) = \frac{9}{4}$, so $N_{WST}(G_1) = 16$. Since the weight of each edge of $G_1$ is 1, we have $N_{ST}(G_1) = N_{WST}(G_1) = 16$.

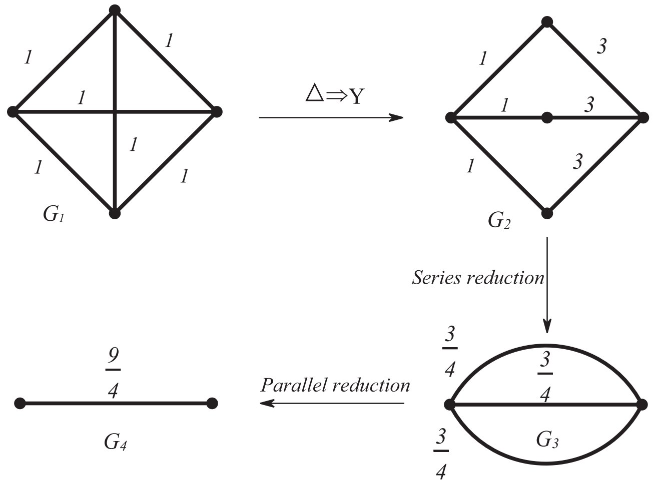


**Figure 4.** A simple example using the technique of those transformations to calculate the number of spanning trees.

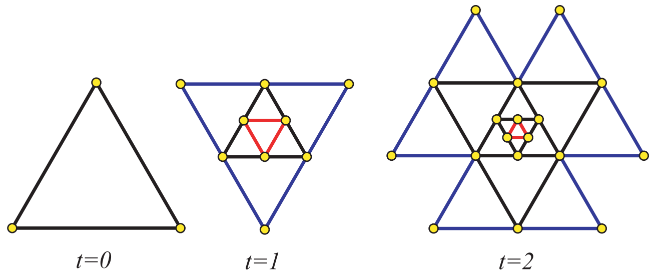


**Figure 5.** Illustration of the growing hybrid network within the first two steps of the iterative process.

*2.2. Construction of the hybrid network*

The hybrid network can be built in an iterative way [25]. Let $H_t\,(t \geqslant 0)$ denote the hybrid network after $t$ iterations. $H_t$ has two special cycles: the outermost cycle and the innermost cycle. For $t = 0$, $H_0$ is a triangle. The outermost cycle of $H_0$ coincides with the innermost cycle. For $t \geqslant 1$, $H_t$ can be obtained from $H_{t-1}$ by performing the following two operations: (i) inserting a new vertex into each edge in the innermost cycle, and connecting these new vertices by new edges; (ii) adding a new vertex for each edge in the outermost cycle, and linking the new vertex to the two endpoints of the corresponding edge. Figure 5 shows the first two steps of the iterative process. It is not difficult to compute that the order and size of $H_t$ are $V_t = |V(H_t)| = 3t + 3 \cdot 2^t$ and $E_t = |E(H_t)| = 6t + 3 \cdot 2^{t+1} - 3$, respectively. Thus, the average degree of the hybrid network $H_t$ is

$$\langle k \rangle = \frac{2E_t}{V_t} = \frac{3 \cdot 2^{t+2} + 12t - 6}{3 \cdot 2^t + 3t},$$

which approaches 4 as $t \to \infty$.

## 3. Counting spanning trees in the hybrid network

In this section, we will determine analytically the number of spanning trees on $H_t$ by applying the above-mentioned electrically equivalent technique [24, 26]. We firstly introduce two particular electrically equivalent transformations which are the combinations of the four basic electrically equivalent transformations in some order. These two particular transformations will help us to simplify the calculation.

### *3.1. Two particular electrically equivalent transformations*

*3.1.1. Delta-edge transformation.* If $e = \{u, v\}$, $f = \{u, w\}$ and $g = \{w, v\}$ are three edges of a triangle, vertex $w$ can be deleted and the edge $e$ can be replaced by a new edge $e' = \{u, v\}$, we call this transformation the *delta-edge transformation*, see figure 6. In order to reduce unnecessary computation in a later section, we assume that the conductances of edges $f$ and $g$ are $a$ and the conductance of edge $e$ is $c$.

It is easy to see that the delta-edge transformation is the combination of two basic transformations in a certain sequence. Since the delta-edge transformation is an electrically equivalent transformation, the conductance of the resulting edge can be calculated as follows:

(1) *Series reduction*: The conductance of the edge in $G_2$ which obtained from the series reduction is $\frac{a}{2}$, and $N_{WST}(G_2) = \frac{1}{2a} N_{WST}(G_1)$.
(2) *Parallel reduction*: The conductance of the edge in $G_3$ which obtained from the parallel reduction is $c + \frac{a}{2}$, and $N_{WST}(G_3) = N_{WST}(G_2)$.

Therefore, the conductance of the new edge $a'$ and the weighted numbers of spanning trees satisfy the following two recurrences:

$$a' = c + \frac{a}{2};$$
$$N_{WST}(G_3) = \frac{1}{2a} N_{WST}(G_1). \tag{2}$$

*3.1.2. Sierpinski-delta transformation.* The Sierpinski graph is the graph obtained from a triangle by inserting an additional vertex in each edge of the triangle, then joining with edges those pairs of new vertices, see figure 7. Note that the edges in the Sierpinski graph can be divided into two disjoint subsets. The edges in the outermost cycle and the edges in the innermost cycle.

If the Sierpinski graph can be replaced by the innermost cycle (triangle), we call this transformation the *Sierpinski-delta* transformation, see figure 8.

It is easy to see that five basic transformations are used in a certain sequence to form the Sierpinski-delta transformation. For the sake of the computation, hereafter, we assume that the conductance of each edge in the outermost cycle is 1 and the conductance of each edge in the innermost cycle is $b$. Since the Sierpinski-delta transformation is an electrically equivalent transformation, the conductances of the resulting edges can be calculated as follows:

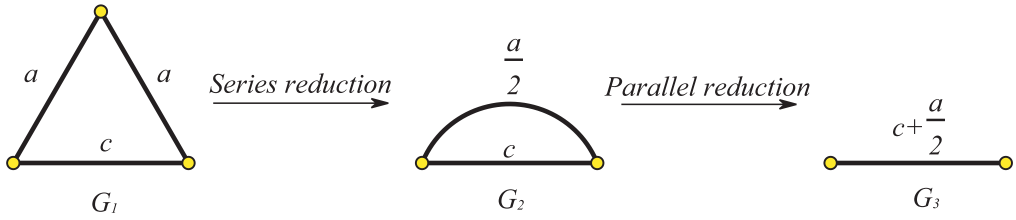


**Figure 6.** Delta-edge transformation.

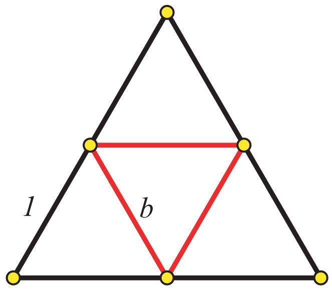


**Figure 7.** Sierpinski graph.

(1) *Delta-wye transform*: The delta-wye transform can be performed on the upmost triangle, the leftmost triangle and the rightmost triangle of the Sierpinski graph $S_1$. Since the conductances of the edges in each of these three triangles are 1,1 and $b$, the conductances of the resulting edges in $S_2$ are $1 + 2b$, $1 + 2b$ and $\frac{1+2b}{b}$. The delta-wye transform has been applied three times, so $N_{WST}(S_2) = \left(\frac{(1+2b)^2}{b}\right)^3 N_{WST}(S_1) = \frac{(1+2b)^6}{b^3} N_{WST}(S_1)$.

(2) *Series reduction*: The conductances of the serial edges in $S_2$ both are $1 + 2b$, the conductance of the new edge in $S_3$ is $\frac{1+2b}{2}$. The Series reduction has been applied three times, then $N_{WST}(S_3) = \frac{1}{8(1+2b)^3} N_{WST}(S_2)$.

(3) *Delta-wye transform*: The three edges in the central triangle of $S_3$ have the same conductance $\frac{1+2b}{2}$. We can apply the delta-wye transform on this triangle and the resulting edges have the same conductance $\frac{3+6b}{2}$. Also, $N_{WST}(S_4) = \frac{9}{2}(1 + 2b) N_{WST}(S_3)$.

(4) *Series reduction*: The conductances of the serial edges in $S_4$ are $\frac{1+2b}{b}$ and $\frac{3+6b}{2}$, the conductance of the new edge in $S_5$ is $\frac{3+6b}{2+3b}$. The series reduction has been applied three times, then $N_{WST}(S_5) = \frac{8b^3}{(1+2b)^3(2+3b)^3} N_{WST}(S_4)$.

(5) *Wye-delta transform*: The three edges in $S_5$ have the same conductance $\frac{3+6b}{2+3b}$. We can apply the wye-delta transform on these three edges and the resulting edges have the same conductance $\frac{1+2b}{2+3b}$. It is easy to compute that $N_{WST}(S_6) = \frac{2\ +\ 3b}{9(1+2b)} N_{WST}(S_5)$.

Therefore, the conductances of the new edges $b'$ and the weighted numbers of spanning trees satisfy the following two recurrences:

$$b' = \frac{1 + 2b}{2 + 3b};$$
$$N_{WST}(S_6) = \frac{1}{2(2 + 3b)^2} N_{WST}(S_1). \tag{3}$$

### 3.2. The weighted numbers of spanning trees of $H_t$

For $t \geqslant 1$, the set of edges in $H_t$ can be divided into three disjoint subsets: the edges in the outermost cycle, the edges in the innermost cycle, and others. So we classify the conductance on each edge of $H_t$ into three categories: the conductance of each edge in the outermost triangle is $a_t$; the conductance of each edge in the innermost edge is $b_t$; all other edges have the same conductance 1. For convenience, we call the edges which are contained in the innermost (outermost) cycle of $H_t$ the innermost (outermost) edges of $H_t$. The triangle which contains outermost edges is called the outermost triangle. In order to calculate the weighted number of spanning trees of $H_t$, we need to find the relationships between $a_t$, $b_t$, and $a_{t-1}$, $b_{t-1}$.

First of all, we have to simplify the structure of $H_t$. According to the construction method of $H_t$, it is easy to see that each outermost edge in $H_{t-1}$ will be replaced by a triangle which will becomes an outermost triangle of $H_t$. Moreover, the innermost cycle of $H_{t-1}$ is a triangle and it will be replaced by the Sierpinski graph which contains the innermost cycle of $H_t$. An example for these two kinds of replacements is given in figure 9.

Next, we can analyze the relationships between the conductances by applying the two particular electrically equivalent transformations. For each outermost triangle of $H_t$, we apply the delta-edge electrically equivalent transformation. Then, we have

$$a_{t-1} = \frac{1}{2} a_t + 1. \tag{4}$$

For the Sierpinski graph which contains the innermost cycle of $H_t$, we apply the Sierpinski-delta electrically equivalent transformation. Hence, we have

$$b_{t-1} = \frac{1 + 2b_t}{2 + 3b_t}. \tag{5}$$

Let $O_t$ be the number of the outermost edges in $H_t$. It is easy to see that $O_t$ satisfies the following recurrence:

$$O_t = 2O_{t-1}. \tag{6}$$

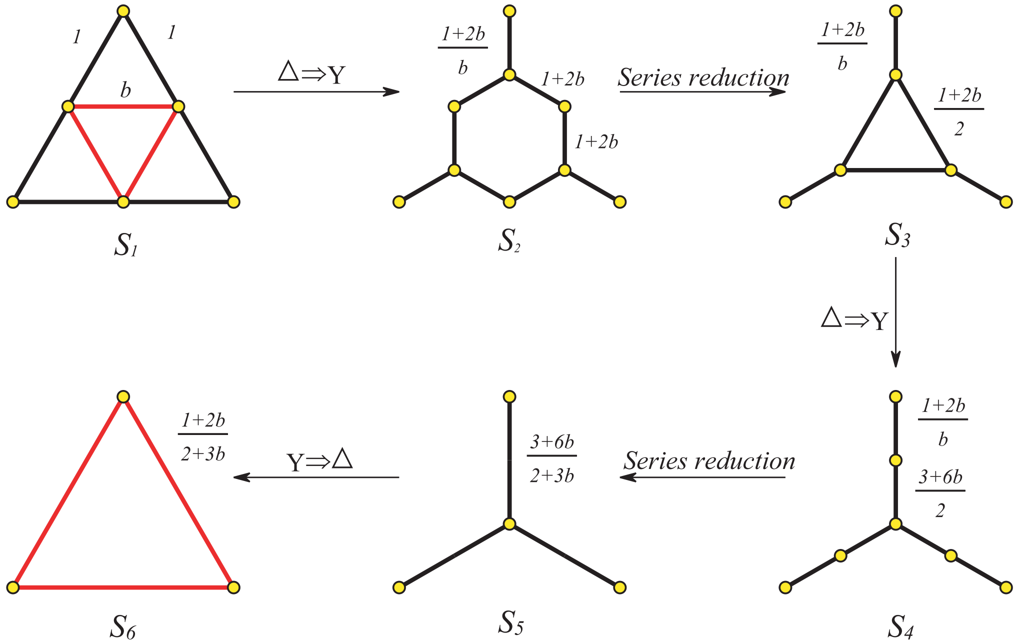


**Figure 8.** Sierpinski-delta transformation.

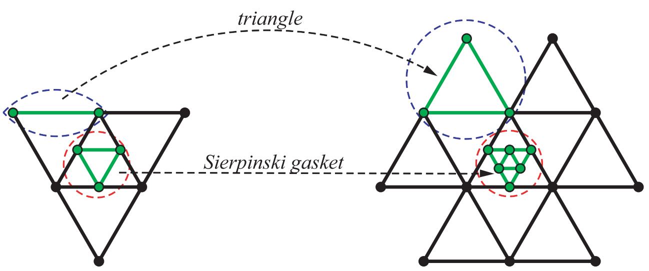


**Figure 9.** Two kinds of replacements.

Considering the initial condition $O_1 = 6$, we can solve equation (6) to obtain

$$O_k = 3 \cdot 2^k. \tag{7}$$

Combining equations (2), (3), and (7), we obtain

$$\begin{aligned} N_{WST}(H_t) &= 2(2+3b_t)^2(2a_t)^{3\cdot 2^{t-1}}N_{WST}(H_{t-1}) \\ &= 2(2+3b_t)^2(2a_t)^{3\cdot 2^{t-1}}[2(2+3b_{t-1})^2 \\ &\quad \times (2a_{t-1})^{3\cdot 2^{t-2}}]N_{WST}(H_{t-2}) \\ &= 2^2[(2+3b_t)(2+3b_{t-1})]^2(2a_t)^{3\cdot 2^{t-1}} \\ &\quad \times (2a_{t-1})^{3\cdot 2^{t-2}}N_{WST}(H_{t-2}) \\ &= 2^{t-1}\prod_{i=2}^{t}(2+3b_i)^2(2a_i)^{3\cdot 2^{i-1}}N_{WST}(H_1). \end{aligned} \tag{8}$$

Then, we will use the two particular electrically equivalent transformations to compute $N_{WST}(H_1)$. Figure 10 shows that $H_1$ can be reduced into a triangle.

(1) *Sierpinski-delta transform*: The resulting three edges have the same conductance $\frac{1+2b_1}{2+3b_1}$, and $N_{WST}(H_{1,1}) = \frac{1}{2(2+3b_1)^2}N_{WST}(H_1)$.
(2) *Delta-edge transform*: The resulting three edges have the same conductance $\frac{1+2b_1}{2+3b_1} + \frac{1}{2}a_1$. The transform has been applied three times, so $N_{WST}(H_{1,2}) = \frac{1}{8a_1^3}N_{WST}(H_{1,1})$. It is easy to see that $N_{WST}(H_{1,2}) = 3\left(\frac{1+2b_1}{2+3b_1} + \frac{1}{2}a_1\right)^2$. Therefore, we have

$$\begin{aligned} N_{WST}(H_1) &= 2(2+3b_1)^2 N_{WST}(H_{1,1}) \\ &= 16a_1^3(2+3b_1)^2 N_{WST}(H_{1,2}) \\ &= 48a_1^3\left(1+2b_1+a_1+\frac{3}{2}a_1b_1\right)^2. \end{aligned} \tag{9}$$

Finally, by substituting equations (9) into (8), we have

$$N_{WST}(H_t) = 3\cdot 2^{t+3}a_1^3\left(1+2b_1+a_1+\frac{3}{2}a_1b_1\right)^2 \times \prod_{i=2}^{t}(2+3b_i)^2(2a_i)^{3\cdot 2^{i-1}}. \tag{10}$$

In particular, when $a_1 = b_1 = 1$, $N_{ST}(H_1) = 1452$; when $a_2 = b_2 = 1$, $N_{ST}(H_2) = 13\ 220\ 496$; when $a_3 = b_3 = 1$, $N_{ST}(H_3) = 32\ 676\ 141\ 960\ 000$, which are consistent with numerical values of $N_{ST}(H_t)$ using the 'matrix-tree theorem' [31, 32] and exhibit an exponential growth trend.

After having an exact expression for the number of spanning trees of $H_t$, we can calculate its spanning tree entropy, which is given by

$$h = lim_{t\to\infty}\frac{lnN_{ST}(H_t)}{V_t}.$$

From figure 11, we obtain $h = 0.945\,8$.

The obtained asymptotic value can be compared with those of other networks which have the same average degree, see table 1. We can see that the spanning tree entropy of the hybrid network is very close to that of the Farey graph.

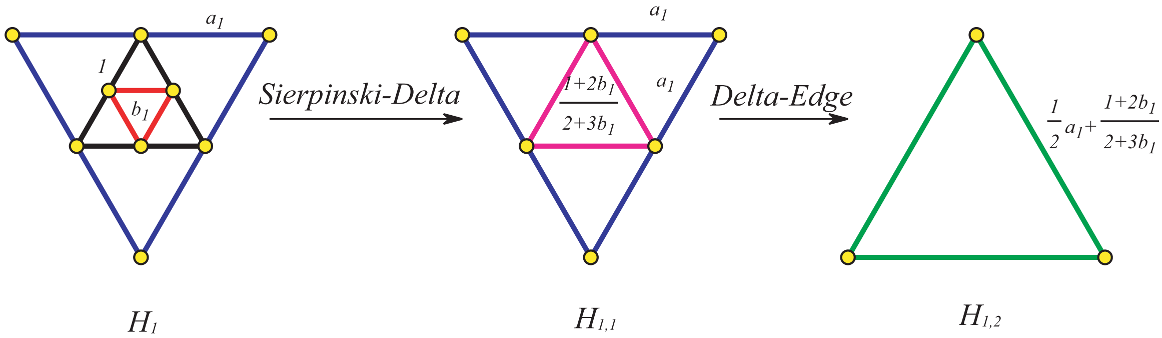


**Figure 10.** Reduce $H_1$ into a triangle.

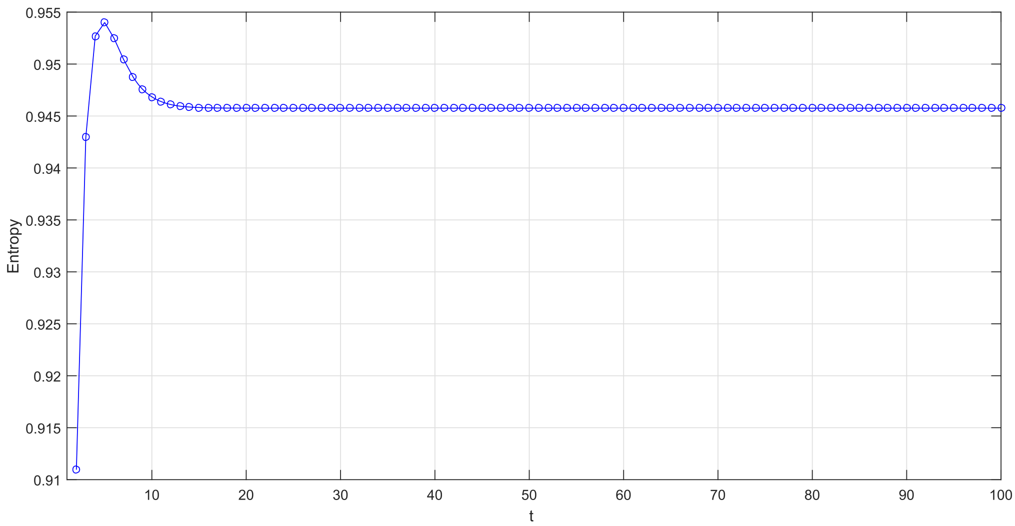


**Figure 11.** The spanning tree entropy of $H_t$.

**Table 1.** Comparison of spanning tree entropies of several networks.

| Type of network | Average degree $\langle k \rangle$ | Spanning tree entropy $h$ |
|---|---|---|
| Pseudofractal scale-free web [23] | 4 | 0.895 9 |
| **The hybrid network** | 4 | **0.945 8** |
| Farey graph [16] | 4 | 0.948 5 |
| Fractal scale-free lattice [29] | 4 | 1.039 7 |
| Sierpinski gasket [22] | 4 | 1.048 6 |
| Square lattice [30] | 4 | 1.166 2 |

Although the six networks in table 1 have the same average degree, their spanning tree entropies are different. Now, we try to explain the difference.

Let $G$ be a graph with $n$ vertices. According to the well-known matrix-tree theorem [31, 32], the number of spanning tree of graph $G$ can be expressed in terms of its Laplacian eigenvalues. Let $L$ represent the Laplacian matrix of graph $G$, then one can obtain $N_{ST}(G)$ by computing the product of all non-zero eigenvalues of $L$,

$$N_{ST}(G) = \frac{1}{n}\prod_{i=1}^{n-1} \lambda_i(G),$$

where $\lambda_i(G)$ $(i = 1, 2, \cdots, n-1)$ denote the $n-1$ non-zero eigenvalues of matrix $L$.

For two networks with the same average degree and order, they have the same number of edges, so the sums of their Laplacian eigenvalues are the same, but the products of their non-zero Laplacian eigenvalues may be different. Thus the numbers of spanning tree of these six networks are determined by the distributions of their Laplacian eigenvalues. It has been shown that there is a strong correlation between the distribution of Laplacian eigenvalues and the degree distribution of a network [33, 34]. Therefore, in order to explain the difference between the spanning tree entropies, we just need to compare the degree distributions of these six networks.

We define $n_k$ as the number of vertices with degree $k$ in graph $G$. Let $p_k = lim_{n\to\infty}\frac{n_k}{n}$. Table 2 shows the degree distributions of these six networks. It is clear that the hybrid network and the Farey graph have the same degree of distribution, so their spanning tree entropies are very close. The Sierpinski gasket and the square lattice are two almost four-regular graphs which implies that the distributions of

**Table 2.** Comparison of degree distributions of several networks.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Pseudofractal scale-free web | $k$ | 2 | 4 | 8 | $\cdots$ | $2^t$ | $2^{t+1}$ |
| | $n_k$ | $3^t$ | $3^{t-1}$ | $3^{t-2}$ | $\cdots$ | 3 | 3 |
| | $p_k$ | 66.7% | 22.2% | 7.4% | $\cdots$ | 0% | 0% |
| The hybrid network | $k$ | 2 | 4 | 6 | $\cdots$ | $2t$ | $2(t+1)$ |
| | $n_k$ | $3\cdot 2^{t-1}$ | $3t{+}3\cdot 2^{t-2}$ | $3\cdot 2^{t-3}$ | $\cdots$ | 3 | 3 |
| | $p_k$ | 50% | 25% | 12.5% | $\cdots$ | 0% | 0% |
| Farey graph | $k$ | 2 | 4 | 6 | $\cdots$ | $2t$ | $2(t+1)$ |
| | $n_k$ | $2^{t-1}$ | $2^{t-2}$ | $2^{t-3}$ | $\cdots$ | 1 | 2 |
| | $p_k$ | 50% | 25% | 12.5% | $\cdots$ | 0% | 0% |
| Fractal scale-free lattice | $k$ | 3 | 5 | 9 | $\cdots$ | $2^t$ | $2^t+1$ |
| | $n_k$ | $2\cdot 4^{t-1}$ | $2\cdot 4^{t-2}$ | $2\cdot 4^{t-3}$ | $\cdots$ | 2 | 2 |
| | $p_k$ | 75% | 18.8% | 4.7% | $\cdots$ | 0% | 0% |
| Sierpinski gasket | $k$ | 2 | 4 | | | | |
| | $n_k$ | 3 | $\frac{3}{2}\cdot(3^t-1)$ | | | | |
| | $p_k$ | 0% | 100% | | | | |
| Square lattice | $k$ | 4 | | | | | |
| | $n_k$ | $t^2$ | | | | | |
| | $p_k$ | 100% | | | | | |

Laplacian eigenvalues of this two graphs are homogeneous [33, 34]. From the famous arithmetic-geometric mean inequality [35] we know that the products of non-zero Laplacian eigenvalues of this two networks are larger than those of other networks. Therefore, the spanning tree entropies of this two networks are very large. The fractal scale-free lattice and the pseudofractal scale-free web are two scale-free networks, but the degree distributions of this two networks are different. Compared with the pseudofractal scale-free web, the fractal scale-free lattice has more vertices whose degree are close to the average degree. So the spanning tree entropy of the fractal scale-free lattice is larger than that of the pseudofractal scale-free web. The hybrid network and the Farey graph are two exponential networks. Table 1 shows that the spanning tree entropies of this two networks are larger than that of the pseudofractal scale-free web, but smaller than that of the fractal scale-free lattice. Therefore, the number of spanning tree of a network is not determined by whether the network is scale-free or not.

For the pseudofractal scale-free web, the percentage of vertices whose degrees are in the interval (1, 10) is 91.4%. For the hybrid network or the Farey graph, the percentage of vertices whose degrees are in the interval (1, 10) is 93.8%. For the fractal scale-free lattice, the percentage of vertices whose degrees are in the interval (1, 10) is 98.4%. Hence, the degree distribution of the fractal scale-free lattice is more concentrated which leads to a more homogeneous distribution of Laplacian eigenvalues. Therefore, the fractal scale-free lattice has larger spanning tree entropy, and the pseudofractal scale-free web has the smallest spanning tree entropy.

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