Opinion Diffusion in Two-Layer Interconnected Networks

Congying Liu, Xiaoqun Wu, Ruiwu Niu, M. A. Aziz-Alaoui, and Jinhu Lü, *Fellow, IEEE*

***Abstract*— In reality, individuals will spread their opinions by word of mouth, meanwhile sharing their opinions on social platforms. To gain a clear insight into this kind of behavior, we propose a diffusion model of various opinions in a two-layer interconnected network using some statistical characteristics of network structures. Theoretical analysis reveals that the final fraction of any opinion in one layer will get identical to that of the same opinion in the other layer. In particular, when the seed fraction of an opinion in one layer is different from that in the other layer, the diffusion behavior and final prevalence of opinions not only rely on the seed fractions of opinions, but also depend on the attributes of individuals that are active on different layers, including their inter-layer linking patterns and linking number. Further analysis shows that mass media will promote the spread of the opinion that is in accordance with its own. Finally, we illustrate the effectiveness of analytical results by simulating the spread of two opinions under four inter-layer linking patterns on three types of two-layer interconnected networks. The findings throw new light on some interesting phenomena in society, and facilitate decision-makers to orientate the prevalence of a special product.**

***Index Terms*— Opinion diffusion, multi-layer network, seed fraction, inter-layer linking patterns, linking number.**

I. Introduction

DURING the past three decades, network science has been of wild applicability and attracted extensive research attention, such as synchronization [1]–[4], importance evaluation of nodes [5]–[7], reconstruction of network structures [8], [9], consensus [10], [11], propagation and controllability of epidemics [12]–[14], among many others. Particularly, there has been a lot of literature [15]–[19], including theoretical analysis and large-scale numerical simulations, focusing on the spreading process of opinions in the context of network science.

Indeed, opinion diffusion, a product of sociology and psychology, describes the spreading process of individuals' opinions influenced by their neighbors. Lots of outstanding results have emerged [18], [20], [21]. For example, DeGroot [22] proved that a group of individuals will reach agreement on a common subject by pooling their individual opinions, Hu [23] studied the impact of social networks, individual fitness and mass media on opinion diffusion, Yang *et al.* [24] modified the Hegselmann-Krause model to investigate the opinion consensus problem and guarantee convergence from general initial conditions, and Meng *et al.* [25] found that network structures have important impact on the convergence of opinions and the number of steady-state opinion groups. Zhai and Zheng [26] investigated a general nonlinear model of opinion dynamics and obtained some sufficient conditions for the stability of the model under three different scenarios.

Above works are about opinion diffusion in single-layer networks. However, due to the diversity of spreading processes and the complexity of individual environment, opinion diffusion occurs not only in single-layer networks, but more frequently in multi-layer networks [27]. Take the spreading process of a certain topic for example. Apart from communicating with friends or colleagues by word of mouth, individuals may chat with netizens about this topic on public social platforms. Therefore, investigating the diffusion process in a single-layer online or offline network cannot accurately depict this practical issue. It is more meaningful and practical to discuss this topic on multi-layer networks than on an isolated one.

Actually, in the past few years, collective behaviors of a group of individuals, such as synchronization, diffusion and epidemic spreading, are increasingly of interest in a multi-layer network setting and many achievements have been made lately [28]–[30]. In 2013, Granell *et al.* [31] combined the spreading processes of a cyclic information awareness (the cycle unaware-aware-unaware (UAU)) and an epidemic (the susceptible-infected-susceptible (SIS)) to propose a seminal UAU-SIS model on multiplex networks, and revealed that information awareness prevented epidemic spreading. In 2017, Wu *et al.* [32] studied the spread of information in which individual decisions were based on trust, and found that the memory of previous behaviors had profound impact on information diffusion. Hu *et al.* [33] proposed a model to

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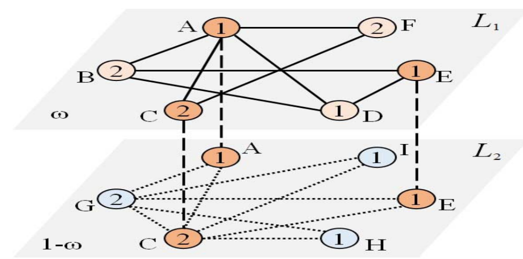
describe the opinion diffusion between the regular and the stubborn people on two different social platforms, and found that the final opinions of regular individuals were confined to the convex combinations of opinions of the stubborn ones. Wang *et al.* [34] reviewed some fascinating and counter-intuitive evolutionary outcome about evolutionary game on multi-layer networks and showed the influence of multi-layer networks on evolutionary game.

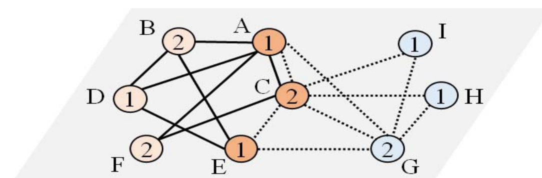
Furthermore, most works of information diffusion focus on evolutionary mechanisms and ignore the influence of network topologies. It is found that slight changes of network structures and dynamical systems may lead to significantly different dynamical behaviors [35]–[37]. Perc [38] presented an overview of information spreading on multi-layer networks and showed that even small unimportant changes in one layer can have catastrophic consequences in other layers. Jia *et al.* [39] focused on schemes for group synchronization in complex dynamical networks and found that group synchronization could be guaranteed by enhancing the external coupling strength between groups. Li *et al.* [40] explored synchronizability of a two-layered star network with two inter-layer links by giving an analytical expression containing the largest and smallest nonzero eigenvalues of the Laplacian matrix, and revealed that connecting two nodes in each layer with positive node degree correlation yielded better synchronizability than negative one. Wei *et al.* [41] investigated synchronizability of two-layered networks with different inter-layer linking patterns, and found that the inter-layer linking weight and linking fraction had a significant impact on synchronizability. Therefore, it naturally follows that inter-layer linking may play important roles in determining dynamics of opinion diffusion.

Based on the above motivations, we establish a probability model to describe a common social behavior in two-layer interconnected networks, such as information spreading on multiple channels, new product marketing, and presidential campaigns with citizens from various classes of the society supporting different candidates. The main contributions of the paper are as follows. Firstly, practical network structures are usually unknown or difficult to accurately identify, so we expect to use some statistical characteristics of network structures, rather than the particular adjacent matrices [23], [33], to model the spreading process of opinions. Therefore, we introduce a probability model to explore the impact of seed (initial) fractions of opinions, inter-layer linking patterns, linking number and mass media on opinion diffusion dynamics. Secondly, in most works [20], [22], [23], [25], [33], it is difficult to analyze the relationships between the parameters and the prevalence of opinions according to the probability model presented without simulations. But in this paper, we gain that the final prevalence of opinions not only rely on the seed fractions of opinions, but also depend on the attributes of individuals that are active in different layers by theoretical analysis.

The rest of the paper is organized as follows. Firstly, Sec. II presents a mathematical model that describes the spreading process of opinions in two-layer interconnected networks, and theoretical analysis reveals the impact of various parameters



(a) The schematics of opinions in a two-layer interconnected network.



(b) The schematics of opinions of (a) unfolded on aggregated network.

Fig. 1. Agents A, D, E, H, I hold Opinion 1, and agents B, C, F, G hold Opinion 2, where agents B, D, F are uncoupled agents in layer L_1, and agents G, H, I are uncoupled agents in layer L_2. Agents A, C, E are coupled agents that are active in both layers, and they are influenced by the neighbors with weight ω $(0 < \omega < 1)$ in layer L_1 and $1 - \omega$ in layer L_2. Solid and dotted lines represent the intra-layer links in layer L_1 and L_2, respectively, and agents connected by dashed lines in panel (a) represent coupled agents.

on the final prevalence of opinions. Next, Sec. III discusses the effect of mass media on opinion diffusion. Then, Sec. IV provides verification of the correctness of the proposed model by Monte Carlo simulations and presents some very interesting findings. Finally, Sec. V concludes the paper and presents some discussions.

II. General Model and Analysis

Consider a two-layer interconnected network, consisting of layer L_1 of size N_1 and layer L_2 of size N_2, where each layer is a degree-degree uncorrelated undirected network with no self-loops and two layers have different intra-layer connectivity. In the network, each node represents an individual, the link between two nodes represents individuals' communication connection, and each agent has an opinion. When opinions spread among people, there are two kinds of individuals in the two-layer interconnected network. One is those who are active and spread their opinions in both layers, termed as coupled agents, and the other is those who are only active in a single layer, termed as uncoupled agents (see Fig. 1 (a)).

Note that a coupled agent holds the same opinion in two layers and can indicate the attributes of networks, such as inter-layer linking patterns (the degree of coupled agents in two layers).

Initially, agents randomly hold Opinion i, $i \in \{1, 2, \ldots, N_o\}$, where N_o (≥ 2) is the total number of opinions. Then, influenced by their neighbors, agents may change their Opinion i to

Opinion j or persist in their previous Opinion i. So, we assume the persuasiveness of an agent is a function of his (her) degree [42], [43]. Specifically, we denote the persuasiveness of an agent with degree k_l as $f(k_l)$, and $f(k_l)$ is a power-law function of degree k_l, that is $f(k_l) = k_l^{\alpha}$, where k_l is the degree of agents in layer L_l, l $(= 1, 2)$ represents the subscript of layers and $0 \leq \alpha \leq 4$. Furthermore, since coupled agents are influenced by their neighbors in two layers, we suppose that coupled agents are influenced by their neighbors from layer L_1 with weight ω $(0 < \omega < 1)$ and the neighbors from layer L_2 with weight $1 - \omega$.

In order to describe the dynamical mechanism of opinion diffusion in a two-layer interconnected network precisely, we unfold the two-layer interconnected network into an aggregated network with size $N_1 + N_2 - N_c$, where N_c is the number of coupled agents (see Fig. 1 (b)). Obviously, there are three kinds of agents, that is, uncoupled agents from layer L_1 (such as agents B, D, F in Fig. 1), uncoupled agents from layer L_2 (such as agents G, H, I in Fig. 1) and coupled agents (such as agents A, C, E in Fig. 1).

Note that the purpose of unfolding the two-layer interconnected network into an aggregated network is to better describe the attributes of coupled agents by statistical characteristics of network structures. Different from the previous works about information diffusion, there could be multiple edges with different characteristics in the aggregated network (see Edge AC in Fig. 1 (b)) in this paper.

A. General Model

In the aggregated network, for uncoupled agents in layer L_l, $l = 1, 2$, the degree distribution of an agent with degree k_l is $P_{l,u}(k_l)$ (it is the fraction of the number of uncoupled agents in layer L_l out of the number of all agents in the aggregated network). At time step $t + 1$, the probability of uncoupled agents having degree k_l and Opinion i is

$$\begin{aligned} &O_{l,u}^{t+1}(k_l, i) \\ &= O_{l,u}^{t}(k_l, j)\frac{\sum_{k_l'} p_l(k_l'|k_l)q_l^t(i|k_l')f(k_l')}{\sum_{k_l'}\sum_j p_l(k_l'|k_l)q_l^t(j|k_l')f(k_l')} \\ &\quad + O_{l,u}^{t}(k_l, i)\left(1 - \frac{\sum_{k_l'} p_l(k_l'|k_l)(1 - q_l^t(i|k_l'))f(k_l')}{\sum_{k_l'}\sum_j p_l(k_l'|k_l)q_l^t(j|k_l')f(k_l')}\right), \quad (1) \end{aligned}$$

where $O_{l,u}^{t}(k_l, j)$ is the probability of uncoupled agents having degree k_l and Opinion non-i, and $O_{l,u}^{t}(k_l, j) = P_{l,u}(k_l) - O_{l,u}^{t}(k_l, i)$, $j \neq i$. $p_l(k_l'|k_l)$ denotes the conditional probability that agents in layer L_l with degree k_l are connected with agents in layer L_l with degree k_l', and $q_l(i|k_l)$ represents the conditional probability that agents in layer L_l with degree k_l hold Opinion i. On the right-hand side of Eq. (1), the first term is the probability that uncoupled agents with degree k_l change their opinions from non-i to i, and the second term is the probability that uncoupled agents with degree k_l stick to their Opinion i.

Note that the denominator of the fractions or probabilities denoted by capital letters is the number of all agents in the aggregated network (i.e. $N_1 + N_2 - N_c$), and that of denoted by lowercase letters is the number of all agents in the layer (i.e. N_1 or N_2).

Since we assume that connections in each layer are uncorrelated with node degrees and opinions are distributed randomly, there are $p_l(k_l'|k_l) = \frac{k_l' p(k_l')}{\langle k_l \rangle}$ and $q_l^t(i|k_l') = \frac{o_l^t(k_l', i)}{p_l(k_l')}$, where $o_l^t(k_l', i)$ represents the probability that agents in layer L_l have Opinion i and degree k_l' at time step t, and $p_l(k_l')$ is the degree distribution of agents in layer L_l, $l = 1, 2$. Then, Eq. (1) can be simplified as

$$O_{l,u}^{t+1}(k_l, i) = P_{l,u}(k_l) \cdot g_l^t(i), \quad (2)$$

where $g_l^t(i) = \frac{\sum_{k_l'} k_l' o_l^t(k_l', i) f(k_l')}{\sum_{k_l'} k_l' p_l(k_l') f(k_l')}$. Apparently, $g_l^t(i)$ represents the influence of Opinion i in layer L_l at time step t.

Similarly, in the aggregated network, for coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2, the probability of them holding Opinion i is

$$\begin{aligned} &O_c^{t+1}(k_1, k_2, i) \\ &= \omega\Bigg[O_c^t(k_1, k_2, j)\frac{\sum_{k_1'} p_1(k_1'|k_1)q_1^t(i|k_1')f(k_1')}{\sum_{k_1'}\sum_j p_1(k_1'|k_1)q_1^t(j|k_1')f(k_1')} \\ &\quad + O_c^t(k_1, k_2, i)\left(1 - \frac{\sum_{k_1'} p_1(k_1'|k_1)(1 - q_1^t(i|k_1'))f(k_1')}{\sum_{k_1'}\sum_j p_1(k_1'|k_1)q_1^t(j|k_1')f(k_1')}\right)\Bigg] \\ &\quad + (1-\omega)\Bigg[O_c^t(k_1, k_2, j)\frac{\sum_{k_2'} p_2(k_2'|k_2)q_2^t(i|k_2')f(k_2')}{\sum_{k_2'}\sum_j p_2(k_2'|k_2)q_2^t(j|k_2')f(k_2')} \\ &\quad + O_c^t(k_1, k_2, i)\left(1 - \frac{\sum_{k_2'} p_2(k_2'|k_2)(1 - q_2^t(i|k_2'))f(k_2')}{\sum_{k_2'}\sum_j p_2(k_2'|k_2)q_2^t(j|k_2')f(k_2')}\right)\Bigg], \quad (3) \end{aligned}$$

where $O_c^t(k_1, k_2, j)$ is the probability of coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2 having Opinion non-i, and $O_c^t(k_1, k_2, j) = P_c(k_1, k_2) - O_c^t(k_1, k_2, i)$. $P_c(k_1, k_2)$ is the degree distribution of coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2. On the right-hand side of Eq. (3), the first term is the probability that coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2 influenced by their neighbors in layer L_1 hold Opinion i, and the second term is the probability that coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2 influenced by their neighbors in layer L_2 hold Opinion i. Eq. (3) can be simplified as

$$O_c^{t+1}(k_1, k_2, i) = \omega P_c(k_1, k_2) g_1^t(i) + (1 - \omega) P_c(k_1, k_2) g_2^t(i). \quad (4)$$

Since $O_{l,u}^t(i) = \sum_{k_l} O_{l,u}^t(k_l, i)$, $l = 1, 2$ and $O_c^t(i) = \sum_{k_1}\sum_{k_2} O_c^t(k_1, k_2, i)$, we can obtain $O_{l,u}^t(i)$ and $O_c^t(i)$, where $O_{l,u}^t(i)$ is the probability of uncoupled agents holding Opinion i in the aggregated network at time step t and $O_c^t(i)$ is the probability of coupled agents holding Opinion i in the aggregated network at time step t. Therefore, in layer L_l of the two-layer interconnected network, we get the fraction of coupled agents holding Opinion i at time step t, denoted by $o_{l,c}^t(i)$, the fraction of uncoupled agents holding Opinion i at time step t, denoted by $o_{l,u}^t(i)$, and the fraction of agents holding Opinion i at time step t, denoted by $o_l^t(i)$, that is,

$$o_{l,c}^t(i)=\eta_l O_c^t(i),\ o_{l,u}^t(i)=\eta_l O_{l,u}^t(i),\ o_l^t(i)=o_{l,c}^t(i)+o_{l,u}^t(i), \tag{5}$$

where $\eta_l = \frac{N_1+N_2-N_c}{N_l}$, $l = 1, 2$ and $i \in \{1, \dots, N_o\}$.

B. Analytical Expression

Suppose that the seed fractions of all agents holding Opinion i in layer L_1 and L_2 are $o_1^0(i) = \rho_0$ and $o_2^0(i) = \rho_0+\epsilon$, respectively. Obviously, $\epsilon = 0$ represents that the seed fractions of Opinion i in two layers are identical, and $\epsilon > 0$ (< 0) represents that the seed fraction of Opinion i in layer L_2 is larger (smaller) than that in layer L_1. According to Eqs. (2) and (4), we can get

$$\begin{aligned} o_1^{t+1}(k_1,i) &= \eta_1 \Bigg(P_{1,u}(k_1)g_1^t(i) + \omega\sum_{k_2} P_c(k_1,k_2)g_1^t(i) \\ &\quad +(1-\omega)\sum_{k_2} P_c(k_1,k_2)g_2^t(i) \Bigg), \\ o_2^{t+1}(k_2,i) &= \eta_2 \Bigg(P_{2,u}(k_2)g_1^t(i) + \omega\sum_{k_1} P_c(k_1,k_2)g_1^t(i) \\ &\quad +(1-\omega)\sum_{k_1} P_c(k_1,k_2)g_2^t(i) \Bigg). \end{aligned} \tag{6}$$

Since $P_{1,c}(k_1) = \sum_{k_2} P_c(k_1,k_2)$ and $P_{2,c}(k_2) = \sum_{k_1} P_c(k_1,k_2)$, there are $p_1(k_1) = \eta_1\left(P_{1,u}(k_1) + P_{1,c}(k_1)\right)$ and $p_2(k_2)=\eta_2\left(P_{2,u}(k_2) + P_{2,c}(k_2)\right)$. Eq. (6) can be written as

$$\begin{aligned} o_1^{t+1}(k_l,i) &= p_1(k_1)g_1^t(i)+\eta_1(1-\omega)P_{1,c}(k_1)\left(g_2^t(i)-g_1^t(i)\right), \\ o_2^{t+1}(k_2,i) &= p_2(k_2)g_2^t(i)+\eta_2\omega P_{2,c}(k_2)\left(g_1^t(i)-g_2^t(i)\right), \end{aligned} \tag{7}$$

where $P_{l,c}(k_l)$ represents the probability of coupled agents having degree k_l in the aggregated network, and $l = 1, 2$.

Initially, opinions are distributed randomly among agents, which implies $o_l^0(k_l,i) = p_l(k_l)o_l^0(i)$. Substituting it into Eq. (7), at time step $t = 1$, we obtain

$$\begin{aligned} o_1^1(k_1,i) &= p_1(k_1)\rho_0 + P_{1,c}(k_1)\cdot\eta_1(1-\omega)\epsilon, \\ o_2^1(k_2,i) &= p_2(k_2)(\rho_0+\epsilon) - P_{2,c}(k_2)\cdot\eta_2\omega\epsilon. \end{aligned}$$

Then, at time step $t = 2$, we have

$$\begin{aligned} o_1^2(k_1,i) &= p_1(k_1)\rho_0 + p_1(k_1)\cdot\eta_1(1-\omega)c_1\epsilon \\ &\quad +P_{1,c}(k_1)\cdot\eta_1(1-\omega)[1-\eta_1(1-\omega)c_1-\eta_1\omega c_2]\epsilon, \\ o_2^2(k_2,i) &= p_2(k_2)(\rho_0+\epsilon) - p_2(k_2)\cdot\eta_2\omega c_2\epsilon \\ &\quad -P_{2,c}(k_2)\cdot\eta_2\omega[1-\eta_2(1-\omega)c_1-\eta_2\omega c_2]\epsilon, \end{aligned}$$

where $c_l = \frac{\sum_{k_l'} k_l' P_{l,c}(k_l')f(k_l')}{\sum_{k_l'} k_l' p_l(k_l')f(k_l')}$, $l = 1, 2$. Obviously, c_l represents the total influence of coupled agents in layer L_l.

By that analogy, we obtain the analytic expressions of $o_1^t(k_1,i)$ and $o_2^t(k_2,i)$, that is,

$$\begin{aligned} o_1^t(k_1,i) &= p_1(k_1)\rho_0+p_1(k_1)\sum_{s=2}^{t}\eta_1(1-\omega)c_1[1-\eta_1(1-\omega)c_1 \\ &\quad -\eta_2\omega c_2]^{s-2}\epsilon+P_{1,c}(k_1)\eta_1(1-\omega)[1-\eta_1(1-\omega)c_1 \\ &\quad -\eta_2\omega c_2]^{t-1}\epsilon, \\ o_2^t(k_2,i) &= p_2(k_2)(\rho_0+\epsilon)-p_2(k_2)\sum_{s=2}^{t}\eta_2\omega c_2[1-\eta_1(1-\omega)c_1 \\ &\quad -\eta_2\omega c_2]^{s-2}\epsilon+P_{2,c}(k_2)\eta_2\omega[1-\eta_1(1-\omega)c_1 \\ &\quad -\eta_2\omega c_2]^{t-1}\epsilon. \end{aligned} \tag{8}$$

From Eq. (8), we obtain that when the seed fraction of Opinion i in layer L_1 (i.e. ρ_0) is approximately equal to that of layer L_2 (i.e. $\rho_0+\epsilon$), that is $\epsilon \approx 0$, the final fraction of Opinion i approximates the seed fraction, regardless of the attributes of coupled agents in the network. However, if the seed fraction of Opinion i in layer L_1 (i.e. ρ_0) differs from that of layer L_2 (i.e. $\rho_0+\epsilon$), that is $\epsilon \neq 0$, the final fraction of Opinion i not only relies on the seed fraction, but also depends on network structures, especially the attributes of coupled agents, including inter-layer linking patterns and linking number.

C. Stability Analysis

Now we wonder whether the fraction of each opinion in two layers tends to a positive constant (i.e. fixed point) as $t \to +\infty$, and if the fixed points exist, what is the relationship between the fraction of one opinion in layer L_1 and that of corresponding one in layer L_2?

Substituting Eq. (5) into Eq. (2), we can obtain

$$\begin{aligned} O_{1,u}^{t+1}(k_1,i) &= p_{1,u}(k_1)\Bigg(\frac{\sum_{k_1'} k_1' O_{1,u}^t(k_1',i)f(k_1')}{\sum_{k_1'} k_1' p_1(k_1')f(k_1')} \\ &\quad +\frac{\sum_{k_1'}\sum_{k_2} k_1' O_c^t(k_1',k_2,i)f(k_1')}{\sum_{k_1'} k_1' p_1(k_1')f(k_1')}\Bigg) \\ O_{2,u}^{t+1}(k_2,i) &= p_{2,u}(k_2)\Bigg(\frac{\sum_{k_2'} k_2' O_{2,u}^t(k_2',i)f(k_2')}{\sum_{k_2'} k_2' p_2(k_2')f(k_2')} \\ &\quad +\frac{\sum_{k_2'}\sum_{k_1} k_2' O_c^t(k_1,k_2',i)f(k_2')}{\sum_{k_2'} k_2' p_2(k_2')f(k_2')}\Bigg), \end{aligned} \tag{9}$$

where $p_{l,u}(k_l)$ is the probability of uncoupled agent in layer L_l with degree k_l and $p_{l,u}(k_l) = \eta_l P_{l,u}(k_l)$, $l = 1, 2$.

Similarly, substituting Eq. (5) into Eq. (4), we can obtain

$$\begin{aligned} &O_c^{t+1}(k_1,k_2,i) \\ &\quad = \omega p_{1,c}(k_1,k_2)\Bigg(\frac{\sum_{k_1'} k_1' O_{1,u}^t(k_1',i)f(k_1')}{\sum_{k_1'} k_1' p_1(k_1')f(k_1')} \\ &\quad +\frac{\sum_{k_1'}\sum_{k_2} k_1' O_c^t(k_1',k_2,i)f(k_1')}{\sum_{k_1'} k_1' p_1(k_1')f(k_1')}\Bigg) \\ &\quad +(1-\omega)p_{2,c}(k_1,k_2)\Bigg(\frac{\sum_{k_2'} k_2' O_{2,u}^t(k_2',i)f(k_2')}{\sum_{k_2'} k_2' p_2(k_2')f(k_2')} \\ &\quad +\frac{\sum_{k_2'}\sum_{k_1} k_2' O_c^t(k_1,k_2',i)f(k_2')}{\sum_{k_2'} k_2' p_2(k_2')f(k_2')}\Bigg) \end{aligned} \tag{10}$$

where $p_{l,c}(k_1,k_2)$ is the probability that coupled agents in layer L_l have degree k_1 in layer L_1 and degree k_2 in layer L_2, and $p_{l,c}(k_1,k_2)=\eta_l P_c(k_1,k_2)$.

Let $S^t_{1,u}(i)=(O^t_{1,u}(k_1^1,i),\ldots,O^t_{1,u}(k_1^{\max},i))$, $S^t_{2,u}(i)=(O^t_{2,u}(k_2^1,i),\ldots,O^t_{2,u}(k_2^{\max},i))$, $S^t_c(i)=(O^t_c(k_1^1,k_2^1,i),\ldots,O^t_c(k_1^1,k_2^{\max},i),\ldots,\ O^t_c(k_1^{\max},k_2^1,i),\ldots,\ O^t_c(k_1^{\max},k_2^{\max},i))$, then $S^t(i)=(S^t_{1,u}(i),S^t_{2,u}(i),S^t_c(i))^\top$. According to Eqs. (9) and (10), we get

$$S^{t+1}(i)=\begin{bmatrix}R_{11} & \mathbf{0}_{|S_{k_1}|\times|S_{k_2}|} & R_{13}\\ \mathbf{0}_{|S_{k_2}|\times|S_{k_1}|} & R_{22} & R_{23}\\ R_{31} & R_{32} & R_{33}\end{bmatrix}S^t(i), \quad (11)$$

where $S_{k_1}=\{k_1^1,\ldots,k_1^{\max}\}$ and $S_{k_1}=\{k_2^1,\ldots,k_2^{\max}\}$ are degree set of layer L_1 and L_2, respectively, $|S_{k_1}|$ and $|S_{k_2}|$ are the cardinal numbers of set S_{k_1} and S_{k_2}, respectively, $\mathbf{0}_{|S_{k_1}|\times|S_{k_2}|}$ ($\mathbf{0}_{|S_{k_2}|\times|S_{k_1}|}$) is the $|S_{k_1}|\times|S_{k_2}|$ ($|S_{k_2}|\times|S_{k_1}|$) matrix with each entry being 0, $\mathbf{1}_{|S_{k_1}|}$ ($\mathbf{1}_{|S_{k_2}|}$) is a $|S_{k_1}|$-dimensional ($|S_{k_2}|$-dimensional) column vector with each entry being 1, $\otimes$ is the Kronecker product, $\top$ is the transpose of a vector or a matrix, $r_1, r_2, R_{11}, R_{22}, R_{31}, R_{32}, R_{13}, R_{23}, R_{33}$, as shown at the bottom of the page.

Thus, the characteristic equation is

$$\lambda^\alpha[(\lambda-r_3-\omega r_4)(\lambda-r_5-(1-\omega)r_6)-\omega(1-\omega)r_4r_6]=0 \quad (12)$$

where $\alpha=|S_{k_1}|+|S_{k_2}|+|S_{k_1}|\cdot|S_{k_2}|-2$, $r_3=\frac{\sum_{k_1'}k_1'p_{1,u}(k_1')f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}$, $r_4=\frac{\sum_{k_1'}\sum_{k_2'}k_1'p_c(k_1',k_2')f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}$, $r_5=\frac{\sum_{k_2'}k_2'p_{2,u}(k_2')f(k_2')}{\sum_{k_2'}k_2'p_2(k_2')f(k_2')}$ and $r_6=\frac{\sum_{k_1'}\sum_{k_2'}k_2'p_c(k_1',k_2')f(k_2')}{\sum_{k_2'}k_2'p_2(k_2')f(k_2')}$.

It follows from Eq. (12) that we only need to consider the eigenvalues of

$$f_1(\lambda)=(\lambda-r_3-\omega r_4)(\lambda-r_5-(1-\omega)r_6)-\omega(1-\omega)r_4r_6=0.$$

And we discover

$$\begin{cases}\Delta=(r_3+\omega r_4+r_5+(1-\omega)r_6)^2+4\omega(1-\omega)r_4r_6>0,\\ f(0)=r_3r_5+(1-\omega)r_3r_6+\omega r_4r_5>0,\\ 0<\frac{r_3+\omega r_4+r_5+(1-\omega)r_6}{2}<1,\\ f(1)=(1-r_3-\omega r_4)(1-r_5-(1-\omega)r_6)\\ \qquad -\omega(1-\omega)r_4r_6=0.\end{cases} \quad (13)$$

In fact, for the third equation of Eq. (13), $p_1(k_1')=p_{1,u}(k_1')+\sum_{k_2'}p_c(k_1',k_2')$ and $0<\omega<1$, so $r_3+\omega r_4=\frac{\sum_{k_1'}k_1'\left(p_{1,u}(k_1')+\omega\sum_{k_2'}p_c(k_1',k_2')\right)f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}<1$. In the same way, $r_5+(1-\omega)r_6<1$. Thus, $\frac{r_3+\omega r_4+r_5+(1-\omega)r_6}{2}<1$. For the fourth equation of Eq. (13), $1-r_3-\omega r_4=\frac{\sum_{k_1'}k_1'\left(p_{1,u}(k_1')+\sum_{k_2'}p_c(k_1',k_2')\right)f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}-\frac{\sum_{k_1'}k_1'p_{1,u}(k_1')f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}-\omega\frac{\sum_{k_1'}k_1'\sum_{k_2'}p_c(k_1',k_2')f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}=(1-\omega)\frac{\sum_{k_1'}\sum_{k_2'}k_1'p_c(k_1',k_2')f(k_1')}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')}=(1-\omega)r_4$. Similarly, $1-r_5-(1-\omega)r_6=\omega r_6$. Thus, $f(1)=0$.

Therefore, we can obtain $|\lambda|\le 1$ and Eq. (11) is stable, which indicates that the fraction of each opinion in two layers trends to a fixed point.

Furthermore, as opinions are stabilized, we can easily derive the following results, that is,

$$o^\infty_{1,c}(i)=o^\infty_{2,c}(i),\ o^\infty_{1,u}(i)=o^\infty_{2,u}(i),\ o^\infty_1(i)=o^\infty_2(i). \quad (14)$$

$$r_1=\frac{1}{\sum_{k_1'}k_1'p_1(k_1')f(k_1')},$$

$$r_2=\frac{1}{\sum_{k_2'}k_2'p_2(k_2')f(k_2')},$$

$$R_{11}=r_1\begin{bmatrix}p_{1,u}(k_1^1)k_1^1f(k_1^1) & \ldots & p_{1,u}(k_1^1)k_1^{\max}f(k_1^{\max})\\ \vdots & \ddots & \vdots\\ p_{1,u}(k_1^{\max})k_1^1f(k_1^1) & \ldots & p_{1,u}(k_1^{\max})k_1^{\max}f(k_1^{\max})\end{bmatrix},$$

$$R_{22}=r_2\begin{bmatrix}p_{2,u}(k_2^1)k_2^1f(k_2^1) & \ldots & p_{2,u}(k_2^1)k_2^{\max}f(k_2^{\max})\\ \vdots & \ddots & \vdots\\ p_{2,u}(k_2^{\max})k_2^1f(k_2^1) & \ldots & p_{2,u}(k_2^{\max})k_2^{\max}f(k_2^{\max})\end{bmatrix},$$

$$R_{31}=\omega r_1\begin{bmatrix}p_c(k_1^1,k_2^1)k_1^1f(k_1^1) & \ldots & p_c(k_1^1,k_2^1)k_1^{\max}f(k_1^{\max})\\ \vdots & \ddots & \vdots\\ p_c(k_1^{\max},k_2^{\max})k_1^1f(k_1^1) & \ldots & p_c(k_1^{\max},k_2^{\max})k_1^{\max}f(k_1^{\max})\end{bmatrix},$$

$$R_{32}=(1-\omega)r_2\begin{bmatrix}p_c(k_1^1,k_2^1)k_2^1f(k_2^1) & \ldots & p_c(k_1^1,k_2^1)k_2^{\max}f(k_2^{\max})\\ \vdots & \ddots & \vdots\\ p_c(k_1^{\max},k_2^{\max})k_2^1f(k_2^1) & \ldots & p_c(k_1^{\max},k_2^{\max})k_2^{\max}f(k_2^{\max})\end{bmatrix},$$

$$R_{13}=R_{11}\otimes\mathbf{1}^\top_{|S_{k_2}|},$$

$$R_{23}=\mathbf{1}^\top_{|S_{k_1}|}\otimes R_{22},$$

$$R_{33}=R_{31}\otimes\mathbf{1}^\top_{|S_{k_2}|}+\mathbf{1}^\top_{|S_{k_1}|}\otimes R_{32}$$

It is analogous to diffusion (mixture) of multiple gases in connected containers. The different gases will not stop diffusing until the fraction of each gas in one container is identical to that in another one.

III. Model With Mass Media

In the real world, besides being influenced by their neighbors, people may also be impacted by mass media, such as information conveyed by TV, and the radio or newspapers. In general, mass media is a global open source of information for people, which indicates that it may impact all agents' opinions [44]–[46]. Therefore, we set that mass media holds opinion $m \in \{1, \ldots N_o\}$ and the impact of mass media on agents is P_m $(0 < P_m < 1)$. Since P_m can be regarded as a measure of the intrinsic individual determination relative to the impact exerted by mass media, the large-P_m scenario implies a social environment with individuals subject to strong social pressure, while the small-P_m case represents a society characterized by loose social pressure.

In the aggregated network, the probability of uncoupled agents in layer L_l, $l = 1, 2$, having degree k_l and holding Opinion i $(i \neq m)$ is

$$O_{l,u}^{t+1}(k_l, i) = (1-P_m)O_{l,u}^{t}(k_l, j)\frac{\sum_{k_l'} p_l(k_l'|k_l)q_l^t(i|k_l')f(k_l')}{\sum_{k_l'}\sum_j p_l(k_l'|k_l)q_l^t(j|k_l')f(k_l')} + (1-P_m)O_{l,u}^{t}(k_l, i) \times \left(1-\frac{\sum_{k_l'} p_l(k_l'|k_l)(1-q_l^t(i|k_l'))f(k_l')}{\sum_{k_l'}\sum_j p_l(k_l'|k_l)q_l^t(j|k_l')f(k_l')}\right). \quad (15)$$

In this case, the agents are actually not influenced by mass media. Specifically, on the right-hand side of Eq. (15), the first term is the probability that uncoupled agents with degree k_l change their opinion from non-i to i and the second term is the probability that they stick to their Opinion i. Eq. (15) can be further simplified into

$$O_{l,u}^{t+1}(k_l, i) = (1 - P_m)P_{l,u}(k_l) \cdot g_l^t(i). \quad (16)$$

For uncoupled agents in layer L_l, $l = 1, 2$, holding Opinion $m(\in \{1, \ldots N_o\})$, the probability of them having degree k_l and Opinion m is

$$O_{l,u}^{t+1}(k_l, m) = O_{l,u}^{t}(k_l, j) \times \left[P_m + (1-P_m)\frac{\sum_{k_l'} p_l(k_l'|k_l)q_l^t(m|k_l')f(k_l')}{\sum_{k_l'}\sum_j p_l(k_l'|k_l)q_l^t(j|k_l')f(k_l')}\right] + O_{l,u}^{t}(k_l, m) \times \left[1-(1-P_m)\frac{\sum_{k_l'} p_l(k_l'|k_l)(1-q_l^t(m|k_l'))f(k_l')}{\sum_{k_l'}\sum_j p_l(k_l'|k_l)q_l^t(j|k_l')f(k_l')}\right]. \quad (17)$$

On the right-hand side of Eq. (17), the first term is the probability that uncoupled agents with degree k_l change their opinion from non-m to m, and in this process mass media impacts on uncoupled agents' opinion. The second term is the probability that they stick to their Opinion m. Then, Eq. (17) can be simplified as

$$O_{l,u}^{t+1}(k_l, m) = P_m P_{l,u}(k_l) + (1 - P_m)P_{l,u}(k_l) \cdot g_l^t(m). \quad (18)$$

Analogously, for coupled agents with degree k_1 in layer L_1 and degree k_2 in layer L_2 in the aggregated network, the probability of them holding Opinion i $(\neq m)$ is

$$O_c^{t+1}(k_1, k_2, i) = \omega(1-P_m)P_c(k_1, k_2)g_1^t(i) + (1-\omega)(1-P_m)P_c(k_1, k_2)g_2^t(i). \quad (19)$$

On the right-hand side of Eq. (19), the first term is the probability that coupled agents with degree k_1 in layer L_1 and degree k_2 in layer L_2 influenced by their neighbors in layer L_1 hold Opinion i, and the second term is the probability that coupled agents with degree k_1 in layer L_1 and degree k_2 in layer L_2 influenced by their neighbors in layer L_2 hold Opinion i. In this situation, we discover that coupled agents are not impacted by mass media.

And for coupled agents with degree k_1 in layer L_1 and degree k_2 in layer L_2, the probability of them holding Opinion m is

$$O_c^{t+1}(k_1, k_2, m) = P_m P_c(k_1, k_2) + \omega(1-P_m)P_c(k_1, k_2)g_1^t(i) + (1-\omega)(1-P_m)P_c(k_1, k_2)g_2^t(i). \quad (20)$$

On the right-hand side of Eq. (20), the first term is the probability that coupled agents with degree k_1 in layer L_1 and degree k_2 in layer L_2 influenced by mass media hold Opinion m, the second term is the probability that coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2 influenced by their neighbors in layer L_1 hold Opinion m, and the third term is the probability that coupled agents with degree k_1 in layer L_1 and k_2 in layer L_2 influenced by their neighbors in layer L_2 hold Opinion m.

Comparing Eqs. (16) with (2), and Eqs. (19) with (4), we can obtain that mass media will restrain the spread of opinions that are different from its own. Similarly, comparing Eqs. (18) with (2) and Eqs. (20) with (4), we can observe that mass media will promote the spreading of opinions that are in accordance with its own.

IV. Numerical Simulation

To confirm the previous analytical prediction, we proceed with numerical simulations in two-layer interconnected networks with the size of each layer being $N_1 = N_2 = 1000$. The average node degree of each independent layer is approximately 4. Consider three topologies: ER-ER networks with each layer being independently formed by the algorithm proposed by Erdös and Rényi for generating random graphs [47], WS-WS networks with each layer being independently formed by the algorithm proposed by Watts and Strogatz for generating small-world graphs [48], and BA-BA networks formed by the algorithm proposed by Barabási and Albert for generating scale-free networks [49]. Specifically, in the ER graphs, each link is included with a probability $p = 0.004$. In the WS graphs, we start with a nearest-neighbor network, where each node is connected to its 4 nearest neighbors, then rewire one end of each link with a probability of $p = 0.05$ and connect

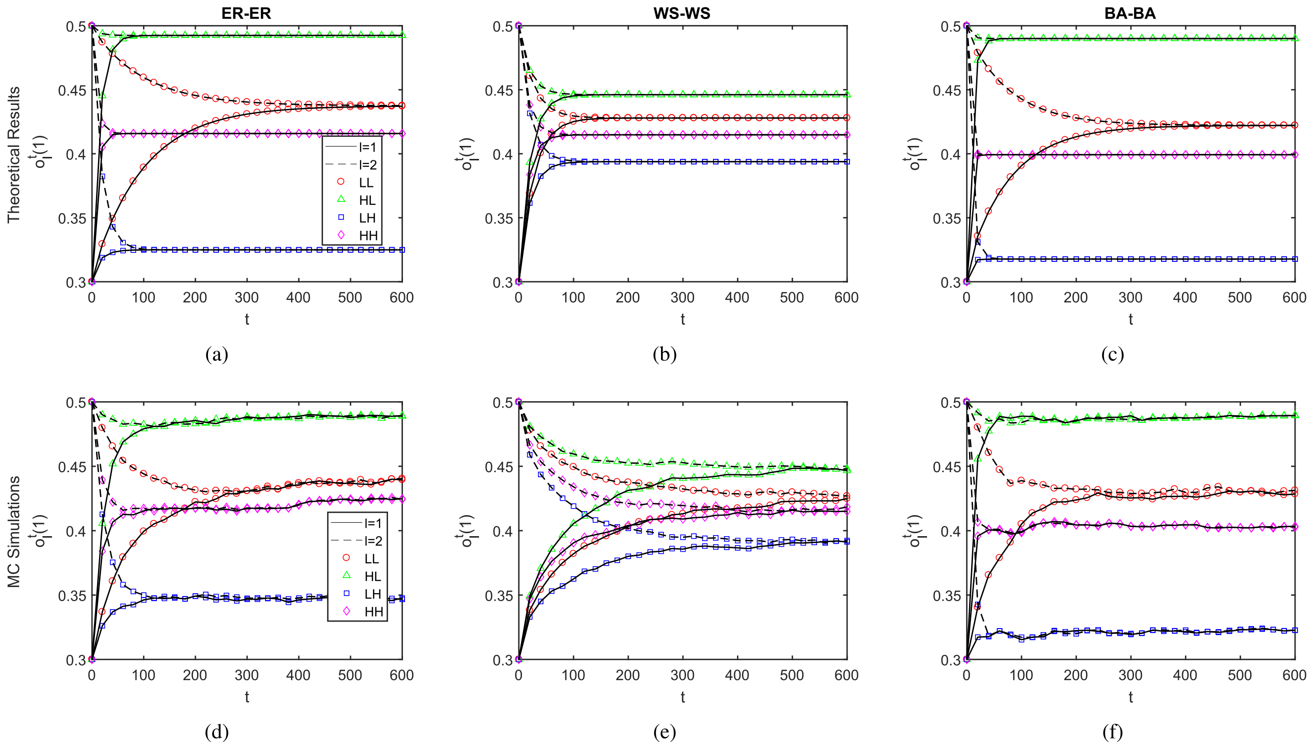


Fig. 2. The validity of the general model. From left to right, the panels are the evolutions of Opinion 1 in two-layer ER-ER (left), WS-WS (middle) and BA-BA (right) networks. The top panels display the theoretical results from Eq. (8), and the bottom panels are the corresponding results obtained from MC simulations. We set the seed fractions $o_1^0(1) = 0.3$ and $o_2^0(1) = 0.5$, and the linking number $N_c = 50$. The solid and dashed curves represent the evolution of Opinion 1 in layer L_1 and L_2, respectively. The circle, triangle, square and rhombus markers represent the LL, HL, LH and HH inter-layer linking patterns, respectively.

it to a new node randomly selected from the network. In the BA graphs, we start with a fully connected network of 100 nodes. Then, we sequentially add the remaining nodes, where each node is connected to 2 existing nodes with probability that is proportional to the number of links that the existing nodes already have.

In a two-layer interconnected network, coupled agents may have diverse attributes. For example, a coupled agent may be a hub or a peripheral node in a layer. Therefore, if coupled agents have high (low) degrees in layer L_1 and high (low) degrees in layer L_2, we call this case the HH (LL) inter-layer linking pattern, whereas if coupled agents have high (low) degrees in layer L_1 and low (high) degrees in layer L_2, we call this case the HL (LH) linking pattern. Specifically, since coupled agents hold the same opinion in two layers, in the HH (LL) linking pattern, we suppose that agents holding Opinion i in each layer are ordered by decreasing (increasing) degrees and the first N_c agents are coupled agents. Similarly, in the HL (LH) linking pattern, agents having Opinion i in layer L_1 and L_2 are ordered by decreasing (increasing) and increasing (decreasing) degrees, respectively.

For simplicity, we discuss the case that there are two opinions spreading ($N_o = 2$) in two-layer interconnected networks. Initially, Opinions 1 and 2 are randomly seeded in all the agents and all coupled agents hold Opinion 1. Since $o_l^t(1) + o_l^t(2) = 1$, we only need to know the dynamics of one term. Therefore, we will focus on the fraction of Opinion 1 and omit that of Opinion 2. Furthermore, we set the parameter in the persuasiveness function $f(k_l)$ as $\alpha = 0.5$ and the weight of influence on coupled agents from their neighbors in layer L_1 as $\omega = 0.4$. Monte Carlo (MC) simulations are employed and repeated 100 times for each fixed two-layer interconnected network with 20 network generations randomly to calculate the final prevalence of each opinion and compare with the theoretical results.

Note that if some of the coupled agents hold Opinion 1 and the others hold Opinion 2, there could be 16 (i.e. 4^2) kinds of inter-layer linking cases in two-layer interconnected networks. These cases could cause more complicated spreading process of opinions, which is adverse to analyse opinion diffusion. Therefore, we assume that all coupled agents hold Opinion 1.

A. The Validity of the Model

First, we verify the correctness of the theoretical results of final prevalence of each opinion in two-layer interconnected networks. Fig. 2 shows the evolutions of Opinion 1 in ER-ER (left), WS-WS (middle) and BA-BA (right) networks, where the seed fractions $o_1^0(1) = 0.3$ and $o_2^0(1) = 0.5$, and the number of coupled agents is $N_c = 50$ (only 50 agents are active in both layers L_1 and L_2). It is obvious that the MC simulations are basically in agreement with the theoretical results, although there are some discrepancies, which should be due to randomness caused by the MC simulation. Furthermore, we can observe that the final prevalence (fraction) of Opinion 1 in layer L_1 is identical with that in layer L_2, which

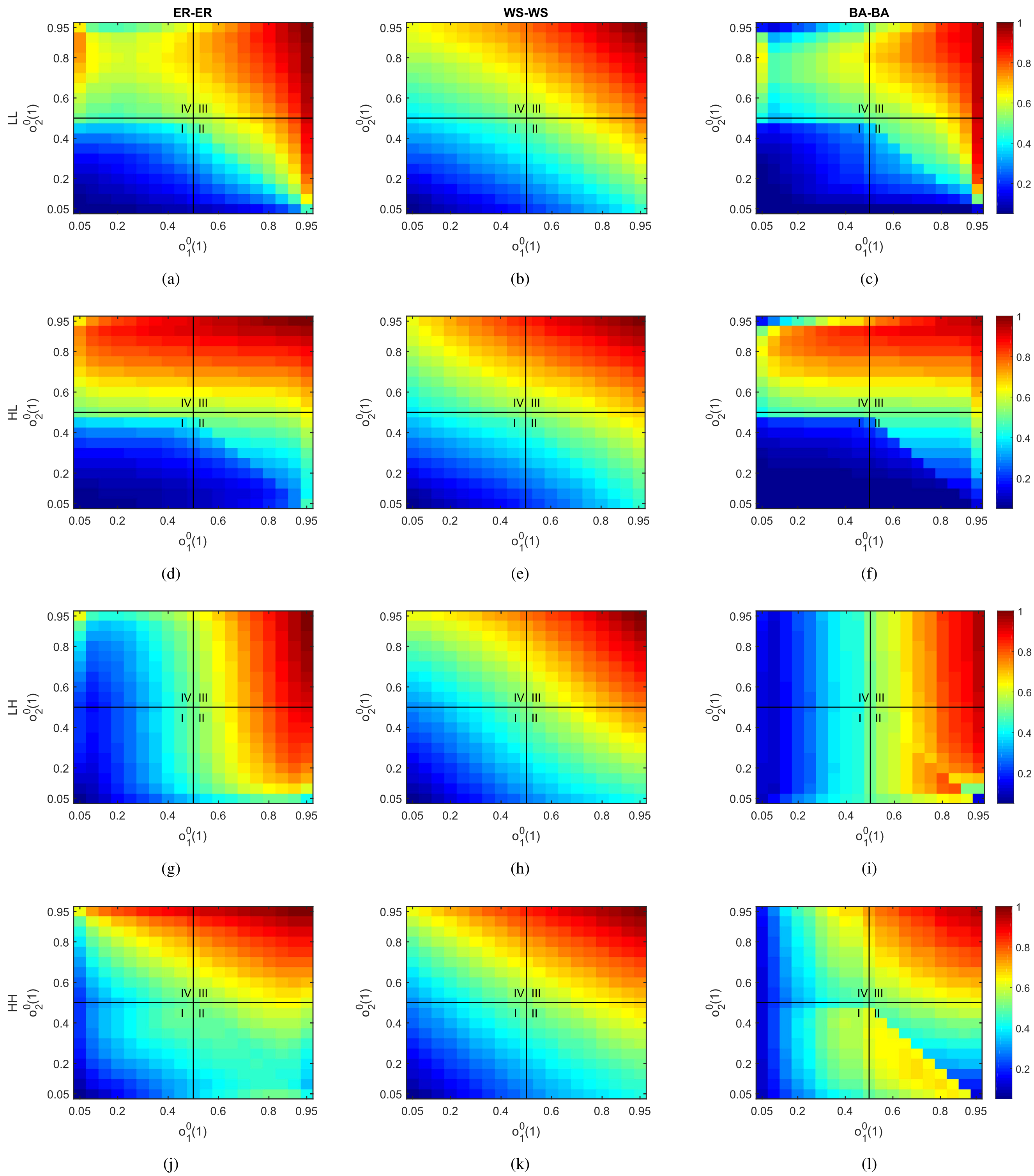


Fig. 3. The impact of the seed fractions of Opinion 1 on the final fraction of Opinion 1. Specifically, with the seed fractions of Opinion 1 varying from 0.05 to 0.95 with step 0.05, the panels from top to bottom are the final fraction of Opinion 1 in the LL (top), HL (top-middle), LH (bottom-middle) and HH (bottom) linking patterns on two-layer ER-ER (left), WS-WS (middle), and BA-BA (right) networks. We set the number of coupled agents $N_c = 50$. The color represents the value of the final fraction of Opinion 1 ($o_1^\infty(1)$ or $o_2^\infty(1)$) as shown in the color bars.

coincide with Eq. (14). In addition, we observe that it takes the least time for opinions to reach a steady state for the HH linking pattern, and it takes the longest time for opinions to get stabilized for the LL pattern, and the other two linking patterns are in between. The phenomenon is more obvious for the two-layer BA-BA networks than for the ER-ER or WS-WS networks. It should be due to the reason that nodes in BA networks are heterogenous whereas nodes in ER or WS

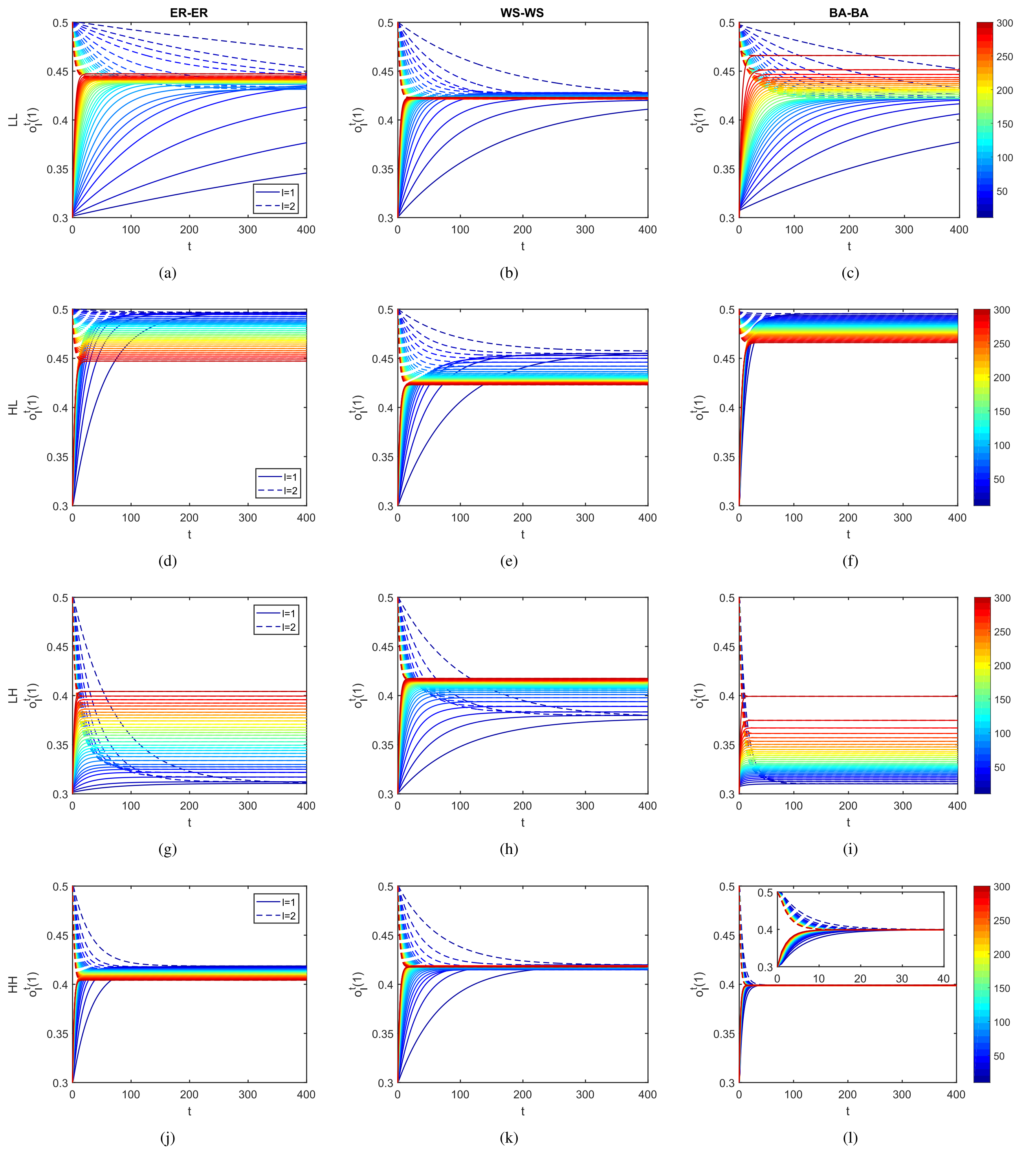


Fig. 4. The impact of linking number on the fractions of Opinion 1. From top to bottom, the panels are the evolutions of Opinion 1 in the LL (top), HL (top-middle), LH (bottom-middle) and HH (bottom) linking patterns on ER-ER (left), WS-WS (middle), and BA-BA (right) networks. We set the seed fractions of Opinion 1 in layer L_1 and L_2 as $o_1^0(1) = 0.3$ and $o_2^0(1) = 0.5$. The color of each curve represents the value of the inter-layer linking number N_c as shown in the color bars. The solid and dashed curves represent the evolution of Opinion 1 in layer L_1 and L_2, respectively.

networks are homogeneous. Thus, different inter-layer linking patterns in BA-BA networks will result in a more obviously different result than that in the other two.

B. The Impact of Seed Fractions

Next, we explore how the seed fractions of Opinion 1 in the two layers, $o_1^0(1)$ and $o_2^0(1)$, impact the final prevalence of Opinion 1 in two-layer interconnected networks. In Fig. 3, we vary the seed fractions of Opinion 1 from 0.05 to 0.95, and divide the plane $(0.05, 0.95) \times (0.05, 0.95)$ into four regions, that is, Region I ($(0.05, 0.5) \times (0.05, 0.5)$), Region II ($(0.5, 0.95) \times (0.05, 0.5)$), Region III ($(0.5, 0.95) \times (0.5, 0.95)$) and Region IV ($(0.05, 0.5) \times (0.5, 0.95)$). On one hand, when the seed fractions of Opinion 1 are in Region I, its final

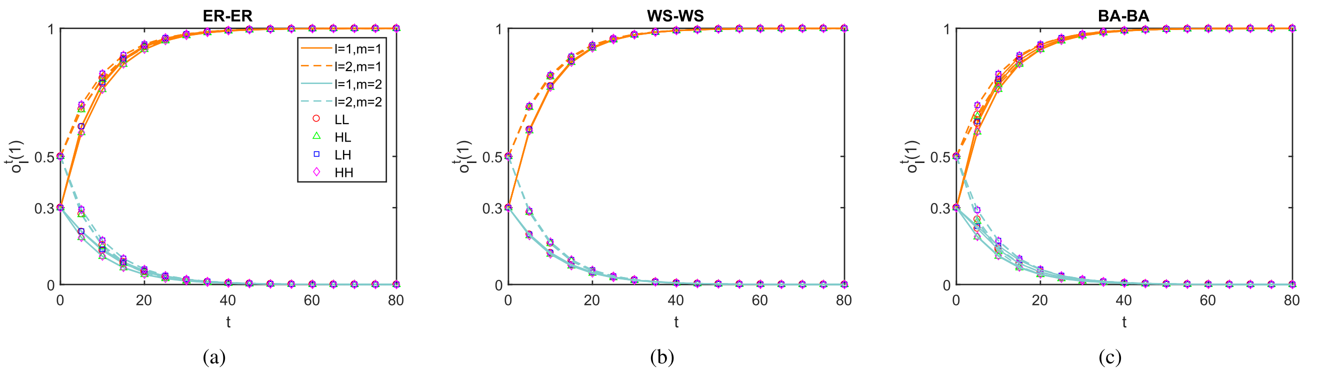


Fig. 5. The impact of mass media on the fractions of Opinion 1. From left to right, the panels are the evolutions of Opinion 1 in two-layer ER-ER (left), WS-WS (middle), and BA-BA (right) networks with the impact of mass media for $P_m = 0.1$. The orange color represents that mass media hold Opinion 1 (that is, $m = 1$), whereas the celeste color represents that mass media hold Opinion 2 (that is, $m = 2$). We set the seed fractions $o_1^0(1) = 0.3$ and $o_2^0(1) = 0.5$, and the linking numbers $N_c = 50$. The solid and dashed curves represent the evolution of Opinion 1 in layer L_1 and L_2, respectively. The circle, triangle, square and rhombus markers represent the LL, HL, LH and HH linking patterns.

fraction is the largest with the HH linking pattern, whereas when the seed fractions are in Region III, the final prevalence of Opinion 1 change slightly for the four different linking patterns. These phenomena reveal that when Opinion 1 are not dominant in either layer (see Region I), coupled agents having high degrees in the two layers are more favorable for the spread of Opinion 1, whereas when Opinion 1 are dominant in both layers (see Region III), the degrees of coupled agents have little impact on the final prevalence of this opinion. Indeed, when a cultural fad or a new product are not of high popularity, celebrities (agents with larger degrees) definitely have greater influence on the spread of them than ordinary people, that is celebrity charm. However, when a culture fad or a new product has become very popular, celebrities might play little role in their spread, that is why some popular products are rarely advertised but they still sell well.

On the other hand, from Fig. 3, we can also observe that when the seed fractions of Opinion 1 are in Region II, the final fraction of the opinion is the largest in the LH linking pattern. Likewise, the final fraction of Opinion 1 is the largest in the HL linking pattern when the seed fractions of Opinion 1 are in Region IV. These phenomena indicate that, if the coupled agents have low degrees in the layer where the seed fraction of the opinion is dominant and have high degrees in the layer where the seed fraction of the opinion is subordinate, the spread of Opinion 1 can be faster than any other case. This result seems to be paradoxical but can explain some social phenomena. Take online celebrities for example. Online celebrities have hundreds of thousands of online fans who are usually civilians, but they may have few friends in the circle of the rich. And luxury goods are popular among the rich, but not so attractive to civilians. Suppose that the group of individuals including the rich and the online celebrities as layer L_1, and the group of civilians and celebrities as layer L_2. Here the online celebrities can be regarded as coupled agents, and the will to buy luxury goods as Opinion 1. According to our result, online celebrities can promote the sale of luxury goods, which coincides with the fact of "online celebrity economy". Moreover, we discover that the tendency of the final fractions of Opinion 1 differ in the three types of two-layer interconnected networks. Particularly, the tendency of the final prevalence of Opinion 1 is approximately the same for the four linking patterns in WS-WS networks, which implies that the inter-layer linking pattern has little impact on diffusion in WS-WS networks.

C. The Impact of Linking Number

In the following, we vary the number of coupled agents N_c, from 10 to 300, and display the evolutions of fractions of Opinion 1, where the color of each curve represents the value of the inter-layer linking number as shown by the color bars. It is obvious that with the number of coupled agents increasing, it takes less time for the opinion to reach a steady state, regardless of network structures and the inter-layer linking patterns. Furthermore, with the number of coupled agents rising, the final fraction of Opinion 1 decreases in the HL linking pattern, while it increases in the LH linking pattern. When coupled agents have low degrees in one layer and high degrees in the other layer, with the increase of the number of coupled agents, the final prevalence of Opinion 1 will tend to the fraction of Opinion 1 of the layer with coupled agents having high degrees. This partially reflects that coupled agents with high degrees play significant role in determining the prevalence of Opinion 1. But, it is not always the best way to improve the spread of an opinion by coupling agents with high degrees in the two layers (see Region IV of panels (d) and (j) in Fig. 3), which is an intriguing result. Additionally, we can observe that in each kind of two-layered interconnected networks, the HH linking pattern basically leads to the fastest stabilization, while the LL leads to the slowest, especially when there are few coupled agents.

D. The Impact of Mass Media

Finally, we discuss the impact of mass media on the dynamics of opinion diffusion. From Fig. 5, we observe that all agents finally reach consensus and take mass media's opinion, regardless of the linking patterns or network structures.

In particular, all agents finally hold Opinion 2, even if coupled agents hold Opinion 1 but the mass media holds Opinion 2. The phenomenon is in agreement with the analysis we get in Sec IV, which further indicates that mass media plays a pivotal role in determining the final prevalence of opinions. It also provides hints for us to deal with some practical problems. Take the spread of rumors for example. When rumors are prevalent, mass media can stand out to suppress the spread of rumors by refuting the rumor or broadcasting the right information.

V. Conclusion and Discussions

In the paper, we propose an opinion diffusion model to describe a practical and generic diffusion process in two-layer interconnected networks. Our results demonstrate that when the seed fractions of opinions in two layers differ from each other, the final prevalence of different opinions not only relies on the seed fractions, but also depends on the network structures, especially the inter-layer linking patterns and linking number. Furthermore, mass media facilitates the spread of the opinion that is the same with its own. In particular, we explore the spread of two opinions in three types of two-layer interconnected networks under all coupled agents holding the same opinion. We discover that coupled agents, who have low degrees in the layer that the seed fraction of an opinion is dominant and have high degrees in the layer that the seed fraction of the same opinion is subordinate, will favor diffusion of the opinion. In this case, with the increase of the number of coupled agents, the final prevalence of the opinion will get closer to the seed fraction of the opinion in the layer with coupled agents having high degrees. The findings reported here are of interest to researchers seeking to better understand the diffusion of various opinions in multi-layer networks, as well as to practitioners (governors) seeking guidance to orientate the prevalence of a special opinion in the real world.

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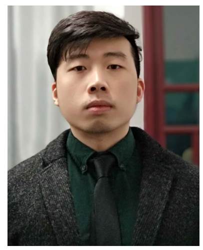

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