



Bifurcation Analysis and Optimal Harvesting of a Delayed Predator–Prey Model

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A delay predator–prey model is formulated with continuous threshold prey harvesting and Holling response function of type III. Global qualitative and bifurcation analyses are combined to determine the global dynamics of the model. The positive invariance of the non-negative orthant is proved and the uniform boundedness of the trajectories. Stability of equilibria is investigated and the existence of some local bifurcations is established: saddle-node bifurcation, Hopf bifurcation. We use optimal control theory to provide the correct approach to natural resource management. Results are also obtained for optimal harvesting. Numerical simulations are given to illustrate the results.

Keywords: Delay; predator–prey; optimal harvesting; optimal control; bifurcation.

1. Introduction

In this paper we consider a system of delay differential equations modeling the predator–prey dynamics with continuous threshold prey harvesting and Holling response function of type III.

The predator–prey model has received much attention due to its practical importance and also due to the rich dynamics observed in the

corresponding mathematical models (see for example [Bohn *et al.*, 2011; Etoua & Rousseau, 2010; Ji & Wu, 2009; Leard *et al.*, 2008; Martin & Ruan, 2001; Gopalsamy, 1992; Kuang, 1993] and the references therein). Profit, overexploitation and extinction of a species being harvested are primary concerns in ecology and commercial harvesting industries.

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Harvesting has generally a strong impact on the population dynamics of a harvested species. The severity of this impact depends on the nature of the implemented harvesting strategy. The study of population dynamics with harvesting is a subject of mathematical bio-economics, and it is related to the optimal management of renewable resources.

Renewable resources management is complicated and constructing accurate mathematical models on the effect of harvesting on animal populations is even more complicated. This is so because to have a perfect model we would have to take into account many factors having an effect on the cost-benefit criterion and on the survival of the harvested population. According to [Clark, 1990], the management of renewal resources is based on the notion of *maximum sustainable yield* (MSY) of harvesting; MSY is the maximum harvesting compatible with survival. Hence, if the harvesting of population exceeds its MSY (i.e. the population is overexploited), then this population will become extinct. Therefore, the best we can do is to look for analyzable models that describe as well as make possible the reality or the effect of harvesting on populations. Most predator-prey models consider either constant or linear harvesting functions ([Etoua & Rousseau, 2010; Ji & Wu, 2009; Leard *et al.*, 2008] and the references therein).

In the literature, one of the predator-prey model most commonly used is the *generalized Gause-type predator-prey model with harvesting of prey* [Gause, 1935]. It has the form

$$\begin{cases} \dot{x}(t) = \varphi(x(t)) - my(t)p(x(t)) - H(x(t)), \\ \dot{y}(t) = [-d + cmp(x(t))]y(t), \end{cases} \quad (1)$$

where x and y represent the population of preys and predators, respectively. d is the natural mortality rate of the predators. c and m are positive constants. The function

$$\varphi(x) = rx \left(1 - \frac{x}{K}\right), \quad (2)$$

models the behavior of preys in the absence of predators: r is the growth rate of preys when x is small, while K is the capacity of the environment to support the preys. The functions $H(x)$ and $p(x)$ are the harvesting function of the preys and the response function of predators to preys, respectively.

On the other hand, time delays of one type or another have been incorporated into biological

models by many researchers, we refer to [Gopal-samy, 1992; Kuang, 1993] for the general delayed biological systems; [Fan *et al.*, 2009] and the references cited therein for studies on delayed predator-prey systems.

Basically a constant time delay can be incorporated into the model in three different ways [Martin & Ruan, 2001]. In this paper, a time delay τ is in the predator response term $p(x(t))$ in the predator equation, that is,

$$\begin{cases} \dot{x}(t) = \varphi(x(t)) - my(t)p(x(t)) - H(x(t)), \\ \dot{y}(t) = [-d + cmp(x(t - \tau))]y(t). \end{cases} \quad (3)$$

This delay can be regarded as a gestation period or reaction time of the predators.

Bohn *et al.* [2011] considered the generalized Gause-type predator-prey model (1) with continuous threshold policy harvesting functions on the prey and a Holling response function of type II. One of the continuous threshold policy harvesting functions on the prey is considered in [Bohn *et al.*, 2011]:

$$H(x) = \begin{cases} 0 & \text{if } x < T, \\ \frac{h(x - T)}{h + x - T} & \text{if } x \geq T, \end{cases} \quad (4)$$

where T is the threshold value. In this way, once the prey population reaches the size $x = T$, then harvesting starts and increases smoothly to a limit value h .

Etoua *et al.* [Etoua & Rousseau, 2010] considered the generalized Gause-type predator-prey model (1) with constant policy harvesting functions on the prey and a Holling response function of type III [Bazykin, 1998]. That is,

$$p(x) = \frac{x^2}{ax^2 + bx + 1}, \quad (5)$$

where a is positive constant and b is non-negative constant. This function is one of the potential response functions of predators to preys, modeling the consumption of preys by predators. It reflects very small predation when the number of preys is small ($p'(0) = 0$), and group advantage for the preys when their number is high ($p(x)$ tends to $\frac{1}{a}$ when x tends to infinity).

The objective of this paper is to study the combined effects of continuous threshold policy harvesting function on the prey [see Eq. (4)] and a Holling response function of type III [see Eq. (5)] into the delay differential equation (3).

System (3) is considered with the following initial values

$$x(\theta) = \phi_0(\theta) \geq 0, \quad \theta \in [-\tau, 0] \quad \text{and} \\ y(0) = y_0 \geq 0.$$

This paper is organized as follows. In Sec. 2 we show that the positive orthant is possessively invariant with respect to the semi-flow of system (3) and every solution of (3) is uniformly bounded. Section 3 is devoted to the equilibria of model (3). Stability analysis of equilibria of model (3) is investigated in Secs. 4 and 5. In Sec. 6, we find some bifurcations present in the model. In Sec. 7 we analyze the optimal harvesting.

In the following, we will make use of the following assumption.

Assumption 1. We assume that:

- (i) $0 < x(0) \leq K$.
- (ii) $T \leq K$.

Let us give some comments on Assumption 1. (i) It is not plausible (the ecological point of view) to have an initial value of the preys $x(0)$ at time $t = 0$ which is greater than the carrying capacity K of the preys. (ii) From item (i) of Assumption 1, we can find that $x(t) \leq K$ for all $t \geq 0$ (see proof of Theorem 1). Thus, if we assume that $K < T$, then the harvesting function H [see Eq. (4)] leads to $H(x(t)) = 0$ for all $x(t) \leq K$. Since $x(t) \leq K$ for all $t \geq 0$, we have $H(x(t)) = 0$ for all $t \geq 0$. That is, there is no harvesting of the preys in system (3).

Let us set

$$d_1 = \frac{d}{cm}. \tag{6}$$

2. Boundedness of Solutions

We start by showing that solutions of system (3) that start in \mathbb{R}_+^2 will remain there and are uniformly bounded. Indeed, we have the following theorem.

Theorem 1. *Let Assumption 1(i) hold. Then, every solution of system (3) that starts in \mathbb{R}_+^2 will remain there and is uniformly bounded.*

Proof. Let $(x_0, y_0) \in \mathbb{R}_+^2$ be given and let us denote for each $t \geq 0$, $(x(t), y(t))$ the orbit of model (3) passing through (x_0, y_0) at $t = 0$. Then, we find that $(x(t), y(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$. Thus, every solution of system (3) that starts in \mathbb{R}_+^2 will remain

there. From the \dot{x} -equation of system (3), we have

$$\dot{x}(t) \leq rx(t) \left(1 - \frac{x(t)}{K} \right).$$

Applying a differential inequality [Hale, 1980], we get

$$x(t) \leq \frac{1}{\frac{1}{K} + \left(\frac{1}{x(0)} - \frac{1}{K} \right) e^{-rt}}$$

for all $t \geq 0$. Since $0 < x(0) \leq K$ [Assumption 1(i)], it follows that $x(t) \leq K$ for all $t \geq 0$. Now, let us check for the boundedness of $y(t)$. If $y(0) = 0$, then the \dot{y} -equation of model (3) leads to $y(t) = 0$ for all $t \geq 0$. If not (i.e. $y(0) > 0$), then $y(t) > 0$ for all $t \geq 0$. Using the fact that the function p defined by Eq. (5) is such that $p(s) \leq \frac{1}{a}$ for all $s \geq 0$; the \dot{y} -equation of system (3) becomes $\frac{dy(t)}{y(t)} \leq (\frac{cm}{a} - d)dt$. Integrating the preceding differential inequality from $t - \tau$ to t , we have

$$y(t) \leq \mu(\tau)y(t - \tau); \quad \text{for all } t \geq \tau, \tag{7}$$

where $\mu(\tau) = e^{-(d - \frac{cm}{a})\tau}$.

Let us set $u(t) = c\mu(\tau)x(t - \tau) + y(t)$ for all $t \geq \tau$. Recalling inequality (7), then for all β ($0 < \beta < d$) we get $\dot{u}(t) + \beta u(t) \leq \frac{c\mu(\tau)K}{4r\beta}(r + \beta)^2$, for all $t \geq \tau$. Integrating the last differential inequality from τ to t , we have

$$c\mu(\tau)x(t - \tau) + y(t) \\ \leq \frac{c\mu(\tau)K}{4r\beta}(r + \beta)^2(1 - e^{-\beta(t-\tau)}) \\ + [c\mu(\tau)x(0) + y(\tau)]e^{-\beta(t-\tau)};$$

for all $t \geq \tau$. Thus, $y(t) \leq \frac{c\mu(\tau)K}{4r\beta}(r + \beta)^2$ as $t \rightarrow \infty$. Therefore, every solution of (3), starting in \mathbb{R}_+^2 , is uniformly bounded. ■

3. Equilibrium of System (3)

The following proposition holds, concerning equilibria of system (3).

Proposition 1

- (1) *If $ad_1 \geq 1$ then $E_0(0, 0)$ and $F_0(x_0, 0) \in \mathbb{R}_+^* \times \{0\}$ are the equilibria of system (3) (the predator free equilibria). Equilibrium E_0 (resp., F_0) is the predator free equilibrium when the number*

of prey is less than (resp., greater than or equal to) the threshold value T .

- (2) If $ad_1 < 1$ then equilibria of the model are $E_0(0, 0)$; $F_0(x_0, 0) \in \mathbb{R}_+^* \times \{0\}$ and $E(x^*, y^*) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ (the predator free equilibrium and the coexistence equilibrium),

where

$$x^* = \frac{bd_1 + \sqrt{(bd_1)^2 + 4d_1(1 - ad_1)}}{2(1 - ad_1)}; \tag{8}$$

and

$$y^* = \frac{\varphi(x^*) - H(x^*)}{md_1}; \tag{9}$$

with $x^* \in]0, T[$ (resp., $x^* \in [T, x_0[$) for the equilibrium E^* (resp., F^*) and $x_0 \in [T, K]$ is the unique positive real root on $[T, K]$ of the following equation:

$$rx \left(1 - \frac{x}{K}\right) = \frac{h(x - T)}{h + x - T}.$$

Proof. An equilibrium (x_e, y_e) of system (3) is the solution of system (10) [resp., system (11)] if $x_e < T$ (resp., $x_e \geq T$)

$$\begin{cases} rx_e \left(1 - \frac{x_e}{K}\right) - mp(x_e)y_e = 0, \\ cmp(x_e)y_e - dy_e = 0, \end{cases} \tag{10}$$

resp.,

$$\begin{cases} rx_e \left(1 - \frac{x_e}{K}\right) - mp(x_e)y_e \\ - \frac{h(x_e - T)}{h + x_e - T} = 0, \\ cmp(x_e)y_e - dy_e = 0. \end{cases} \tag{11}$$

It is easily found that $E_0(0, 0)$ and $E_K(K, 0)$ (resp., $F_0(x_0, 0) \in \mathbb{R}_+^* \times \{0\}$) are always solutions of system (10) [resp., system (11)], where $x_0 \in [T, K]$ is the unique positive real root on $[T, K]$ of the following equation:

$$rx \left(1 - \frac{x}{K}\right) = \frac{h(x - T)}{h + x - T}.$$

Let us note that $E_K(K, 0)$ is an equilibrium of system (3) if $K < T$, which is a contradiction with Assumption 1.

If $ad_1 < 1$, then equation $cmp(x_e) - d = 0$ has unique positive real root x^* defined by (8), from where we obtain the value of y^* defined by (9). For

$E(x^*, y^*)$ to be a coexisting equilibrium, in addition to $ad_1 < 1$, we also need $x^* \in]0; x_0[$. ■

4. Stability Analysis of Model (3) for $\tau = 0$

The general Jacobian matrix of model (3) for $x < T$ and $\tau = 0$ is

$$J_0(x, y) = \begin{pmatrix} \varphi'(x) - myp'(x) & -mp(x) \\ mcy p'(x) & -d + mcp(x) \end{pmatrix}.$$

The eigenvalues of J_0 at equilibrium E_0 are $r > 0$ and $-d$. So, E_0 is always unstable (a saddle type).

The general Jacobian matrix of model (3) for $x \geq T$ and $\tau = 0$ is

$$J(x, y) = \begin{pmatrix} \varphi'(x) - mp'(x)y - H'(x) & -mp(x) \\ cmp'(x)y & cmp(x) - d \end{pmatrix}.$$

Thus at equilibrium F_0 we find that the eigenvalues of $J(x_0, 0)$ are $\lambda_1 = \varphi'(x_0) - H'(x_0)$ and $\lambda_2 = cmp(x_0) - d$. Thus, F_0 is asymptotically stable if and only if $(\varphi'(x_0) - H'(x_0) < 0$ and $p(x_0) - d_1 < 0)$ and F_0 is unstable if and only if $(\varphi'(x_0) - H'(x_0) \geq 0$ or $p(x_0) - d_1 \geq 0)$. More precisely;

- (a) F_0 is a saddle if $\varphi'(x_0) - H'(x_0) < 0$ and $p(x_0) - d_1 > 0$.
- (b) F_0 is unstable node if $\varphi'(x_0) - H'(x_0) > 0$ and $p(x_0) - d_1 > 0$.
- (c) F_0 is stable node if $\varphi'(x_0) - H'(x_0) < 0$ and $p(x_0) - d_1 < 0$.
- (d) F_0 is stable (resp., unstable) improper node if $(\varphi'(x_0) - H'(x_0) < 0; p(x_0) - d_1 = 0)$ or $(\varphi'(x_0) - H'(x_0) = 0; p(x_0) - d_1 < 0)$ [resp., $(\varphi'(x_0) - H'(x_0) > 0; p(x_0) - d_1 = 0)$ or $(\varphi'(x_0) - H'(x_0) = 0; p(x_0) - d_1 > 0)$].

We refer to Tables 1 and 2 for the sign of $\varphi'(x_0) - H'(x_0)$ and $p(x_0) - d_1$, where

$$x_u = \frac{K}{2} \left[1 - \frac{H' \left(\frac{K}{2} \right)}{r} \right]. \tag{12}$$

The stability of the coexisting equilibrium $E(x^*, y^*)$ of model (3) for $\tau = 0$ is given by Theorem 2.

Theorem 2. *Let*

$$\Delta_E = [\varphi'(x^*) - mp'(x^*)y^* - H'(x^*)]^2 - 4mdp'(x^*)y^*. \tag{13}$$

Table 1. Sign of $\varphi'(x) - H'(x)$ on $[T, K]$.

Region	Sign of $\varphi'(x) - H'(x)$
$T \geq \frac{K}{2}$	$\varphi'(x) - H'(x) < 0$ for all $x \in [T, K]$
$T < \frac{K}{2}$ and $\frac{h^2}{\left(h + \frac{k}{2} - T\right)^2} \geq r \left(1 - \frac{2T}{K}\right)$	$\varphi'(x) - H'(x) < 0$ for all $x \in [T, K]$
$T < \frac{K}{2}$ and $\frac{h^2}{\left(h + \frac{k}{2} - T\right)^2} < r \left(1 - \frac{2T}{K}\right)$	(i) $\varphi'(x) - H'(x) < 0$ for all $x \in [x_u, K]$ (ii) for $x \in [T, x_u[$, $\varphi'(x) - H'(x)$ can be negative or not.

The stability of equilibrium $E(x^*, y^*)$ for $\tau = 0$ is given as follows:

- (a) If $\Delta_E \geq 0$ then E is a node. The node is stable if $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) < 0$ and unstable if the inequality is reversed.
- (b) If $\Delta_E < 0$ then E is a focus. The focus is stable if $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) < 0$ and unstable if the inequality is reversed.
- (c) If $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) = 0$ then E is center-type.

Proof. At equilibrium E , $J(x^*, y^*)$ becomes

$$\begin{pmatrix} \varphi'(x^*) - mp'(x^*)y^* - H'(x^*) & -md_1 \\ cmp'(x^*)y^* & 0 \end{pmatrix}.$$

The determinant $cm^2d_1p'(x^*)y^*$ of $J(x^*, y^*)$ is always positive. The trace and the discriminant of the characteristic equation of $J(x^*, y^*)$ are respectively defined by $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*)$ and Δ_E . Hence, the conclusions (a)–(c) follow. ■

Remark 4.1. The importance of this section is due to the fact that, if an equilibrium of system (3) is unstable for $\tau = 0$, it remains unstable for $\tau > 0$ [Culshaw & Ruan, 2000; Martin & Ruan, 2001].

Table 2. Sign of $p(x) - d_1$ on $[T, K]$.

Region	Sign of $p(x) - d_1$
$1 - ad_1 \leq 0$	$p(x) - d_1 < 0$ for all $x \in [T, K]$
$1 - ad_1 > 0$	$p(x) - d_1 < 0$ for all $x \in [T, x^*[$
$1 - ad_1 > 0$	$p(x) - d_1 \geq 0$ for all $x \in [x^*, K]$

5. Stability Analysis of System (3) for $\tau > 0$

Following Remark 4.1, this section deals with the stability analysis of the equilibria F_0 and E on the following respective conditions:

$$\text{(COND}_{F_0}\text{): } \varphi'(x_0) - H'(x_0) < 0 \quad \text{and} \\ p(x_0) - d_1 < 0.$$

$$\text{(COND}_E\text{): } \varphi'(x^*) - mp'(x^*)y^* - H'(x^*) < 0.$$

The general characteristic equation of system (3) for $\tau > 0$ at (x, y) is defined by

$$\begin{aligned} \mathcal{Q}(x, y, \tau, \lambda) = & \lambda^2 - [\varphi'(x) - mp'(x)y - H'(x) \\ & + cmp(x) - d]\lambda + [\varphi'(x) - mp'(x)y \\ & - H'(x)](cmp(x) - d) \\ & + cm^2p(x)p'(x)y \exp(-\tau\lambda). \end{aligned} \quad (14)$$

Since $y_{F_0} = 0$, we deduce that the stability of the equilibrium F_0 with respect to system (3) for $\tau > 0$ is similar to the stability of F_0 , with respect to system (3) for $\tau = 0$, given in Sec. 4. That is, there is no stability change for the equilibrium F_0 of system (3) due to the delay $\tau > 0$.

For the coexistence equilibrium $E(x^*, y^*)$; applying results in [Fan *et al.*, 2009] to the model (3), we obtain the following lemma.

Lemma 1. Assume that (COND_E) holds and let

$$\text{tr}(E) = \varphi'(x^*) - H'(x^*) - mp'(x^*)y^* < 0, \quad (15)$$

$$\begin{aligned} \bar{\omega} = & \left\{ \frac{1}{2} \left[\sqrt{\text{tr}(E)^4 + 4m^2d^2p'(x^*)^2y^{*2}} \right. \right. \\ & \left. \left. - (\text{tr}(E))^2 \right] \right\}^{1/2}, \end{aligned} \quad (16)$$

and

$$\bar{\tau} = \frac{1}{\bar{\omega}} \arccos\left(\frac{\bar{\omega}^2}{mdp'(x^*)y^*}\right). \quad (17)$$

Then, the characteristic equation (14) of system (3) at the coexisting equilibrium $E(x^*, y^*)$ has a pair of pure imaginary roots $\pm i\bar{\omega}$ when the delay τ takes the value $\bar{\tau} + \frac{2n\pi}{\bar{\omega}}$ (with $n \in \mathbb{N}$).

Proof. According to Eq. (14), the characteristic equation of model (3) at equilibrium $E(x^*, y^*)$ can be written as

$$\lambda^2 - \text{tr}(E)\lambda + mdp'(x^*)y^* \exp(-\tau\lambda) = 0,$$

where $\text{tr}(E)$ is defined by inequality (15). Moreover, $-\text{tr}(E) > 0$ because of (COND_E) . Then applying successively Lemma 1, Theorem 1, and Theorem 2(iv.a) in [Fan et al., 2009], the result of the lemma follows. ■

The stability results of system (3) for $\tau > 0$ is summarized in Theorem 3.

Theorem 3

- (1) *The equilibrium E_0 is always unstable.*
- (2) *Stability results of equilibrium F_0 :*
 - (a) F_0 is a saddle if $\varphi'(x_0) - H'(x_0) < 0$ and $p(x_0) - d_1 > 0$.
 - (b) F_0 is an unstable node if $\varphi'(x_0) - H'(x_0) > 0$ and $p(x_0) - d_1 > 0$.
 - (c) F_0 is a stable node if $\varphi'(x_0) - H'(x_0) < 0$ and $p(x_0) - d_1 < 0$.
 - (d) F_0 is a stable (resp., unstable) improper node if $(\varphi'(x_0) - H'(x_0) < 0; p(x_0) - d_1 = 0)$ or $(\varphi'(x_0) - H'(x_0) = 0; p(x_0) - d_1 < 0)$ [resp., $(\varphi'(x_0) - H'(x_0) > 0; p(x_0) - d_1 = 0)$ or $(\varphi'(x_0) - H'(x_0) = 0; p(x_0) - d_1 > 0)$].
- (3) *Stability results of equilibrium $E(x^*, y^*)$: Recalling Eqs. (13), (17) and (16), we define the following conditions:*

$$\left\{ \begin{array}{l} \varphi'(x^*) - mp'(x^*)y^* - H'(x^*) > 0 \quad \text{or} \\ \varphi'(x^*) - mp'(x^*)y^* - H'(x^*) < 0 \quad \text{and} \\ \tau \in \left] \bar{\tau}, \bar{\tau} + \frac{2\pi}{\bar{\omega}} \right[\end{array} \right. \quad (18)$$

$$\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) < 0 \quad \text{and} \quad \tau \in [0, \bar{\tau}[. \quad (19)$$

- (a) *If $\Delta_E \geq 0$ then $E(x^*, y^*)$ is a node. The node is stable if condition (19) holds and unstable if condition (18) holds.*
- (b) *If $\Delta_E < 0$ then $E(x^*, y^*)$ is a focus. The focus is stable if condition (19) holds and unstable if condition (18) holds.*
- (c) *If $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) = 0$ then $E(x^*, y^*)$ is center-type.*

Proof. Since $y_{E_0} = 0$ and $y_{F_0} = 0$, we deduce that the stability of the equilibria E_0 and F_0 with respect to system (3) for $\tau > 0$ is similar to their stability with respect to the model (3) for $\tau = 0$ given in Sec. 4. That is, there is no stability change for the equilibria E_0 and F_0 of system (3) due to the delay $\tau > 0$. This ends the proof of items 1 and 2 of the theorem.

For item 3 of the theorem, the first case of condition (18), $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) > 0$, is a consequence of Remark 4.1 and Theorem 2. In the following, we assume that (COND_E) holds. Recalling the transcendental equation (14), let us define

$$\Delta(\tau, \lambda) := \mathcal{Q}(x^*, y^*, \tau, \lambda). \quad (20)$$

Setting $\tau_0 = \bar{\tau}$ and $\lambda_0 = i\bar{\omega}$, where $\bar{\tau}$ and $\bar{\omega}$ are respectively defined by (17) and (16), we have $\Delta(\tau_0, \lambda_0) = 0$ and τ_0 is the unique value of $\tau \in [0, \tau_0 + \frac{2\pi}{\bar{\omega}}[$ such that Δ has purely imaginary roots (Lemma 1). We also find that

$$\frac{\partial \Delta}{\partial \lambda}(\tau, \lambda) \Big|_{(\tau_0, \lambda_0)} = -\text{tr}(E) - \tau_0 \bar{\omega}^2 + i(2\bar{\omega} - \tau_0 \text{tr}(E)\bar{\omega}) \neq 0; \quad (21)$$

where $\text{tr}(E)$ is defined by (15) and $\text{tr}(E) < 0$, since (COND_E) holds.

Thus, implicit function theorem [Fritzsche & Grauert, 2002] leads to $\lambda := \lambda(\tau)$ for $\tau \in [0, \tau_0 + \frac{2\pi}{\bar{\omega}}[$.

Denote $\lambda = \alpha(\tau) + i\omega(\tau)$ ($\omega > 0$), the roots of the characteristic equation $\Delta(\tau, \lambda) = 0$, where the real part $\alpha(\tau)$ and the imaginary part $\omega(\tau)$ depend on the delay τ . Since the equilibrium $E(x^*, y^*)$ of the model (3) with $\tau = 0$ is stable when (COND_E) holds (Theorem 2); it follows that $\alpha(0) < 0$ for $\tau = 0$. By continuity of $\alpha(\tau)$, with respect to τ , and the fact that τ_0 is the unique root of equation $\alpha(\tau) = 0$ on $[0, \tau_0 + \frac{2\pi}{\bar{\omega}}[$, we see that $\alpha(\tau) < 0$ for $\tau \in [0, \tau_0[$

and $E(x^*, y^*)$ is still stable. Since $\alpha(\tau_0) = 0$ (so that $\lambda = i\omega_0$ is a purely imaginary root of characteristic equation $\Delta(\tau, \lambda) = 0$), then the steady state $E(x^*, y^*)$ loses its stability and becomes unstable. In fact, the characteristic equation $\Delta(\tau, \lambda) = 0$ has roots with positive real parts if and only if it has purely imaginary roots. (Rouché’s Theorem; [Dieudonne, 1960, Theorem 9.17.4].) ■

6. Bifurcation

We wish to find some bifurcations that are present in our system (3). Theorem 4 (resp., Theorem 5) leads with the saddle-node (resp., Hopf) bifurcation of system (3) around the equilibrium $F_0(x_0, 0)$ (resp., $E(x^*, y^*)$).

6.1. Saddle-node bifurcation

We have the following theorem concerning saddle-node bifurcations.

Theorem 4. Assume that $\frac{h^2}{(x_0 - T + h)^3} - \frac{r}{K} \neq 0$; $\varphi'(x_0) - H'(x_0) = 0$ and $p(x_0) - d_1 \neq 0$. Then equilibrium F_0 (when it is well defined) is a saddle-node.

- (i) If $p(x_0) - d_1 < 0$, then F_0 is attractive saddle-node.
- (ii) If $p(x_0) - d_1 > 0$, then F_0 is repelling saddle-node.

Proof. By the translation $(u_1, v_1) = (x - x_0, y)$, we bring equilibrium $F_0(x_0, 0)$ to the origin. Around the origin and using the fact that $\varphi'(x_0) - H'(x_0) = 0$, system (3) becomes

$$\begin{cases} \dot{u}_1(t) = \frac{1}{2}(\varphi''(x_0) - H''(x_0))u_1^2(t) - m \left[p'(x_0)u_1(t) + \frac{p''(x_0)}{2}u_1^2(t) \right] v_1(t) + O(|u_1, v_1|^4), \\ \dot{v}_1(t) = (cmp(x_0) - d)v_1(t) + cmp'(x_0)u_1(t - \tau)v_1(t) + \frac{cm}{2}p''(x_0)u_1^2(t - \tau)v_1(t) + O(|u_1, v_1|^4). \end{cases} \tag{22}$$

Setting $u(s) = u_1(t - s)$ and $v(s) = v_1(t - s)$ for $s \in [0, \tau]$; system (22) can be rewritten on the space $\mathcal{X} := C^0([0, \tau], \mathbb{R}^2)$ as follows

$$\begin{cases} u'(0) = \frac{1}{2}(\varphi''(x_0) - H''(x_0))u^2(0) - m \left[p'(x_0)u(0) + \frac{p''(x_0)}{2}u^2(0) \right] v(0) + O(|u, v|^4), \\ v'(0) = (cmp(x_0) - d)v(0) + cmp'(x_0)u(\tau)v(0) + \frac{cm}{2}p''(x_0)u^2(\tau)v(0) + O(|u, v|^4). \end{cases} \tag{23}$$

At the neighborhood of the origin, system (23) is topologically equivalent to

$$u'(0) = \frac{1}{2}(\varphi''(x_0) - H''(x_0))u^2(0) + O(|u|^3); \quad v'(0) = (cmp(x_0) - d)v(0) + O(|v|^2).$$

Since $\varphi''(x_0) - H''(x_0) = 2[\frac{h^2}{(x_0 - T + h)^3} - \frac{r}{K}] \neq 0$, we see that F_0 is a saddle-node point. We easily find that, if $cmp(x_0) - d = cm(p(x_0) - d_1) > 0$ then the equilibrium F_0 is a repelling saddle-node. This ends the proof of part (i) of the theorem. For part (ii), we will use the center manifold theorem. The reader can consult [Kelley, 1967; Carr, 1981; Hale, 1985; Diekmann & Van Gils, 1991].

To do so, let us rewrite system (23) as

$$(u'(0); v'(0))^T = L(u, v) + N(u, v),$$

where L and N are the operators defined on \mathcal{X} by

$$\begin{aligned} L(u, v) &= (0; (cmp(x_0) - d)v(0))^T \quad \text{and} \\ N(u, v) &= \begin{pmatrix} \frac{1}{2}(\varphi''(x_0) - H''(x_0))u^2(0) - m \left[p'(x_0)u(0) + \frac{p''(x_0)}{2}u^2(0) \right] v(0) \\ cmp'(x_0)u(\tau)v(0) + \frac{cm}{2}p''(x_0)u^2(\tau)v(0) \end{pmatrix}. \end{aligned}$$

Let us remark that $\mathcal{X} = \mathcal{X}_c \oplus \mathcal{X}_s$, where

$$\mathcal{X}_c = \{(u, v) \in \mathcal{X} : v(0) = 0\} \quad \text{and}$$

$$\mathcal{X}_s = \{(u, v) \in \mathcal{X} : v = v(0); u = 0\}.$$

It is easily found that $L\mathcal{X}_c \subset \mathcal{X}_c$; $\sigma(L|_{\mathcal{X}_c}) \subset \{\lambda \in \mathbb{C} : R_e(\lambda) = 0\}$ and $\sigma(L|_{\mathcal{X}_s}) \subset \{\lambda \in \mathbb{C} : R_e(\lambda) < 0\}$, where $\sigma(A)$ is the spectrum of operator A . The central manifold theorem leads to the fact that the central manifold, at the neighborhood of the origin, is defined by $M_0 := \{v = 0\}$.

Let Q be the projection of the space \mathcal{X} on \mathcal{X}_s ; then system $(u'(0); v'(0))^T = (I_{\mathcal{X}} - Q)[L(u, v) + N(u, v)]$ on the central manifold M_0 becomes

$$u'(0) = \frac{1}{2}(\varphi''(x_0) - H''(x_0))u^2(0) + O(|u|^3);$$

$$v'(0) = (cmp(x_0) - d)v(0) + O(|v|^2).$$

Since $\varphi''(x_0) - H''(x_0) \neq 0$ and $cmp(x_0) - d < 0$, it follows that F_0 is an attractive saddle-node. ■

6.2. Hopf bifurcation

For more details on the Hopf bifurcation, the reader can consult [Hassard et al., 1981].

Let us set $z(t) = (x(t); y(t)) - (x^*; y^*)$, this translation brings the coexisting equilibrium $E(x^*, y^*)$ to the origin. Then system (3) becomes

$$\dot{z}(t) = L(\tau)z_t + R(z_t, \tau) := F(z_t, \tau), \quad (24)$$

where $F(\cdot, \tau)$; $L(\tau)$ and $R(\cdot, \tau)$ are operators defined on $\mathcal{X}_\tau := \mathcal{C}^0([0, \tau], \mathbb{R}^2)$ by

$$F(\psi, \tau) = \begin{pmatrix} \varphi(x^* + \psi_1(0)) - mp(x^* + \psi_1(0))(y^* + \psi_2(0)) - H(x^* + \psi_1(0)) \\ cmp(x^* + \psi_1(\tau))(y^* + \psi_2(0)) - d(y^* + \psi_2(0)) \end{pmatrix};$$

$$L(\tau)\psi = \begin{pmatrix} \varphi'(x^*)\psi_1(0) - mp'(x^*)y^*\psi_1(0) - mp(x^*)\psi_2(0) - H'(x^*)\psi_1(0) \\ mp'(x^*)y^*\psi_1(\tau) + cmp(x^*)\psi_2(0) - d\psi_2(0) \end{pmatrix};$$

and $R(\cdot, \tau) = F(\cdot, \tau) - L(\tau)$.

Note that z_t is a function defined on $[0, \bar{\tau}]$ by $z_t(s) = z(t - s)$.

Recalling Eq. (20), we find that the characteristic equation $\Delta(\tau, \lambda) = 0$ at the equilibrium E has a pair of pure imaginary roots $\pm i\bar{\omega}$ at $\tau = \bar{\tau}$ (Lemma 1).

To see if Hopf bifurcation occurs, we need to verify the transversality condition. Following the proof of Lemma 1, the characteristic equation $\Delta(\tau, \lambda) = 0$ leads to $\Delta(\tau, \lambda(\tau)) = 0$, for τ in the neighborhood of $\bar{\tau}$. Differentiating equation $\Delta(\tau, \lambda(\tau)) = 0$ with respect to τ we obtain

$$[2\lambda - \text{tr}(E) - mdp'(x^*)y^*\tau e^{-\tau\lambda}] \frac{d\lambda(\tau)}{d\tau}$$

$$= mdp'(x^*)y^*\lambda e^{-\tau\lambda},$$

where $\text{tr}(E)$ is defined by (15) and $-\text{tr}(E) > 0$ since condition (COND $_E$) holds. Therefore

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = \frac{2\lambda - \text{tr}(E) - mdp'(x^*)y^*\tau e^{-\tau\lambda}}{mdp'(x^*)y^*\lambda e^{-\tau\lambda}}.$$

Following Cooke and Grossman [1982], we therefore obtain (using equation $\Delta(\bar{\tau}_2, i\bar{\omega}_2) = 0$)

$$\text{sign} \left\{ R_e \left(\frac{d\lambda(\tau)}{d\tau} \right)^{-1} \right\} \Big|_{\tau=\bar{\tau}}$$

$$= \text{sign} \left\{ R_e \left(\frac{d\lambda(\tau)}{d\tau} \right)^{-1} \right\} \Big|_{\tau=\bar{\tau}}$$

$$= \frac{(\text{tr}(E) + \bar{\tau}\bar{\omega})^2 + \bar{\omega}^2(2 - \bar{\tau}\text{tr}(E))^2}{\bar{\omega}^2(\text{tr}(E)^2 + 2\bar{\omega}^2)}$$

$$> 0.$$

Thus the transversality condition is satisfied and Hopf bifurcation occurs at $\tau = \bar{\tau}$ around equilibrium $E(x^*, y^*)$.

Now we proceed as in [Diekmann & Van Gils, 1991] to determine the Hopf bifurcation direction. This direction is given by the nonzero first term from the Taylor development of τ around the critical value $\tau_0 := \bar{\tau}$; that is

$$\tau(\varepsilon) = \tau_0 + \tau_2\varepsilon^2 + o(\varepsilon^2).$$

If $\tau_2 > 0$, then the bifurcation is supercritical as the periodic orbit appears when $\tau > \tau_0$.

If $\tau_2 < 0$, then the bifurcation is subcritical as the periodic orbit appears when $\tau < \tau_0$.

The parameter τ_2 is calculated as in [Diekmann & Van Gils, 1991] using the following formula

$$\tau_2 = \frac{R_e(c_2)}{R_e(q_2 D_\tau M_0(\lambda_0, 0) p_2)}; \quad (25)$$

where $\lambda_0 = i\bar{\omega}$, $M_0(\lambda, \tau)$ is the characteristic matrix of Eq. (24) defined by

$$M_0(\lambda, \tau) = \begin{pmatrix} \text{tr}(E) - \lambda & -md_1 \\ c m p'(x^*) y^* e^{-\tau\lambda} & -\lambda \end{pmatrix};$$

$D_\tau M_0(\lambda_0, 0)$ is the derivative of M_0 at $(\lambda_0, 0)$. The complex constant c_2 is defined by

$$\begin{aligned} c_2 = & \frac{1}{2} q_2 D_1^3 R(0, \tau_0)(P_2(\theta), P_2(\theta), \overline{P_2(\theta)}) \\ & + q_2 D_1^2 R(0, \tau_0)(e^0 \cdot M_0^{-1}(0, \tau_0) \\ & \times D_1^2 R(0, \tau_0)(P_2(\theta), \overline{P_2(\theta)}); P_2(\theta)) \\ & + \frac{1}{2} q_2 D_1^2 R(0, \tau_0)(e^{2\lambda_0} \cdot M_0^{-1}(2\lambda_0, \tau_0) \\ & \times D_1^2 R(0, \tau_0)(P_2(\theta), P_2(\theta)); \overline{P_2(\theta)}), \end{aligned}$$

where R is the nonlinear term of Eq. (24); $D_1^i R$ ($i = 2, 3$) is the i th-derivative of R with respect to ψ . Denote by $P_2(\theta)$ the eigenvector of the linear operator A (the infinitesimal generator of the semi-group associated to the linear part $L_0 = L(\tau_0)$ of Eq. (24) with respect to the eigenvalue λ_0).

More precisely, the parameter c_2 is evaluated as follows:

Let $L_0 = L(\tau_0) : \mathcal{X}_0 \rightarrow \mathbb{R}^2$ the linear term of Eq. (24). From the Riesz theorem [Hale & Verduyn Lunel, 1993] we obtain $L_0 \psi = \int_0^{\tau_0} d\eta(s) \psi(s)$, where

$$d\eta(s) = \begin{pmatrix} \text{tr}(E) \delta(s) & -m p(x^*) \delta(s) \\ c m p'(x^*) y^* \delta(\tau_0 - s) & 0 \end{pmatrix},$$

and δ is the Dirac delta function. Then, the infinitesimal generator A associated to the linear part L_0 of Eq. (24) is

$$A\psi(s) = \begin{cases} \frac{d\psi(s)}{ds} & \text{if } s \in]0, \tau_0], \\ L_0 \psi & \text{if } s = 0, \end{cases} \quad (26)$$

with the domain

$$D(A) = \left\{ \phi \in \mathcal{X}_0 : \frac{d\phi}{ds} \in \mathcal{X}_0; \frac{d\phi(0)}{ds} = L_0 \phi \right\}.$$

The derivatives $D_1^2 R(0, \tau_0)$ and $D_1^3 R(0, \tau_0)$ are respectively given by

$$D_1^2 R(0, \tau_0) \psi \chi = \begin{pmatrix} D_1^2 R_1(0, \tau_0) \psi \chi \\ D_1^2 R_2(0, \tau_0) \psi \chi \end{pmatrix},$$

where

$$\begin{aligned} D_1^2 R_1(0, \tau_0) \psi \chi &= \varphi''(x^*) \psi_1(0) \chi_1(0) - m p''(x^*) y^* \psi_1(0) \chi_1(0) \\ &\quad - m p'(x^*) \psi_1(0) \chi_2(0) - m p'(x^*) \psi_2(0) \chi_1(0) \\ &\quad - H''(x^*) \psi_1(0) \chi_1(0), \end{aligned}$$

$$\begin{aligned} D_1^2 R_2(0, \tau_0) \psi \chi &= c m p''(x^*) y^* \psi_1(\tau) \chi_1(\tau) \\ &\quad + c m p'(x^*) \psi_1(\tau) \chi_2(0) \\ &\quad + c m p'(x^*) \psi_2(0) \chi_1(\tau) \end{aligned}$$

and

$$D_1^3 R(0, \tau_0) \psi \chi \xi = 0,$$

with $\psi = (\psi_1, \psi_2)$, $\chi = (\chi_1, \chi_2)$, $\xi = (\xi_1, \xi_2) \in \mathcal{X}_0$.

Recalling Eq. (20) and since $\Delta(\tau_0, \lambda_0) = 0$, then λ_0 is an eigenvalue of A . So, there exist an eigenvector of the form $P_2(\theta) = p_2 e^{i\lambda_0 \theta}$ where the components (p_{21}, p_{22}) of p_2 are solutions of the system $M(\lambda_0, \tau_0) p = 0$. By taking $p_{21} = 1$, we obtain $p_{22} = \frac{\text{tr}(E) - \lambda_0}{m d_1}$.

Now, the adjoint operator A^* of A is defined by $A^* : \mathcal{X}_0^* := \mathcal{C}^0([-\tau_0, 0[, \mathbb{R}^2) \rightarrow \mathbb{R}^2$, where

$$A^* \psi(s) = \begin{cases} -\frac{d\psi}{ds}(s) & \text{for } s \in [-\tau_0, 0[, \\ -\int_0^{\tau_0} \psi(-\theta) d\eta(\theta) & \text{for } s = 0, \end{cases}$$

with the domain

$$\begin{aligned} D(A^*) = & \left\{ \phi \in \mathcal{X}_0^* : \frac{d\phi}{ds} \in \mathcal{X}_0^*; \frac{d\phi(0)}{ds} \right. \\ & \left. = -\int_0^{\tau_0} \phi(-\theta) d\eta(\theta) \right\}. \end{aligned}$$

Let $Q_2(s) = q_2 e^{s\lambda_0}$ be the eigenvector of A^* associated to the eigenvalue λ_0 with $q_2 = (q_{21}, q_{22})$. q_2 is chosen such that the duality product (see

[Hale & Verduyn Lunel, 1993])

$$\langle Q_2, \overline{P_2} \rangle = Q_2(0) \overline{P_2(0)} - \int_0^{\tau_0} \int_0^{\theta} Q_2(\xi - \theta) d\eta(\theta) \overline{P_2(\xi)} d\xi$$

takes the value 1. Taking $q_{22} = 0$, we obtain $q_{21} = \frac{1 + \tau_0 \lambda_0}{1 + \tau_0^2 |\lambda_0|^2}$.

Since $R_e(q_2 D_\tau M_0(\lambda_0, 0) p_2) = -\frac{|\lambda_0|^2}{1 + |\lambda_0|^2} < 0$, it appears that $\text{sign}(\tau_2) = \text{sign}\{-R_e(c_2)\}$, where c_2 is defined by

$$c_2 = \frac{1 + i\bar{\tau}\bar{\omega}}{mdp'(x^*)(1 + \bar{\tau}^2\bar{\omega}^2)} \times \left\{ R_e(B_1) \left[\bar{A}_1 + i\frac{p'(x^*)}{d_1}\bar{\omega} \right] + cm^2 p'(x^*)^2 y^* \bar{A}_1 \right\} + \frac{1 + i\bar{\tau}\bar{\omega}}{2W_0(1 + \bar{\tau}^2\bar{\omega}^2)} \times \{ [i\bar{\omega}mp''(x^*)y^* - 2i\bar{\omega}\bar{A}_1 - cm^2 p'(x^*)y^* e^{-2i\bar{\omega}\bar{\tau}}] A_1 + [md_1\bar{A}_1 + 3i\bar{\omega}mp'(x^*)] B_1 \}, \tag{27}$$

with

$$A_1 = \varphi''(x^*) - mp''(x^*)y^* - H''(x^*) - \frac{2p'(x^*)}{d_1}(\text{tr}(E) - i\bar{\omega}),$$

$$B_1 = cmp''(x^*)y^* + \frac{2cp'(x^*)}{d_1}(\text{tr}(E) - i\bar{\omega}),$$

$$W_0 = 2i\bar{\omega}(2i\bar{\omega} - \text{tr}(E)) + md_1 p'(x^*) y^* e^{-2i\bar{\omega}\bar{\tau}}. \tag{28}$$

Regrouping our results, we have the following theorem.

Theorem 5. *Recalling Eqs. (8) and (9), let $E(x^*, y^*)$ be the coexistence equilibrium of system (3) such that $\varphi'(x^*) - mp'(x^*)y^* - H'(x^*) < 0$. Let c_2 be the complex constant defined by (27). Then, system (3) undergoes a Hopf bifurcation when $\tau = \bar{\tau}$ [see Eq. (17)]; that is, a family of periodic solutions bifurcates from $E(x^*, y^*)$ as the delay τ passes through the critical value $\bar{\tau}$. Moreover,*

- (i) *If $R_e(c_2) < 0$, the Hopf bifurcation is supercritical and periodic solution appears for $\tau > \bar{\tau}$.*
- (ii) *If $R_e(c_2) > 0$, the Hopf bifurcation is subcritical and periodic solution appears for $\tau < \bar{\tau}$.*

7. Optimal Harvesting

The first part of this section deals with the *bionomic equilibrium* of system (3).

The term *bionomic equilibrium* is an amalgamation of the concepts of ecological equilibrium and economic equilibrium. As we have already seen, an ecological equilibrium is given by $\dot{x} = 0 = \dot{y}$. The economic equilibrium is said to be achieved when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for effort devoted to harvesting).

Let ε = cost per unit effort for prey; ξ = price per unit biomass for the prey. We assume that $\varepsilon < \xi$.

Assuming that $x \geq T$, we have the economic rent:

$$\pi(x, h) = \xi H(x, h) - \varepsilon h; \tag{29}$$

where H is the function defined by Eq. (4). Then, the bionomic equilibria $B_i(x_e, y_e, h_e)$, $i = 1, 2, 3, 4$, are obtained by solving the system (with $x_e \geq T$)

$$rx_e \left(1 - \frac{x_e}{K} \right) - mp(x_e)y_e - \frac{h_e(x_e - T)}{h_e + x_e - T} = 0; \tag{30}$$

$$(cmp(x_e) - d)y_e = 0; \tag{31}$$

$$\xi \frac{h_e(x_e - T)}{h_e + x_e - T} - \varepsilon h_e = 0. \tag{32}$$

The trivial cases are $B_1(K, 0, 0)$ and $B_2(x^*, y_2, 0)$, where x^* is given by (8) and $y_2 = \frac{\varphi(x^*)}{md_1}$.

Disregarding the trivial cases, there are two equilibria of interest.

Case 1. $y_3 = 0$; $h_3 > 0$. This corresponds to the removal of predators from the system. Solving Eq. (32) for x_3 gives

$$x_3 = T + \frac{\varepsilon h_3}{\xi - \varepsilon}.$$

Substituting x_3 into Eq. (30) gives

$$h_3 = \frac{K(\xi - \varepsilon)^2}{r\xi\varepsilon} \left[\sqrt{\Delta_3} - \frac{T\varepsilon}{K(\xi - \varepsilon)} \right];$$

with

$$\Delta_3 = \left(\frac{T\varepsilon}{K(\xi - \varepsilon)} \right)^2 + \frac{T}{K} \left(\frac{r\xi}{\xi - \varepsilon} \right)^2 \left(1 - \frac{T}{K} \right).$$

Thus, we completely have the bionomic equilibrium $B_3(x_3, 0, h_3)$.

Case 2. Assuming $x_e, y_e, h_e > 0$; Eq. (31) gives $p(x_e) = d_1$, i.e. $x_4 = x^*$, where x^* is defined by (8). Substituting x^* into Eq. (32) gives

$$h_4 = (x^* - T) \left(\frac{\xi}{\varepsilon} - 1 \right).$$

From Eq. (30), we deduce that

$$y_4 = \frac{\varphi(x^*) - \frac{h_4 \varepsilon}{\xi}}{m d_1}.$$

Thus, we have equilibrium $B_4(x_4, y_4, h_4)$.

Let us notice that bionomic equilibria B_2 and B_4 are plausible if and only if $x^* \in [T, x_0[$.

The second part of this section deals with the optimal harvesting.

Wilen [1985] pointed out that, from the point of view of humans, the ultimate users of natural resources, “population of natural organisms are not conveniently viewed as stocks of capital or assets which provide potential flows of services. Determining how to maximize benefits from these resources thus becomes a problem of capital theory deciding

mainly how to use this portfolio of stocks over time”. Hence optimal control theory provides the correct approach to natural resource management.

The present value of a continuous time-stream or revenue is given by

$$J(h) = \int_0^{t_f} \pi(x(t), h(t)) e^{-\delta t} dt, \quad (33)$$

where $\pi(x(t), h(t))$ is defined by (29) and δ denotes the instantaneous annual rate of discount. Our problem now is to find h^* such that

$$J(h^*) = \max_{h \in \Omega} J(h), \quad (34)$$

where $\Omega = \{h \in L^1(0, t_f); 0 \leq h \leq K\}$.

The existence of an optimal harvesting is due to the concavity of integrand of J with respect to h , a boundedness of the state solutions $(x(t), y(t))$, and the Lipschitz property of the state system (3) with respect to the state variables (see [Pontryagin *et al.*, 1992]).

Using the Pontryagin’s maximum principle for delayed control problem [Göllmann *et al.*, 2008], problem (34) is reduced to maximize the Hamiltonian \mathcal{H} defined by:

$$\begin{aligned} \mathcal{H}(x(t), y(t), x(t - \tau), h(t), \lambda(t)) = & \pi(x(t), h(t)) e^{-\delta t} + \lambda_2(t) [cmp(x(t - \tau))y(t) - dy(t)] \\ & + \lambda_1(t) [\varphi(x(t)) - my(t)p(x(t)) - H(x(t), h(t))], \end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2)$.

The necessary conditions for the existence of solution to problem (34) are:

$$\left\{ \begin{aligned} \frac{d\lambda_1(t)}{dt} = & -\frac{\partial \mathcal{H}}{\partial x(t)}(x(t), y(t), x(t - \tau), h(t), \lambda(t)) \\ & - \chi_{[0, t_f - \tau]}(t) \frac{\partial \mathcal{H}}{\partial x(t - \tau)}(x(t + \tau), y(t + \tau), x(t), h(t + \tau), \lambda(t + \tau)), \\ \frac{d\lambda_2(t)}{dt} = & -\frac{\partial \mathcal{H}}{\partial y(t)}(x(t), y(t), x(t - \tau), h(t), \lambda(t)) \end{aligned} \right. \quad (35)$$

and

$$\frac{\partial \mathcal{H}}{\partial h(t)}(x(t), y(t), x(t - \tau), h(t), \lambda(t)) = 0, \quad (36)$$

where $\chi_{[0, t_f - \tau]}(t)$ is the “indicatrice” function on $[0, t_f - \tau]$.

Therefore, we obtain the adjoint system:

$$\left\{ \begin{aligned} \dot{\lambda}_1(t) = & -\xi e^{-\delta t} \frac{\partial H(x(t), h(t))}{\partial x(t)} - cmp'(x(t)) \chi_{[0, t_f - \tau]}(t) y(t + \tau) \lambda_2(t + \tau) \\ & - \left[\varphi'(x(t)) - mp'(x(t))y(t) - \frac{\partial H(x(t), h(t))}{\partial x(t)} \right] \lambda_1(t), \\ \dot{\lambda}_2(t) = & mp(x(t)) \lambda_1(t) - [cmp(x(t - \tau)) - d] \lambda_2(t), \quad \text{for } t \leq t_f. \end{aligned} \right. \quad (37)$$

The transversality conditions of system (37) are

$$\lambda_1(t) = \lambda_2(t) = 0, \quad \text{for } t \in [t_f; t_f + \tau].$$

By considering the optimality condition (36), and solving for h^* ; subject to the constraints (Ω set), we have the following characterization of optimal harvesting:

$$h^*(t) = \min \left\{ \max \left(0; (x(t) - T) \times \left(\left(\frac{\xi - \lambda_1(t)e^{\delta t}}{\varepsilon} \right)^{1/2} - 1 \right) \right); T \right\}; \quad (38)$$

(when $\lambda_1(t)e^{\delta t} \leq \xi$ for all $t \in [0; t_f]$).

8. Numerical Simulations

Now, let us give numerical illustrations of our stability results. To do so, we give some examples of parameters for system (3).

Example 8.1. The logistic growth function is $\varphi(x) = x(1 - \frac{x}{40})$ (i.e. $r = 1$ and $K = 40$). The

parameters h and T of the harvesting function (4) are $h = 0.2 * 40$ and $T = 0.4 * 40$. The parameters of the Holling response function of type III are $a = 0.1$ and $b = 0.6$. Other parameters are $d = 0.67$, $m = 0.1$ and $c = 1$.

There is a predator free equilibrium $F_0 = (33.44; 0)$ and the coexistence equilibrium $E = (13.48; 13.40)$. Recalling Eqs. (15) and (13), we have $\text{tr}(E) = 0.08 > 0$ and $\Delta_E = -0.97 < 0$. Then Theorem 3 implies that equilibrium E is unstable (a periodic stable solution appeared) and there is no change of stability due to the delay τ (see Fig. 1).

Figure 2 illustrates the effect of the continuous threshold policy harvesting function $H(x)$ on the prey. We observe that increasing the value of the harvesting rate on the prey h does not change the behavior in time of the model.

We also notice that for a constant harvesting on the prey (that is, $H = 0.2 * 40$) the prey species is driven to extinction and the system collapses whereas this is not the case using the continuous threshold harvesting function $H(x)$ with harvesting rate $h = 0.2 * 40$.

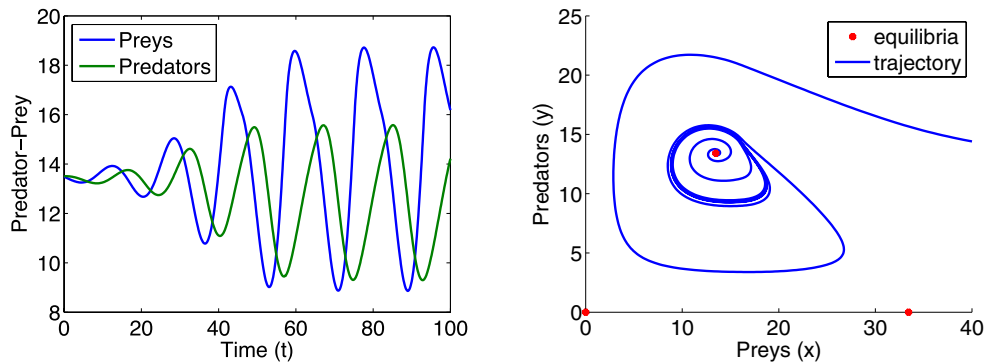


Fig. 1. Behaviors of the prey and predator populations for $x(0) = 40$, $y(0) = 14.40$ and $\tau = 0.49$. For the right one, we also compute another trajectory with initial condition $(40; 13.41)$.

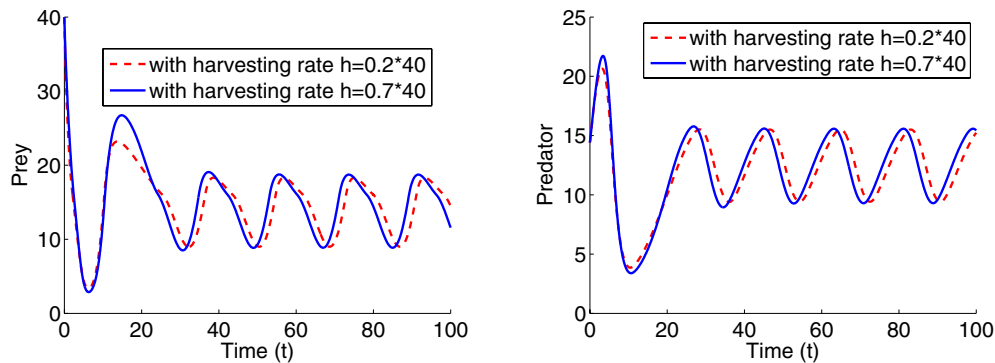


Fig. 2. Behaviors of the prey and predator populations for different harvesting rate h into the continuous threshold policy harvesting function. $x(0) = 40$, $y(0) = 14.40$ and $\tau = 0.49$.

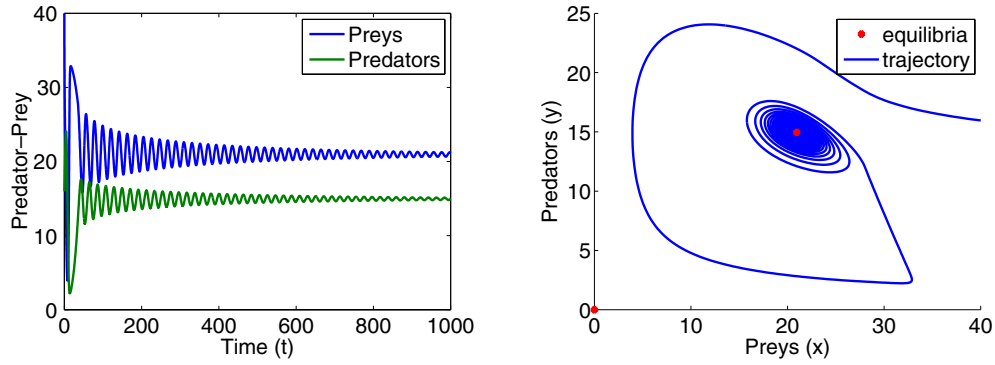


Fig. 3. Behaviors of the prey and predator populations for $x(0) = 40$, $y(0) = 15.97$ and $\tau = 2 < \bar{\tau}$.

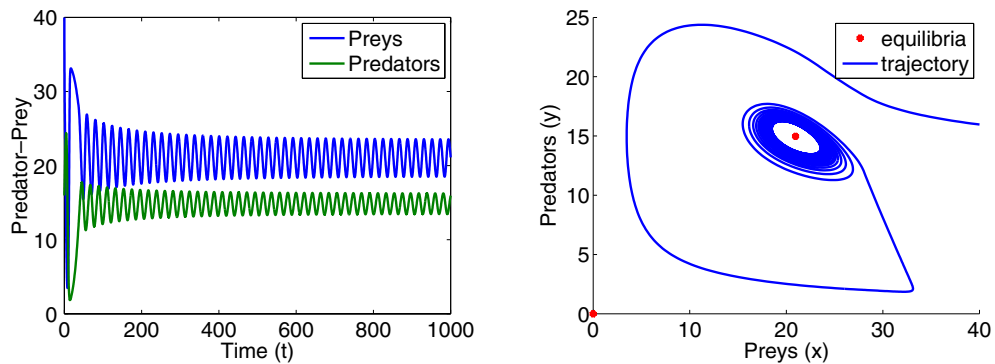


Fig. 4. Behaviors of the prey and predator populations for $x(0) = 40$, $y(0) = 15.97$ and $\tau = 2.1 > \bar{\tau}$.

Example 8.2. In this example, we are interested in studying the combined effect of the delay τ and the continuous threshold harvesting function $H(x)$ on the dynamics of the model. The logistic growth function and the harvesting function (4) are the same as in Example 8.1, i.e. ($r = 1$; $K = 40$) and ($h = 0.5 * 40$; $T = 0.7 * 40$). The parameters of the Holling response function of type III are $a = 0.1$ and $b = 1$. Other parameters are the same as in Example 8.1 ($d = 0.67$, $m = 0.1$, $c = 1$).

The predator free equilibrium F_0 is not plausible and the coexistence equilibrium is $E = (20.95; 14.97)$. Recalling (15) and (13), we have $\text{tr}(E) = -0.21 < 0$ and $\Delta_E = -0.61 < 0$. By Lemma 1, there is a critical value $\bar{\tau} = 2.0603$. By Theorem 3, the coexistence equilibrium E is stable when $\tau < 2.0603$ (Fig. 3); Hopf bifurcation occurs when $\tau = 2.0603$, and the equilibrium becomes unstable and a bifurcating periodic solution exists when $\tau > 2.0603$ (see Fig. 4). Figure 4

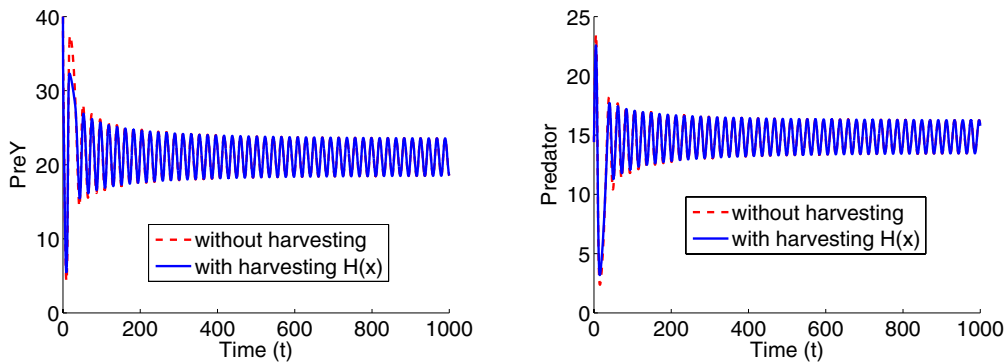


Fig. 5. Behaviors of the prey and predator populations with continuous threshold policy harvesting function and without harvesting on the prey. $x(0) = 40$, $y(0) = 15.97$ and $\tau = 2.1$.

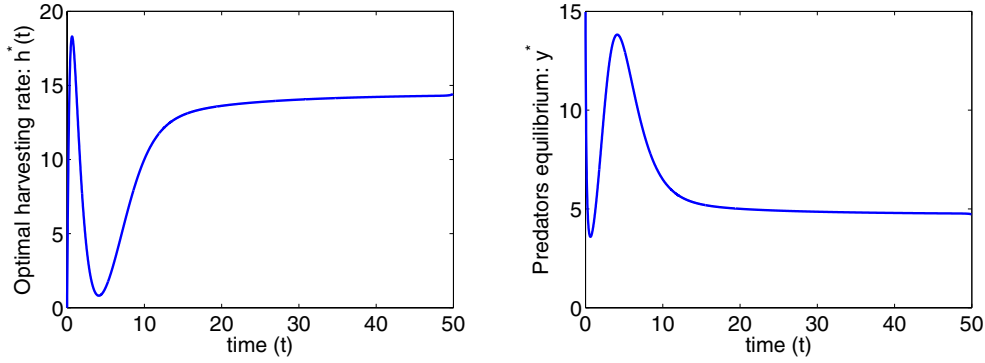


Fig. 6. Optimal harvesting on the preys and optimal equilibrium of the predators for the threshold value $T = 0.2 * 40$.

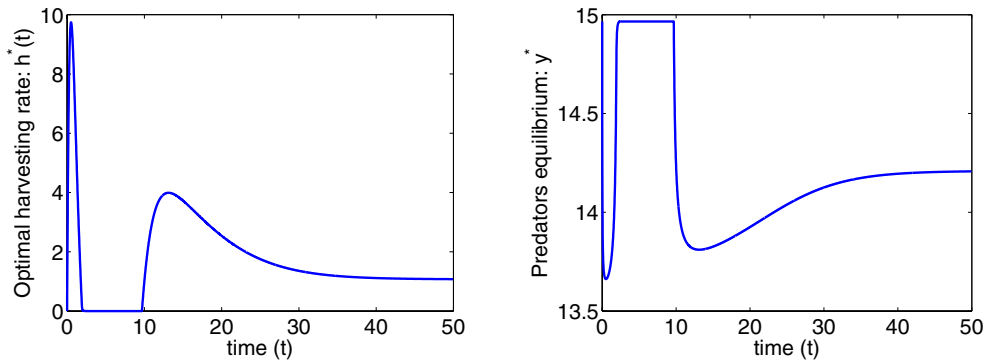


Fig. 7. Optimal harvesting on the preys and optimal equilibrium of the predators for the threshold value $T = 0.5 * 40$.

shows that both the prey and predator populations reach periodic oscillations around the equilibrium $E = (20.95, 14.97)$ in finite time.

To highlight the fact that the harvesting function $H(x)$ is more sound from a biological viewpoint, we illustrate that the behavior of the preys and predators with continuous threshold policy harvesting function on the prey is close to that without harvesting on the prey (see Fig. 5). We also notice that for a constant harvesting on the prey (that is, $H = 0.5 * 40$), the predator species is driven to extinction and the system collapses whereas this is not the case using the continuous threshold harvesting function $H(x)$ with harvesting rate $h = 0.5 * 40$.

Example 8.3 deals with numerical illustration of the optimal harvesting. To determine the optimal harvesting, we proceed following the different steps described in [Emvudu et al., 2011].

Example 8.3. Let $a = 0.1$; $r = 1$; $k = 40$; $m = 0.1$; $c = 1$; $b = 1$; $d = 2/3$; $\tau = 0.11$; $\delta = 1.5$; $t_f = 50$; $\xi = 45$; $\varepsilon = 10$. For these values of parameters, we find the optimal equilibrium (x^*, y^*) . We find

that the optimal equilibrium x^* of the preys is constant with respect to the time, i.e. $x^* = 20.95$. The optimal equilibrium of the predators y^* could not be constant with respect to time, it depends on the time variation of the optimal harvesting $h^*(t)$ on the preys. This is illustrated for two different threshold values T (see Figs. 6 and 7).

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