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Chaos in Fractional Order Cubic Chua System and Synchronization

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This paper deals with fractional order Chua system with cubic nonlinearity (cubic Chua system), which is a modification of Chua system. The aim is, first, to study the chaotic behavior in fractional order cubic Chua system. We found that chaos indeed exists in the fractional version of this system. The necessary condition for exhibiting chaotic attractors similar to its integer order counterpart is presented. This condition is used to distinguish for which parameters and orders the system generates double scroll chaotic attractors. The synchronization problem between two coupled chaotic fractional order cubic Chua systems is then addressed. An adaptive feedback control scheme for the synchronization with suitable feedback nonlinear controller applied to the response system is presented. Numerical simulations are performed to verify the theoretical analysis.

Keywords: Chaos; cubic Chua system; double scroll chaotic attractor; feedback control; chaos synchronization; fractional order system.

1. Introduction

Chaos is a very interesting nonlinear phenomenon which has been intensively studied in the last three decades. It is found to be useful or has great potential in many fields such as secure communication, data encryption, flow dynamics and biomedical engineering [Chen & Yu, 2003]. A chaotic system has complex dynamical behaviors with special features, such as an extreme sensitivity to tiny variations of initial conditions or bounded trajectories in the phase space. Despite this fact, control and synchronization of chaotic dynamical systems have attracted a wide range of research activity in recent years [Yamada & Fujisaka, 1983; Pecora & Carroll, 1990; Ott *et al.*, 1990; Aziz-Alaoui, 2006].

On the other hand, fractional calculus, as generalization of integer order integration and differentiation to its noninteger (fractional) order counterpart, has proved to be a valuable tool in the modeling of many physical phenomena [Samko *et al.*, 1993; Podlubny, 1999]. This mathematical phenomenon allows to describe a real object more

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accurately than the classical integer methods. Fractional derivatives provide an excellent instrument for describing systems with long-term memory [Hilfer, 2000; Wei *et al.*, 2016; Caputo & Cametti, 2017; Sandev *et al.*, 2017], nonlocal spatial [Tarasov, 2011] and fractal properties [Li, 2010]. The advantages or the real objects of the fractional order systems are that we have more degrees of freedom in the model and that a "memory" is included in the model [Petráš, 2008].

Recently, studying fractional order systems has become an active research area. The chaotic dynamics of fractional order systems began to attract much attention in recent years. It has been shown that the fractional order systems, as generalizations of many well-known systems, can also behave chaotically [Arneodo *et al.*, 1985; Hartley *et al.*, 1995; Grigorenko & Grigorenko, 2003; Li & Chen, 2004a, 2004b; Deng & Li, 2005; Lu & Chen, 2006; Ge & Ou, 2007; Baleanu *et al.*, 2015; Hu *et al.*, 2017; Palanivel *et al.*, 2017], such as the fractional Duffing system [Ge & Ou, 2007], the fractional Chua system [Hartley *et al.*, 1995; Petráš, 2008], the fractional Rössler system [Li & Chen, 2004a], the fractional Chen system [Li & Chen, 2004b; Lu & Chen, 2006], the fractional Lorenz system [Grigorenko & Grigorenko, 2003], the fractional Arneodo system [Arneodo *et al.*, 1985] and the fractional Lü system [Deng & Li, 2005]. In [Hartley *et al.*, 1995; Li & Chen, 2004a, 2004b], it has been shown that some fractional order systems can produce chaotic attractors with order less than 3. Moreover, other studies show that chaotic fractional order systems can also be synchronized [Li & Zhou, 2005; Wang *et al.*, 2006; Li *et al.*, 2006; Peng, 2007; Sheu *et al.*, 2007; Yan & Li, 2007; Li & Yan, 2007; Zhou *et al.*, 2008; Zhu *et al.*, 2009; Wu & Lu, 2009; Odibat *et al.*, 2010; Odibat, 2010; Wu & Wan, 2010; Zeng *et al.*, 2011; Zhang & Yang, 2011; Taghvafard & Erjaee, 2011; Yang *et al.*, 2011; Li *et al.*, 2012; Wang *et al.*, 2012; Machado, 2012; Si *et al.*, 2012; Yuan & Yang, 2012; Odibat, 2012; Kuntanapreeda, 2012; Moaddy *et al.*, 2012; Hegazi *et al.*, 2013; Danca & Garrappa, 2015; Wu *et al.*, 2016; El-Sayed *et al.*, 2013; Singh *et al.*, 2017]. In many literatures, synchronization among fractional order systems is only investigated through numerical simulations that are based on stability criteria of linear fractional order systems, such as the work presented in [Sheu *et al.*, 2007; Yan & Li, 2007; Li & Yan, 2007] or on Laplace transform theory, such as the work presented in [Wang *et al.*, 2006; Li *et al.*, 2006; Zhu *et al.*, 2009].

The objectives of this work are twofold. Firstly, we aim to study the chaotic behavior in fractional order cubic Chua system. We investigate theoretically and numerically necessary conditions on parameters and fractional orders for the system to display double scroll chaotic attractors. Secondly, using the drive-response concept and based on stability results for linear fractional order systems, we construct an adaptive nonlinear feedback control to achieve synchronization between two chaotic fractional order cubic Chua systems.

1.1. *Basic concepts*

In this paper, we mainly use the Caputo definition for a fractional derivative which is a modification of the Riemann–Liouville definition. This has the advantage of being more appropriate for initial value problems in which the initial conditions are given in terms of the field variables and their integer order, which is the case in most physical processes. The Caputo fractional differential operator of order α, $\alpha > 0$, is defined as

$$D^{\alpha} f(t) = J^{m-\alpha} D^{m} f(t), \tag{1}$$

where $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$ [Caputo, 1967; Gorenflo & Mainardi, 1997]. Here D^m is the usual integer differential operator of order m and J^{μ} is the Riemann–Liouville integral operator of order $\mu > 0$, defined by

$$J^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, \quad t > 0. \tag{2}$$

Details and properties of Caputo fractional differential operator and Riemann–Liouville fractional integral operator can be found in [Caputo, 1967; Samko *et al.*, 1993; Gorenflo & Mainardi, 1997; Podlubny, 1999; Hilfer, 2000].

1.2. *Stability analysis of fractional systems*

Stability analysis of fractional order systems, which is of main interest in control theory, has been thoroughly investigated where necessary and sufficient conditions have been derived [Matignon, 1996; Deng *et al.*, 2007; Ahmed *et al.*, 2007; Tavazoei & Haeri, 2009], see also the references therein. In this section, we recall the main stability results. For this object, we consider the following n dimensional fractional order system,

$$\begin{cases} \dfrac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = f_1(x_1, x_2, \ldots, x_n), \\ \dfrac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = f_2(x_1, x_2, \ldots, x_n), \\ \qquad \vdots \\ \dfrac{d^{\alpha_n} x_n}{dt^{\alpha_n}} = f_n(x_1, x_2, \ldots, x_n), \end{cases} \tag{3}$$

where α_i's are equal to real number or rational numbers between 0 and 1 and $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the Caputo fractional derivative of order α_i, for $i = 1, 2, \ldots, n$. If function f_i has second continuous partial derivatives in a ball centered at an equilibrium point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, that is $f_i(x_1^*, x_2^*, \ldots, x_n^*) = 0$, for $i = 1, 2, \ldots, n$, then we have the following results,

- If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then the equilibrium point $\mathbf{x}^*$ of system (3) is asymptotically stable if $|\arg(\text{spec}(J|_{\mathbf{x}^*}))| > \alpha\pi/2$, where the matrix J is the Jacobian matrix of the system (3) that is defined as $J = [\frac{\partial f_i}{\partial x_j}]_{i,j=1}^n$ [Ahmed *et al.*, 2007].
- If α_i's are rational numbers between 0 and 1 such that $\alpha_i = l_i/m_i$, $(l_i, m_i) = 1$, $l_i, m_i \in \mathbb{N}$, for $i = 1, 2, \ldots, n$, then the equilibrium point $\mathbf{x}^*$ of system (3) is asymptotically stable if all roots λ of the equation $\det(\text{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \ldots, \lambda^{m\alpha_n}) - J|_{\mathbf{x}^*}) = 0$ satisfy $|\arg(\lambda)| > q\pi/2$, where $q = 1/m$ and m be the least common multiple of the denominators m_i's of α_i's [Tavazoei & Haeri, 2009].

If the integer order system is stable then any fractional version of this system is also stable, indeed systems with memory are typically stable when their memory-less counterpart is stable [Ahmed *et al.*, 2007]. Fractional order systems may have some different properties from the classical integer order systems. For example, fractional order systems described by Caputo definition can not produce exact periodic solutions [Tavazoei, 2010]. Also, in [Shen & Lam, 2014], it has been shown that any equilibrium of a fractional order nonlinear system described by either Caputo or Riemann–Liouville definition can never be finite-time stable. The previous stability results play an important role in studying the existence of chaotic attractors and the synchronization between fractional order systems.

2. Cubic Chua System

Chua's circuit system is one of the paradigms of chaos since it exhibits a wide variety of nonlinear dynamics phenomena such as bifurcations and chaos [Chua *et al.*, 1986; Chua & Lin, 1990; Chua *et al.*, 1993]. It contains three energy-store elements (an inductor and two capacitors), a linear resistor and a single nonlinear resistor. Hartley [1989] suggested to replace the piecewise linear function by a cubic polynomial. This system is known, later, as cubic Chua system. Many studies for this modified system can also be found in [Huang *et al.*, 1996; Hwang *et al.*, 1996; Wu & Chen, 2002; Yassen, 2003]. It is described by the following dynamical system,

$$\begin{cases} \dot{x} = a(y + bx + cx^3) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y - \gamma z. \end{cases} \tag{4}$$

We find numerically that for some range of parameters, cubic Chua system (4) behaves chaotically. Indeed the study of bifurcation diagrams numerically evidence the presence of chaos. Figures 1–3 give the bifurcation diagrams with respect to the parameters $a \in [8.5, 10.9]$, $b \in [0.08, 0.4]$ and $\beta \in [14.5, 19]$, respectively. From these figures, we can observe numerically that cubic Chua system (4) seems chaotic, for example, when,

$$(b, c, \beta, \gamma) = (0.15, -0.3, 14, 0.02), \quad a \in I_a, \tag{5}$$

$$(a, c, \beta, \gamma) = (10, -0.3, 14, 0.02), \quad b \in I_b, \tag{6}$$

$$(a, b, c, \gamma) = (10, 0.2, -0.3, 0.02), \quad \beta \in I_\beta, \tag{7}$$

where

$$\begin{aligned} I_a &= [9.42, 9.65] \cup [10.1, 10.29] \cup [10.43, 10.53] \\ &\quad \cup [10.54, 10.72] \cup [10.74, 10.9], \\ I_b &= [0.09, 0.12] \cup [0.114, 0.122] \cup [0.124, 0.146] \\ &\quad \cup [0.154, 0.265] \cup [0.275, 0.312] \quad \text{and} \\ I_\beta &= [15.1, 15.2] \cup [15.215, 15.32] \cup [15.34, 15.408] \\ &\quad \cup [15.425, 15.6] \end{aligned}$$

(Obviously, to have a better numerical evidence of the chaotic behavior requires at least the computation of Lyapunov exponents). For the above parameters, cubic Chua system (4), in most cases, displays double scroll chaotic attractors, see Figs. 4–6 in which we plot the (x, y, z) phase portrait with

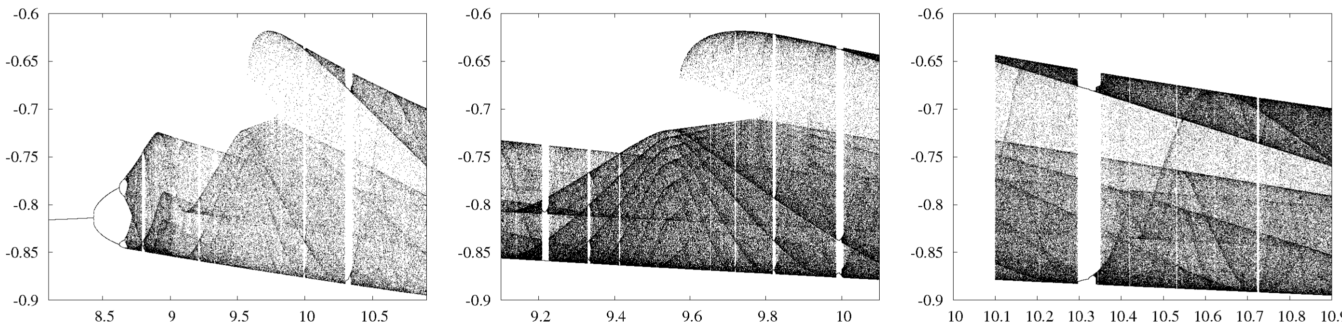

Fig. 1. Bifurcation diagrams of cubic Chua system (4), when $(b, c, \beta, \gamma) = (0.15, -0.3, 14, 0.02)$; x-axis represents the parameter a.

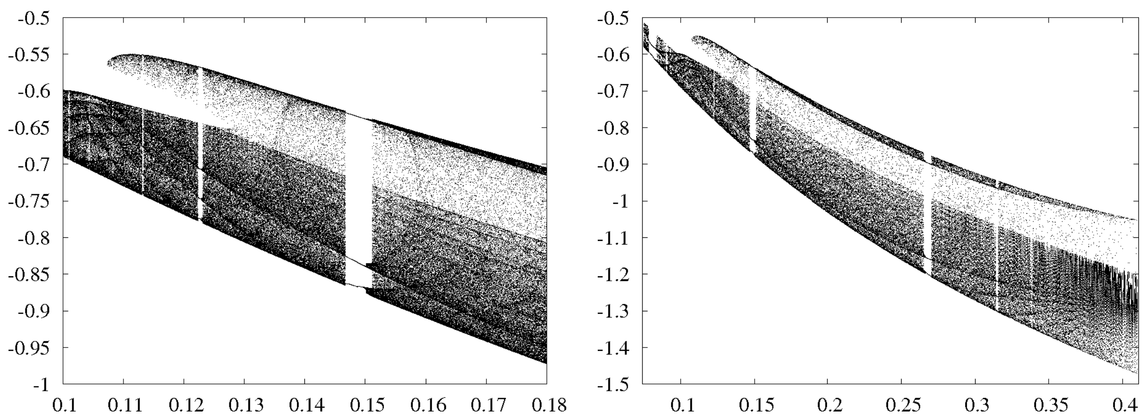

Fig. 2. Bifurcation diagrams of cubic Chua system (4), when $(a, c, \beta, \gamma) = (10, -0.3, 14, 0.02)$; x-axis represents the parameter b.

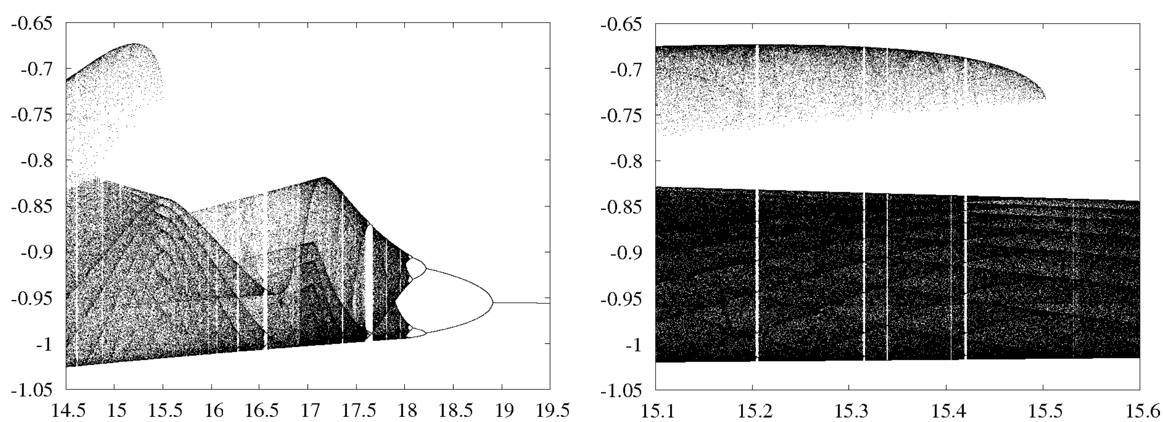

Fig. 3. Bifurcation diagrams of cubic Chua system (4), when $(a, b, c, \gamma) = (10, 0.2, -0.3, 0.02)$; x-axis represents the parameter β.

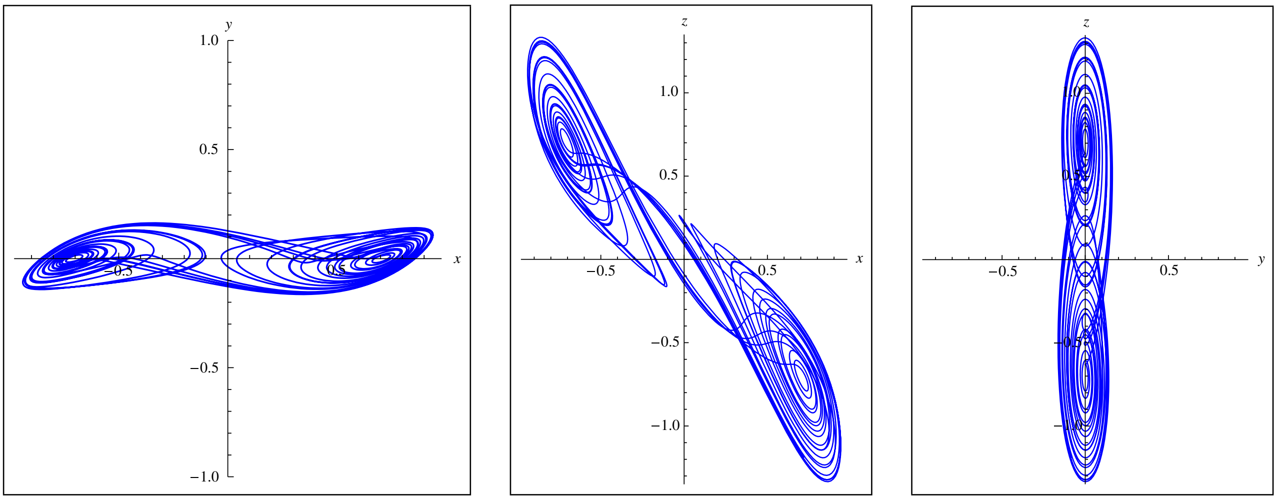

Fig. 4. Chaotic attractor of cubic Chua system (4), when $(a, b, c, \beta, \gamma) = (9.5, 0.15, -0.3, 14, 0.02)$.

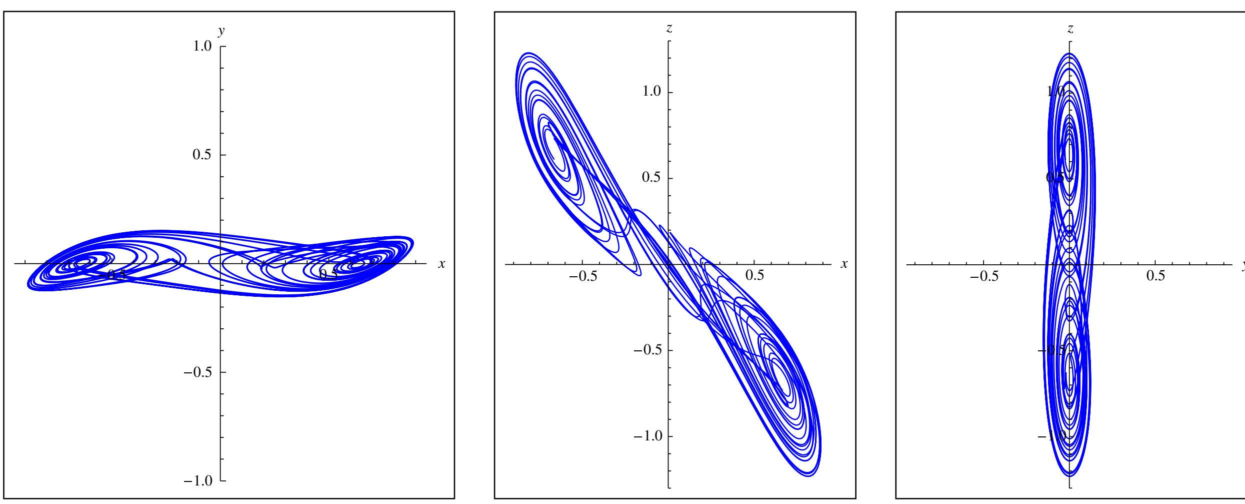

Fig. 5. Chaotic attractor of cubic Chua system (4), when $(a, b, c, \beta, \gamma) = (10, 0.13, -0.3, 14, 0.02)$.

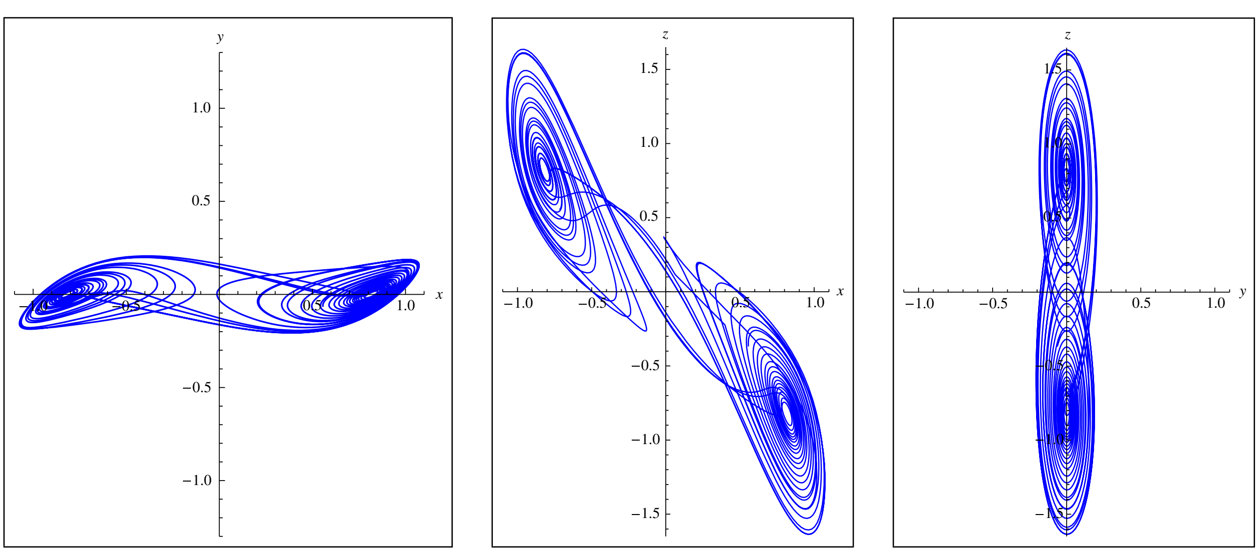

Fig. 6. Chaotic attractor of cubic Chua system (4), when $(a, b, c, \beta, \gamma) = (10, 0.2, -0.3, 15.45, 0.02)$.

parameters $(a, b, c, \beta, \gamma) = (9.5, 0.15, -0.3, 14, 0.02)$, $(a, b, c, \beta, \gamma) = (10, 0.13, -0.3, 14, 0.02)$ and $(a, b, c, \beta, \gamma) = (10, 0.2, -0.3, 15.45, 0.02)$, satisfying (5)–(7), respectively. It is clear, from these figures, that cubic Chua system (4) produces double scroll chaotic attractors.

3. Chaotic Dynamics of Fractional Cubic Chua System

In this section, we study the chaotic dynamics of fractional cubic Chua system. It is obtained from the classical system, described in (4), by replacing the first time derivative d/dt by a fractional derivative d^α/dt^α, where the last denotes the differential operator in the sense of Caputo. The fractional version of cubic Chua system reads as,

$$\begin{cases} \dfrac{d^{\alpha_1}x}{dt^{\alpha_1}} = a(y + bx + cx^3) \\ \dfrac{d^{\alpha_2}y}{dt^{\alpha_2}} = x - y + z \\ \dfrac{d^{\alpha_3}z}{dt^{\alpha_3}} = -\beta y - \gamma z \end{cases} \tag{8}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is subject to $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$. In general, if the integer order cubic Chua system, when $\alpha = 1$, displays a double scroll chaotic attractor then this system has a saddle point of index 2 and other two unstable equilibrium points. It is obvious that the fractional order and the integer order cubic Chua system have the same equilibrium points. Hence, a necessary condition for fractional order cubic Chua system to exhibit the chaotic attractor similar to its integer order counterpart is instability of the equilibrium points of the system surrounded by scrolls.

Assume that $\alpha_i = l_i/m_i$, $(l_i, m_i) = 1$, $l_i, m_i \in \mathbb{N}$, for $i = 1, 2, 3$. Let m be the least common multiple of the denominators m_i's of α_i's. According to stability results of fractional order systems and the results presented in [Tavazoei & Haeri, 2007], a necessary condition for fractional cubic Chua system to exhibit chaotic attractor similar to its integer order counterpart is,

$$\min_i\{|\arg(\lambda_i)|\} \leq \frac{q\pi}{2}, \tag{9}$$

where $q = 1/m$ and λ_i's are the roots of the equation $\det(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J|_{\mathbf{x}^*}) = 0$, $J|_{\mathbf{x}^*}$ is the Jacobian matrix of system (8) evaluated at the equilibrium point $\mathbf{x}^*$, for every equilibrium point $\mathbf{x}^*$. Otherwise, one of these equilibria becomes asymptotically stable and attracts the nearby trajectories. In case of $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, where α is a real number between 0 and 1, then condition (9) reduces to $\min\{|\arg(\lambda_i)|, i = 1, 2, 3\} \leq \alpha\pi/2$. Now, we fix the parameters as given in (5)–(7) and perform numerical simulations to study the chaotic behavior of fractional cubic Chua system (8). Section 3.1 deals with the commensurate fractional order cubic Chua system, where the fractional orders are equal to real number (i.e. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$). Section 3.2 deals with the incommensurate fractional order cubic Chua system, where the fractional orders are rational numbers between zero and one.

3.1. *Commensurate fractional cubic Chua system*

We first consider parameters $(b, c, \beta, \gamma) = (0.15, -0.3, 14, 0.02)$ and $a \in I_a$, for which system (8) has three equilibria given by,

$$\begin{cases} P_1 : (0, 0, 0), \\ P_2 : (0.71046119, 0.00101349, -0.70944769), \\ P_3 : (-0.71046119, -0.00101349, 0.709447699). \end{cases} \tag{10}$$

The Jacobian matrix of system (8) evaluated at the equilibrium point $\mathbf{x}^* = (x^*, y^*, z^*)$ is,

$$J_a(\mathbf{x}^*) = \begin{pmatrix} -0.9ax^{*2} + 0.15a & a & 0 \\ 1 & -1 & 1 \\ 0 & -14 & -0.02 \end{pmatrix}. \tag{11}$$

If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and, for example, $a = 9.5$, then the eigenvalues of the equilibrium points are,

$$\begin{cases} P_1 : \lambda_1 = 2.46978760, \\ \qquad \lambda_{2,3} = -1.03239380 \pm 2.66462940i, \\ P_2 : \lambda_1 = -4.33455977, \\ \qquad \lambda_{2,3} = 0.21195178 \pm 3.04318601i, \\ P_3 : \lambda_1 = -4.33455977, \\ \qquad \lambda_{2,3} = 0.21195178 \pm 3.04318601i \end{cases} \tag{12}$$

and so, in view of P_2 and P_3, we have,

$$|\arg(\lambda_{2,3})| = \tan^{-1}\frac{3.04318601}{0.21195178} = 1.50126063. \tag{13}$$

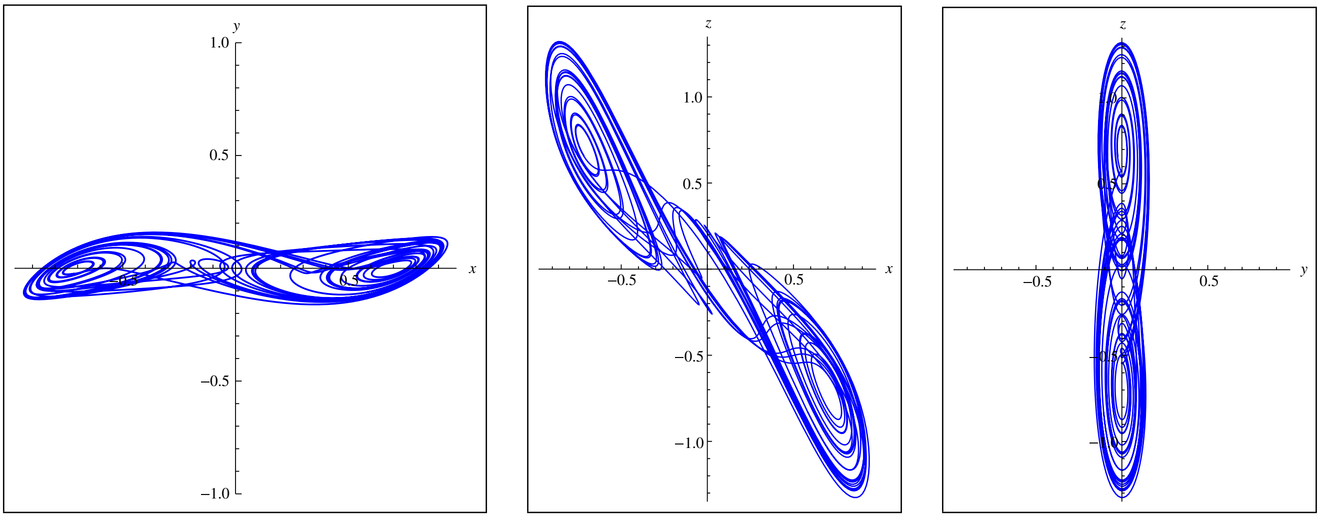

Fig. 7. Simulation results for fractional cubic Chua system (8), when $\alpha = 0.98$ and $(a, b, c, \beta, \gamma) = (9.5, 0.15, -0.3, 14, 0.02)$.

Thus, if $\alpha < (\frac{2}{\pi}) * 1.50126063 = 0.95573220$, then the equilibria P_2 and P_3 become asymptotically stable. Therefore, system (8) has the necessary condition $\alpha > 0.95573220$ for exhibiting double scroll chaotic attractor. Figure 7 shows numerical simulation results for $\alpha = 0.98$, which satisfies the necessary condition, and $(a, b, c, \beta, \gamma) = (9.5, 0.15, -0.3, 14, 0.02)$. In this case, from Fig. 7, it is clear that system (8) is chaotic and produces double scroll attractor. Now, if $a \in I_a$ and (a, α) does not lie in the shaded region shown in Fig. 8 then system (8) has the necessary condition to exhibit double scroll chaotic attractor.

Next, we consider the parameters $(a, c, \beta, \gamma) = (10, -0.3, 14, 0.02)$ and $b \in I_b$, then system (8) has

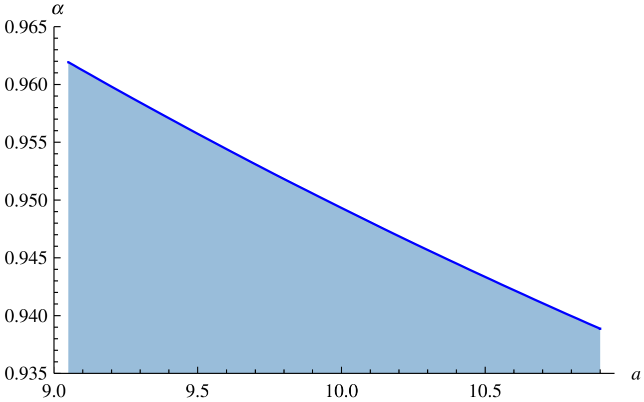

Fig. 8. Critical region for fractional cubic Chua system (8), when $(b, c, \beta, \gamma) = (0.15, -0.3, 14, 0.02)$. If $a \in I_a$ and (a, α) lie in the shaded region, then system (8) does not satisfy the necessary condition for exhibiting double scroll chaotic attractor.

three equilibria given by,

$$\begin{cases} P_1 : (0, 0, 0), \\ P_2 : (1.824439915\delta, 0.002602620\delta, -1.821836530\delta), \\ P_3 : (-1.824439915\delta, -0.002602620\delta, 1.821836530\delta), \end{cases} \tag{14}$$

where,

$$\delta = \sqrt{0.00142857 + 1.00142857b} \tag{15}$$

and the Jacobian matrix of system (3), evaluated at the equilibrium $\mathbf{x}^* = (x^*, y^*, z^*)$, is,

$$J_b(\mathbf{x}^*) = \begin{pmatrix} -9x^{*2} + 10b & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14 & -0.02 \end{pmatrix}. \tag{16}$$

If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and, for example, $b = 0.12$, then the eigenvalues of the equilibria are,

$$\begin{cases} P_1 : \lambda_1 = 2.26468570, \\ \qquad \lambda_{2,3} = -1.04234285 \pm 2.53587840i, \\ P_2 : \lambda_1 = -3.97769020, \\ \qquad \lambda_{2,3} = 0.25744709 \pm 2.91435456i, \\ P_3 : \lambda_1 = -3.97769020, \\ \qquad \lambda_{2,3} = 0.25744709 \pm 2.91435456i \end{cases} \tag{17}$$

and so, in view of P_2 and P_3, we have

$$|\arg(\lambda_{2,3})| = \tan^{-1} \frac{2.91435456}{0.25744709} = 1.48268743. \tag{18}$$

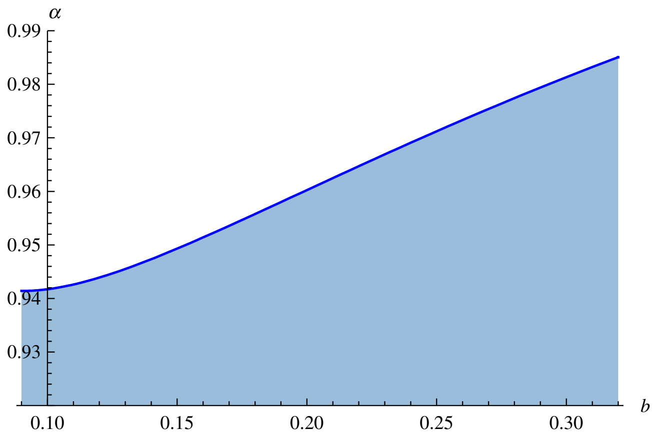

Fig. 9. Critical region for fractional cubic Chua system (8), when $(a, c, \beta, \gamma) = (10, -0.3, 14, 0.02)$. If $b \in I_b$ and (b, α) lie in the shaded region, then system (8) does not satisfy the necessary condition for exhibiting double scroll chaotic attractor.

Thus, if $\alpha < (\frac{2}{\pi}) * 1.48268743 = 0.94390814$, then the equilibria P_2 and P_3 become asymptotically stable. Therefore, system (8) has the necessary condition $\alpha > 0.94390814$ to exhibit double scroll chaotic attractor. In general, if $b \in I_b$ and (b, α) does not lie in the shaded region shown in Fig. 9 then system (8) has the necessary condition to exhibit double scroll chaotic attractor.

Now we consider the parameters $(a, b, c, \gamma) = (10, 0.15, -0.3, 0.02)$ and $\beta \in I_\beta$, then, if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, system (8) has the necessary condition that (β, α) does not lie in the shaded region shown in Fig. 10 for exhibiting double scroll chaotic attractor. From Fig. 10, we can see that as β varys in $[14.5, 16]$ the necessary condition to observe a double scroll chaotic attractor is $\alpha > \alpha_0$, where $\alpha_0 \in [0.96259843, 0.96899785]$.

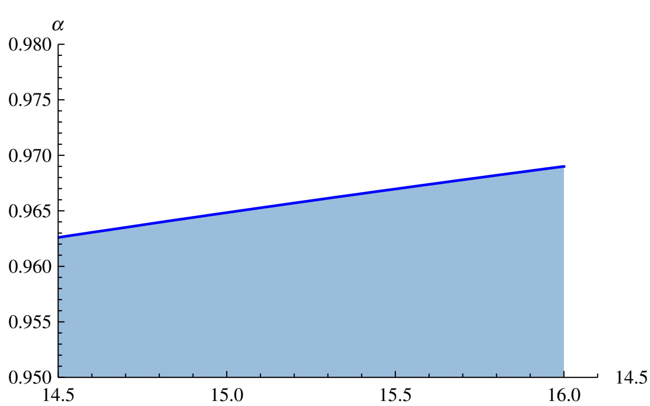

Fig. 10. Critical region for fractional cubic Chua system (8), when $(a, b, c, \gamma) = (10, 0.15, -0.3, 0.02)$. If $\beta \in I_\beta$ and (β, α) lie in the shaded region, then system (8) does not satisfy the necessary condition for exhibiting double scroll chaotic attractor.

3.2. *Incommensurate fractional cubic Chua system*

If α_1, α_2 and α_3 are rational numbers between zero and one, which are not necessarily equal, then it is not easy to find a region for the fractional orders that satisfies the necessary condition for exhibiting double scroll chaotic attractor. But we can investigate, numerically, the chaotic behavior for each given fractional order. Here we study two special cases.

First, we take $(\alpha_1, \alpha_2, \alpha_3) = (1, 0.95, 0.975)$ and $(a, b, c, \beta, \gamma) = (10.5, 0.15, -0.3, 14, 0.02)$. Then we have $l_1 = 40$, $l_2 = 38$, $l_3 = 39$ and $m = 40$. According to the last two equilibria P_2 and P_3, given in Eq. (10), the equation $\det(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_a|_{\mathbf{x}^*}) = 0$ becomes,

$$\begin{aligned}(\lambda^{40} + 3.19493581)[(\lambda^{38} + 1)(\lambda^{39} + 0.02) + 14]\\ - 10.5(\lambda^{39} + 0.02) = 0 \qquad (19)\end{aligned}$$

and simple calculations give,

$$\min_i\{|\arg(\lambda_i)|\} = 0 < \frac{\pi}{80}. \qquad (20)$$

Hence, system (8) satisfies the necessary condition for exhibiting a double scroll attractor in this case. The numerical simulation results shown in Fig. 11 confirm this result.

Second, we take $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.9, 0.8)$ and $(a, b, c, \beta, \gamma) = (10, 0.16, -0.3, 14, 0.02)$. Here, we have $l_1 = 17$, $l_2 = 18$, $l_3 = 16$ and $m = 20$. According to the last two equilibria P_2 and P_3, given in Eq. (14), the equation $\det(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_b|_{\mathbf{x}^*}) = 0$ becomes,

$$\begin{aligned}(\lambda^{17} + 3.24279601)[(\lambda^{18} + 1)(\lambda^{16} + 0.02) + 14]\\ - 10(\lambda^{16} + 0.02) = 0 \qquad (21)\end{aligned}$$

and so we get,

$$\min_i\{|\arg(\lambda_i)|\} = 0.08919554 > \frac{\pi}{40}. \qquad (22)$$

Hence, system (8) does not satisfy the necessary condition for exhibiting a double scroll attractor in this case. Therefore, in this case, system (8) does not display chaotic attractor.

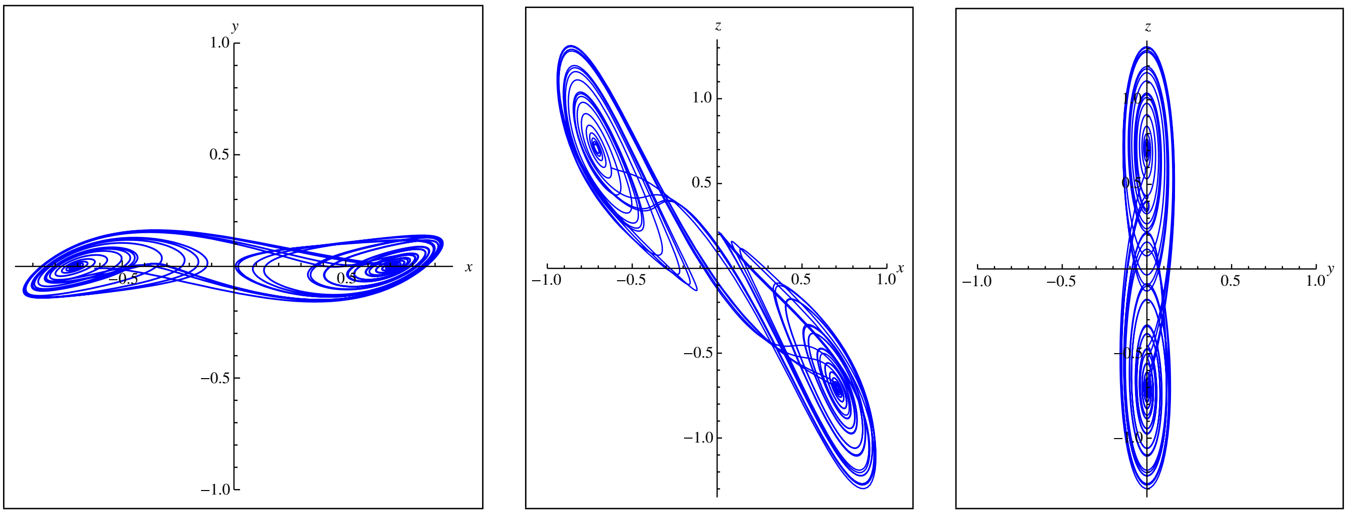

Fig. 11. Simulation results for fractional cubic Chua system (8), when $(\alpha_1, \alpha_2, \alpha_3) = (1, 0.95, 0.975)$ and $(a, b, c, \beta, \gamma) = (10.5, 0.15, -0.3, 14, 0.02)$.

4. Synchronization of Fractional Cubic Chua System

A way to study synchronization is to use a controller to make the output of the slave (response) system copy in some manner the master (drive) system one. In this section, based on the stability results of fractional differential equations, we study the synchronization of two identical chaotic fractional order cubic Chua systems via nonlinear control. Using the drive-response concept, an adaptive feedback control is constructed to achieve synchronization between the systems. In order to observe synchronization behavior, we construct the master system and the slave system as,

$$\text{Master}\begin{cases}\dfrac{d^{\alpha_1}x_m}{dt^{\alpha_1}} = a(y_m + bx_m + cx_m^3)\\[2ex] \dfrac{d^{\alpha_2}y_m}{dt^{\alpha_2}} = x_m - y_m + z_m\\[2ex] \dfrac{d^{\alpha_3}z_m}{dt^{\alpha_3}} = -\beta y_m - \gamma z_m\end{cases} \tag{23}$$

$$\text{Slave}\begin{cases}\dfrac{d^{\alpha_1}x_s}{dt^{\alpha_1}} = a(y_s + bx_s + cx_s^3) + u_1(t)\\[2ex] \dfrac{d^{\alpha_2}y_s}{dt^{\alpha_2}} = x_s - y_s + z_s + u_2(t)\\[2ex] \dfrac{d^{\alpha_3}z_s}{dt^{\alpha_3}} = -\beta y_s - \gamma z_s + u_3(t)\end{cases} \tag{24}$$

where $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the fractional differential operator in Caputo sense, $0 < \alpha_i \leq 1$, $j = 1, 2, 3$. Subscripts m and s stand for the master system and slave system, respectively, and $\mathbf{u}(t) = [u_1(t), u_2(t), u_3(t)]^T$ is the nonlinear controller to be designed for the global synchronization of fractional order systems (23) and (24). For this purpose, we define the synchronization error as,

$$\begin{cases}e_1(t) = x_s(t) - x_m(t)\\ e_2(t) = y_s(t) - y_m(t)\\ e_3(t) = z_s(t) - z_m(t)\end{cases} \tag{25}$$

Now, define the controller $\mathbf{u}(t)$ as follows,

$$\begin{cases}u_1(t) = -3ac(x_m e_1^2 + x_m^2 e_1) + k_1(e_1, e_2, e_3)\\ u_2(t) = k_2(e_1, e_2, e_3)\\ u_3(t) = k_3(e_1, e_2, e_3)\end{cases} \tag{26}$$

where

$$k_i(e_1, e_2, e_3) = k_{i1}e_1 + k_{i2}e_2 + k_{i3}e_3, \quad i = 1, 2, 3, \tag{27}$$

$k_{ij} \in \mathbb{R}$, for $1 \leq i, j \leq 3$, $\mathbf{k} = (k_1(e_1, e_2, e_3), k_2(e_1, e_2, e_3), k_3(e_1, e_2, e_3))$ is the linear part of coupling matrix. Our aim is to determine the possible matrices $\mathbf{k}$ such that the drive system (23) and the response system (24) are synchronized ($\|\mathbf{e}(t)\| \to 0$, as $t \to +\infty$). With this controller, according to the synchronization error, the error system becomes,

$$\begin{cases} \dfrac{d^{\alpha_1}e_1}{dt^{\alpha_1}} = ae_2 + abe_1 + ace_1^3 + k_1(e_1, e_2, e_3) \\ \dfrac{d^{\alpha_2}e_2}{dt^{\alpha_2}} = e_1 - e_2 + e_3 + k_2(e_1, e_2, e_3) \\ \dfrac{d^{\alpha_3}e_3}{dt^{\alpha_3}} = -\beta e_2 - \gamma e_3 + k_3(e_1, e_2, e_3). \end{cases} \tag{28}$$

The Jacobian matrix J_e, evaluated at the equilibrium point $\mathbf{e}^* = (e_1^*, e_2^*, e_3^*)$, for the error system (28) is given by,

$$J_e(\mathbf{e}^*) = \begin{pmatrix} ab + 3ac(e_1^*)^2 & a & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix} + \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}. \tag{29}$$

According to the stability analysis, we have the following two cases,

Case 1. If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ then the drive system (23) and the response system (24) with the nonlinear control law (26) are synchronized if all eigenvalues λ of the matrix J_e, evaluated at each equilibrium point $\mathbf{e}^*$, lie in the region $|\arg(\lambda)| > \alpha\pi/2$.

Case 2. If α_i's are rational numbers such that $\alpha_i = l_i/m_i$, $(l_i, m_i) = 1$, $q = 1/m$ where m is the least common multiple of the denominators m_i of α_i's, $l_i, m_i \in \mathbb{N}$, for $i = 1, 2, 3$, then the drive system (23) and the response system (24) with the nonlinear control law (26) are synchronized if all roots λ of the characteristic equation $\det(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_e) = 0$, for every equilibrium point $\mathbf{e}^*$, satisfy $|\arg(\lambda)| > q\pi/2$.

Example 1. Taking $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.95, 0.95)$ and $(a, b, c, \beta, \gamma) = (10, 0.1, -0.3, 14, 0.02)$, then system (8) displays double scroll chaotic attractor.

If we take $\mathbf{k} = (k_1, k_2, k_3) = (-3e_1, -2e_2, 0.64e_3)$, then error system (29) has one real equilibrium point $\mathbf{e}^* = (0, 0, 0)$. According to Case 1, the roots of the equation $\det(\lambda I - J_e) = 0$ or the eigenvalues

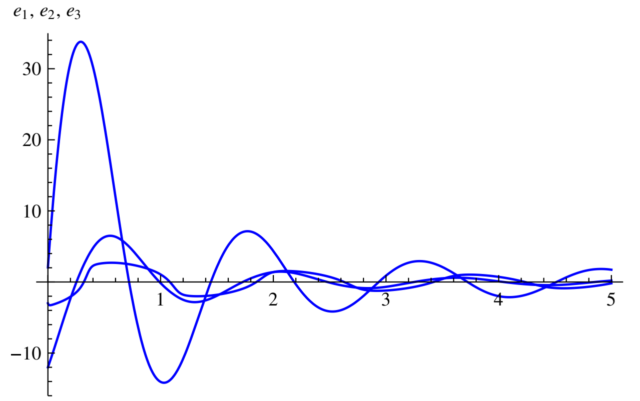

Fig. 12. Synchronization of fractional cubic Chua systems (23) and (24), when $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.95, 0.95)$, $(a, b, c, \beta, \gamma) = (10, 0.1, -0.3, 14, 0.02)$ and $\mathbf{k} = (k_1, k_2, k_3) = (-3e_1, -2e_2, 0.64e_3)$.

of the equilibrium point $\mathbf{e}^* = (0, 0, 0)$ are,

$$\begin{cases} \lambda_1 = -4.38985829, \\ \lambda_2 = 0.00492914 + 2.63500516i, \\ \lambda_3 = 0.00492914 - 2.63500516i. \end{cases} \tag{30}$$

In this case, we have $|\arg(\lambda_{2,3})| = 1.56892568 > (0.95)\pi/2$. Therefore, systems (23) and (24) are synchronized. The error functions evolution is shown in Fig. 12. From Fig. 12, it is obvious that the error system (29) decays towards zero as t goes to $+\infty$. As a result, we can numerically conclude that the design controller can effectively control the chaotic fractional order cubic Chua system to achieve synchronization between drive system (23) and response system (24).

Example 2. If we take $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 1, 0.95)$ and $(a, b, c, \beta, \gamma) = (10.5, 0.15, -0.3, 14, 0.02)$, then, as shown in the previous section, we can verify numerically that, in this case, system (8) exhibits double scroll chaotic attractor.

If we select the matrix $\mathbf{k}$ as $\mathbf{k} = (k_1, k_2, k_3) = (-2e_1 - 10.5e_2, 0.5e_2, 0)$, then we have $l_1 = 19$, $l_2 = 20$, $l_3 = 19$, $m = 20$ and the error system (29) has one equilibrium point $\mathbf{e}^* = (0, 0, 0)$. According to Case 2, the characteristic equation $\det(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \lambda^{m\alpha_3}) - J_e) = 0$ becomes,

$$(\lambda^{19} + 0.425)[(\lambda^{20} + 0.5)(\lambda^{19} + 0.02) + 14] = 0. \tag{31}$$

We can simply show that all roots of the above equation lie in the region $|\arg(\lambda)| > \pi/40$. Therefore,

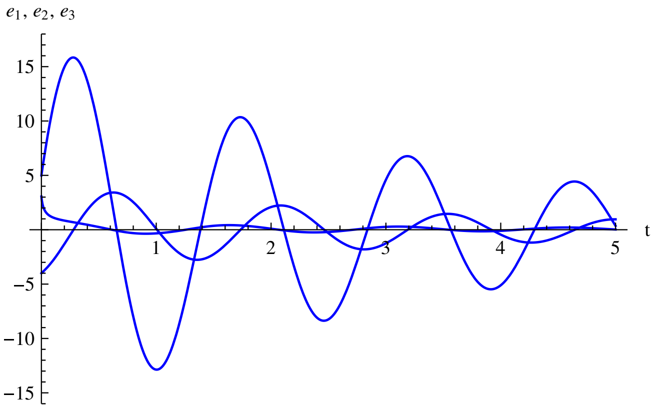

Fig. 13. Synchronization of fractional cubic Chua systems (23) and (24), when $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 1, 0.95)$, $(a, b, c, \beta, \gamma) = (10.5, 0.15, -0.3, 14, 0.02)$ and $\mathbf{k} = (k_1, k_2, k_3) = (-2e_1 - 10.5e_2, -0.5e_2, 0)$.

systems (23) and (24) are synchronized. The error functions evolution is shown in Fig. 13. As observed in the previous example, in which the error system (29) decays towards zero as t goes to $+\infty$, we can numerically conclude that the design controller can effectively control the chaotic fractional order cubic Chua system to achieve synchronization between drive system (23) and response system (24).

Example 3. If we take $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, where $\alpha \in [0.95, 1]$, and $(a, b, c, \beta, \gamma) = (10.8, 0.15, -0.3, 14, 0.02)$, then, in view of Fig. 8, system (8) satisfies the necessary condition for exhibiting double scroll chaotic attractor.

If we select the matrix $\mathbf{k}$ as $\mathbf{k} = (k_1, k_2, k_3) = (-4.5e_1, 2e_2, -1.8e_3)$, then error system (29) has one equilibrium point $\mathbf{e}^* = (0, 0, 0)$, which gives the eigenvalues $\lambda_1 = -3.78728071$ and $\lambda_{2,3} = 0.04364035 \pm 2.01748657i$. So, we get,

$$\frac{2}{\pi}|\arg(\lambda_{2,3})| = \frac{2}{\pi}\tan^{-1}\frac{2.01748657}{0.04364035} = 0.98623139. \quad (32)$$

Therefore, according to Case 1, systems (23) and (24) are synchronized under the control law (26) if $0.98623139 < \alpha < 1$. If $0.95 < \alpha < 0.98623139$, then the equilibrium point $\mathbf{e}^*$ is not stable. Hence, the error system (29) does not decay toward zero as t goes to $+\infty$ and so, systems (23) and (24) are not synchronized.

5. Conclusion

The chaotic dynamics of cubic Chua system with fractional order has been studied in this paper. It is shown that chaos exists in this system with order less than 3. Some necessary conditions for fractional cubic Chua system to exhibit chaotic attractor similar to its integer order counterpart are investigated. This condition is used to distinguish for what parameters and orders, the system generates double scroll chaotic attractors.

Moreover, the control and synchronization problems of chaotic fractional cubic Chua system are addressed. Based on stability results of fractional order systems and using master-slave synchronization scheme, sufficient conditions for global synchronization of fractional cubic Chua systems are given. The designed adaptive nonlinear controller applied to the response system affects the system dynamics to realize synchronization. Numerical simulations are provided to confirm the effectiveness of the presented chaotic and synchronization schemes.

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