Optimal harvesting and stability for a prey–predator model

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ABSTRACT

This paper describes a fish prey–predator model with a new functional response. The dynamics of the system is discussed mainly from the point of view of permanence and stability. We obtain conditions that affect the persistence of the system. Local asymptotic stability of various equilibrium solutions is explored to understand the dynamics of the model system. The global asymptotic stability of positive interior equilibrium solution is established using suitable Lyapunov functional. We then examine possibilities of the existence of bionomic equilibrium. Lastly, the optimal harvesting policy is obtained by using the Pontryagin's maximum principle. The objective is to maximize the monetary social benefit as well as conservation of the ecosystem.

1. Introduction and mathematical model

Predator–prey dynamics are usually represented by a functional response, which is the amount of prey eaten per predator and per unit of time. This functional is a proxy of the flux of matter from one trophic level to another as it determines the transfer of biomass in the food chain [1]. Typically, a predator–prey model focuses on interactions between two species taking into account some aspects that are considered nodal to explain the dynamics. These interactions depend on the nature of the studied species [2–4]. Recently, in [5], authors proposed a new response functional in order to explain the influence of changing water level fluctuations in an artificial lake on fish predator–prey dynamics. In the studied lake, two interdependent species are considered; the pike (brochet in French) which is the most important predator and the roach (gardon in French) which is the prey. This response functional is based on the following general considerations. When a predator attacks a prey, it has access to a certain quantity of food depending on the water level. When the water level is low, during the autumn, the predator is more in contact with the prey, and the predation increases. Conversely, when the water level is high, in the spring, it is more difficult for the predator to find a prey and the predation decreases. It is assumed that the accessibility function $b(t)$ for

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the prey is continuous and 1-periodic, the minimum value b_1 is reached in spring and the maximum value b_2 is attained during autumn. The predator needs a quantity γ as food, but it has access to a quantity

$$g(x,y) = \frac{b(t)x}{y+D},$$

which depends on the water level, where D measures other causes of mortality outside of predation. Thus, if

$$g(x,y) \geq \gamma,$$

then the predator will be satisfied with the quantity γ for his food. Otherwise, if

$$g(x,y) < \gamma,$$

the predator will content himself with

$$g(x,y) = \frac{b(t)x}{y+D}.$$

To summarize, the quantity of food received per predator and per unit of time is

$$\min\left(\frac{b(t)x}{y+D}, \gamma\right). \tag{1}$$

The authors in [5] studied the following non-autonomous prey–predator model

$$\begin{cases} \dot{x} = ax(t)\left(1 - \dfrac{x(t)}{K}\right) - \min\left(\dfrac{b(t)x(t)}{y(t)+D}, \gamma\right)y(t), \\ \dot{y} = -dy(t) + e\min\left(\dfrac{b(t)x(t)}{y(t)+D}, \gamma\right)y(t). \end{cases} \tag{2}$$

The constants mentioned above are all positive. The prey grow logistically with carrying capacity K and intrinsic growth rate a. By using Gaines and Mawhin's continuation theorem of coincidence degree theory [6], the authors have established sufficient conditions for the existence of positive periodic solutions of the prey–predator system (2). Such a solution describes an equilibrium situation consistent with the variability of environmental conditions, such that both populations survive. The trajectories in the phase plane of these solutions of the nonautonomous system take the place of the equilibria points of the autonomous system. In the numerical simulations given in [5], the periodic predation rate function $b(t) = b(1 + 0.5cos(2\pi t))$ is used, for more details, see [5,7–9].

In the present work, we focus on the autonomous case and use as predation rate, the mean function $b = \int_0^1 b(t)dt$. Moreover, to investigate the effects of harvesting on the prey–predator ecosystem, we incorporate and extend the work done by [5]. We aim to obtain some results which are theoretically beneficial to maintaining the sustainable development of the prey–predator system as well as keeping the economic interest of harvesting at an ideal level. Therefore, we study the following prey–predator model:

$$\begin{cases} \dot{x} = ax(t)\left(1 - \dfrac{x(t)}{K}\right) - \min\left(\dfrac{bx(t)}{y(t)+D}, \gamma\right)y(t) - qEx(t) := F_1(x,y), \\ \dot{y} = -dy(t) + e\min\left(\dfrac{bx(t)}{y(t)+D}, \gamma\right)y(t) := F_2(x,y), \end{cases} \tag{3}$$

where q is the catchability coefficient of the prey species and E denotes the effort devoted to the harvesting.

The present article is organized as follows: In Section 2, we focus on the dynamics of the system (3), specifically, we establish sufficient criteria for the boundedness, permanence, and predator extinction. The local and the global stability of the dynamical system for the model are studied in Section 3. In Section 4, the existence of a bionomic equilibrium is investigated. The optimal harvesting policy is studied with the help of Pontryagin's maximum principle in Section 5. Some numerical examples are taken up to illustrate the results. Brief concluding remarks are given in Section 6 to close this work.

2. Mathematical analysis and main result

In this section, we give a qualitative analysis of system (3). From the standpoint of biology, we are only interested in the dynamics of model (3) in the closed first octant $\mathbb{R}_+^2$. Thus, we consider the biologically meaningful initial condition $x(0) = x_0 \geq 0$ and $y(0) = y_0 \geq 0$.

It is easy to see that if $E > \frac{a}{q}$ then $\dot{x} < 0$, hence, throughout this analysis, we make the following assumption on the fishing effort:

$$E < \frac{a}{q}. \tag{H_0}$$

We also assume throughout this work that the predation rate b satisfies:

$$b < min \left\{ \frac{\gamma(y_0 + D)}{x_0}, \frac{4a\gamma Dd}{K(a+d-qE)^2} \right\} := \tilde{b}. \tag{H_1}$$

Biologically, assumption (H_0) means that if the fishing effort increases beyond a threshold value (that is if $E > \frac{a}{q}$), then the two species vanish eventually.

Assumption (H_1) will be used in Subsection 2.1 to prove the persistence of system (3), and means that if the predation rate is less than a threshold value $\tilde{b}$, all the species are present and none of them will go to extinction.

We start by showing that solutions of (3) starting into $\mathbb{R}_+^2$, exist, will remain there and are uniformly bounded. First of all, we state the following lemma:

Lemma 1. *Let $h : (x, y) \to min(g(x, y), \gamma)$. If g is locally Lipschitz, then it is also goes for h.*

Proof. It is easy to see that

$$h(x, y) = min\left(g(x, y), \gamma\right) = \frac{g(x, y) + \gamma - |g(x, y) - \gamma|}{2}.$$

The form of h with respect to g obviously shows that if g is locally Lipschitz, then h is locally Lipschitz. Hence, local existence and uniqueness of solutions of system (3) are obtained for the corresponding Cauchy problem [10]. $\square$

Regarding the positivity and boundedness of the solution for system (3) we have the following lemma:

Lemma 2. 1. *The positive cone $\mathbb{R}_+^2$ is positively invariant for (3).*
2. *All the solutions of system (3) which initiate in $\mathbb{R}_+^2$ are bounded, with ultimate bound.*

Proof. Let the interval $[0, T_{max})$ be the maximal interval of existence of solutions of system (3).

1. From system (3), it follows that $x = 0$ (resp. $y = 0$) is an invariant subset, that is, $x = 0$ (resp. $y = 0$) if and only if $x(t) = 0$ (resp. $y(t) = 0$) for some t. Thus if $x(0) > 0$ (resp. $y(0) > 0$), then $x(t) > 0$ (resp. $y(t) > 0$) for all $t \in [0, T_{max})$.

2. Let us consider $w(t) = ex(t) + y(t)$, then the time derivative along the solutions of the system (3) is given by

$$\frac{dw}{dt} = e\frac{dx}{dt} + \frac{dy}{dt} = eax\left(1 - \frac{x}{K}\right) - eqEx - dy.$$

Hence

$$\frac{dw}{dt} + dw = ex\left[(a - qE + d) - \frac{a}{K}x\right] \leq e\frac{K}{4a}(a - qE + d)^2 := \mu$$

where μ is the maximum value of the function $ex\left[(a-qE+d)-\frac{a}{K}x\right]$. By using the differential inequality [10], we obtain

$$0 \leq w(t) \leq e^{-dt}w(0) + \frac{\mu}{d}(1 - e^{-dt})$$
$$\leq max\left(w(0), \frac{\mu}{d}\right).$$

From the well known extension theorem, we have $T_{max} = \infty$, therefore the solutions are bounded. Moreover, we have $\lim_{t\to\infty}w(t) \leq \frac{\mu}{d}$, which is independent of the initial condition.

Hence all the solutions of (3) that initiate in $\mathbb{R}^2_+$ are confined in the region

$$B = \left\{(x,y) \in \mathbb{R}^2_+ : 0 \leq w = ex + y \leq \frac{\mu}{d} + \varepsilon\right\},$$

for any $\epsilon > 0$ as $t \to \infty$.

2.1. Persistence and permanence

In this subsection, we analyze the persistence (weak and strong) and permanence behavior of system (3).

Definition 1 (*Persistence*). System (3) is said to be weakly persistent if every solution $(x(t), y(t))$ satisfies two conditions:

(i) $x(t) \geq 0, y(t) \geq 0, \forall t \geq 0$.

(ii) $\limsup_{t\to+\infty} x(t) > 0, \limsup_{t\to+\infty} y(t) > 0$.

System (3) is said to be strongly persistent if every solution $(x(t), y(t))$ satisfies the following condition along with the first condition of the weak persistence:

(iii) $\liminf_{t\to+\infty} x(t) > 0, \liminf_{t\to+\infty} y(t) > 0$.

Definition 2 (*Permanence and non-permanence*). System (3) is said to be permanent if there exist positive constants $0 < m \leq M$ such that,

$$min\left\{\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right\} \geq m, \qquad max\left\{\limsup_{t\to+\infty} x(t), \limsup_{t\to+\infty} y(t)\right\} \leq M$$

for all solutions $(x(t), y(t))$ of system (3) with positive initial values.

System (3) is said to be non-permanent if there is a positive solution $(x(t), y(t))$ of (3) and such that,

$$min\left\{\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right\} = 0.$$

Geometrically, persistence means that trajectories that initiate in a positive cone are eventually bounded away from coordinate planes. On the other hand, permanently coexistence (uniform persistence) implies the existence of a region in the phase space at a non-zero distance from boundary in which population vectors must lie ultimately. The last ensures the survival of species in biological sense.

To establish the persistence for system (3), we need to recall the following lemma, whose proof can be found in [11].

Lemma 3. *If $a, b > 0$ and $\frac{dX}{dt} \leq (resp. \geq)X(t)(a - bX(t))$, with $X(0) > 0$, then we have*

$$\limsup_{t\to\infty} X(t) \leq \frac{a}{b}\left(resp. \liminf_{t\to\infty} X(t) \geq \frac{a}{b}\right).$$

Let us denote:

$$m_1 \stackrel{def}{=} \frac{K}{a}(a - qE - b),$$
$$m_2 \stackrel{def}{=} \frac{ebm_1}{d} - D,$$
$$M_1 \stackrel{def}{=} K,$$
$$M_2 \stackrel{def}{=} \frac{ebM_1}{d} - D,$$

then $M_i > m_i, i = 1, 2$. We will show that $max\{m_i, 0\}(i = 1, 2)$ are the lower bounds for the limiting bounds of species, as time t goes to infinity. This is obvious when $m_i \leq 0$. Therefore, it is assumed that:

$$m_i > 0, i = 1, 2. \tag{H_2}$$

Our main result is stated in the following proposition.

Proposition 1. *In addition to* (H_0), (H_1), *assume further that* (H_2) *holds. Then system* (3) *is permanent, i.e., any positive solution* $(x(t), y(t))$ *of system* (3) *satisfies*

$$0 < m_1 \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) \leq M_1,$$
$$0 < m_2 \leq \liminf_{t \to \infty} y(t) \leq \limsup_{t \to \infty} y(t) \leq M_2.$$

Proof. As the variables x, y are positive, from the first equation of system (3), it follows that:

$$\frac{dx}{dt} \leq ax\left(1 - \frac{x}{K}\right),$$

using Lemma 3, we get

$$\limsup_{t \to \infty} x(t) \leq M_1. \tag{4}$$

Thus, for arbitrary $\varepsilon_1 > 0$, there exists a positive real number T_1 such that

$$x(t) \leq M_1 + \varepsilon_1, \forall t \geq T_1.$$

Further, from the predator equation, it follows that for $t \geq T_1$,

$$\begin{aligned}
\frac{dy}{dt} &\leq y\left(-d + \frac{eb(M_1 + \varepsilon_1)}{y + D}\right), \\
&= \frac{y}{y + D}\left(eb(M_1 + \varepsilon_1) - dD - dy\right), \\
&\leq \frac{1}{D}y\left(eb(M_1 + \varepsilon_1) - dD - dy\right).
\end{aligned}$$

Using Lemma 3 and the arbitrariness of ε_1, we obtain

$$\limsup_{t \to \infty} y(t) \leq M_2. \tag{5}$$

Thus, for arbitrary $\varepsilon_2 > 0$, there exists a positive real number $T_2 > T_1$, such that

$$y(t) \leq M_2 + \varepsilon_2, \forall t \geq T_2.$$

Hence, system (3) is dissipative.

Before proving the strongly persistence of system (3), we give the following result.

Proposition 2. *Under hypothesis* (H_1), *we have for all* $t \geq 0$,

$$bx(t) < \gamma(y(t) + D).$$

Proof. Let

$$u(t) = bx(t) - \gamma(y(t) + D),$$

note that $u(0) < 0$ by assumption (H_1). It is claimed that $u(t) < 0$ for all $t \geq 0$. If this was not the case, there exists $t_0 > 0$ such that:

$$u(t_0) = 0 \quad \text{and} \quad \frac{du}{dt}(t_0) \geq 0.$$

The condition $u(t_0) = 0$ implies that

$$y(t_0) = \frac{bx(t_0)}{\gamma} - D.$$

From (3), we get

$$\begin{aligned}
\frac{du}{dt}(t_0) &= b\frac{dx}{dt}(t_0) - \gamma\frac{dy}{dt}(t_0) \\
&= -b(b + e\gamma)\frac{y(t_0)}{y(t_0) + D}x(t_0) + b(a + d - qE)x(t_0) - \gamma dD - \frac{ba}{K}(x(t_0))^2,
\end{aligned}$$

it follows that

$$\frac{du}{dt}(t_0) \leq -\frac{ba}{K}(x(t_0))^2 + b(a + d - qE)x(t_0) - \gamma dD.$$

Condition (H_1) implies that

$$\frac{du}{dt}(t_0) < 0,$$

which leads to a contradiction. Therefore $u(t) < 0$ for all $t \geq 0$. $\square$

Consequently under hypothesis (H_1) system (3) is reduced to the simple form

$$\begin{cases}
\dfrac{dx}{dt} = ax\left(1 - \dfrac{x}{K}\right) - \dfrac{bxy}{y + D} - qEx, \\
\dfrac{dy}{dt} = -dy + \dfrac{ebxy}{y + D}.
\end{cases} \tag{6}$$

Now, we come back to the proof of the (strongly) persistence of system (6) (which is equivalent to system (3) under hypothesis (H_1)).

According to the first equation of system (6), it is easy to see that

$$\begin{aligned}
\frac{dx}{dt} &= x\left(a - \frac{ax}{K} - \frac{by}{y + D} - qE\right) \\
&\geq x(a - qE - b - \frac{ax}{K}).
\end{aligned}$$

Using Lemma 3, we obtain

$$\liminf_{t \to \infty} x(t) \geq m_1. \tag{7}$$

For arbitrary $\varepsilon_3 > 0$, there exists a positive real number T_3 such that

$$x(t) \geq m_1 - \varepsilon_3, \forall t \geq T_3. \tag{8}$$

Thus, by applying (8) to the second equation of system (6), we obtain

$$\frac{dy}{dt} \geq \frac{y}{y+D}\Big(eb(m_1 - \epsilon_3) - dD - dy\Big),$$

and for $t \geq T_4 = max\{T_2, T_3\}$, we get

$$\frac{dy}{dt} \geq \frac{1}{M_2 + \epsilon_2 + D} y\Big(eb(m_1 - \epsilon_3) - dD - dy\Big), \forall t \geq T_4.$$

Using Lemma 3 and the arbitrariness of ε_2 and ε_3, we obtain

$$\liminf_{t \to \infty} y(t) \geq \frac{ebm_1}{d} - D := m_2. \tag{9}$$

Eqs. (4), (5), (7) and (9), show that under the assumption of Proposition 1, system (3) is permanent. □
    In the next proposition, we are able to give sufficient conditions under which the given system is not persistent.

Proposition 3. *If $M_2 < 0$, then $\lim_{t \to \infty} y(t) = 0$, that is, the predator goes to extinction.*

Proof. Using the upper bounds for x, from the predator equation, we have

$$\frac{dy}{dt} \leq y\Big(-d + \frac{ebM_1}{D}\Big),$$

then

$$y(t) \leq y_0 e^{\left(-d + \frac{ebM_1}{D}\right)t}.$$

Thus, under the given hypothesis, $y(t) \to 0$ as $t \to \infty$. That is predator goes to extinction. □

Remark 1. Biologically, it means that, when the predation rate is enough small, the predator disappears.

3. Steady states and their existence

    System (6) possesses the following three equilibria:
    (i) The trivial equilibrium $P^0 = (0,0)$.
    (ii) The predator free equilibrium $P^1 = (\bar{x}, 0)$, where $\bar{x} = \frac{K}{a}(a - qE)$.
    (iii) The steady state of coexistence (interior equilibrium point) $P^* = (x^*, y^*)$. The last is the point of intersection of the prey zero growth rate isocline (i.e., $\frac{dx}{dt} = 0$) and the predator zero growth rate isocline (i.e., $\frac{dy}{dt} = 0$) given by

$$\begin{cases} a\left(1 - \dfrac{x^*}{K}\right) - \dfrac{by^*}{y^*+D} - qE = 0, \\ -d + \dfrac{ebx^*}{y^*+D} = 0, \end{cases} \tag{10}$$

where $y^* = \frac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right)$, $x^* = \frac{d}{eb}(y^* + D)$

and
$$B = 2D - \frac{ebK(a-qE-b)}{ad} = D - m_2, C = D^2 - \frac{ebKD(a-qE)}{ad}.$$

Note that $\sqrt{B^2 - 4C}$ is always positive, then, the   interior equilibrium is positive if one of the two cases holds:

1. $B < 0$, $(m_2 > D)$ which is equivalent to $0 < E < \frac{a}{q}(1 - \frac{2dD}{ebK}) - \frac{b}{q}$,
2. $B \geq 0$ and $C < 0$, that is $0 < \frac{a}{q}(1 - \frac{2dD}{ebK}) - \frac{b}{q} \leq E < \frac{a}{q}(1 - \frac{dD}{ebK})$.

Hence, the interior equilibrium is positive if and only if condition

$$0 < E < \frac{a}{q}\left(1 - \frac{dD}{ebK}\right), \tag{H_3}$$

holds.

3.1. Dynamical behavior: stability analysis

The stability of the equilibrium state is determined by the nature of the eigenvalues of the Jacobian matrix around the equilibrium point.

Proposition 4. 1. *The equilibrium point P^0 is always a saddle point.*
    2. *The point P^1 is stable iff*

$$E > \frac{a}{q}\left(1 - \frac{dD}{ebK}\right). \tag{H_3^c}$$

3. *The Steady state P^* is locally asymptotically stable when it exists.*

Proof.
To obtain the local stability results, we use the Jacobian matrix associated to system (6)

$$J(x,y) = \begin{pmatrix} a - qE - \dfrac{2ax}{K} - \dfrac{by}{y+D} & \dfrac{-bDx}{(y+D)^2} \\ \dfrac{eby}{y+D} & -d + \dfrac{ebDx}{(y+D)^2} \end{pmatrix}.$$

The Jacobian matrix of the equilibrium $P_0 = (0,0)$ is

$$J(0,0) = \begin{pmatrix} a - qE & 0 \\ 0 & -d \end{pmatrix}.$$

Hence, P_0 is a saddle point.

• The Jacobian matrix of the equilibrium $P_1 = (\bar{x}, 0)$ is

$$J(\bar{x},0) = \begin{pmatrix} -(a-qE) & -\dfrac{bK(a-qE)}{Da} \\ 0 & -d + \dfrac{ebK(a-qE)}{Da} \end{pmatrix}.$$

If (H_3^c) holds, then P_1 is stable, and there is no interior equilibrium, otherwise, if

$$E < \frac{a}{q}\left(1 - \frac{dD}{ebK}\right),$$
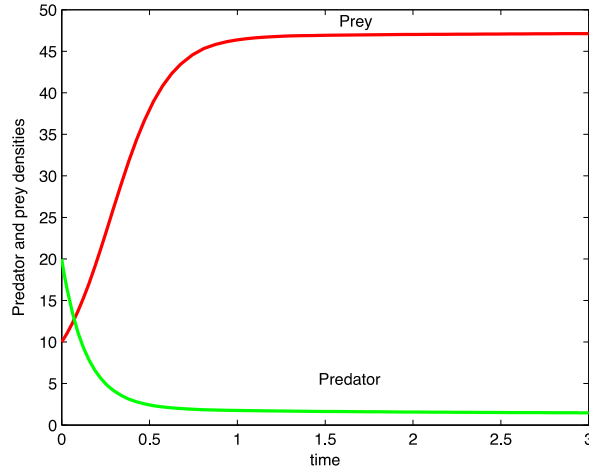
then P_1 is unstable.

Fig. 1. Both the prey and predator populations converge to their equilibrium values. $a = 12, b = 10, K = 20, e = 1.25, p = 1, D = 4, q = 6, E = 0.15$.

● The Jacobian matrix for P^* is

$$J\left(x^*, y^*\right) = \begin{pmatrix} -\dfrac{ax^*}{K} & -\dfrac{bDx^*}{(y^* + D)^2} \\ \dfrac{eby^*}{y^* + D} & -\dfrac{ebx^*y^*}{(y^* + D)^2} \end{pmatrix}.$$

It is easy to see that the trace of $J\left(x^*, y^*\right)$ is

$$trJ\left(x^*, y^*\right) = -\frac{ax^*}{K} - \frac{ebx^*y^*}{(y^* + D)^2} < 0,$$

and its determinant is

$$detJ\left(x^*, y^*\right) = \frac{aebx^{*2}y^*}{K(y^* + D)^2} + \frac{eb^2Dx^*y^*}{(y^* + D)^3} > 0.$$

Hence (x^*, y^*) is locally asymptotically stable whenever it exists (see Figs. 1 and 2). □

3.2. Global stability

In this subsection, we shall establish the global asymptotic stability of the co-existing equilibrium point P^* by constructing a suitable Lyapunov function.

Theorem 1. *If conditions (H_0)–(H_3) hold, then, the co-existing equilibrium point P^* is globally asymptotically stable.*

Proof. Let

$$V(x, y) = \left[(x - x^*) - x^* \log(\frac{x}{x^*})\right] + \alpha \left[(y - y^*) - y^* \log\left(\frac{y}{y^*}\right)\right],$$

where α is a positive constant to be chosen suitably in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium (x^*, y^*) and is positive for all other positive values of x, y.
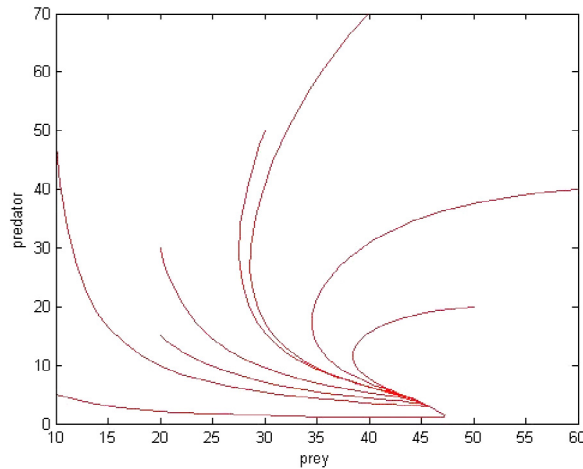
Fig. 2. Phase space trajectories corresponding to different initial conditions.

The time derivative of V along the trajectories of (6) is given by

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{x - x^*}{x}\frac{dx}{dt} + \alpha\frac{y - y^*}{y}\frac{dy}{dt} \\
&= (x - x^*)\left[(a - qE) - \frac{a}{K}x - \frac{by}{y + D}\right] + \alpha\,(y - y^*)\left[-d + \frac{ebx}{y + D}\right].
\end{aligned}
\tag{11}
$$

Also we have the set of equilibrium equations corresponding to the steady state $P_2 = (x^*, y^*)$:

$$
\begin{aligned}
a\left(1 - \frac{x^*}{K}\right) - \frac{by^*}{y^* + D} - qE &= 0, \\
-d + \frac{ebx^*}{y^* + D} &= 0.
\end{aligned}
\tag{12}
$$

We can write Eq. (11) together with the above two equations in the form:

$$
\frac{dV}{dt} = (x - x^*)\left[-\frac{a}{K}x - \frac{by}{y + D} + \frac{a}{K}x^* + \frac{by^*}{y^* + D}\right] + \alpha\,(y - y^*)\left[\frac{ebx}{y + D} - \frac{ebx^*}{y^* + D}\right].
$$

By choosing

$$
\alpha = \frac{D}{e(y^* + D)},
$$

we obtain

$$
\frac{dV}{dt} = -\left[\frac{a}{K}(x - x^*)^2 + \frac{ebhx^*}{(y + D)(y^* + D)}(y - y^*)^2\right]
$$

thus, $\frac{dV}{dt} < 0$ strictly for all $x, y > 0$ except the interior equilibrium point (x^*, y^*) where $\frac{dV}{dt} = 0$. Thus $V(x, y)$ satisfies Lyapunov's asymptotic stability theorem [10], and the interior equilibrium point P^* of system (6) is globally asymptotically stable. $\square$

Remark 2. The consequence of global stability is that exploitation will not irreversibly change the system. As long as the prey are not made extinct by excessive exploitation of their food supply, the system is able to recover.

4. Bionomic equilibrium

As we have already seen, a biological equilibrium is given by $\dot{x} = \dot{y} = 0$. The bionomic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost utilized in harvesting it [12].

Let c be a constant fishing cost per unit effort and p the constant price per unit biomass of prey fish. Then, the economic rent (net revenue) at any time is given by,

$$\pi = (pqx - c) E. \tag{13}$$

The bionomic equilibrium is $P_\infty(x_\infty, y_\infty, E_\infty)$, where $x_\infty, y_\infty, E_\infty$ are the positive solutions of

$$\begin{aligned} a\left(1 - \frac{x}{K}\right) - \frac{by}{y + D} - qE &= 0, \\ -d + \frac{ebx}{y + D} &= 0, \\ \pi = (pqx - c)E &= 0. \end{aligned} \tag{14}$$

It may be noted here that if $c > pqx$, i.e., if fishing cost exceeds the revenue obtained from it, then the economic rent obtained from the fishery becomes negative. Hence the fishery will be closed and no bionomic equilibrium exists. Therefore, in order to achieve the bionomic equilibrium, the following inequality must hold $c < pqx$.

Solving the above equations, we get,

$$x_\infty = \frac{c}{pq}, \tag{15}$$

$$y_\infty = \frac{ebc}{pqd} - D, \tag{16}$$

where $y_\infty > 0$, if

$$\frac{ebc}{pqd} - D > 0 \tag{H_4}$$

and

$$E_\infty = \frac{1}{q}\left[a\left(1 - \frac{c}{pqK}\right) - \frac{by_\infty}{y_\infty + D}\right]. \tag{17}$$

Since $c < pqx < pqK$, therefore $1 - \frac{c}{pqK} > 0$, and since (H_4) is satisfied, hence
$E_\infty > 0$, if

$$a\left(1 - \frac{c}{pqK}\right) > \frac{by_\infty}{y_\infty + D}. \tag{H_5}$$

Therefore, we have the following theorem.

Theorem 2. *The bionomic equilibrium* $P_\infty(x_\infty, y_\infty, E_\infty)$ *exists if in addition to* (H_0)–(H_3), *conditions* (H_4) *and* (H_5) *hold together.*

Interpretation. If $E > E_\infty$, then the total cost utilized in harvesting the prey population would exceed the total revenues obtained from the fishery. Hence, some of the fishermen would be in loss and, naturally, they would withdraw their participation from the fishery. Hence $E > E_\infty$ cannot be maintained indefinitely. If conversely $E < E_\infty$, then the fishery is more profitable, and hence in an open access fishery it would attract more and more fishermen. This will have an increasing effect on the harvesting effort. Hence $E < E_\infty$ also cannot be maintained indefinitely (see Fig. 3).
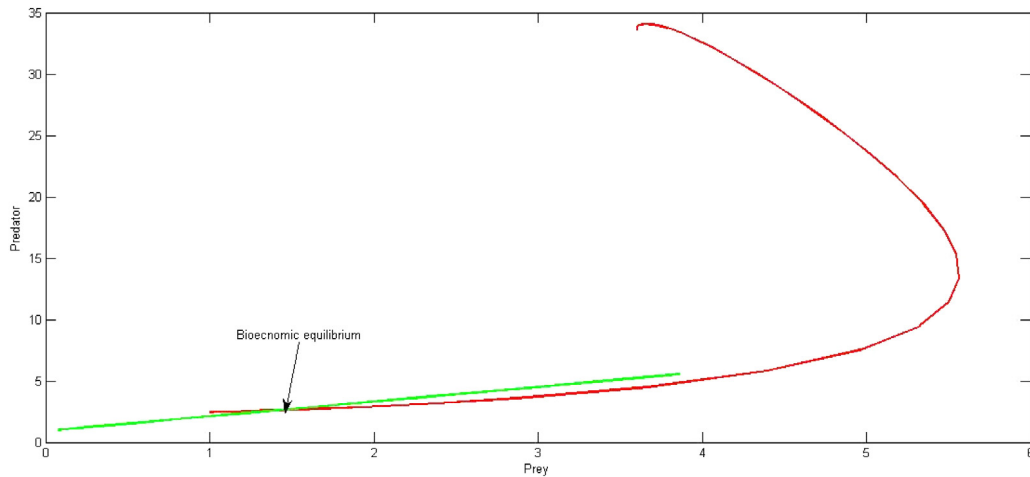
Fig. 3. Phase diagram showing the unique bionomic equilibrium. $a = 12, b = 10, K = 20, e = 1.25, p = 1, D = 4, q = 6, E = 0.15$.

5. Optimal harvesting policy

The fundamental problem in the determination of an optimal harvest policy in a commercial fishery is to determine the optimal trade-off between the current and future harvests [12–18]. As observed by Clark [12], this problem, which is the very essence of resource conservation, is an exceedingly difficult one, not from a mathematical point of view perhaps, but certainly from a political and philosophical viewpoint. The standard device used to handle questions of inter temporal economic benefits is time discounting. Although there is considerable controversy as to the social justifiability of this concept [19], time discounting is a normal practice in business management.

To determine an optimal harvesting policy, we consider the present value J of a continuous time-stream of revenues, given by

$$J(E) = \int_0^\infty e^{-\delta t} \pi(x, y, E) dt, \tag{18}$$

where π is given by $\pi = (pqx - c)E$ and δ is the instantaneous annual rate of discount.

Let us denote by E_δ an optimal control with corresponding states x_δ and y_δ. We take $A_\delta = (x_\delta, x_\delta)$ as optimal equilibrium point. Here we intend to derive optimal control E_δ such that

$$J(E_\delta) = max\{J(E), E \in V\},$$

where $V = [0, E_{max}]$ is the control set, and E_{max} is a feasible upper limit for the harvesting effort.

Now the Hamiltonian of this optimal control problem is

$$H = e^{-\delta t} \left(pqx - c\right) E + \lambda_1 \left(ax \left(1 - \frac{x}{K} \right) - \frac{bxy}{y + D} - qEx \right)$$

$$+ \lambda_2 \left(-dy + \frac{ebxy}{y + D} \right), \tag{19}$$

where λ_1 and λ_2 are the adjoint variables.

The objective functional and the differential equations are linear in the control with bounded states, and one can show by standard results that an optimal control and corresponding optimal states exist [20].

Since the Hamiltonian is linear in the control, we must consider if the optimal control is bang–bang (at its lower or upper bound), singular or a combination. The singular case could occur if the slope or the switching function

$$\frac{\partial H}{\partial E} = e^{-\delta t}\left(pqx - c\right) - \lambda_1 qx := \sigma(t) \tag{20}$$

is zero on non-trivial interval of time. Note that the optimal control would be at its upper bound or its lower bound according to:

$$\sigma(t) > 0 \text{ or } < 0.$$

To investigate the singular case, let us suppose $\sigma(t) = 0$ on some non-trivial interval. In this case, the optimal harvesting policy is

$$E(t) = \begin{cases} E_{max} & \text{if } \sigma(t) > 0, \\ E^* & \text{if } \sigma(t) = 0, \\ 0 & \text{if } \sigma(t) < 0. \end{cases}$$

When $\sigma(t) = 0$, it follows that

$$\lambda_1 qx = e^{-\delta t}\left(pqx - c\right) = e^{-\delta t}\frac{\partial \pi}{\partial E}. \tag{21}$$

This implies that the users cost of harvest per unit of effort equals the discounted value of the future marginal profit of the effort at the steady-state level.

By the maximum principle [21], the adjoint variables satisfy

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x}$$
$$= -\left[e^{-\delta t}pqE + \lambda_1\left\{a - \frac{2ax}{K} - \frac{by}{y+D} - qE\right\} + \lambda_2\left\{\frac{eby}{y+D}\right\}\right].$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y}$$
$$= -\left[\lambda_1\left\{\frac{-bDx}{(y+D)^2}\right\} + \lambda_2\left\{-d + \frac{ebDx}{(y+D)^2}\right\}\right]. \tag{22}$$

We seek to find the optimal equilibrium solution of the problem, hence x, y and E are to be treated as constants in the subsequent steps.

Considering the interior equilibrium $P_2 = (x^*, y^*)$, for singular control, we have $\frac{\partial H}{\partial E} = 0$. This gives

$$\lambda_1 = e^{-\delta t}\left(p - \frac{c}{qx^*}\right). \tag{23}$$

The shadow price $\lambda_1 e^{\delta t}$ remains bounded as $t \to \infty$, hence it satisfies the transversality condition at ∞.

Now, (23) can be written as $\lambda_1 = A_1 e^{-\delta t}$, where $A_1 = p - \frac{c}{qx^*}$.

Similarly, considering the interior equilibrium $P_2 = (x^*, y^*)$, from (22), we get

$$\frac{d\lambda_2}{dt} - A_2\lambda_2 = -A_1 A_3 e^{-\delta t}, \tag{24}$$

whose solution is given by

$$\lambda_2(t) = \frac{A_1 A_3}{A_2 + \delta}e^{-\delta t}, \tag{25}$$

where $A_1 = p - \frac{c}{qx^*}$, $A_2 = \frac{ebDx^*}{(y^*+D)^2} - d$ and $A_3 = -\frac{bDx^*}{(y^*+D)^2}$,
which also satisfies the transversality condition.

The singular path is given by

$$\left(p - \frac{c}{qx^*}\right) = A_1, \tag{26}$$

which can be written as $G(x^*) = \left(p - \frac{c}{qx^*}\right) - A_1 = 0$.

There exists a unique positive root $x^* = x_\delta$ of $G(x^*) = 0$ in the interval $0 < x_\delta < K$ if the following inequalities hold:

$$\lim_{x \to 0^+} G(x) < 0, \quad G(K) > 0, \quad G'(x) > 0 \quad for \quad x > 0.$$

For $x^* = x_\delta$, we get

$$y_\delta = \frac{eb}{d} x_\delta - D,$$

where

$$y_\delta > 0 \quad if \quad x_\delta > \frac{dD}{eb}$$

and

$$E_\delta = \frac{1}{q}\left[a\left(1 - \frac{x_\delta}{K}\right) - \frac{by_\delta}{y_\delta + D}\right],$$

which verifies

$$E_\delta > 0 \quad if \quad a\left(1 - \frac{x_\delta}{K}\right) > \frac{by_\delta}{y_\delta + D}.$$

6. Concluding remarks

In the present paper, we have considered and analyzed a predator–prey system with harvesting taking into account the quantity of food received by the predator. This work can be looked upon as an extension of the work in [5,7]. The main modification here is that the predation rate in the functional response is chosen as the average of predation rate in one year and prey population is subjected to harvesting. The objectives were the analysis of the dynamical properties of different equilibrium points of the system, and studying the harvesting strategy that results in maximizing the profit without leading to extinction. Our investigation indicates that fishing effort and predation rate play an important role to change different steady state behaviors.

Firstly, we showed that if the fishing effort increases beyond a threshold value, that is, if $E > \frac{a}{q}$, then both species will become extinct and the system will not be permanent. To avoid this situation, hypothesis (H_0) was assumed.

Also, in order to simplify our study, we have made few assumptions, assumed that the predation rate is less than a critical value given by condition (H_1) which depends on biological parameters.

Secondly, with use of the stability theory of ordinary differential equations, we proved that the interior equilibrium of the reduced model exists under certain conditions, and it is globally asymptotically stable. On the other hand, it is important to note that the presence of harvesting can impact the existence and the behavior of the positive equilibrium, that is for $0 \le E \le \frac{a}{q}\left(1 - \frac{dD}{ebK}\right)$, the two fish populations can be maintained at an appropriate equilibrium level in the habitat. As harvesting becomes larger than the level $\frac{a}{q}\left(1 - \frac{dD}{ebK}\right)$, overharvesting can lead to the extinction of the predators. Indeed, the density of the prey

population decreases with increasing effort used to harvesting, which leads to a more difficult situation for the predator to find a prey, and to possible extinction. The major contributions of this paper are as follows:
1. The persistence of the species depends on two factors. The first is biological and concerns the predation rate b, indeed, we have shown that the threshold between persistence and extinction depends critically on the predation rate b, if the predation rate is between two levels (i.e., (H_1), (H_2) hold), then both prey and predator population coexist. However, when predation rate is less than a critical value given by Proposition 3, then the predator goes to extinction. The second is linked to the exploitation and mechanisms that reduce the fishing effort (i.e., (H_3) holds). The last can be reduced by limiting the time for fishing, reducing the capacity that a vessel can carry, or installing protected area.
2. The ecosystem is often altered by human activities. We analyzed the harvesting strategy that results in maximizing the profit and does not lead to extinction. We obtained the optimal harvesting equation by using Pontryagin's maximum principle.

Finally, we can easily check the compatibility of all hypotheses used in this work. The results and the methodological framework outlined here will provide a useful tool to investigate the consequences for particular real systems in future work. It may also be pointed out that in this paper several important parameters such as refuge, interaction with other species, etc. are disregarded. Hence, further research is necessary to accomplish the needs in this field.

Acknowledgments

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