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Boundedness and Global Stability for a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes

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Abstract—We present a two-dimensional continuous time dynamical system modeling a predator-prey food chain, and based on a modified version of the Leslie-Gower scheme and on the Holling-type II scheme. The main result is given in terms of boundedness of solutions, existence of an attracting set and global stability of the coexisting interior equilibrium.

Keywords—Differential equations, Leslie-Gower, Holling-type-II, Boundedness, Lyapunov function, Stability.

1. INTRODUCTION AND MATHEMATICAL MODEL

The goal of this paper is to introduce and to give a first study of a two-dimensional system of autonomous differential equations modeling a predator-prey system. This model incorporates a modified version of the Leslie-Gower functional response as well as that of the Holling-type II. Recently, the Leslie-Gower scheme has recovered some interest; see [1–3] in which only numerical studies for a Leslie-Gower-type tritrophic model is done. A clear analytical study is harder to get, because of the complex mathematical expressions involved in the analysis. However, in [4], a beginning of the analytical study of the system numerically studied in [1] is established. We also note that in [5] a global stability of a simpler predator-prey model with the Leslie-Gower term has been done.

The system presented here is rather different. Given some reasonable restrictions on the model, we determine conditions and establish results for boundedness, existence of a positively invariant and attracting set and the global stability of the coexisting interior equilibrium.

This two-species food chain model describes a prey population x which serves as food for a predator y. The rate equations for the two components of the chain population can be written

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as follows:

$$\begin{aligned}\frac{dx}{dt} &= \left(r_1 - b_1 x - \frac{a_1 y}{x + k_1}\right) x,\\ \frac{dy}{dt} &= \left(r_2 - \frac{a_2 y}{x + k_2}\right) y,\end{aligned} \tag{1}$$

with $x(0) \geq 0$ and $y(0) \geq 0$, where x and y represent the population densities at time t; r_1, a_1, b_1, k_1, r_2, a_2, and k_2 are model parameters assuming only positive values. These parameters are defined as follows: r_1 is the growth rate of prey x, b_1 measures the strength of competition among individuals of species x, a_1 is the maximum value which *per capita* reduction rate of x can attain, k_1 (respectively, k_2) measures the extent to which environment provides protection to prey x (respectively, to predator y), r_2 describes the growth rate of y, and a_2 has a similar meaning to a_1.

The system we study in the present paper may, for example, be considered as a representation of an insect pest-spider food chain, nature abounds in systems which exemplify this model; see [3,6].

Let us mention that the first equation of system (1) is standard. By contrast, the second equation is absolutely not standard. It contains a modified Leslie-Gower term, that is the second term on the right-hand side in the second equation of (1); the last depicts the loss in the predator population.

The Leslie-Gower formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Indeed, Leslie [7] introduced a predator-prey model where the carrying capacity of the predator's environment is proportional to the number of prey. This interesting formulation for the predator dynamics has been discussed by Leslie and Gower in [8] and by Pielou in [9]. It is $\frac{dy}{dt} = r_2 y(1 - y/\alpha x)$, in which the growth of the predator population is of logistic form (i.e., $\frac{dy}{dt} = r_2 y(1 - y/C)$), but the conventional 'C', which measures the carrying capacity set by the environmental resources is $C = \alpha x$, proportional to prey abundance (α is the conversion factor of prey into predators). The term $y/\alpha x$ of this equation is called the Leslie-Gower term. It measures the loss in the predator population due to rarity (per capita y/x) of its favorite food. In the case of severe scarcity, y can switch over to other populations but its growth will be limited by the fact that its most favorite food (x) is not available in abundance. This situation can be taken care of by adding a positive constant d to the denominator. Hence, the equation above becomes $\frac{dy}{dt} = r_2 y(1 - y/(\alpha x + d))$, and thus, $\frac{dy}{dt} = y(r_2 - (r_2/\alpha)(y/(x + d/\alpha)))$; that is, the third equation of system (1), $\frac{dy}{dt} = (r_2 - a_2 y/(x + k_2))y$.

2. BOUNDEDNESS OF THE MODEL AND EXISTENCE OF A POSITIVELY INVARIANT ATTRACTING SET

We denote by $\mathbb{R}^2_+$ the nonnegative quadrant, and by $\mathrm{Int}(\mathbb{R}^2_+)$ the positive quadrant.

LEMMA 1. *Positive quadrant* $\mathrm{Int}(\mathbb{R}^2_+)$ *is invariant for system (1).*

PROOF. We first observe that the boundaries of the nonnegative quadrant $\mathbb{R}^2_+$ are invariant; this is obvious from system (1). Therefore, the densities $x(t)$ and $y(t)$ are positive: for $t \geq 0$, if $x(0) > 0$ and $y(0) > 0$, then $x(t) > 0$ and $y(t) > 0$. The basic existence and uniqueness theorem for differential equations ensures that positive solutions and the axis cannot intersect. ∎

We will show that, under some assumptions, the solutions of system (1) which start in $\mathbb{R}^2_+$ are ultimately bounded. First let us give the following (classical) comparison lemma the proof of which may be found, for example, in [4].

LEMMA 2. *Let ϕ be an absolutely-continuous function satisfying the differential inequality*

$$\frac{d\phi(t)}{dt} + \alpha_1 \phi(t) \leq \alpha_2, \quad t \geq 0, \qquad \text{where } (\alpha_1, \alpha_2) \in \mathbb{R}^2, \quad \alpha_1 \neq 0.$$

Then,

$$\forall t \geq \tilde{T} \geq 0, \qquad \phi(t) \leq \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1} - \phi\left(\tilde{T}\right)\right) e^{-\alpha_1 (t-\tilde{T})}.$$

DEFINITION 3. *A solution $\phi(t, t_0, x_0, y_0)$ of system (1) is said to be ultimately bounded with respect to $\mathbb{R}_+^2$, if there exists a compact region $\mathcal{A} \in \mathbb{R}_+^2$ and a finite time T ($T = T(t_0, x_0, y_0)$) such that, for any $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+^2$, $\phi(t, t_0, x_0, y_0) \in \mathcal{A}$ for all $t > T$.*

THEOREM 4. *Let $\mathcal{A}$ be the set defined by*

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}_+^2 : 0 \leq x \leq \frac{r_1}{b_1},\ 0 \leq x + y \leq L_1\right\},$$

where

$$L_1 = \frac{1}{4a_2 b_1}\left(a_2 r_1 (r_1 + 4) + (r_2 + 1)^2 (r_1 + b_1 k_2)\right).$$

Then,

(a) *$\mathcal{A}$ is positively invariant, and*
(b) *all solutions of (1) initiating in $\mathbb{R}_+^2$ are ultimately bounded with respect to $\mathbb{R}_+^2$ and eventually enter the attracting set $\mathcal{A}$.*

PROOF.

(i) Letting $(x(0), y(0)) \in \mathcal{A}$, we will show that $(x(t), y(t)) \in \mathcal{A}$ for all $t \geq 0$. Obviously, from Lemma 1, as $(x(0), y(0)) \in \mathcal{A}$, $(x(t), y(t))$ remain nonnegative. We then have to show that for all $t \geq 0$, $x(t) \leq r_1/b_1$ and $x(t) + y(t) \leq L_1$.

(a1) We first prove that $x(t) \leq r_1/b_1$, for all $t \geq 0$. Since $x > 0$ and $y > 0$ in $\mathrm{Int}(\mathbb{R}_+^2)$, every solution $\phi(t) = (x(t), y(t))$ of (1), which starts in $\mathrm{Int}(\mathbb{R}_+^2)$, satisfies the differential inequality $\frac{dx}{dt} \leq x(t)(r_1 - b_1 x(t))$. This is obvious by considering the first equation of (1). Thus, $x(t)$ may be compared with solutions of $\frac{du(t)}{dt} = u(t)(r_1 - b_1 u(t))$, $u(0) = x(0) > 0$, which are $u(t) = 1/(b_1/r_1 + ce^{-r_1 t})$ with $c = 1/u(0) - b_1/r_1$. It follows that every nonnegative solution $\phi(t)$ of (1) satisfies

$$x(t) \leq \frac{r_1}{b_1}, \qquad \text{for all } t \geq 0. \tag{2}$$

(a2) We prove now that $x(t) + y(t) \leq L_1$, for all $t \geq 0$.

We define the function $\sigma(t) = x(t) + y(t)$, the time derivative of which is

$$\frac{d\sigma}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \left(r_1 - b_1 x - \frac{a_1 y}{x + k_1}\right) x + \left(r_2 - \frac{a_2 y}{x + k_2}\right) y.$$

Since all parameters are positive and solutions initiating in $\mathbb{R}_+^2$ remain in the nonnegative quadrant, then

$$\frac{d\sigma}{dt} \leq (r_1 - b_1 x)x + \left(r_2 - \frac{a_2 y}{x + k_2}\right) y$$

holds for all x and y nonnegative. Thus, as $\max_{\mathbb{R}_+}(r_1 - b_1 x)x = r_1^2/4b_1$, then $\frac{d\sigma}{dt} \leq r_1^2/4b_1 - \sigma(t) + x + y + (r_2 - a_2 y/(x + k_2))y$, and thus, $\frac{d\sigma(t)}{dt} + \sigma(t) \leq r_1^2/4b_1 + x + (1 + r_2 - a_2 y/(x + k_2))y$, and then $\frac{d\sigma(t)}{dt} + \sigma(t) \leq r_1^2/4b_1 + r_1/b_1 + (1 + r_2 - a_2 b_1 y/(r_1 + b_1 k_2))y$, since, in $\mathcal{A}$, $0 \leq x \leq r_1/b_1$.

Moreover, it can be easily verified that $\max_{\mathbb{R}_+}(1 + r_2 - a_2 b_1 y/(r_1 + b_1 k_2))y = (1/4a_2 b_1) \times ((r_2 + 1)^2 (r_1 + b_1 k_2))$.

Consequently, $\frac{d\sigma(t)}{dt} + \sigma(t) \leq L_1$. Using Lemma 2, (with $\alpha_1 = 1$ and $\alpha_2 = L_1$), we then get, for all $t \geq \tilde{T} \geq 0$,

$$\sigma(t) \leq L_1 - \left(L_1 - \sigma\left(\tilde{T}\right)\right) e^{-(t-\tilde{T})}. \tag{3}$$

Then, if $\tilde{T} = 0$, $\sigma(t) \le L_1 - (L_1 - \sigma(0))e^{-t}$. Hence, since $(x(0), y(0)) \in \mathcal{A}$,

$$\sigma(t) = x(t) + y(t) \le L_1, \qquad \text{for all } t \ge 0. \tag{4}$$

(b) We have to prove that, for $(x(0), y(0)) \in \mathbb{R}_+^2$, $(x(t), y(t)) \longrightarrow \mathcal{A}$ as $t \longrightarrow +\infty$. We then will show that $\overline{\lim}_{t\longrightarrow+\infty} x(t) \le r_1/b_1$ and $\overline{\lim}_{t\longrightarrow+\infty} (x(t) + y(t)) \le L_1$..

(b1) First, the result $\overline{\lim}_{t\longrightarrow+\infty} x(t) \le r_1/b_1$ follows directly from (a1) and Lemma 2, since solutions of the initial value problem $\frac{dx}{dt} = x(t)(r_1 - b_1 x(t))$, $x(0) \ge 0$, satisfies $\overline{\lim}_{t\longrightarrow+\infty} x(t) \le r_1/b_1$.

(b2) Now, for the second result, let $\varepsilon > 0$ be given, and then there exists a $T_1 > 0$ such that $x(t) \le 1 + \varepsilon/2$ for all $t \ge T_1$. From equation (3) with $\tilde{T} = T_1$, we get, for all $t \ge T_1 \ge 0$,

$$\begin{aligned}\sigma(t) = x(t) + y(t) &\le L_1 - (L_1 - \sigma(T_1))\, e^{-(t-T_1)} \\ &\le L_1 - \left[L_1 e^{T_1} - (x(T_1) + y(T_1))e^{T_1}\right] e^{-t} \le L_1 - \left[L_1 - (x(T_1) + y(T_1))e^{T_1}\right] e^{-t}.\end{aligned}$$

Then

$$\sigma(t) = x(t) + y(t) \le \left(L_1 + \frac{\varepsilon}{2}\right) - \left[\left(L_1 + \frac{\varepsilon}{2}\right) - (x(T_1) + y(T_1))e^{T_1}\right] e^{-t},$$

for all $t \ge T_1 \ge 0$. Let $T_2 \ge T_1$ be such that

$$\left|\left(L_1 + \frac{\varepsilon}{2}\right) - (x(T_1) + y(T_1))\, e^{T_1}\right| e^{-t} \le \frac{\varepsilon}{2}, \qquad \text{for all } t \ge T_2.$$

Then $x(t) + y(t) \le L_1 + \varepsilon$ for all $t \ge T_2$. Hence,

$$\overline{\lim}_{t\longrightarrow+\infty} (x(t) + y(t)) \le L_1.$$

This completes the proof, and we also conclude that system (1) is dissipative in $\mathbb{R}_+^2$. ∎

3. GLOBAL STABILITY

In this section we shall prove the global stability of system (1) by constructing a suitable Lyapunov function. First of all, it is easy to verify that this system has three trivial equilibria (belonging to the boundary of $\mathbb{R}_+^2$, i.e., at which one or more of populations has zero density or is extinct)

$$E_0 = (0,0), \qquad E_1 = \left(\frac{r_1}{b_1}, 0\right), \qquad \text{and} \qquad E_2 = \left(0, \frac{r_2 k_2}{a_2}\right).$$

PROPOSITION 5. *Let us assume the following condition:*

$$\frac{r_2 k_2}{a_2} < \frac{r_1 k_1}{a_1}. \tag{5}$$

Then system (1) has a unique interior equilibrium $E^*(x^*, y^*)$ (i.e., $x^* > 0$ and $y^* > 0$).

PROOF. From system (1), such a point satisfies

$$(r_1 - b_1 x^*)(x^* + k_1) = a_1 y^*, \tag{6}$$

$$y^* = \frac{r_2 (x^* + k_2)}{a_2}. \tag{7}$$

We easily get $a_2 b_1 {x^*}^2 + (a_1 r_2 - a_2 r_1 + a_2 b_1 k_1)x^* + a_1 r_2 k_2 - a_2 r_1 k_1 = 0$, and thus,

$$x_\pm^* = \frac{1}{2 a_2 b_1}\left(-(a_1 r_2 - a_2 r_1 + a_2 b_1 k_1) \pm \Delta^{1/2}\right),$$

where $\Delta = (a_1r_2 - a_2r_1 + a_2b_1k_1)^2 - 4a_2b_1(a_1r_2k_2 - a_2r_1k_1)$. Δ is nonnegative if (5) holds. Moreover, simple algebraic computations show that, under condition (5), $x_+^* > 0$ and $x_-^* < 0$. Therefore, system (1) possesses a unique interior equilibrium $E^*(x^*, y^*)$ given by

$$x^* = x_+^* = \frac{1}{2a_2b_1}\left(-(a_1r_2 - a_2r_1 + a_2b_1k_1) + \Delta^{1/2}\right), \tag{8}$$

$$y^* = \frac{r_2(x^* + k_2)}{a_2}. \tag{9}$$ ∎

It is easy to verify that this fixed point belongs to $\mathcal{A}$. Linear analysis of model (1) shows that if $r_1 \le r_2$ and $k_1 \ge k_2$, then $E^*(x^*, y^*)$ is locally stable. We prove now that, under some assumptions, this steady state is globally asymptotically stable.

THEOREM 6. *The interior equilibrium $E^*(x^*, y^*)$ is globally asymptotically stable if*

$$L_1 < \frac{r_1k_1}{2a_1}, \tag{10}$$

$$k_1 < 2k_2, \tag{11}$$

$$4(r_1 + b_1k_1) < a_1. \tag{12}$$

PROOF. The proof is based on a positive definite Lyapunov function. Let $V(x,y) = V_1(x,y) + V_2(x,y)$ where $V_1(x,y) = (x^* + k_1)(x - x^* - x^*\ln(x/x^*))$ and

$$V_2(x,y) = \frac{a_1(x^* + k_2)}{a_2}\left(y - y^* - y^*\ln\left(\frac{y}{y^*}\right)\right).$$

This function is defined and continuous on $\mathrm{Int}(\mathbb{R}_+^2)$. It can be easily verified that the function $V(x,y)$ is zero at the equilibrium (x^*, y^*) and is positive for all other positive values of x and y, and thus, $E^*(x^*, y^*)$ is the global minimum of V.

Since the solutions of the system are bounded and ultimately enter the set $\mathcal{A}$, we restrict the study for this set. The time derivative of V_1 along the solutions of system (1) is

$$\frac{dV_1}{dt} = \frac{(x^* + k_1)(x - x^*)}{x}\left(r_1 - b_1x - \frac{a_1y}{x + k_1}\right)x,$$

and using equation (6), we get

$$\begin{aligned}\frac{dV_1}{dt} &= (x^* + k_1)(x - x^*)\left(-b_1(x - x^*) + \frac{a_1y^*}{x^* + k_1} - \frac{a_1y}{x + k_1}\right)\\ &= (x^* + k_1)(x - x^*)\left(-b_1(x - x^*) + \frac{a_1y^*(x + k_1) - a_1y(x^* + k_1)}{(x + k_1)(x^* + k_1)}\right)\\ &= (x^* + k_1)(x - x^*)\left(-b_1(x - x^*) + \frac{-a_1k_1(y - y^*) - a_1x(y - y^*) + a_1y(x - x^*)}{(x + k_1)(x^* + k_1)}\right).\end{aligned}$$

Similarly,

$$\frac{dV_2}{dt} = \frac{a_1(x^* + k_2)(y - y^*)}{a_2y}\left(r_2 - \frac{a_2y}{x + k_2}\right)y,$$

and we use equation (7) to get

$$\begin{aligned}\frac{dV_2}{dt} &= \frac{a_1(x^* + k_2)(y - y^*)}{a_2}\left(\frac{a_2y^*}{x^* + k_2} - \frac{a_2y}{x + k_2}\right)\\ &= a_1(x^* + k_2)(y - y^*)\left(\frac{y^*(x + k_2) - y(x^* + k_2)}{(x + k_2)(x^* + k_2)}\right)\\ &= a_1(x^* + k_2)(y - y^*)\left(\frac{-k_2(y - y^*) - x(y - y^*) + y(x - x^*)}{(x + k_2)(x^* + k_2)}\right).\end{aligned}$$

Therefore, computing $\frac{dV}{dt}$ via $\frac{dV_1}{dt}$ and $\frac{dV_2}{dt}$ yields

$$\begin{aligned} \frac{dV}{dt} &= \left(-b_1\left(x^*+k_1\right)+\frac{a_1 y}{x+k_1}\right)\left(x-x^*\right)^2 \\ &+ \left(-a_1+\frac{a_1 y}{x+k_2}\right)\left(x-x^*\right)\left(y-y^*\right)-a_1\left(y-y^*\right)^2. \end{aligned} \tag{13}$$

The above equation can be written as

$$\frac{dV}{dt} = -\left(x-x^*,\quad y-y^*\right)\begin{pmatrix} -g(x,y) & -h(x,y) \\ -h(x,y) & a_1 \end{pmatrix}\begin{pmatrix} x-x^* \\ y-y^* \end{pmatrix},$$

where

$$g(x,y) = -b_1(x^*+k_1)+\frac{a_1 y}{x+k_1}$$

and

$$h(x,y) = \frac{1}{2}\left(-a_1+\frac{a_1 y}{x+k_2}\right).$$

From equation (13), it is obvious that $\frac{dV}{dt} < 0$ if the matrix above is positive definite. This matrix is positive definite iff all upper-left submatrices are positive (Sylvester's criterion), that is, since $a_1 > 0$, iff

(i) $g(x,y) < 0$ and
(ii) $\Phi(x,y) = -a_1 g(x,y) - h^2(x,y) < 0$.

PROOF OF (i).

$$g(x,y) = -b_1\left(x^*+k_1\right)+\frac{a_1 y}{x+k_1} < 0.$$

Due to equation (6),

$$g(x,y) = -r_1+\frac{a_1 y^*}{x^*+k_1}+\frac{a_1 y}{x+k_1}.$$

So, as $\mathcal{A}$ is an attracting positively invariant set, and in $\mathcal{A}$, all solutions satisfy $0 \le x \le r_1/b_1$ and $0 \le x+y \le L_1$, then

$$g(x,y) \le -r_1+\frac{a_1}{k_1}\left(y+y^*\right) \le -r_1+\frac{2a_1 L_1}{k_1}.$$

Therefore, if (10) holds, then $\forall\,(x,y) \in \mathcal{A}^2$, $g(x,y) < 0$, for all $t \ge 0$. ∎

PROOF OF (ii).

$$\Phi(x,y) = -a_1\left(-b_1\left(x^*+k_1\right)+\frac{a_1 y}{x+k_1}\right)-\frac{1}{4}\left(-a_1+\frac{a_1 y}{x+k_2}\right)^2 < 0.$$

Since (for x fixed)

$$\frac{\partial\Phi(x,y)}{\partial y} = \frac{-a_1^2}{x+k_1}-\frac{1}{2}\left(\frac{-a_1^2}{x+k_2}+\frac{a_1^2 y}{(x+k_2)^2}\right),$$

then

$$\frac{\partial^2\Phi(x,y)}{\partial y^2} = \frac{-a_1^2}{2(x+k_2)^2} < 0.$$

Hence, $\frac{\partial\Phi(x,y)}{\partial y}$ is strictly decreasing, in $\mathbb{R}_+$, with respect to y.

Now,

$$\left.\frac{\partial\Phi(x,y)}{\partial y}\right|_{y=0} = \frac{-a_1^2}{x+k_1}+\frac{a_1^2}{2(x+k_2)} = \frac{a_1^2(-x-2k_2+k_1)}{2(x+k_1)(x+k_2)}.$$

Consequently, if (11) holds,

$$\left.\frac{\partial \Phi(x,y)}{\partial y}\right|_{y=0} < 0$$

in $\mathbb{R}_+$, and so $\Phi(x,y)$ is strictly decreasing in $\mathbb{R}_+$. This yields $\Phi(x,y) < D(x,0)$ for $(x,y) \in \mathcal{A}$; that is, $\Phi(x,y) < a_1 b_1(x^* + k_1) - (1/4)a_1^2$.

As $0 < x^* \leq r_1/b_1$, then, $\Phi(x,y) < a_1(r_1 + b_1 k_1 - (1/4)a_1)$, and due to (12), $\forall\,(x,y) \in \mathcal{A}^2$, $\Phi(x,y) < 0$.

It follows that if the hypotheses of Theorem 6 are satisfied, then $\frac{dV}{dt} < 0$ along all trajectories in the first quadrant except (x^*, y^*), so that $E^*(x^*, y^*)$ is globally asymptotically stable. ∎

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