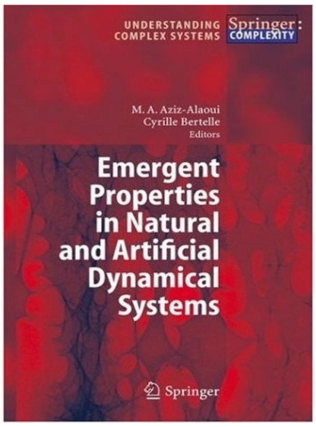
Book Chapter:

Complex Emergent Properties and Chaos (De-)synchronization

To cite this article:

Aziz-Alaoui M.A. (2006),
Complex Emergent Properties and Chaos (De-)synchronization,
in M.A.A. and C.Bertelle (eds.): "Emergent Properties in Natural and Artificial Dynamical Systems", Understanding Complex Systems. Springer-Verlag, Heidelberg (2006), pp: 129-147.



Complex Emergent Properties and Chaos (De)synchronization

Aziz-Alaoui M.A.[1]

Laboratoire de Mathématiques Appliquées,
Université du Havre, BP 540, 76058 Le Havre Cedex, France
aziz.alaoui@univ-lehavre.fr

Summary. Emergent properties are typically novel and unanticipated. In this paper, using chaos synchronization tools and ideas, we demonstrate, via two examples of three-dimensional autonomous differential systems, that, by simple uni- or bi-directional coupling, regular (resp. chaotic) behaviour can emerge from chaotic (resp. regular) behaviour.

Key words: complex systems, emergence, chaos, synchronization, Lorenz-type system, predator-prey.

1 Introduction : Complexity and Emergent Properties

First of all, let us start with the citation below.

> "I think the next century (21th) will be the century of complexity", (Stephen Hawking).

But what is complexity? An extremely difficult "I know it when I see it" concept to define, see [1]. However, intuitively, complexity is usually greater in systems whose components are arranged in some intricate difficult-to-understand pattern or, in the case of a dynamical system, when the outcome of a process is difficult to predict from its initial state (sensitive dependence on initial conditions, see below). A complex system is an animate or inanimate system composed of many interacting components whose behaviour or structure is difficult to understand. Sometimes a system may be structurally complex, like a mechanical clock, but behave very simply.

While several measures of complexity have been proposed in the research literature, they all fall into two general classes:

(1) Static Complexity which addresses the question of how an object or system is put together (i.e. only pure structural informational aspects of an object).
(2) Dynamic Complexity which addresses the question of how much dynamical or computational effort is required to describe the informational content of an object or state of a system.

These two measures are clearly not equivalent. In this paper we embrace the following definition.

Complexity is a scientific theory that asserts that some systems display behavioural phenomena completely inexplicable by any conventional analysis of the systems' constituent parts.

Besides, **emergence** refers to the appearance of higher-level properties and behaviours of a system that while obviously originating from the collective dynamics of that system's components -are neither to be found in nor are directly deductable from the lower-level properties of that system. Emergent properties are properties of the 'whole' that are not possessed by any of the individual parts making up that whole. For example, an air molecule is not a cyclone, an isolated species doesn't form a food chain and an 'isolated' neuron is not conscious: emergent behaviours are typically novel and unanticipated.

Moreover, it is becoming a commonplace that, if the 20th was the century of physics, the 21st will be the century of biology, and, more specifically, mathematical biology, see [8]. We will concentrate our attention in demonstrating the emergence of complex (chaotic) behaviour in coupled (non chaotic) systems. That is to show that for uni- or bi-directionally coupled non chaotic systems, chaos can appear even for large values of coupling parameter. This discussion is based on continuous autonomous differential systems, firstly of Lorenz-type illustrating identical chaos synchronization and regular behaviour emerging from chaotic one, and, secondly, systems modeling predator-prey food-chain showing chaotic behaviours emerging from regular ones.

2 Synchronization and Desynchronization

Synchronization is a ubiquitous phenomenon characteristic of many processes in natural systems and (nonlinear) science. It has permanently remained an object of intensive research and is today considered as one of the basic nonlinear phenomena studied in mathematics, physics, engineering or life science. Synchronization of two dynamical systems generally means that one system somehow traces the motion of another. Indeed, it is well known that many coupled oscillators have the ability to adjust some common relation that they have between them due to weak interaction, which yields to a situation in which a synchronization-like phenomenon takes place, see [2].

Since this discovery, periodic synchronization has found numerous applications in various fields, for instance in biological systems and living nature where synchronization is encountered on differents levels. Examples range from the modeling of the heart to the investigation of the circardian rhythm, phase locking of respiration with a mechanical ventilator, synchronization of oscillations of human insulin secretion and glucose infusion, neuronal information processing within a brain area and communication between different brain areas. Synchronization also plays an important role in several neurological diseases such as epilepsies and pathological tremors, or in different forms of cooperative behaviour of insects, animals or humans. For more details, see [10]. This process may also be encountered in other areas, celestical mechanics or radio engineering and acoustics.

But, even though original notion and theory of synchronization implies periodicity of oscillators, during the last decades, the notion of synchronization has been generalized to the case of interacting chaotic oscillators.

Roughly speaking, a system is **chaotic** if it is deterministic, has a long-term aperiodic behaviour, and shows sensitive dependence on initial conditions on a closed invariant set.

Chaotic oscillators are found in many dynamical systems of various origins, the behaviour of such systems is characterized by instability and, as a result, limited predictability in time.

Despite this, in the two last decades, the research for synchronization moved to chaotic systems. A lot of research has been done and, as a result, researchers showed that two chaotic systems could be synchronized by coupling them : synchronization of chaos is actual and chaos could then be exploitable, see [9], and for a review see [2]. Ever since, many researchers have discussed the theory, the design or applications of synchronized motion in coupled chaotic systems. A broad variety of applications have emerged, for example to increase the power of lasers, to synchronize the output of electronic circuits, to control oscillations in chemical reactions or to encode electronic messages to secure communications. Moreover, in the topics of coupled chaotic systems, many different phenomena, which are usually referred to as *synchronization*, exist and have been studied for more than a decade.

2.1 Synchronization and stability : definitions

For the basic *master-slave* configuration where an autonomous chaotic system (the master) :

$$\frac{dX}{dt} = F(X), \quad X \in \mathbb{R}^n \tag{1}$$

drives another system (the slave):

$$\frac{dY}{dt} = G(X, Y), \quad Y \in \mathbb{R}^m, \tag{2}$$

synchronization takes place when Y asymptotically copies, in a certain manner, a subset X_p of X. That is to say, it exists a relation between the two coupled systems, which could be a smooth invertible function ψ, the last carries trajectories on the attractor of a first system on the attractor of a second system. In other words, if we know, after a transient regime, the state of the first system, it allows us to predict the state of the second: $Y(t) = \psi(X(t))$. Generally, it is assumed $n \geq m$, however, for the sake of easy readability, we will reduce, even if it is not a necessary restriction, to the case $n = m$, and thus $X_p = X$. Henceforth, if we denote the difference $Y - \psi(X)$ by $X_\perp$, in order to reach at a synchronized motion, one expects to have:

$$||X_\perp|| \longrightarrow 0, \;\; as \;\; t \longrightarrow +\infty. \tag{3}$$

If ψ is the identity function, the process is called *identical synchronization* (IS hereafter).

Definition of IS. System (2) synchronizes with system (1), if the set $M = \{(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n, Y = X\}$ is an attracting set with a basin of attraction B ($M \subset B$) such that $\lim_{t\to}\infty||X(t) - Y(t)|| = 0$, forall $(X(0), Y(0)) \in B$.

Thus, this regime corresponds to the situation where all the variables of two (or more) coupled chaotic systems converge. If ψ is not the identity function, the phenomenon is more general and is referred to as *generalized synchronization* (GS).

Definition of GS. System (2) synchronizes with system (1), in the generalized sense, if it exists a transformation $\psi \;\; : \;\; \mathbb{R}^n \longrightarrow \mathbb{R}^m$, a manifold $M = \{(X, Y) \in \mathbb{R}^{n+m}, Y = \psi(X)\}$ and a subset B ($M \subset B$), such that for all $(X_o, Y_o) \in B$, the trajectory based on the initial conditions (X_o, Y_o) approaches M as time goes to infinity.

Henceforth, in the case of identical synchronization, equation (3) above means that a certain hyperplane M, called *synchronization manifold*, within $\mathbb{R}^{2n}$, is asymptotically stable. Consequently, for the sake of synchrony motion, we have to prove that the origin of the transverse system $X_\perp = Y - X$ is asymptotically stable. That is, to prove that the motion transversal to the synchronization manifold dies out.

The Lyapunov exponents associated with the variational equation corresponding to the transverse system $X_\perp$:

$$\frac{dX_\perp}{dt} = DF(X)X_\perp \tag{4}$$

where $DF(X)$ is the Jacobian of the vector field evaluated onto the driving trajectory X, are referred to as transverse or conditional Lyapunov exponents (CLE hereafter).

In the case of IS it appears that the condition $L^\perp_{max} < 0$ is sufficient to insure synchronization, where $L^\perp_{max}$ is the largest CLE. Indeed, Equation (4) gives the dynamics of the motion transverse to the synchronization manifold, therefore CLE will tell us if this motion die out or not and hence, whether the synchronization state is stable or not. Consequently, if $L^\perp_{max}$ is negative, it will insure the stability of the synchronized state. This will be better explained using the two examples below.

2.2 Identical synchronization

The simplest form of chaos synchronization and the best way to explain it is *identical synchronization* (IS), also referred to as *Conventional* or *Complete synchronization*, see [4]. It is also the most typical form of chaotic synchronization often observable in two identical systems.

There are various processes leading to synchronization, depending on the used particular coupling configuration they could be very different. Thus, one has to distinguish the following two main situations, even if they are, in some sense, similar: the **uni-directional** and the **bi-directional** coupling . Indeed, synchronization of chaotic systems is often studied for schemes of the form:

$$\begin{aligned} \frac{dX}{dt} &= F(X) + kN(X - Y) \\ \frac{dY}{dt} &= G(Y) + kM(X - Y) \end{aligned} \tag{5}$$

where F and G act in $\mathbb{R}^n$, $(X, Y) \in (\mathbb{R}^n)^2$, k is a scalar and M and N are coupling matrices belonging to $\mathbb{R}^{n\times n}$. If $F = G$ the two subsystems X and Y are identical. Moreover, when both matrices are nonzero then the coupling is called *bi-directional*, while it is referred to as *uni-directional* if one is the zero matrix, and the other being nonzero.

Other names were given in the literature of this type of synchronization, such as *one-way diffusive* coupling, *drive-response* coupling, *master-slave* coupling or *negative feedback control*.

System (5) above with $F = G$ and $N = 0$ becomes uni-directionlly coupled, and reads:

$$\begin{aligned}\frac{dX}{dt} &= F(X) \\ \frac{dY}{dt} &= F(Y) + kM(X-Y)\end{aligned} \tag{6}$$

M is then a matrix that determines the linear combination of X components that will be used in the difference, and k determines the strength of the coupling.

In uni-directional synchronization, the evolution of the first system (the drive) is unaltered by the coupling, the second system (the response) is then constrained to copy the dynamics of the first.

By contrast to the uni-directional coupling, for the bi-directionally coupling (also called *mutual* or *two-way*), both drive and response systems are connected in such a way that they mutually influence each other's behaviour. Many biological or physical systems consist in bi-directionally interacting elements or components, examples range from cardiac and respiratory systems to coupled lasers with feedback.

3 Emergence of Regular Properties: A Lorenz-type Example

3.1 Uni- and bi-directional identical synchronization for a Lorenz-type system

Let us give an example, and for the sake of simplicity, let us develop the idea on the following 3-dimensional simple autonomous system, which belongs to the class of dynamical systems called *generalized Lorenz systems*, see [7] and references therein:

$$\begin{cases}\dot{x} = -9x - 9y \\ \dot{y} = -17x - y - xz \\ \dot{z} = -z + xy\ .\end{cases} \tag{7}$$

The signs used differentiate system (7) from the well-known Lorenz system:

$$\dot{x} = -10x + 10y,\ \dot{y} = 28x - y - xz,\ \dot{z} = -\frac{8}{3}z + xy.$$

From previous observations, it has been shown that system (7) oscillates chaotically, its Lyapunov exponents are $+0.601$, 0.000 and -16.470, it shows the chaotic attractor of figure 1, with a 3D feature very similar to that of Lorenz attractor.

Uni-directional coupling
Let us consider an example with two copies of system (7), and for

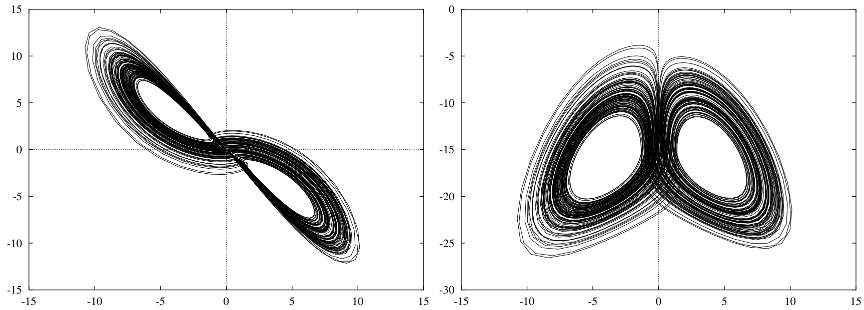


Fig. 1. The chaotic attractor of system (7) : xy and xz-plane projections.

$$M = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix} \tag{8}$$

that is, by adding a damping term to the first equation of the response system, we get a following uni-directionally coupled system, coupled through a linear term $k > 0$ according to variables $x_{1,2}$:

$$\begin{cases} \dot{x_1} = -9x_1 - 9y_1 \\ \dot{y_1} = -17x_1 - y_1 - x_1 z_1 \\ \dot{z_1} = -z_1 + x_1 y_1 \\ \dot{x_2} = -9x_2 - 9y_2 - k(x_2 - x_1) \\ \dot{y_2} = -17x_2 - y_2 - x_2 z_2 \\ \dot{z_2} = -z_2 + x_2 y_2 \end{cases} \tag{9}$$

For $k = 0$ the two subsystems are uncoupled, for $k > 0$ both subsystems are uni-directionally coupled. Our numerical computations yield the optimal value $\tilde{k}$ for the synchronization , we found that for $k \geq \tilde{k} = 4.999$ both subsystems of (9) synchronize. That is to say, starting from random initial conditions, and after some transient time, system (9) generates the same attractor as for system (7), see figure 1. Consequently, all the variables of the coupled chaotic subsystems converge, that are x_2 converges to x_1, y_2 to y_1 and z_2 to z_1, see figure 2. Thus, the second system (the response) is locked to the first one (the drive). One could also give correlation plots, that are the amplitudes x_1 against x_2, y_1 against y_2 and z_1 against z_2, and observe diagonal lines, meaning also that the system synchronizes.

Bi-directional coupling

Let us then take two copies of the same system (7) as given above, but two-way coupled through a linear constant term $k > 0$ according to variables $x_{1,2}$:

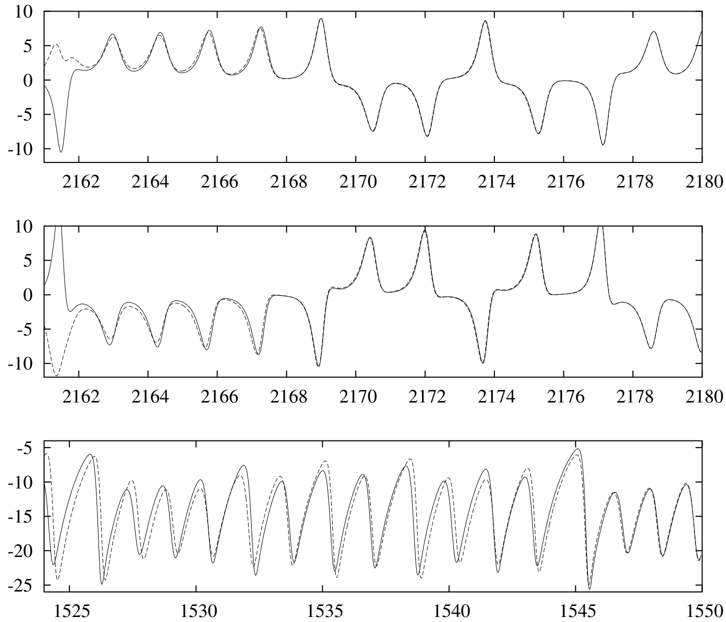


Fig. 2. Time series for $x_i(t)$, $y_i(t)$ and $z_i(t)$ in system (9), $(i = 1, 2)$, for the coupling constant k $= 5.0$, that is beyond the threshold necessary for synchronization. After transients die down, the two subsystems synchronize perfectly: Regular behaviour emerges from chaotic behaviours.

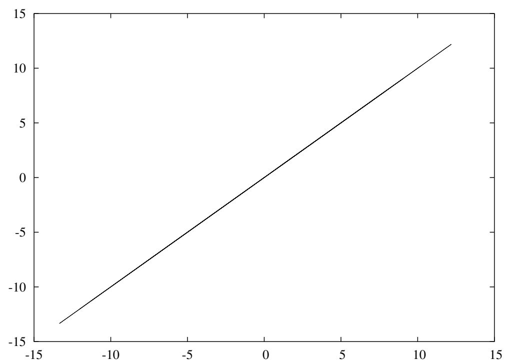


Fig. 3. The plot of amplitudes y_1 against y_2, after transients die down, shows a diagonal line, which also indicates that the receiver and the transmitter are maintaining synchronization. The plot of z_1 against z_2 shows a similar figure: Regular behaviour emerges from chaotic behaviours.

$$\begin{cases} \dot{x_1} = -9x_1 - 9y_1 - k(x_1 - x_2) \\ \dot{y_1} = -17x_1 - y_1 - x_1 z_1 \\ \dot{z_1} = -z_1 + x_1 y_1 \\ \dot{x_2} = -9x_2 - 9y_2 - k(x_2 - x_1) \\ \dot{y_2} = -17x_2 - y_2 - x_2 z_2 \\ \dot{z_2} = -z_2 + x_2 y_2 \end{cases} \tag{10}$$

We can get an idea of the onset of synchronization by plotting for example x_1 against x_2 for various values of the coupling strength parameter k. Our numerical computations yield the optimal value $\tilde{k}$ for the synchronization : $\tilde{k} \simeq 2.50$, see figure 4, both (x_i, y_i, z_i)-subsystems synchronize and system (10) also generates the attractor of figure 1.

These results also show that, for sufficiently lage values of the coupling parameter k, simple uni- or bi-directional coupling of two chaotic systems does not increase the chaoticity of the new system, unlike what one might expect. Thus, in some sense (see synchronization manifold below), regular behaviour emerges from chaotic behaviour (the motion is confined in some manifold).

3.2 Remark on the stability manifold

Geometrically, the fact that systems (9) and (10) beyond synchronization generate the same attractor as system (7), implies that the attractors of these combined drive-response 6-dimensional systems are confined to a 3-dimensional hyperplane (the *synchronization manifold*) defined by $Y = X$. This hyperplane is stable since small perturbations which take the trajectory off the synchronization manifold will decay in time. Indeed, as we said before, conditional Lyapunov exponents of the linearization of the system around the synchronous state could determine the stability of the synchronized solution. This means that the origin of the transverse system $X_\perp$ is asymptotically stable. To see this, for both systems (9) and (10), we switch to the new set of coordinates, $X_\perp = Y - X$, that is $x_\perp = x_2 - x_1$, $y_\perp = y_2 - y_1$ and $z_\perp = z_2 - z_1$. The origin $(0,0,0)$ is obviously a fixed point for this transverse system, within the synchronization manifold. Therefore, for small deviations from the synchronization manifold, this system reduces to a typical variational equation:

$$\frac{dX_\perp}{dt} = DF(X)X_\perp \tag{11}$$

where $DF(X)$ is the Jacobian of the vector field evaluated onto the driving trajectory X. Previous analysis, see [2], shows that $L^{\perp}_{max}$ becomes negative as k increases, for both uni- or bi-directionally coupling, which insures the stability of the synchronized state for systems (9) and (10), figure 5.

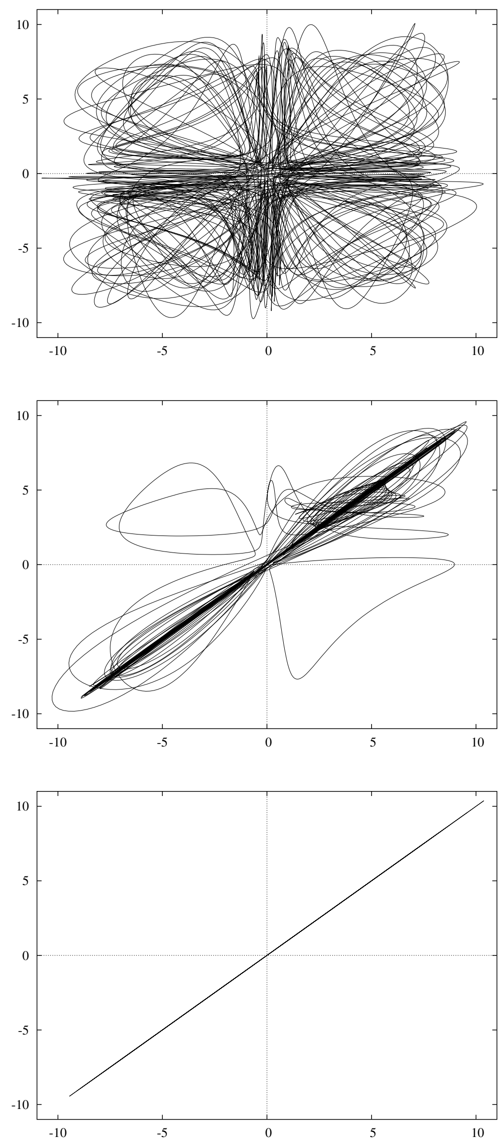


Fig. 4. Illustration of the synchronization onset of system (10). (a), (b) and (c) plot the amplitudes x_1 against x_2 for values of the coupling parameter $k = 0.5$, $k = 1.5$ and $k = 2.8$ respectively. The system synchronizes for $k \geq 2.5$.

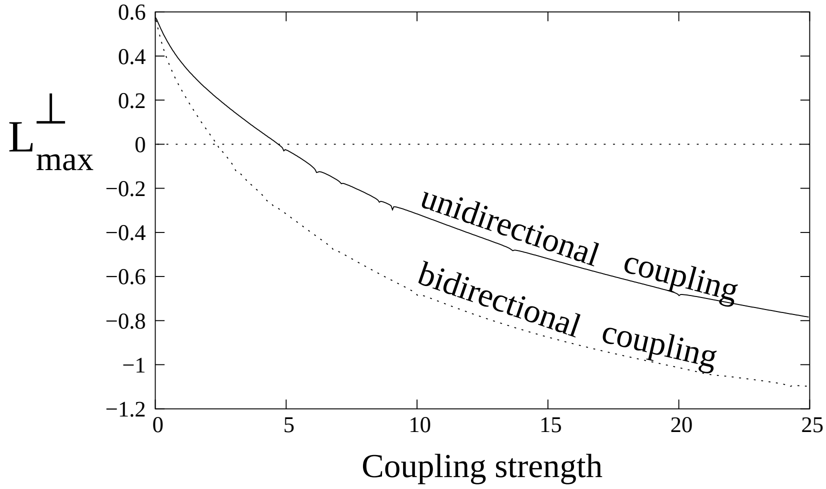


Fig. 5. The largest transverse Lyapunov exponents $L^{\perp}_{max}$ as a function of coupling strength k in the uni-directional system (9) (solid) and the bi-directional system (10) (dotted).

Let us note that this can also be proved analytically, as done in [7], by using a suitable Lyapunov function, and using a new extended version of LaSalle invariance principle.

Desynchronization motion. Synchronization depends on the coupling strength, but also on the vector field and the coupling function. For a choice of these quantities, synchronization may occur only within a finite range $[k_1, k_2]$ of coupling strength, in such a case a *desynchronization* phenomenon occurs: thus, increasing k beyond the critical value k_2 yields loss of the synchronized motion ($L^{\perp}_{max}$ becomes positive).

4 Emergence of Chaotic Properties : A Predator-prey Example

This example, contrary to the first, shows a situation where the larger is the coupling coefficient the weaker is the synchronization.

4.1 The model

As we said in the introduction, it is becoming a commonplace that, if the 20th was the century of physics, the 21st will be the century of biology and more specifically, mathematical biology, ecology, ... and in general, nonlinear dynamics and complexity in life sciences.

In this last part, we will hence focus ourselves on emergent (chaotic or regular) properties that arise when we couple uni- or bi-directionally two 3-dimensional autonomous differential systems, modeling predator-prey food-chains.

Let us then consider a continuous time dynamical system, model for a tritrophic food chain, based on a modified version of the Leslie-Gower scheme, see [3, 6], for which the rate equations of the three components of the chain population can be written as follows:

$$\begin{cases} \dfrac{dX}{dT} = a_o X - b_o X^2 - \dfrac{v_o XY}{d_o + X} \\ \dfrac{dY}{dT} = -a_1 Y + \dfrac{v_1 XY}{d_1 + X} - \dfrac{v_2 YZ}{d_2 + Y} \\ \dfrac{dZ}{dT} = c_3 Z - \dfrac{v_3 Z^2}{d_3 + Y}, \end{cases} \tag{12}$$

with $X(0) \geq 0$, $Y(0) \geq 0$ and $Z(0) \geq 0$, where X, Y and Z represent the population densities at time T ; $a_0, b_0, v_0, d_0,\ a_1, v_1, d_1, v_2, d_2,\ c_3, v_3$ and d_3 are model parameters assuming only positive values and defined as follows: a_o is the growth rate of prey X, b_o measures the strength of competition among individuals of species X, v_o is the maximum value which *per capita* reduction rate of X can attain, d_o measures the extent to which environment provides protection to prey X, a_1 represents the rate at which Y will die out when there is no X, v_1 and d_1 have a similar meaning to v_0 and d_o, v_2 and v_3 have a similar biological connotation as that of v_o and v_1, d_2 is the value of Y at which the *per capita* removal rate of Y becomes $v_2/2$, c_3 describes the growth rate of Z, assuming that the number of males and females is equal, d_3 represents the residual loss in species Z due to severe scarcity of its favourite food Y ; the second term on the right hand side in the third equation of (12) depicts the loss in the predator population.

For the origin of this system and for some theoretical results, boundedness of solutions, existence of an attracting set, existence and local or global stability of equilibria, etc ..., see [3, 6]. In these works, using intensive numerical qualitative analysis, it has been demonstrated that the model could show periodic solutions, figure (6), and quasi-periodic or chaotic dynamics, figure (7), for the following parameters and state values:

$$\begin{cases} b_o = 0.06, \quad v_o = 1.0, \quad d_0 = d_1 = d_2 = 10.0, \\ a_1 = 1.0,\ v_1 = 2.0,\ v_2 = 0.9, \\ c_3 = 0.02,\ v_3 = 0.01,\ d_3 = 0.3. \end{cases} \tag{13}$$

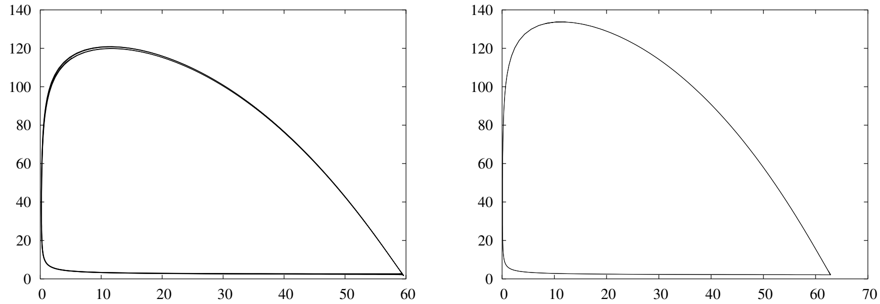


Fig. 6. Limit cycles of period one and two, for $a_0 = 3.6$ and $a_0 = 3.8$ respectively, found for system (12) and for parameters given by (13).

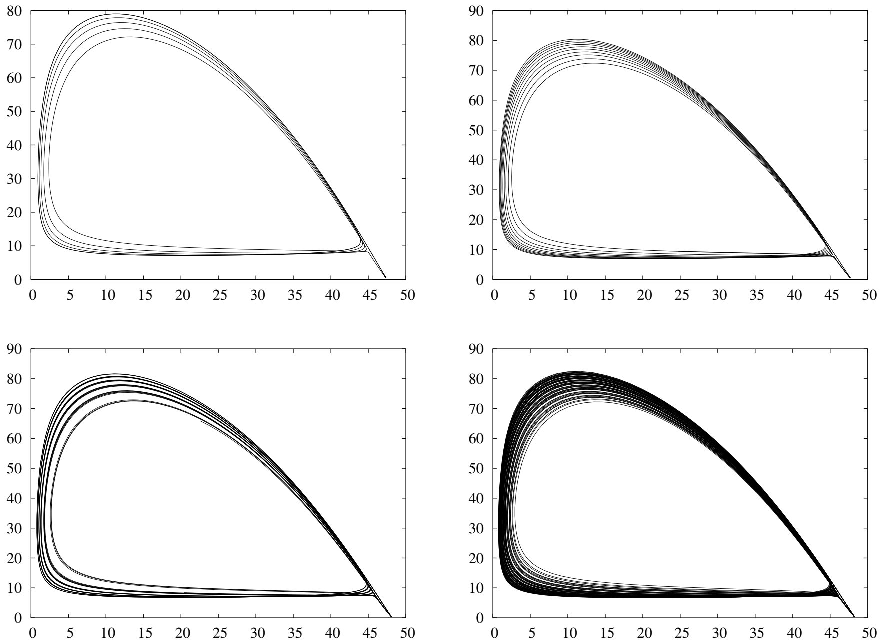


Fig. 7. Transition to chaotic (or quasi-periodic) behaviour found for system (12), it is established via period doubling bifurcation, respectively for $a_0 = 2.85$, $a_0 = 2.87$, $a_0 = 2.89$ and $a_0 = 2.90$, with parameters given by (13).

We will set, for the rest of the paper, the system parameters as given in (13) in order that system (13) oscillates in a regular way around a stable limit cycle of period one, figure 6(b).

4.2 Uni-directional desynchronization : Predator-prey system

Usually, as we have seen in the first example, section 2, in uni-directional synchronization, while the evolution of the first system (the drive) is unaltered by the coupling, the second system (the response) is constrained to copy the dynamics of the first.

However, it is not the case for our example below, for which, as we will see, while both subsystems evolve periodically (limit cycles of figure 6(b)), the coupled system behaviour is extremely complex.

Let us then consider two copies of system (12). By adding a damping term to the first equation of the response system, we get a following uni-directionally coupled system, coupled through a linear term $k > 0$ according to variables $x_{1,2}$:

$$\left\{\begin{aligned}
\dot{X}_1 &= a_o X_1 - b_o X_1^2 - \frac{v_o X_1 Y_1}{d_o + X_1}\\
\dot{Y}_1 &= -a_1 Y_1 + \frac{v_1 X_1 Y_1}{d_1 + X_1} - \frac{v_2 Y_1 Z_1}{d_2 + Y_1}\\
\dot{Z}_1 &= c_3 Z_1 - \frac{v_3 Z_1^2}{d_3 + Y_1}\\
\\
\dot{X}_2 &= a_o X_2 - b_o X_2^2 - \frac{v_o X_2 Y_2}{d_o + X_2} - k(X_2 - X_1)\\
\dot{Y}_2 &= -a_1 Y_2 + \frac{v_1 X_2 Y_2}{d_1 + X_2} - \frac{v_2 Y_2 Z_2}{d_2 + Y_2}\\
\dot{Z}_2 &= c_3 Z_2 - \frac{v_3 Z_2^2}{d_3 + Y_2}
\end{aligned}\right. \tag{14}$$

For $k = 0$ the two subsystems are uncoupled, for $k > 0$ both subsystems are uni-directionally coupled, and for $k \longrightarrow +\infty$ one can expect to obtain the same results as those obtained for the previous example in section 2, that is strong synchronization and two subsystems which evolve identically.

We have chosen, for the coupled system, a range of parameters for which both subsystem constituent parts evolve periodically, as figure 6(b) shows.

However, our numerical computations show that both subsystems of (14) never synchronize nor identically neither *generally*, unless the coupling parameter k is very small. In such a case a certain generalized synchronization form takes place, see figure 8(a). That is, starting from random initial conditions, and after some transient time, system (14) generates an attractor different from those showed by system (12) in figure 6(b). Consequently, all the variables of the coupled limit cycle subsystems surprisingly do not converge, as, at first sight, one may intuitively expect, see figure 8.

These results show that uni-directional coupling of these two non-chaotic systems (that are the subsystem constituents of system (14)) increases the bahaviour complexity, and transforms a periodic situation into a chaotic one.

Emergent chaotic properties are typically novel and unanticipated, for this example.

In fact, this phenomenon corresponds to the classical cascade of periodic-doubling bifurcation processus, with a sequence of order and disorder windows.

4.3 Bi-directional desynchronization : Predator-prey system

As many biological or physical systems consist in bi-directional interacting elements or components, let us use a bi-directionally (*mutual*) coupling, in order that both drive and response subsystems are connected in such a way that they mutually influence each other's behaviour. Let us then take two copies of the same system (12) given above, but two-way coupled through a linear constant term $k > 0$ according to variables $x_{1,2}$:

$$\begin{cases} \dot{X_1} = a_o X_1 - b_o X_1^2 - \dfrac{v_o X_1 Y_1}{d_o + X_1} - k(X_1 - X_2) \\ \dot{Y_1} = -a_1 Y_1 + \dfrac{v_1 X_1 Y_1}{d_1 + X_1} - \dfrac{v_2 Y_1 Z_1}{d_2 + Y_1} \\ \dot{Z_1} = c_3 Z_1 - \dfrac{v_3 Z_1^2}{d_3 + Y_1} \\ \\ \dot{X_2} = a_o X_2 - b_o X_2^2 - \dfrac{v_o X_2 Y_2}{d_o + X_2} - k(X_2 - X_1) \\ \dot{Y_2} = -a_1 Y_2 + \dfrac{v_1 X_2 Y_2}{d_1 + X_2} - \dfrac{v_2 Y_2 Z_2}{d_2 + Y_2} \\ \dot{Z_2} = c_3 Z_2 - \dfrac{v_3 Z_2^2}{d_3 + Y_2} \end{cases} \tag{15}$$

We have also chosen, for this bi-directionally coupled system, the same range of parameters for which the subsystem constituent parts evolve periodically, as figure 6(b) shows.

Figure 9 demonstrates also, for some interval of parameter k, that the larger is this coupling coefficient the weaker is the synchronization. Thus, we

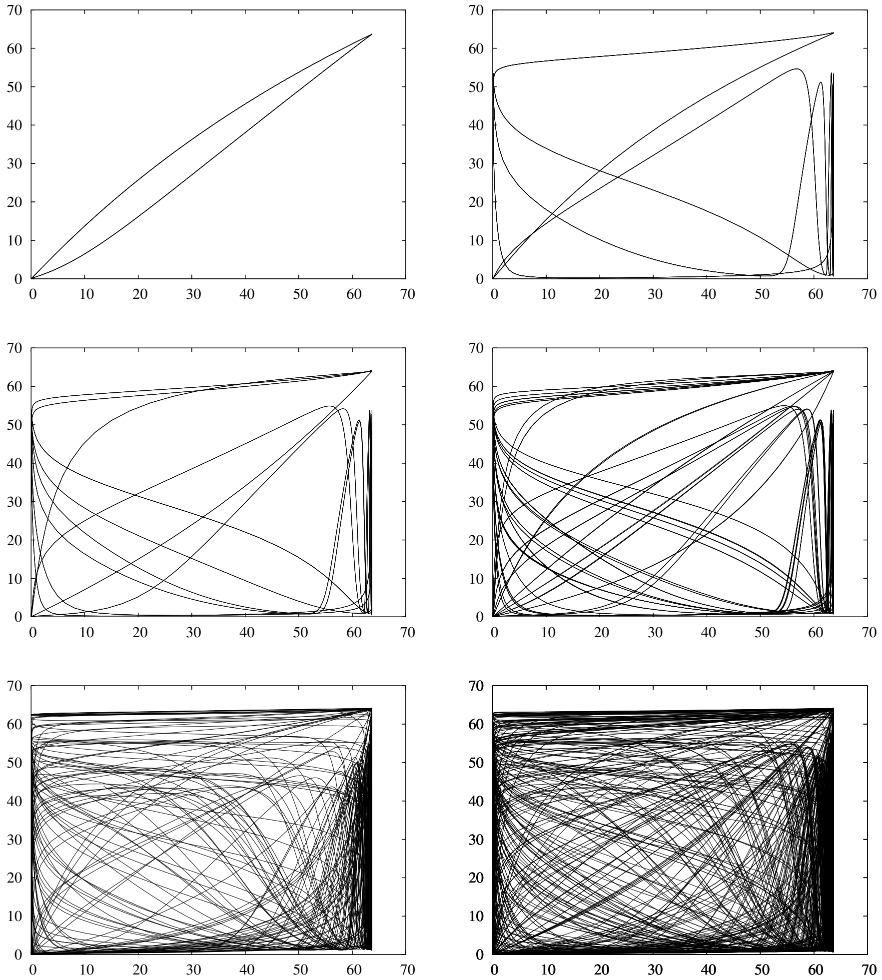


Fig. 8. Illustration of the desynchronization onset of the unidirectional coupled system (14). Figures (a), (b), ... (f) are done from left to right and up to down, and plot the amplitudes x_1 against x_2 for values of the coupling parameter (a) $k = 0.01$, (b) $k = 0.055$, (c) $k = 0.056$, (d) $k = 0.0565$, (e) $k = 0.057$ and (f) $k = 0.1$. Figure (a) shows a generalized synchronization phenomenon: the system synchronizes (in the generalized sense) for very small values of k. But a desynchronization processus quickly arises by increasing k, figures (b,c,d,e,f): in some interval for k, the larger is the coupling coefficient the weaker is the synchronization. Hence, we have the emergence of chaotic properties: the coupled system displays behavioural chaotic phenomena which are not showed by the systems' constituent parts (that are the two predator-prey systems without coupling) which point out the limit-cycle of figure 6(b), for the same parameters and the same initial conditions. This phenomenon is robust with respect to small parameters variations.

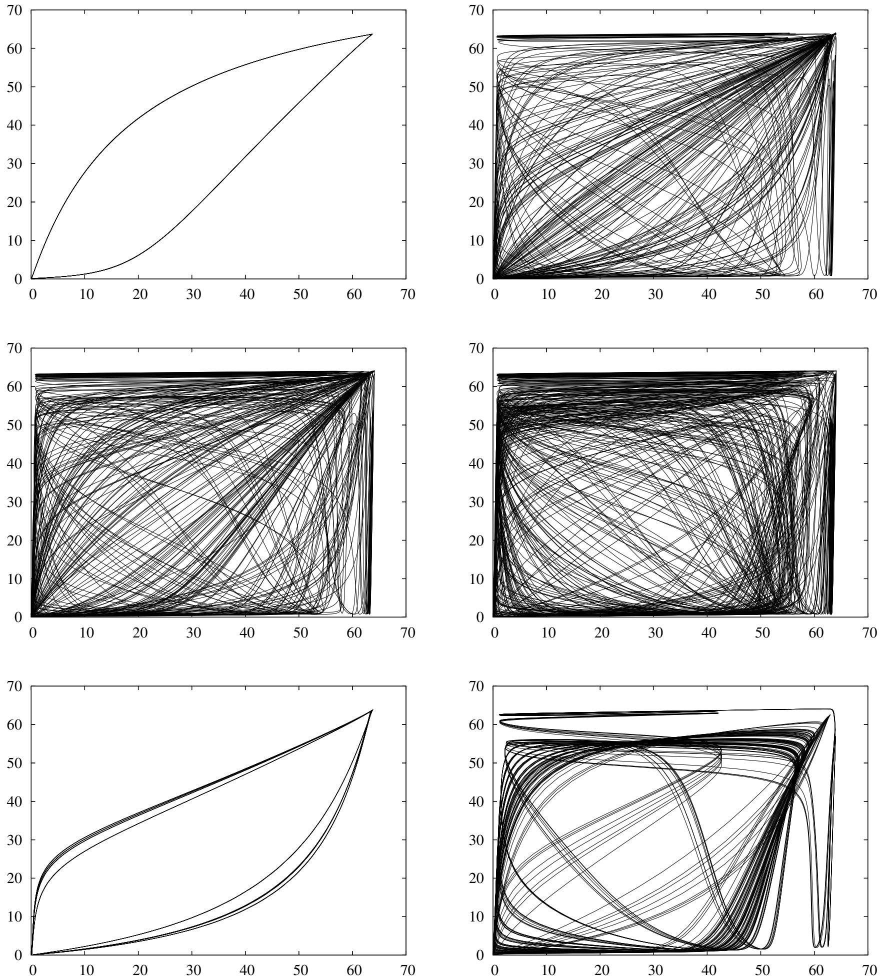


Fig. 9. Bi-directional coupling. Figures are done from left to right and up to down and plot amplitudes x_1 against x_2 for the same values as done in the previous figure, respectively for $k = 0.01$, $k = 0.055$ and $k = 0.056$ $k = 0.0565$, $k = 0.057$ and $k = 0.1$. These figures, Illustrate a window of generalized synchronization and desynchronization of system (12). (a), (b) and (c) plot The system synchronizes (in the generalized sense) for $k \leq 0.01$, as it has been shown in the uinidirectional case. But the desynchronization processus arises by increasing k, quickly in comparison with the unidirectional case.

have emergence of new properties for the coupled system. The latter displays behavioural chaotic phenomenon which is not showed by systems' constituent parts (that are the two predator-prey systems without coupling) and for the same parameters and the same initial conditions. A robust phenomenon with respect to small variations of parameter values.

Furthermore, the bi-directional case enhances the desynchronization processus which allows occurence of new complex phenomenon. The latter occurs quickly in comparison to the uni-directinal case. For the same interval $k \in J =]0, 0.1]$, chaotic properites take place for $k = 0.055$ in the unidirectional case and for $k = 0.057$ in the uni-directional case. This complex behaviour remains observable in the whole interval J for the last, but for the first, it disappears after $k = 0.056$ -some regular generalized synchronization takes place- and appears again for $k \in]0.057, 0.1]$

Thus, we can conclude that, the larger is the coupling coefficient k the weaker is the synchronization (within some interval for k).

All these numerical results show that the whole predator-prey food chain in 6-dimensional space, displays behavioural phenomena which are completely inexplicable by any conventional analysis of the 3-dimensional systems' constituent parts, which have for the same ranges of parameters a one-peridoic solutions. They have to be compared to the results obtained in the previous section, in which it has been shown that the larger is the coupling coefficient the stronger is the synchronization

Therefore, our predator-prey system is an example pointing out new emergent properties, which are properties of the "whole" 6-dimensional system, being not possessed by any of the individual parts (which are the two 3-dimensional subsystems).

5 Conclusion and Discussion

Identical chaotic systems synchronize by following the same chaotic trajectory (IS). However, in the real world systems are in general not identical. For instance, when the parameters of two-coupled identical systems do not match, or when these coupled systems belong to different classes, complete IS is not to be expected, because it does not exist such an invariant manifold $Y = X$, as for identical synchronisation. For non-identical systems, the possibility of some other types of synchronization has been investigated (see [2] and references within cited). It has been showed [11] that when two different systems are coupled with sufficient strong coupling strenght, a general synchronous relation between their states could exist and be expressed by a smooth invertible function, $Y(t) = \psi(X(t))$, as we have done in the previous section.

But, for coupled non-identical chaotic systems, other types of synchronization exist. For example *phase synchronization* (PS hereafter) which is rather a weak degree of synchronization, see [10]. It is a hidden phenomenon, in the sense that the synchronous motion is not visible. Indeed, in case of PS, the phases of chaotic systems are locked, that is to say that it exists a certain relation between them, whereas the amplitudes vary chaotically and are practically uncorrelated. Thus, it is mostly close to synchronization of periodic oscillators.

Let us note that such a phenomenon occurs when a zero Lyapunov exponent of the response system becomes negative, while, as explained above, identical chaotic systems synchronize by following the same chaotic trajectory, when their largest transverse Lyapunov exponent of the synchronized manifold decreases from positive to negative values.

This processus deserves to be investigated for our predator-prey food chain case. A more detailed analysis of such phenomena will be provided in the near future.

References

1. L'Enigme de l'Emergence (2005) *Sciences et Avenir*, 143, Juillet-Août.
2. Aziz-Alaoui M.A. (2006) Synchronization of Chaos *Encyclopedia of Mathematical Physics*. Elsevier, 5:213-226
3. Aziz-Alaoui M.A. (2002) Study of a LeslieGower-type tritrophic population model, *Chaos Solitons and Fractals*, 14:12751293.
4. Boccaletti S., Kurths J., Osipov G., Valladares D. and Zhou (2002) The synchronization of chaotic systems, *Physics Reports* 366: 1-101.
5. Chen G. and Dong X.(1998) *From chaos to order*, Singapore: World Scientific.
6. Daher Okiye M.. (2004) Etude et Analyse Asymptotique de certains Systèmes Dynamiques Non-Linéaires, *PhD Dissertation* University of Le Havre, France.
7. Derivière S. and Aziz-Alaoui M.A. (2003) Estimation of attractors and synchronization of generalized Lorenz systems, *Dynamics of Continuous, Discrete and Impulsive Systems series B: Applications and Algorithms* 10(6): 833-852.
8. Holmes P. (2005) Ninety plus thirty years of nonlinear dynamics: Less is more and more is different, *International Journal of Bifurcation and Chaos*, 15(9): 2703-2716.
9. Pecora L. and Carroll T. (1990) Synchronization in chaotic systems, *Physical Review letters* 64, pp. 821-824.
10. Pikovsky A., Rosenblum M. and Kurths J. (2001) *Synchronization, A Universal Concept in Nonlinear Science*, Cambridge University Press.
11. Rulkov N., Sushchik M., Tsimring L. and Abarbanel H. (1995) Generalized synchronization of chaos in directionally coupled chaotic systems, *Phys. Rev. E* 51(2): 980-994.