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COMPLEX EMERGENT PROPERTIES AND CHAOS (DE-)SYNCHRONIZATION

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ABSTRACT

Emergent properties are typically novel and unanticipated. In this paper, using chaos synchronization tools and ideas, we demonstrate, via two examples of three-dimensional autonomous differential systems, that, by simple uni- or bi-directional coupling, regular (resp. chaotic) behavior can emerge from chaotic (resp. regular) behavior.

Keywords : Complex Systems, Emergence, Chaos, Synchronization, Lorenz-type system, Predator-prey.

1. Introduction : Complexity and Emergent Properties

First of all, let us open by the citation below.

”I think the next century (21th) will be the century of complexity”, (Stephen Hawking)

But what is complexity ?

An extremely difficult ”I know it when I see it” concept to define, see [1].

However, intuitively, complexity is usually greatest in systems whose components are arranged in some intricate difficult-to-understand pattern or, in the case of a dynamical system, when the outcome of some process is difficult to predict from its initial state (sensitive dependence on initial conditions, see below). And a complex system is an animate or inanimate system composed of many interacting components whose behavior or structure is difficult to understand. Sometimes a system may be structurally complex, like a mechanical clock, but behave very simply.

While several measures of complexity have been proposed in the research literature, they all fall into two general classes:

(1) Static Complexity -which addresses the question of how an object or system is put together (i.e. only purely structural informational aspects of an object), and is independent of the processes by which information is encoded and decoded;

(2) Dynamic Complexity -which addresses the question of how much dynamical or computational effort is required to describe the information content of an object or state of a system.

These two measures are clearly not equivalent. In this paper we embrace the following defintion.

Complexity is a scientific theory that asserts that some systems display behavioral phenomena is completely inexplicable by any conventional analysis of the systems’ constituent parts.

Besides, **emergence** refers to the appearance of higher-level properties and behaviors of a system that while obviously originating from the collective dynamics of that system’s components -are neither to be found in nor are directly deducable from the lower-level properties of that system. Emergent properties are properties of the ”whole” that are not possessed by any of the individual parts making up that whole. For example, an air molecule is not a tornado and a neuron is not conscious : emergent behaviors are typically novel and unanticipated.

Moreover, it is becoming a commonplace that, if the 20th was the century of physics, the 21st will be the century of biology, and, more specifically, mathematical biology, see [8]. We will concentrate our attention to demonstrate the emergence of complex (chaotic) behavior in coupled (non chaotic) systems. That is to show that for uni- or bi-directionally coupled non chaotic systems, chaos can appear even for large values of coupling parameter. This discussion is based on continuous autonomous differential systems, first of type-Lorenz illustrating identical chaos synchronization and regular behavior emerging from chaotic one, and, second, systems modeling predator-preys food-chain showing chaotic behaviors emerging from regular ones.

2. Synchronization and Desynchronization

Synchronization is a ubiquitous phenomenon characteristic of many processes in natural systems and (nonlinear) science, it has permanently remained an objectif of intensive research and is today considered as one of the basic nonlinear phenomena studied in mathematics, physics, engineering or life science. Synchronization of two dynamical systems generally means that one system somehow traces the motion of another. Indeed, it is well known that many coupled oscillators have the ability to adjust

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some common relation that they have between them due to weak interaction, which yields to a situation in which a synchronization-like phenomenon takes place, see [2].

Since this discovery, periodic synchronization has found numerous applications in various domains, for instance in biological systems and living nature where synchronization is encountered on differents levels. Examples range from the modeling of the heart to the investigation of the circardian rhythm, phase locking of respiration with a mechanical ventilator, synchronization of oscillations of human insulin secretion and glucose infusion, neuronal information processing within a brain area and communication between different brain areas. Also synchronization plays an important role in several neurological diseases such as epilepsies and pathological tremors, or in differents forms of cooperative behavior of insects, animals or humans. For more details, see [10].

This process may also be encountered in other areas, celestical mechanics or radio engineering and acoustics.

But, even though original notion and theory of synchronization implies periodicity of oscillators, during the last decades, the notion of synchronization has been generalized to the case of interacting chaotic oscillators.

Roughly speaking, a system is chaotic if it is deterministic, has a long-term aperiodic behavior, and exhibits sensitive dependence on initial conditions on a closed invariant set.
Chaotic oscillators are found in many dynamical systems of various origins, the behavior of such systems is characterized by instability and, as the result, limited predictability in time.
Despite this, in the two last decades, the search for synchronization has moved to chaotic systems. A lot of research has been done and, as a result, researchers showed that two chaotic systems could be synchronized by coupling them : synchronization of chaos is actual and chaos could then be expoitable, see [9], and for a review see [2]. Ever since, many researchers have discussed the theory, the design or applications of synchronized motion in coupled chaotic systems. A broad variety of applications have emerged, for example to increase the power of lasers, to synchronize the output of electronic circuits, to control oscillations in chemical reactions or to encode electronic messages for secure communications. Moreover, in the topics of coupled chaotic systems, many different phenomena, which are usually referred to as *synchronization*, exist and have been studied now for over a decade.

2.1. Synchronization and stability : definitions

For the basic *master-slave* configuration where an autonomous chaotic system (the master) :

$$\frac{dX}{dt} = F(X), \quad X \in \mathbb{R}^n \tag{1}$$

drives another system (the slave) :

$$\frac{dY}{dt} = G(X,Y), \quad Y \in \mathbb{R}^m, \tag{2}$$

synchronization takes place when Y asymptotically copies, in a certain manner, a subset X_p of X. That is, there exists a relation between the two coupled systems, which could be a smooth invertible function ψ, the last carries trajectories on the attractor of a first system on the attractor of a second system. In other words, if we know, after a transient regime, the state of the first system, it allows us to predict the state of the second : $Y(t) = \psi(X(t))$. Generally, it is assumed $n \geq m$, however, for the sake of easy readability, we will reduce, even if this is not a necessary restriction, to the case $n = m$, and thus $X_p = X$. Henceforth, if we denote the difference $Y - \psi(X)$ by $X_\perp$, in order to arrive at a synchronized motion, it is expected to have :

$$||X_\perp|| \longrightarrow 0, \;\; as \;\; t \longrightarrow +\infty. \tag{3}$$

If ψ is the identity function, the process is called *identical synchronization* (IS hereafter).

Definition of IS. System (2) synchronizes with system (1), if the set $M = \{(X,Y) \in \mathbb{R}^n \times \mathbb{R}^n, Y = X\}$ is an attracting set with a basin of attraction B ($M \subset B$) such that $\lim_{t\to} \infty ||X(t) - Y(t)|| = 0$, forall $(X(0), Y(0)) \in B$.

Thus, this regime corresponds to the situation where all the variables of two (or more) coupled chaotic systems converge.

If ψ is not the identity function, the phenomenon is more general and is referred to as *generalized synchronization* (GS).

Definition of GS. System (2) synchronizes with system (1), in the generalized sense, if there exists a transformation $\psi \;:\; \mathbb{R}^n \longrightarrow \mathbb{R}^m$, a manifold $M = \{(X,Y) \in \mathbb{R}^{n+m}, Y = \psi(X)\}$ and a subset B ($M \subset B$), such that for all $(X_o, Y_o) \in B$, the trajectory based on the initial conditions (X_o, Y_o) approaches M as time goes to infinity.

Henceforth, in the case of identical synchronization, equation (3) above means that a certain hyperplane M, called *synchronization manifold*, within $\mathbb{R}^{2n}$, is asymptotically stable. Consequently, for the sake of synchrony motion, we have to prove that the origin of the transverse system $X_\perp = Y - X$ is asymptotically stable. That is, to prove that the motion transversal to the synchronization manifold dies out.

The Lyapunov exponents associated with the variational equation corresponding to the tranverse system $X_\perp$:

$$\frac{dX_\perp}{dt} = DF(X)X_\perp \tag{4}$$

where $DF(X)$ is the Jacobian of the vector field evaluated onto the driving trajectory X, are referred to as transverse or conditional Lyapunov exponents (CLE hereafter).

In the case of IS it appears that the condition $L^\perp_{max} < 0$, is sufficient to insure synchronization, where $L^\perp_{max}$ is the largest CLE. Indeed, Equation (4) gives the dynamics of the motion transverse to the synchronization manifold, therefore CLE will tell us if this motion die out or not, and hence, whether the synchronization state is stable or not. Consequently, if $L^{\perp}_{max}$ is negative, it will insure the stability of the synchronized state. This will be best explained using two examples below.

2.2. Identical synchronization

(Or, the larger is the coupling coefficient the stronger is the synchronization)

Thet simplest form of chaos synchronization and the best way to explain it, is *identical synchronization* (IS), also referred to as *Conventional* or *Complete synchronization* (see [4]). It is also the most typical form of chaotic synchronization often observable in two identical systems.

There are various processes leading to synchronization, depending on the used particular coupling configuration they could be very different. So, one has to distinguish between the two following main situations, even if they are, in some sense, similar : the **uni-directional** and the **bi-directional** coupling. Indeed, synchronization of chaotic systems is often studied for schemes of the form :

$$\begin{aligned} \frac{dX}{dt} &= F(X) + kN(X-Y) \\ \frac{dY}{dt} &= G(Y) + kM(X-Y) \end{aligned} \qquad (5)$$

where F and G act in $\mathbb{R}^n$, $(X, Y) \in (\mathbb{R}^n)^2$, k is a scalar and M and N are coupling matrices belonging to $\mathbb{R}^{n\times n}$. If $F = G$ the two subsystems X and Y are identical. Moreover, when both matrices are nonzero then the coupling is called *bi-directional*, while it is referred to as *uni-directional* if one is the zero matrix, and the other being nonzero.

Other names were given in the litterature of this type of synchronization, such as *one-way diffusive* coupling, *drive-response* coupling, *master-slave* coupling or *negative feedback control*.

System (5) above with $F = G$ and $N = 0$ becomes uni-directionlly coupled, and reads :

$$\begin{aligned} \frac{dX}{dt} &= F(X) \\ \frac{dY}{dt} &= F(Y) + kM(X-Y) \end{aligned} \qquad (6)$$

M is then a matrix that determines the linear combination of X components that will be used in the difference, and k determines the strength of the coupling.

In uni-directional synchronization, the evolution of the first system (the drive) is unaltered by the coupling, the second system (the response) is then constrained to copy the dynamics of the first.

In contrast to the uni-directional coupling, for the bi-directionally (also called *mutual* or *two-way*) coupling, both drive and response systems are connected in such a way that they mutually influence each other's behavior. Many biological or physical systems consist of bi-directionally interacting elements or components, examples range from cardiac and respiratory systems to coupled lasers with feedback.

2.3. Uni- and bi-directional identical synchronization for a Lorenz-type system

(Or the larger is the coupling coefficient the stronger is the synchronization)

Let us give an example, and for the sake of simplicity, let us develop the idea on the following 3-dimensional simple autonomous system, which belongs to the class of dynamical systems called *generalized Lorenz systems*, see [7] and references therein :

$$\begin{cases} \dot{x} &= -9x - 9y \\ \dot{y} &= -17x - y - xz \\ \dot{z} &= -z + xy\,. \end{cases} \qquad (7)$$

The signs used differentiate system (7) from the well-known Lorenz system :

$$\dot{x} = -10x + 10y,\ \dot{y} = 28x - y - xz,\ \dot{z} = -\frac{8}{3}z + xy.$$

From previous observations, it was shown that system (7) oscillate chaotically, its Lyapunov exponents are $+0.601$, 0.000 and -16.470, it exhibits the chaotic attractor of figure 1, with a 3D feature very similar to that of Lorenz attractor.

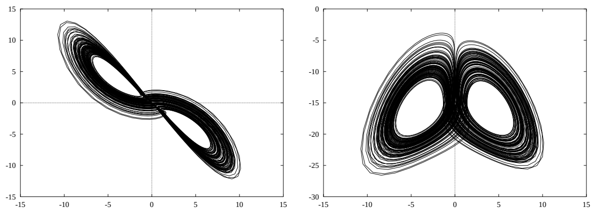

Figure 1. The chaotic attractor of system (7) : xy and xz-plane projections.

Uni-directional coupling

Let us consider an example with two copies of system (7), and for

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (8)$$

that is, by adding a damping term to the first equation of the response system, we get a following uni-directionally coupled system, coupled through a linear term $k > 0$ according to variables $x_{1,2}$:

$$\begin{cases} \dot{x_1} &= -9x_1 - 9y_1 \\ \dot{y_1} &= -17x_1 - y_1 - x_1z_1 \\ \dot{z_1} &= -z_1 + x_1y_1 \\ \dot{x_2} &= -9x_2 - 9y_2 - k(x_2 - x_1) \\ \dot{y_2} &= -17x_2 - y_2 - x_2z_2 \\ \dot{z_2} &= -z_2 + x_2y_2 \end{cases} \tag{9}$$

For $k = 0$ the two subsystems are uncoupled, for $k > 0$ both subsystems are uni-directionally coupled. Our numerical computations yield the optimal value $\tilde{k}$ for the synchronization, we found that for $k \geq \tilde{k} = 4.999$ both subsystems of (9) synchronize. That is, starting from random initial conditions, and after some transient time, system (9) generates the same attractor as for system (7), see figure 1. Consequently, all the variables of the coupled chaotic subsystems converge, that are x_2 converges to x_1, y_2 to y_1 and z_2 to z_1, see figure 2. Thus, the second system (the response) is locked to the first one (the drive). One

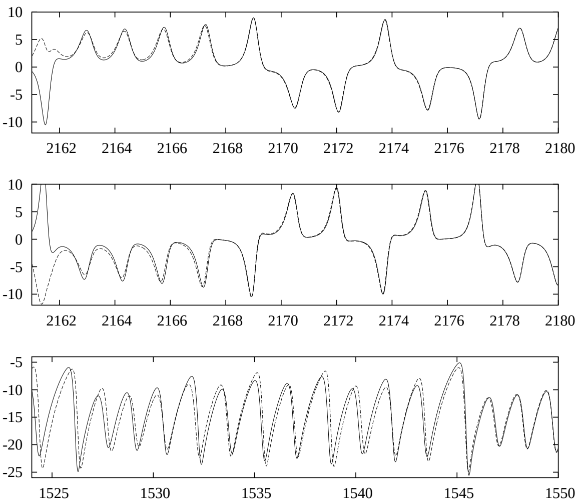

Figure 2. Time series for $x_i(t)$, $y_i(t)$ and $z_i(t)$ in system (9), $(i = 1, 2)$, for the coupling constant k = 5.0, that is beyond the threshold necessary for synchronization. After transients die down, the two subsystems synchronize perfectly.

also could give correlation plots that are the amplitudes x_1 against x_2, y_1 against y_2 and z_1 against z_2, and observe diagonal lines, meaning also that the system synchronizes.

Bi-directional coupling

Let us then take two copies of the same system (7) as given above, but two-way coupled through a linear constant term $k > 0$ according to variables $x_{1,2}$:

$$\begin{cases} \dot{x_1} &= -9x_1 - 9y_1 - k(x_1 - x_2) \\ \dot{y_1} &= -17x_1 - y_1 - x_1z_1 \\ \dot{z_1} &= -z_1 + x_1y_1 \\ \dot{x_2} &= -9x_2 - 9y_2 - k(x_2 - x_1) \\ \dot{y_2} &= -17x_2 - y_2 - x_2z_2 \\ \dot{z_2} &= -z_2 + x_2y_2 \end{cases} \tag{10}$$

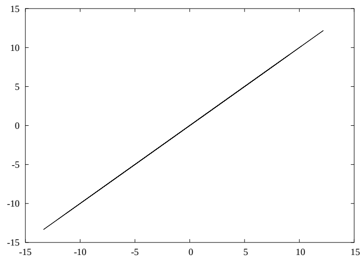

Figure 3. Regular behavior emerges from chaotic behaviors. The plot of amplitudes y_1 against y_2, after transients die down, shows a diagonal line, which also indicates that the receiver and the transmitter are maintaining synchronization. The plot of z_1 against z_2 shows a similar figure.

We can get an idea of the onset of synchronization by plotting for example x_1 against x_2 for various values of the coupling strength parameter k. Our numerical computations yield the optimal value $\tilde{k}$ for the synchronization : $\tilde{k} \simeq 2.50$, see figure 4, both (x_i, y_i, z_i)-subsystems synchronize and system (10) also generates the attractor of figure 1.

These results show also that simple bi-directional coupling of two chaotic systems do not increase the chaoticity of the new system, unlike what one might expect. Thus, in some sense (see synchronization manifolf below), regular behavior emerges from chaotic behavior, (i.e. the motion is confined in some manifold).

2.4. Remark on the stability manifold

Geometrically, the fact that systems (9) and (10), beyond synchronization, generate the same attractor as for system (7), implies that the attractors of these combined drive-response 6-dimensional systems are confined to a 3-dimensional hyperplane (the *synchronization manifold*) defined by $Y = X$. This hyperplane is stable since small perturbations which take the trajectory off the synchronization manifold will decay in time. Indeed, as said before, conditional Lyapunov exponents of the linearization of the system around the synchronous state could determine the stability of the synchronized solution. This leads to requiring that the origin of the transverse system, $X_\perp$, is asymptotically stable. To see this, for both systems (9) and (10), we then switch to the new set of coordinates, $X_\perp = Y - X$, that is $x_\perp = x_2 - x_1$, $y_\perp = y_2 - y_1$ and $z_\perp = z_2 - z_1$. The origin $(0, 0, 0)$ is obviously a fixed point for this transverse system, within the synchronization manifold. Therefore, for small deviations from the synchronization manifold, this system reduces to a typical variational equation :

$$\frac{dX_\perp}{dt} = DF(X)X_\perp \tag{11}$$

where $DF(X)$ is the Jacobian of the vector field evaluated onto the driving trajectory X. Previous analysis, see [2], shows that $L^\perp_{max}$ becomes negative as k increases, for both uni- or bi-directionally coupling, which insure the stability of the synchronized state, for systems (9) and (10), figure 5.

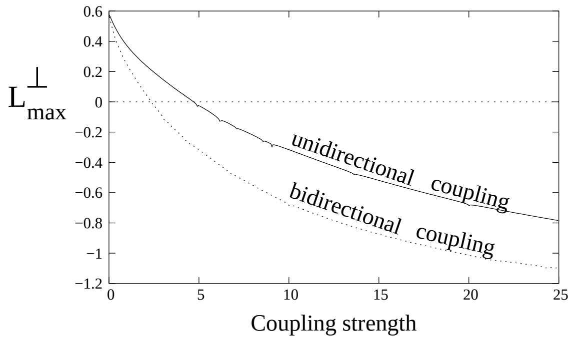

Figure 5. The largest transverse Lyapunov exponents $L^{\perp}_{max}$ as a function of coupling strength k in the unidirectional system (9) (solid) and the bi-directional system (10) (dotted).

Let us note that this can also be proved analytically as done in [7], by using a suitable Lyapunov function, and using some new extended version of LaSalle invariance principle.

Desynchronization motion. Synchronization depends on the coupling strength, but also on the vector field and the coupling function. For some choice of these quantities, synchronization may occur only within a finite range $[k_1, k_2]$ of coupling strength, in such a case a *desynchronization* phenomenon occurs. Thus, increasing k beyond the critical value k_2 yields loss of the synchronized motion ($L^{\perp}_{max}$ becomes positive).

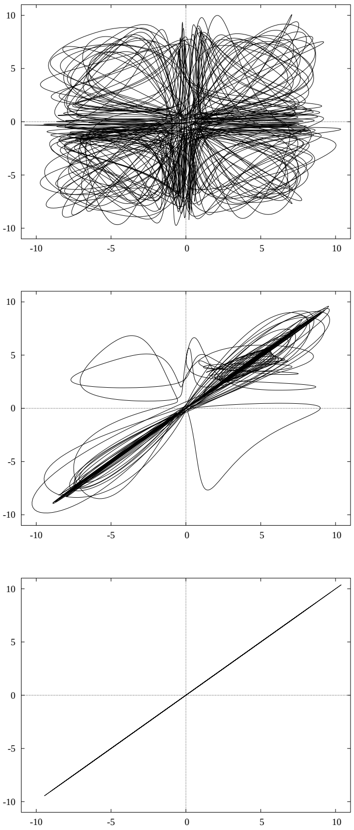

Figure 4. Illustration of the onset of synchronization of system (10). (a), (b) and (c) plot the amplitudes x_1 against x_2 for values of the coupling parameter $k = 0.5$, $k = 1.5$ and $k = 2.8$ respectively. The system synchronizes for $k \geq 2.5$.

3. Emergence of Chaotic Properties : A Predator-Prey Example

(Or, the larger is the coupling coefficient the weaker is the synchronization).

3.1. The model

As we said in the introduction, it is becoming a commonplace that, if the 20th was the century of physics, the 21st will be the century of biology, and, more specifically, mathematical biology, ecology, ... and, in general, nonlinear dynamics and complexity in life sciences.

In this last part, we will hence focus ourself on emergent (chaotic or regular) properties that arrise when we couple uni- or bi-directionally two 3-dimensional autonomous differential systems, modeling predator-prey food-chains.

Le us then consider a continuous time dynamical system, model for a tritrophic food chain, based on a modified version of the Leslie-Gower scheme, see [3, 6], for which the rate equations for the three components of the chain population can be written as follows :

$$\begin{cases} \dfrac{dX}{dT} &=& a_o X - b_o X^2 - \dfrac{v_o X Y}{d_o + X} \\ \dfrac{dY}{dT} &=& -a_1 Y + \dfrac{v_1 X Y}{d_1 + X} - \dfrac{v_2 Y Z}{d_2 + Y} \\ \dfrac{dZ}{dT} &=& c_3 Z - \dfrac{v_3 Z^2}{d_3 + Y}, \end{cases} \quad (12)$$

with $X(0) \geq 0$, $Y(0) \geq 0$ and $Z(0) \geq 0$, where X, Y and Z represent the population densities at time T ; a_0, b_0, v_0, d_0, a_1, v_1, d_1, v_2, d_2, c_3, v_3 and d_3 are model parameters assuming only positive values and are defined as follows : a_o is the growth rate of prey X, b_o measures the strength of competition among individuals of species X, v_o is the maximum value which *per capita* reduction rate of X can attain, d_o measures the extent to which environment provides protection to prey X, a_1 represents the rate at which Y will die out when there is no X, v_1 and d_1 have a similar meaning as v_0 and d_o, v_2 and v_3 have a similar biological connotation as that of v_o and v_1, d_2 is the value of Y at which the *per capita* removal rate of Y becomes $v_2/2$, c_3 describes the growth rate of Z, assuming that the number of males and females is equal, d_3 represents the residual loss in species Z due to severe scarcity of its favorite food Y ; the second term on the right hand side in the third equation of (12) depicts the loss in the predator population.

For the origin of this system and for some theoretical results, boundedness of solutions, existence of an attracting set, existence and local or global stability of equilibria, etc ..., see [3, 6]. In these works, using intensive numerical qualitative analysis, it has been demonstrated that the model could exhibit periodic solutions, figure (6), and quasi-periodic or chaotic dynamics, figure (7), for the following parameters and state values :

$$\begin{cases} b_o = 0.06, \quad v_o = 1.0, \quad d_0 = d_1 = d_2 = 10.0, \\ a_1 = 1.0, \; v_1 = 2.0, \; v_2 = 0.9, \\ c_3 = 0.02, \; v_3 = 0.01, \; d_3 = 0.3. \end{cases} \quad (13)$$

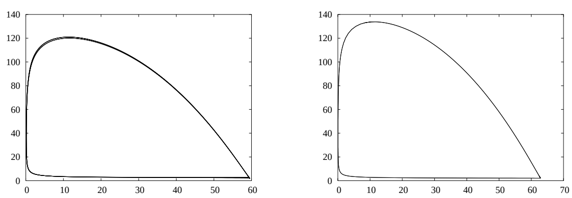

Figure 6. Limit cycles of periood one and two, for $a_0 = 3.6$ and $a_0 = 3.8$ respectively, found for system (12) and for parameters given by (13).

We will fix, for the rest of the paper, the system parameters as given in (13) in order that system (13) oscillates in a regular maneer around a stable limit cycle of period one figure 6(b).

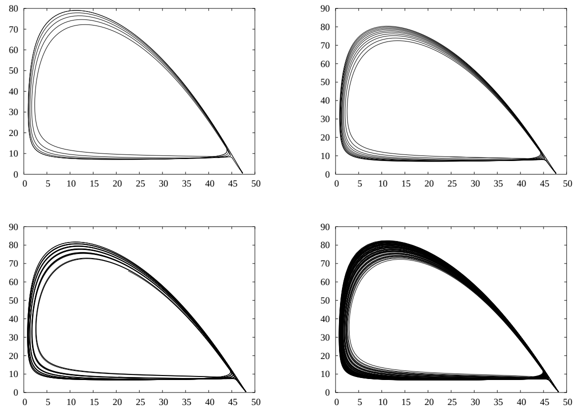

Figure 7. Transition to chaotic (or quasi-periodic) behavior is established via period doubling bifurcation, for respectively $a_0 = 2.85$, $a_0 = 2.87$, $a_0 = 2.89$ and $a_0 = 2.90$, found for system (12) and for parameters given by (13).

3.2. Uni-directional desynchronization : Predator-Prey system

Usually, as we have seen in the previous section, in uni-directional synchronization, while the evolution of the first system (the drive) is unaltered by the coupling, the second system (the response) is constrained to copy the dynamics of the first.

However, this is not the case for our example below, for which, as we will see, while both subsystems evolve periodically (limit cycles of figure 6(b)), the coupled system' behavior is extremely complex.

Let us then consider two copies of system (12). By adding a damping term to the first equation of the response system, we get a following uni-directionally coupled system, coupled through a linear term $k > 0$ according to variables $x_{1,2}$:

$$\begin{cases} \dot{X_1} &=& a_o X_1 - b_o X_1^2 - \dfrac{v_o X_1 Y_1}{d_o + X_1} \\ \dot{Y_1} &=& -a_1 Y_1 + \dfrac{v_1 X_1 Y_1}{d_1 + X_1} - \dfrac{v_2 Y_1 Z_1}{d_2 + Y_1} \\ \dot{Z_1} &=& c_3 Z_1 - \dfrac{v_3 Z_1^2}{d_3 + Y_1} \\ & & \\ \dot{X_2} &=& a_o X_2 - b_o X_2^2 - \dfrac{v_o X_2 Y_2}{d_o + X_2} - k(X_2 - X_1) \\ \dot{Y_2} &=& -a_1 Y_2 + \dfrac{v_1 X_2 Y_2}{d_1 + X_2} - \dfrac{v_2 Y_2 Z_2}{d_2 + Y_2} \\ \dot{Z_2} &=& c_3 Z_2 - \dfrac{v_3 Z_2^2}{d_3 + Y_2} \end{cases}$$

(14)

For $k = 0$ the two subsystems are uncoupled, for $k > 0$ both subsystems are uni-directionally coupled, and for $k \longrightarrow +\infty$ one can expect to obtain the same results as those obtained for the previous example in the previous section, that is strong synchronization and two subsystems which evolve identically.

We have choose, for the coupled system, a range of parameters for which both subsystems constituents parts evolve periodically, as figure 6(b) shows.

However, our numerical computations show that both subsystems of (14) never synchronize nor identically neither *generally*, unless the coupling parameter k is very small. In such a case a certain generalized synchronization form takes place, see figure 8(a). That is, starting from random initial conditions, and after some transient time, system (14) generates attractor different from those exhibited by system (12) in figure 6(b). Consequently, all the variables of the coupled limit cycles subsystems surprisingly do not converge, as, at first sight, one may intuitively expect, see figure 8.

These results show that uni-directional coupling of these two non-chaotic systems, that are the subsystems constituent of system (14), increase the bahavior complexity, and transforms a periodic situation in a chaotic one.

Emergent chaotic properties are typically novel and unanticipated, for this example.

This corresponds to the classical cascade of periodic-doubl- ing bifurcation processus, with a sequence of order and disorder windows.

3.3. Bidirectional desynchronization : Predator-Prey system

As, many biological or physical systems consist of bi-directionally interacting elements or components, let us use a bi-directionally (*mutual*) coupling, in order that both drive and response subsystems are connected in such a way that they mutually influence each other's behavior. Let us then take two copies of the same system (12) as given above, but two-way coupled through a linear constant term $k > 0$ according to variables $x_{1,2}$:

$$\left\{\begin{array}{rcl}\dot{X_1} &=& a_oX_1 - b_oX_1^2 - \dfrac{v_oX_1Y_1}{d_o+X_1} - k(X_1-X_2)\\ \dot{Y_1} &=& -a_1Y_1 + \dfrac{v_1X_1Y_1}{d_1+X_1} - \dfrac{v_2Y_1Z_1}{d_2+Y_1}\\ \dot{Z_1} &=& c_3Z_1 - \dfrac{v_3Z_1^2}{d_3+Y_1}\\ && \\ \dot{X_2} &=& a_oX_2 - b_oX_2^2 - \dfrac{v_oX_2Y_2}{d_o+X_2} - k(X_2-X_1)\\ \dot{Y_2} &=& -a_1Y_2 + \dfrac{v_1X_2Y_2}{d_1+X_2} - \dfrac{v_2Y_2Z_2}{d_2+Y_2}\\ \dot{Z_2} &=& c_3Z_2 - \dfrac{v_3Z_2^2}{d_3+Y_2}\end{array}\right. \tag{15}$$

We have also choose, for this bi-directionally coupled system, the same range of parameters for which the sub-

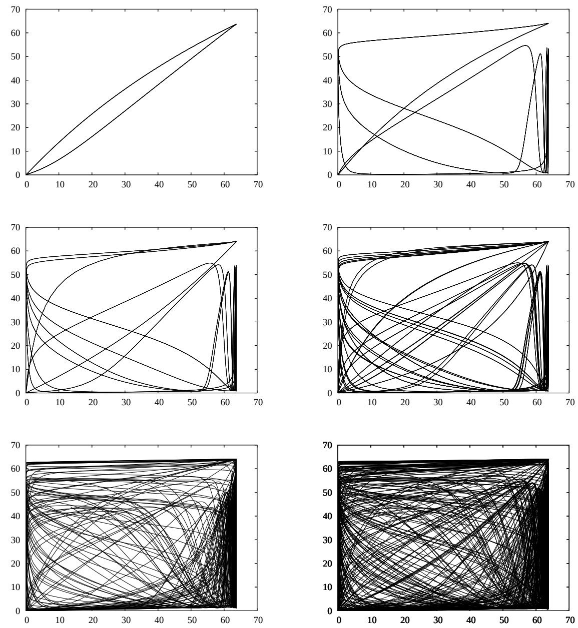

Figure 8. Illustration of the onset of desynchronization of the unidirectionel coupled system (14). Figures (a), (b), ... (f) are done left-right and up-down, and plot the amplitudes x_1 against x_2 for values of the coupling parameter (a) $k = 0.01$, (b) $k = 0.055$, (c) $k = 0.056$, (d) $k = 0.0565$, (e) $k = 0.057$ and (f) $k = 0.1$. Figure (a,b,c) show a generalized synchronization phenomenon : the system synchronizes (in the generalized sense) for very small values of k. But a desynchronization processus fastly arrises by increasing k, figures (d,e,f) : -in some interval for k, the larger is the coupling coefficient the weaker is the synchronization-. Hence, we have emergence of chaotic properties : The coupled system displays behavioral chaotic phenomena which is not exhibited by systems' constituent parts, that are the two predator-prey systems before coupling, which exhibit the limit-cycle of figure 6(b), and for the same parameters, same initial conditions. This phenomenon is robust with respect to small parameters variations.

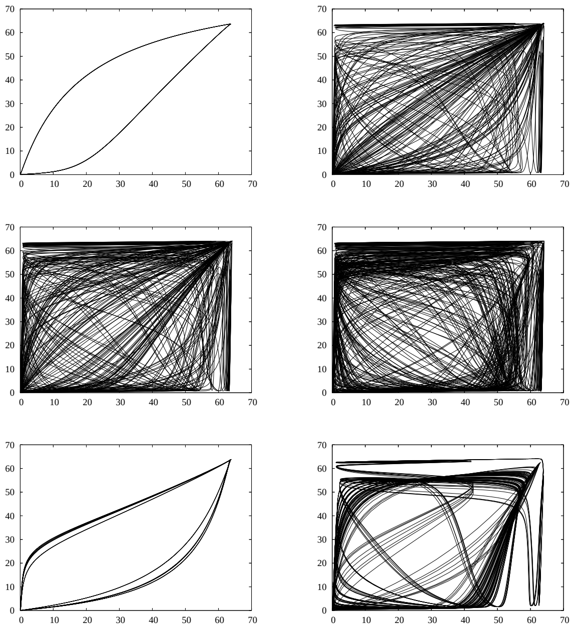

Figure 9. Bidirectionel coupling. Figures plot amplitudes x_1 against x_2 for the same values as done for the previous figure, that are, respectively, for $k = 0.01$, $k = 0.055$ and $k = 0.056$ $k = 0.0565$, $k = 0.057$ and $k = 0.1$. These figures, Illustrate a window of generalized synchronization and desynchronization of system (12). (a), (b) and (c) plot The system synchronizes (in the generalized sense) for $k \leq 0.01$, as it has been shown in the uinidirectional case. But the desynchronization processus arrises by increasing k, fastly in comparison with the unidirectional case.

systems constituents parts evolve periodically, as figure 6(b) shows.

Figure 9 demonstrates also, for some interval of parameter k, that the larger is this coupling coefficient the weaker is the synchronization. That is, we have emergence of new properties for the coupled system. The last displays behavioral chaotic phenomena which is not exhibited by systems' constituent parts, that are the two predator-prey systems before coupling, and for the same parameters, same initial conditions. A robust phenomenon with respect to small variations of parameters values.

Furthermore, the bidirectional case enhances the desynchronization processus that is the occurence of new complex phenomenon, and makes it occuring fastly in comparison to the unidirectinal case ; for the same interval $k \in J =]0, 0.1]$, chaotic properites take place for $k = 0.055$ in the unidirectional case, and for $k = 0.057$ in the unidirectional case. This complex behavior remains observable in whole interval J for the last, but for the first, it disappears after $k = 0.056$ -some regular generalized synchronization takes place- and appears again for $k \in]0.057, 0.1]$

Thus, one may conclude that, the larger is the coupling coefficient k the weaker is the synchronization (within some interval for k).

All these numerical results show that the whole predator-prey food chain in 6-dimensional space, displayes behavioral phenomena which is completely inexplicable by any conventional analysis of the 3-dimensional systems' constituent parts, which have for the same ranges of parameters a 1-peridoic solutions. They have to be compared to the obtained results in the previous section, in which it has been shown that : the larger is the coupling coefficient the stronger is the synchronization

Therefore, our predator-prey system is an example exhibiting new emergent properties, which are properties of the "whole" 6-dimensional system that are not possessed by any of the individual parts (that are the two 3-dimensional subsystems).

4. Conclusion and Discussion

Identical chaotic systems synchronize by following the same chaotic trajectory (IS). However, in the real world systems are in general not identical. For instance, when the parameters of two-coupled identical systems do not match, or when those coupled systems belong to different classes, complete IS may not be expected, because there does not exist such an invariant manifold $Y = X$, as for identical synchronisation. For nonidentical systems, the possibility of some other types of synchronization has been investigated (see references cited in [2]). It was showed, [11], that when two different systems are coupled with sufficiently strong coupling strenght, a general synchronous relation between their states could be exist and expressed by a smooth invertible function, $Y(t) = \psi(X(t))$, as we have done in the previos section.

But, for coupled nonidentical chaotic systems, other type of synchronization exist. For example *phase synchronization* (PS hereafter) which is a rather weak degree of synchronization, see [10]. It is a hidden phenomenon, in the sense that the synchronous motion is not visible. Indeed, in case of PS, the phases of chaotic systems are locked, that is there exists a certain relation between them, whereas the amplitudes vary chaotically and are practically uncorrelated. Thus, it is mostly close to synchronization of periodic oscillators.

Let us note that such a phenomenon occurs when a zero Lyapunov exponent of the response system becomes negative, while, as explained above, identical chaotic systems synchronize by following the same chaotic trajectory, when their largest transverse Lyapunov exponent of the synchronized manifold decreases from positive to negative values.

This processus deserves to be investigated for our predator-prey food chain case. More detailed analysis for such phenomena will be provided in the near future.

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