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Mathematical Modeling of Human Behaviors During Catastrophic Events: Stability and Bifurcations

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The aim of this paper is to present some mathematical results concerning the PCR system (Panic-Control-Reflex), which is a model for human behaviors during catastrophic events. This model has been proposed to better understand and predict human reactions of individuals facing a brutal catastrophe, in a context of an established increase of natural and industrial disasters. After stating some basic properties, that is positiveness, boundedness, and stability of the solutions, we analyze the transitional dynamic. We then focus on the bifurcation that occurs in the system, when one behavioral evolution parameter passes through a critical value. We exhibit a degeneracy case of a saddle-node bifurcation, in a larger context of classical saddle-node bifurcations and saddle-node bifurcations at infinity, and we study the inhibition effect of higher order terms.

Keywords: Bifurcation; stability; catastrophic event; mathematical modeling; panic.

1. Introduction

Aristotle used to think that the brain does not play any particular role in the process of adopting a certain behavior [Gross, 1995]. He pretended that its action was only devoted to the control of some basic organic functions. Nowadays, the knowledge of the brain has widely improved. Biologists and neuroscientists have understood that some regions of the brain are dedicated to some particular behaviors, or decisions that are to be made by an individual facing a non-normal event, and particularly a catastrophic event [Noto *et al.*, 1994; Crocq, 1994; Laborit, 1994].

The PCR system (Panic-Control-Reflex) is a model which was built to better predict the human behavior during catastrophic events [Verdière *et al.*, 2014; Provitolo *et al.*, 2015]. Dividing into three groups of behaviors for a population affected by a disaster, the model takes into account the links between the different behavior phases, distinguishing evolution processes and imitation phenomena. The catastrophic events can have a natural origin (tsunami, earthquakes, fires…), or can correspond to an industrial disaster (nuclear blast, factory explosion…). We consider only sudden disasters, with no alert to the population. The complete PCR system integrates some mortality terms and some domino effect terms. Indeed, in this paper, we shall focus on a situation with a constant population. Verdière *et al.* [2014] and Provitolo *et al.* [2015] clarified the initial choices made to build the model, which present similarities with epidemiological models [Murray, 2003, 2002; Thieme, 2003; Provitolo, 2005; Zhou *et al.*, 2006] or prey–predator models [Mukherjee, 2003; Roy & Roy, 2016]. They precise the form of the imitation functions chosen to model the *emotion contagion* phenomenon, that can act symmetrically [Hatfield *et al.*, 1994; George & Gamond, 2011], whose flow depends on the relative proportions on each behavior subgroup. Some numerical simulations are also shown, as a first step in the validation process of the model, confronted with rare available data [Boyd, 1981; Vermeiren, 2007].

In this paper, we shall present some mathematical results of the qualitative study of the PCR system, giving a rigorous frame to numerical simulations. It will occasionally be a necessity to simplify the form of the equations in the PCR system, considering that modeling is a difficult task which implies a constant dilemma between, on the first hand, the desire to take into account numerous phenomena to approach reality, or at least the perception that we have from, and on the other hand, the obligation to propose a simple model that can be studied with a qualitative point of view [Thieme, 2003].

The outline is the following. In the first section, we will recall some basics about the PCR system, presenting its components and parameters, and we will prove the positiveness and the boundedness of the solutions, which are obvious properties to be satisfied by a population dynamic model. Then, we shall study the asymptotic stability of the trivial equilibrium, using Poincaré–Lyapunov classical methods, and the transitional dynamic of the system, that presents an attractive node. The last section is devoted to the analysis of the bifurcation identified when some evolution parameter passes through a critical value, exhibiting a larger context of saddle-node bifurcations in which the solutions evolve. Finally, we study the inhibition effect of higher order terms.

2. Problem Statement and Preliminaries

2.1. *PCR system*

We consider the following system of ordinary differential equations, resulting from the modeling of human behaviors during catastrophic events [Verdière *et al.*, 2014; Provitolo *et al.*, 2015]:

$$\dot{X} = \Phi(t, X) \tag{1}$$

with $\dot{X} = \frac{\mathrm{d}X}{\mathrm{d}t}$, $X = (r, c, p, q, b)^T \in \mathbb{R}^5$ and Φ given by

$$\Phi(t, X) = (\Phi_i(t, X))^T, \quad i \in [\![1, 5]\!],$$

where the functions Φ_i are real valued functions defined on $\mathbb{R} \times \mathbb{R}^5$ by

$$\begin{cases} \Phi_1(t, X) = \gamma(t)q\left(1 - \dfrac{r}{r_m}\right) - (B_1 + B_2)r \\ \qquad\qquad + s_1(t)c + s_2(t)p + F(r, c)rc \\ \qquad\qquad + G(r, p)rp \\ \Phi_2(t, X) = -\varphi(t)c(1 - b) + B_1 r + C_1 p - C_2 c \\ \qquad\qquad - s_1(t)c - F(r, c)rc + H(c, p)cp \\ \Phi_3(t, X) = B_2 r - C_1 p + C_2 c - s_2(t)p \\ \qquad\qquad - G(r, p)rp - H(c, p)cp \\ \Phi_4(t, X) = -\gamma(t)q\left(1 - \dfrac{r}{r_m}\right) \\ \Phi_5(t, X) = \varphi(t)c(1 - b). \end{cases}$$

The imitation functions F, G and H are real valued functions defined on $\mathbb{R} \times \mathbb{R}$ by

$$F(r, c) = -\alpha_1 f_1\left(\frac{r}{c + \epsilon}\right) + \alpha_2 f_2\left(\frac{c}{r + \epsilon}\right)$$

$$G(r, p) = -\delta_1 g_1\left(\frac{r}{p + \epsilon}\right) + \delta_2 g_2\left(\frac{p}{r + \epsilon}\right)$$

$$H(c, p) = \mu_1 h_1\left(\frac{c}{p + \epsilon}\right) - \mu_2 h_2\left(\frac{p}{c + \epsilon}\right),$$

where ϵ is a positive number, and f_i, g_i, h_i, $i \in \{1, 2\}$, real valued functions defined on $\mathbb{R}$, with a decreasing shape chosen to model the possibility

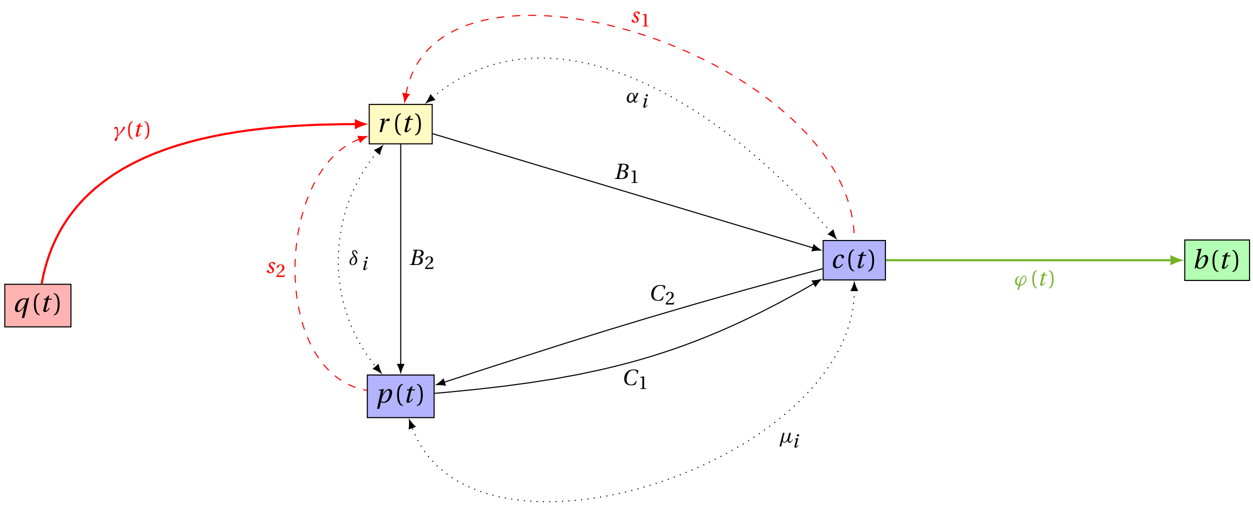


Fig. 1. Schema for the PCR system, showing the evolution parameters B_i and C_i, the imitation parameters α_i, δ_i and μ_i, and the domino effect parameters s_i, $i \in \{1, 2\}$. The beginning of the disaster and the return to a daily behavior are respectively modeled by $\gamma(t)$ and $\varphi(t)$.

that a behavior imitation can act symmetrically. Those functions satisfy the property

$$0 \leq f_i(s) \leq 1, \quad 0 \leq g_i(s) \leq 1, \quad 0 \leq h_i(s) \leq 1, \quad \forall\, s \in \mathbb{R}. \tag{2}$$

In the last section, we will reduce the analysis to the case with constant imitation functions.

This model is a nonlinear, adimensional differential system, and the variables r, c, p, q and b denote respectively the densities of people being in a *reflex*, *control*, *panic*, *daily*[1] or *back to daily* behavior (see Fig. 1 and Table 1). We will consider an initial time $t_0 \geq 0$, and an initial condition

$$(r(t_0), c(t_0), p(t_0), q(t_0), b(t_0)) = (r_0, c_0, p_0, q_0, b_0), \tag{3}$$

that satisfies the properties

$$\begin{cases} r_0 + c_0 + p_0 + q_0 + b_0 = 1 \\ (r_0, c_0, p_0, q_0, b_0) \in (\mathbb{R}^+)^5. \end{cases} \tag{4}$$

We will often choose

$$(r(t_0), c(t_0), p(t_0), q(t_0), b(t_0)) = (0, 0, 0, 1, 0), \tag{5}$$

which corresponds to the situation when all the individuals are in a daily behavior before the beginning of the disaster. In order to study the stability of the steady states, we will relax this initial condition when necessary.

[1] The letter q corresponds to the french word *quotidien*.

Remark 2.1. The sum of the five equations is null, which means that the considered population is constant. In other words, this model does not take into account mortality rate, as mentioned in our introduction. However, it is easy to enhance the system, adding linear terms on each equation to consider that a part of the population affected by a brutal

Table 1. Notations for the main functions and parameters in the PCR system.

Function	Notation
Daily behaviors	$q(t)$
Reflex behaviors	$r(t)$
Control behaviors	$c(t)$
Panic behaviors	$p(t)$
Back to daily behaviors	$b(t)$
Beginning of the disaster	$\gamma(t)$
Return to a daily behavior	$\varphi(t)$
Imitation functions	F, G, H
Parameter	**Notation**
Evolution from reflex to control	B_1
Evolution from reflex to panic	B_2
Evolution from panic to control	C_1
Evolution from control to panic	C_2
Imitation between reflex and control	α_1, α_2
Imitation between reflex and panic	δ_1, δ_2
Imitation between panic and control	μ_1, μ_2
Domino effect	s_1, s_2
Reflex behavior maximum size	r_m

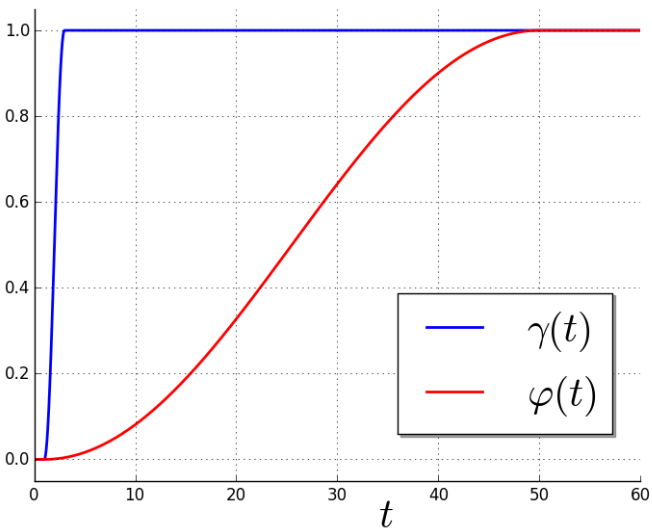


Fig. 2. A possible shape for the functions γ and φ, that respectively model the beginning of the catastrophe and the return to a daily behavior, in the case of an abrupt disaster and a smooth exit of catastrophe behaviors.

disaster is concerned with death. The qualitative study is quite similar, and we have chosen to focus in this paper on a constant population model.

The parameters of the PCR system are the real coefficients $r_m > 0$ (reflex behavior maximum size), $B_i > 0$, $C_i \geq 0$, $i \in \{1, 2\}$ (evolution coefficients), $\alpha_i \geq 0$, $\delta_i \geq 0$, $\mu_i \geq 0$, $i \in \{1, 2\}$ (interaction coefficients involved in the functions F, G and H), $s_i \geq 0$, $i \in \{1, 2\}$ (domino effect coefficients), which can also be built in a periodic form in order to model a succession of disasters. For more convenience, we introduce the vector of parameters

$$\Lambda = (r_m, B_1, B_2, C_1, C_2, s_1, s_2, \alpha_1, \alpha_2, \delta_1, \delta_2, \mu_1, \mu_2)$$

and its domain $\mathscr{D} = (\mathbb{R}^{+*})^3 \times (\mathbb{R}^+)^{10}$.

Remark 2.2. The functions γ and φ respectively model the beginning of the disaster and the return to a daily behavior (see Fig. 2). Their shape can be adapted to various scenarios and they satisfy $\gamma(t) = \varphi(t) = 1$ for t sufficiently large. Furthermore, φ and γ are supposed to be increasing functions.

It will sometimes be more convenient, for technical reasons, to consider the four equations system

$$\dot{X} = \Psi(t, X), \quad X = (r, c, p, q)^T,$$

whose equations read

$$\begin{cases} \dot{r} = \gamma(t)q\left(1 - \dfrac{r}{r_m}\right) - (B_1 + B_2)r + s_1(t)c \\ \qquad + s_2(t)p + F(r,c)rc + G(r,p)rp \\ \dot{c} = -\varphi(t)c(r + c + p + q) + B_1 r + C_1 p - C_2 c \\ \qquad - s_1(t)c - F(r,c)rc + H(c,p)cp \\ \dot{p} = B_2 r - C_1 p + C_2 c - s_2(t)p - G(r,p)rp \\ \qquad - H(c,p)cp \\ \dot{q} = -\gamma(t)q\left(1 - \dfrac{r}{r_m}\right). \end{cases} \tag{6}$$

This system is simply obtained by substituting $r + c + p + q$ by $1 - b$, considering that the total population $r + c + p + q + b$ is constant, equal to 1, from the moment that the initial condition satisfies property (4). The initial condition corresponding to $(0, 0, 0, 1, 0)$ becomes:

$$(r(t_0), c(t_0), p(t_0), q(t_0)) = (0, 0, 0, 1). \tag{7}$$

2.2. *Positiveness and boundedness*

The first proposition guarantees the positiveness of the solutions. We then prove that they lie in a compact set (see Fig. 3).

Proposition 2.1. *We consider the Cauchy problem (1)–(3). For any value of the parameters*

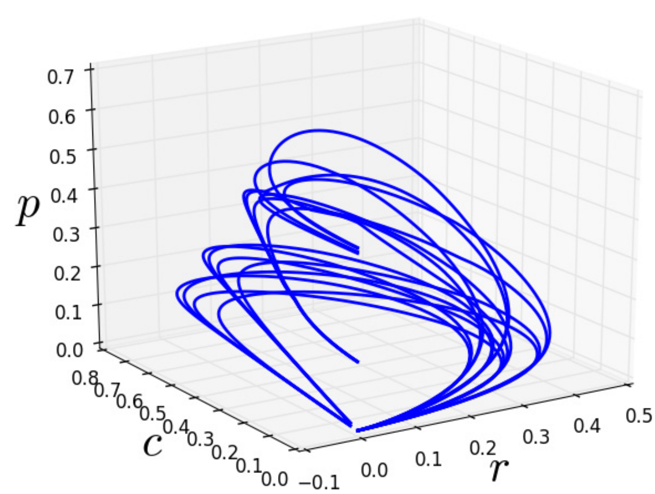


Fig. 3. Several orbits of the PCR system for various values of the parameters B_1, B_2, C_1, C_2, and the same initial condition $(0, 0, 0, 1, 0)$, projected in the (r, c, p) space. The solutions lie in the compact set $[0, 1]^5$.

$\Lambda \in \mathscr{D}$, there exists a unique maximal solution. Furthermore, its components are non-negative.

Proof. The existence and uniqueness of a maximal solution is guaranteed by the fundamental Cauchy–Lipschitz theorem, as the function Φ is regular. We have to prove that the components are non-negative. We first consider $q(t)$, and integrate the fourth equation in system (1):

$$q(t) = q(t_0)e^{-\int_{t_0}^{t} \gamma(s)(1-r(s)/r_m)\mathrm{d}s}, \tag{8}$$

which directly implies $q(t) > 0$ for all $t \geq t_0$ if $q(t_0) > 0$, and $q(t) = 0$ for all $t \geq t_0$ if $q(t_0) = 0$. Then, we suppose that there exists $\theta > t_0$ such that:

$$(r(\theta), c(\theta), p(\theta), b(\theta)) \notin]0, +\infty[^4$$

and we consider

$$t_1 = \inf\{\theta > t_0; (r(\theta), c(\theta), p(\theta), b(\theta)) \notin]0, +\infty[^4\}.$$

If $(r(t_1), c(t_1), p(t_1), b(t_1)) = (0, \bar{c}, \bar{p}, \bar{b})$, with $\bar{c} > 0$, $\bar{p} > 0$ and $\bar{b} > 0$, then

$$\dot{r}(t_1) = \gamma(t_1)q(t_1) + s_1(t_1)c(t_1) + s_2(t_1)p(t_1) > 0.$$

We can here invoke the simple following real analysis lemma, whose proof can be made with a simple Taylor expansion of order 1.

Lemma 2.1. *Let f be a real valued function defined on a non-empty interval $]a, b[\subset \mathbb{R}$. We suppose that f is differentiable on $]a, b[$, and that there exists $c \in]a, b[$ such that $f(x) > 0$ for all $x \in]a, c[$, $f(c) = 0$, and $f(x) < 0$ for all $x \in]c, b[$. Then $\frac{df}{dt}(c) \leq 0$.*

The function r satisfies the hypothesis of the lemma, but $\dot{r}(t_1) > 0$, which yields a contradiction. The other cases are treated in the same way. ■

Corollary 2.1. *For any value of the parameters $\Lambda \in \mathscr{D}$, the compact set $[0, 1]^5$ is invariant under the flow induced by the PCR system (1) and the initial condition (3).*

Proof. We have already remarked that the total population $r + c + p + q + b$ is constant. Since we suppose the initial condition to satisfy $(r + c + p + q + b)(t_0) = 1$, we directly conclude that the solution is global and lies within the compact set $[0, 1]^5$. ■

2.3. *Critical points*

We conclude this section with the research of the critical points of the PCR system (1). To that aim, we solve

$$\Phi(t, \overline{X}) = 0, \quad \forall\, t \geq t_0.$$

Some basic algebraic computations produce the following result. For all values of the parameters $\Lambda \in \mathscr{D}$, $\mathscr{O}(0, 0, 0, 0, 1)$ is a critical point, that we will call *trivial equilibrium* in what follows. If $C_1 > 0$, or if $s_2 > 0$, it is the only equilibrium point. Else if $C_1 = s_2 = 0$, then for all $\bar{p} \in [0, 1]$, $\mathscr{P}_{\bar{p}}(0, 0, \bar{p}, 0, 1 - \bar{p})$ is another critical point, that we will name *persistence of panic*.

The parameters C_1 and s_2 appear in a particular role, letting the panic behavior in a plug, when approaching zero. In the next section, we will study the stability of the trivial equilibrium. The analysis of the stability of $\mathscr{P}_{\bar{p}}$ will be postponed to Sec. 5. As the linearization of the PCR system leads to one zero eigenvalue, this stability is tightly linked to a bifurcation that occurs in the system when C_1 vanishes. The research of the center manifold will highlight the role of the total population $r + c + p + q$ involved in the disaster, that can be considered as a potential. The Lyapunov function used in the next section was built with an energy point of view that confirms this potential role.

3. Stability of the Trivial Equilibrium

In this section, we study the stability of the trivial equilibrium. The next proposition focuses on its local stability. We then focus in detail on the orbit of the PCR system (1) stemming from the initial condition $(0, 0, 0, 1, 0)$, in which all the individuals affected by the catastrophic event are in a daily behavior before the disaster.

Proposition 3.1. *For any value of the parameters $\Lambda \in \mathscr{D}$, the equilibrium point $\mathscr{O}$ is locally stable.*

Proof. To study the local stability of $\mathscr{O}$, we consider the four equations PCR system (6) presented in Sec. 2.1,

$$\dot{X} = \Psi(t, X)$$

with $X = (r, c, p, q)^T$, and we introduce $\mathscr{O}^*$, the projection of $\mathscr{O}$ in $\mathbb{R}^4$, whose coordinates are $(0, 0, 0, 0)$.

We consider the function V defined by

$$V(t,X)=\frac{1}{2}(r+c+p+q)^2,$$

with $X=(r,c,p,q)\in\mathbb{R}^4$, and $t\in[t_0,+\infty[$. It is clear that V is definite positive. Furthermore, its orbital derivative is given by

$$\begin{aligned}\dot{V}(t,X(t)) &= (r(t)+c(t)+p(t)+q(t))\\ &\quad\times(\dot{r}(t)+\dot{c}(t)+\dot{p}(t)+\dot{q}(t))\\ &= -\varphi(t)c(t)(r(t)+c(t)+p(t)+q(t))^2,\end{aligned}$$

so $\dot{V}$ is semi-definite negative. Hence, the Lyapunov theorem guarantees that the critical point $\mathscr{O}^*$ is locally stable. Since the solution of the five equations system (1) lies in the hyperplane of equation $r+c+p+q+b=1$, we can conclude that $\mathscr{O}$ is also locally stable. ■

After stating the local stability of the critical point $\mathscr{O}^*$, we are now going to study in detail the orbit of system (6) stemming from the initial condition (7), which cannot be considered as being close to $\mathscr{O}^*$. This study leads us to a global result about the asymptotic behavior of the solution, which is not surprising, considering the dissipative character of the PCR system (6). Indeed, assume for the sake of simplicity that $\gamma(t)=\varphi(t)=1$ for all $t\geq t_0$ (see Remark 2.2), and that the imitation terms $\alpha_i,\delta_i,\mu_i, i\in\{1,2\}$ are null. Then the divergence is given by

$$\begin{aligned}\operatorname{div}\Psi(t,\ X) &= \sum_{i=1}^{4}\frac{\partial\Psi_i}{\partial x_i}\\ &\leq -B_1-B_2-C_1-C_2-s_1-s_2\\ &< 0.\end{aligned}$$

Yet it is well known that dynamical systems admitting a negative divergence often exhibit attractors [Dang-Vu & Delcarte, 2000]. We continue with two lemmas that clarify the behavior of the components c and q, whose convergence is determined by the action of φ and γ respectively.

Lemma 3.1. *Let c denote the control behavior component of the solution of the PCR system (6). Then*

$$\lim_{t\to+\infty}c(t)=0.$$

Proof. We have seen before that

$$\frac{\mathrm{d}(r+c+p+q)}{\mathrm{d}t}(t)=-\varphi(t)c(t)(r+c+p+q)(t),$$

for all $t\geq t_0$. The positiveness of φ, c and $r+c+p+q$ then guarantees that $r+c+p+q$ is a decreasing function on $[t_0,+\infty[$. As it is also positive, it converges to a non-negative limit ℓ. Let us suppose that $\dot{r}+\dot{c}+\dot{p}+\dot{q}$ does not converge to 0. As $(\dot{r}+\dot{c}+\dot{p}+\dot{q})(t)<0$ for all $t\geq t_0$, it follows that a real positive number η can be found, such that

$$(\dot{r}+\dot{c}+\dot{p}+\dot{q})(t)\leq-\eta,$$

for all $t\geq t_0$. Integrating from t_0 to t yields

$$\begin{aligned}(r+c+p+q)(t)-(r+c+p+q)(t_0)\\ \leq -\eta(t-t_0),\end{aligned}$$

which produces a contradiction, as

$$\lim_{t\to+\infty}-\eta(t-t_0)=-\infty,$$

while

$$\lim_{t\to+\infty}(r+c+p+q)(t)=\ell.$$

Consequently, we have

$$\lim_{t\to+\infty}\varphi(t)c(t)(r+c+p+q)(t)=0.$$

We recall that $\varphi(t)=1$ for t sufficiently large, so if $\ell=0$, then $r+c+p+q$ converges to 0, and obviously c also does. If $\ell>0$, then c must converge to 0, and this completes the proof. ■

Lemma 3.2. *We assume that $r_m=1$. Let q denote the daily behavior component of the solution of the PCR system (6) stemming from the initial condition $(0,0,0,1)$. Then there exist $\beta>0$, $k>0$ and $\tau>0$ such that*

$$q(t)\leq ke^{-\beta t},$$

for all $t\geq t_0+\tau$.

Remark 3.1. At the cost of technical arguments, the hypothesis $r_m=1$ can be partially ignored, provided that there are some sufficient conditions for r to stay in the compact interval $[0,r_m]$, that is $B_1+B_2\geq\alpha_2+\delta_2$. The next proposition achieves the study of the asymptotic convergence of the solution of the PCR system. Once again, the parameter C_1 appears in a particular role. We choose to focus on a situation without interaction of domino effects, so we assume the parameters s_i, α_i, δ_i and μ_i, $i \in \{1, 2\}$ to be null, and we again fix $r_m = 1$. Nonetheless, the effect of higher order terms will be studied in Sec. 5.

Proof. We first examine the behavior of r when approaching the upper boundary of the compact interval $[0, 1]$. Indeed, if there exists $\theta > 0$ such that $r(\theta) = 1$, then necessarily

$$c(\theta) = p(\theta) = q(\theta) = 0,$$

thus

$$\dot{r}(\theta) = -B_1 - B_2 < 0.$$

As $r(t_0) = 0$, this excludes the possibility $r(\theta) = 1$. A similar reasoning excludes the possibility

$$\lim_{t \to +\infty} r(t) = 1.$$

Consequently, there exists $\beta > 0$ such that

$$r(t) \leq 1 - \beta$$

for all $t \geq t_0$. Afterwards, we consider $\tau > 0$ such that $\gamma(t) = 1$ for all $t \geq t_0 + \tau$. Thus

$$\dot{q}(t) = -q(t)(1 - r(t))$$

for all $t \geq t_0 + \tau$. We write

$$\begin{aligned} q(t) &= q(\tau)e^{-\int_\tau^t (1 - r(s))\mathrm{d}s} \\ &\leq q(\tau)e^{-\int_\tau^t \beta \mathrm{d}s} \\ &\leq k e^{-\beta t} \end{aligned}$$

for all $t \geq \tau$, with $k = q(\tau)e^{\beta\tau}$. ■

Proposition 3.2. *Let us consider the PCR system with the following assumptions on the parameters*:

$$\begin{cases} s_i = \alpha_i = \delta_i = \mu_i = 0, \quad i \in \{1, 2\} \\ r_m = 1. \end{cases}$$

If $C_1 > 0$, then the orbit stemming from the initial condition $(0, 0, 0, 1)$ converges to $\mathscr{O}^*$.

Proof. The hypothesis on the parameters allow us to consider the following subsystem of two equations, separating r and q from the rest of the system,

$$\begin{cases} \dot{r}(t) = -(B_1 + B_2)r(t) + \gamma(t)q(t)(1 - r(t)) \\ \dot{q}(t) = -\gamma(t)q(t)(1 - r(t)). \end{cases} \tag{9}$$

It can be rewritten

$$\dot{x}(t) = Ax(t) + B(t)x(t) + \psi(t, x), \tag{10}$$

with $x = (r, q)^T$, and A, $B(t)$ two squared matrices whose coefficients are

$$A = \begin{pmatrix} -B_1 - B_2 & 1 \\ 0 & -1 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & \gamma(t) - 1 \\ 0 & -\gamma(t) + 1 \end{pmatrix}$$

and ψ defined by

$$\psi(t) = \begin{pmatrix} -\gamma(t)q(t)r(t) \\ \gamma(t)q(t)r(t) \end{pmatrix}.$$

Let $S(t)$ denote the fundamental matrix of the linear system $\dot{x} = Ax$. As A has negative eigenvalues -1 and $-B_1 - B_2$, we know from the theory of linear differential systems, that there exist $\xi > 0$ and $C > 0$ such that

$$\|S(t)\| \leq Ce^{-\xi(t - t_0)},$$

where $\|x\| = \sum_{i=1}^2 |x_i|$ for all $x = (x_1,\ x_2)^T \in \mathbb{R}^2$. We write system (10) as an integral equation

$$\begin{aligned} x(t) = S(t)x(t_0) + \int_{t_0}^t & S(t - s + t_0) \\ & \times [B(s)x(s) + \psi(s, x(s))]\mathrm{d}s. \end{aligned}$$

The previous lemma guarantees that

$$\begin{cases} q(t) \leq k e^{-\beta t} \\ \gamma(t) = 1, \end{cases}$$

for all $t \geq t_0 + \tau$ with $\tau > 0$, $k > 0$ and $\beta > 0$. Let $t_1 = t_0 + \tau$. We obtain

$$\begin{aligned} \|x(t)\| &\leq \tilde{C}e^{-\xi(t - t_1)} + \int_{t_1}^t \tilde{C}e^{-\xi(t - s)}\|\psi(s, x(s))\|\mathrm{d}s \\ &\leq \tilde{C}e^{-\xi(t - t_1)} + \int_{t_1}^t \tilde{C}e^{-\xi(t - s)}|q(s)||r(s)|\mathrm{d}s \\ &\leq \tilde{C}e^{-\xi(t - t_1)} + \int_{t_1}^t \tilde{C}e^{-\xi(t - s)}k e^{-\beta s}\|x(s)\|\mathrm{d}s \end{aligned}$$

thus

$$e^{\xi(t - t_1)}\|x(t)\| \leq \tilde{C} + \int_{t_1}^t \tilde{C}e^{\xi(s - t_1)}\|x(s)\|k e^{-\beta s}\mathrm{d}s$$

which produces, using Gronwall's inequality

$$e^{\xi(t - t_1)}\|x(t)\| \leq \tilde{C}e^{\int_{t_1}^t \tilde{C}k e^{-\beta s}\mathrm{d}s}$$

and finally

$$\|x(t)\| \le \tilde{C} e^{-\xi(t-t_1)} e^{\frac{-k\tilde{C}}{\beta} e^{-\beta t}}$$

for all $t \ge t_1$. Hence we can conclude that $x(t)$ converges to 0. Finally, we consider the following subsystem

$$\begin{cases} \dot{c}(t) = B_1 r(t) - C_2 c(t) + C_1 p(t) \\ \qquad\quad - \varphi(t) c(t)(r+c+p+q)(t) \\ \dot{p}(t) = B_2 r(t) + C_2 c(t) - C_1 p(t). \end{cases} \tag{11}$$

We have previously shown that $r(t)$, $c(t)$ and $q(t)$ converge to 0, and $(r+c+p+q)(t)$ converges to ℓ, so $p(t)$ converges to ℓ, and consequently $\dot{p}(t)$ also converges to a finite limit, which is necessarily 0 (we recall that if f is a real valued smooth function defined on $\mathbb{R}$, such that $f(t)$ converges to a finite limit ℓ_1, while $\dot{f}(t)$ converges to a finite limit ℓ_2, then necessarily $\ell_2 = 0$). If $C_1 > 0$, we obtain $0 = -C_1 \ell$ thus $\ell = 0$. ■

Remark 3.2. The geographical meaning of this asymptotic stability resides in the fact that numerous observations record a return of all the affected individuals to daily behavior after the disaster. In other words, this qualitative result represents a new step in the validation process of the PCR system.

4. Transitional Dynamic

The dynamic of the PCR system is governed by many parameters. In this section, we show that the function φ, that models the return to a daily behavior, plays an important role, by emptying the control behavior subpopulation c. To that aim, we consider a time interval $[t_1, t_2]$ on which

$$\begin{cases} \gamma(t) = 1 \\ \varphi(t) = 0 \end{cases} \tag{12}$$

that we consider as the transitional phase of the PCR system. Consequently, the unknown function b, that models the subgroup of individuals who return to daily behavior, can be eliminated from system (1). Furthermore, as previously, we choose to focus on a situation without interaction effects, so we assume:

$$\alpha_i = 0, \quad \mu_i = 0, \quad \delta_i = 0, \quad i \in \{1, 2\}$$

and we again put $r_m = 1$. Finally, we study the following differential system

$$\begin{cases} \dot{r} = q(1-r) - (B_1 + B_2) r + s_1 c + s_2 p \\ \dot{c} = B_1 r + C_1 p - C_2 c - s_1 c \\ \dot{p} = B_2 r - C_1 p + C_2 c - s_2 p \\ \dot{q} = -q(1-r). \end{cases} \tag{13}$$

Proposition 4.1. *The transitional dynamic of the PCR system defined by (12) and (13) exhibits an attractive equilibrium point.*

This attractive point is depicted in Figs. 4 and 5. Figure 4 shows several orbits of the PCR system, projected in the (r, c, p) space, for some varying initial conditions, whereas Fig. 5 presents each component of the solution stemming from the initial condition $(0, 0, 0, 1, 0)$ for a given set of parameters, during the transitional phase ($t_1 \le t \le t_2$), and after the transitional phase ($t \ge t_2$).

Proof. We first look for the critical points of system (13), which are given by

$$\begin{cases} q(1-r) - (B_1 + B_2) r + s_1 c + s_2 p = 0 \\ B_1 r + C_1 p - C_2 c - s_1 c = 0 \\ B_2 r - C_1 p + C_2 c - s_2 p = 0 \\ -q(1-r) = 0. \end{cases}$$

As the sum $r + c + p + q$ is constant, equal to 1, we obtain

$$\begin{cases} 1 - (2 + B_1 + B_2) r + (s_1 - 1) c \\ \qquad + (s_2 - 1) p + r^2 + rc + rp = 0 \\ B_1 r + C_1 p - C_2 c - s_1 c = 0 \\ B_2 r - C_1 p + C_2 c - s_2 p = 0. \end{cases} \tag{14}$$

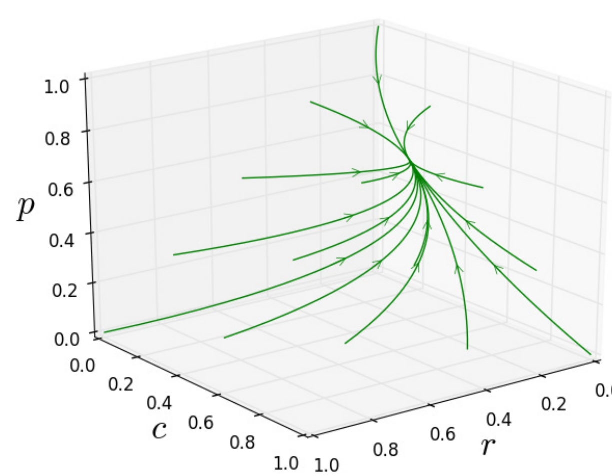
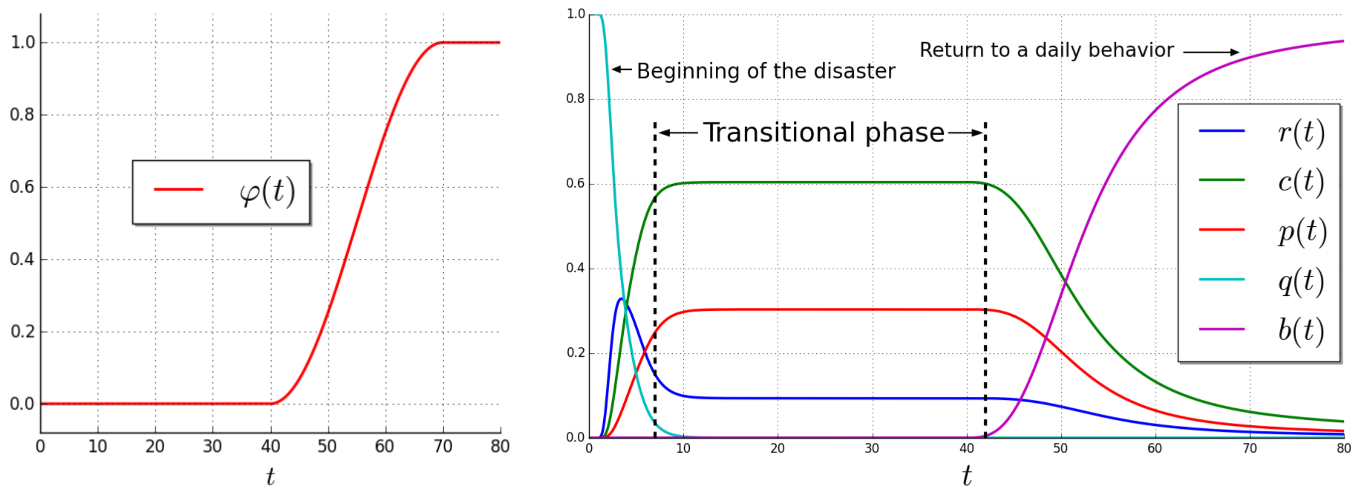


Fig. 4. Numerical results projected in the (r, c, p) space for the simplified system (13), showing a stable equilibrium in the transitional dynamic of the PCR system.



(a) (b)

Fig. 5. Transitional dynamic in the PCR system. (a) A delay introduced in the function φ, lets (b) a transitional equilibrium appear among the three subgroups of behaviors.

Some basic computations lead to the solution

$$\begin{cases} r^* = \dfrac{C_1 s_1 + C_2 s_2 + s_1 s_2}{N} \\ c^* = \dfrac{B_1 s_2 + C_1 B_1 + C_1 B_2}{N} \\ p^* = \dfrac{B_1 C_2 + B_2 C_2 + B_2 s_1}{N}, \end{cases}$$

where

$$N = C_1 s_1 + C_2 s_2 + s_1 s_2 + B_1 s_2 + C_1 B_1 + C_1 B_2 + B_1 C_2 + B_2 C_2 + B_2 s_1,$$

that satisfies

$$r^* + c^* + p^* = 1.$$

The Jacobian matrix of system (14), evaluated in (r^*, c^*, p^*), admits three eigenvalues

$$\begin{cases} \lambda_1 = \dfrac{-B_1 - B_2 - C_1 - C_2 - s_1 - s_2 + \sqrt{\Delta}}{2} \\ \lambda_2 = \dfrac{-B_1 - B_2 - C_1 - C_2 - s_1 - s_2 - \sqrt{\Delta}}{2} \\ \lambda_3 = \dfrac{-(B_1 C_1 + B_1 C_2 + B_1 s_1 + B_2 C_1 + B_2 C_2 + B_2 s_1)}{N}, \end{cases}$$

with

$$\begin{aligned} \Delta = {} & B_1^2 + 2B_1 B_2 - 2B_1 C_1 - 2B_1 C_2 + 2B_1 s_1 \\ & - 2B_1 s_2 + B_2^2 - 2B_2 C_1 - 2B_2 C_2 - 2B_2 s_1 \\ & + 2B_2 s_2 + C_1^2 + 2C_1 C_2 - 2C_1 s_1 + 2C_1 s_2 \\ & + C_2^2 + 2C_2 s_1 - 2C_2 s_2 + s_1^2 - 2 s_1 s_2 + s_2^2. \end{aligned}$$

The stability of (r^*, c^*, p^*) is given by the sign of λ_1, λ_2 and λ_3. Obviously, we have $\lambda_3 < 0$ and $\lambda_2 < 0$. Furthermore, the sum and the product of λ_1 and λ_2 are given by

$$\lambda_1 + \lambda_2 = -(B_1 + B_2 + C_1 + C_2 + s_1 + s_2)$$

$$\begin{aligned} \lambda_1 \lambda_2 = {} & B_1 C_1 + B_1 C_2 + B_1 s_2 + B_2 C_1 + B_2 C_2 \\ & + B_2 s_1 + C_1 s_1 + C_2 s_2 + s_1 s_2 \end{aligned}$$

so we also have $\lambda_1 < 0$. This demonstrates that the nontrivial critical point (r^*, c^*, p^*) is an attractive node for system (14). Finally, as the sum $r + c + p + q$ is constant, equal to 1, $(r^*, c^*, p^*, 0)$ is also an attractive equilibrium point for system (13). ∎

The latter proof uses a linearization method, that produces negative eigenvalues, whatever be the values of the parameters. It means that the stability of the transitional dynamic is structural. In particular, it is independent of the asymptotic behavior of the solution towards one or another equilibrium point.

Remark 4.1. We have previously mentioned that a succession of disasters could be modeled by choosing a periodic form for the domino effect parameters s_1 and s_2. In that case, it can be shown that the transitional dynamic reveals the existence of an attractive cycle whose diameter increases with the intensity of the catastrophic events. After this transitional dynamic, the emptying role of the function $\varphi(t)$ takes place, and the attractive cycle vanishes. This analysis of domino effect will be presented in a forthcoming paper.

5. Bifurcation Analysis in a Reduced Case

5.1. *Reduction to center manifold and calculation of normal form*

The research of the equilibrium points exhibits a particular role for the parameter C_1, that lets new equilibrium points appear when tending to 0. In that case, numerical experiments show a persistence of panic behavior (see Fig. 6). In this section, we shall study the dynamic related to the parameter C_1, by stating a local equation of the center

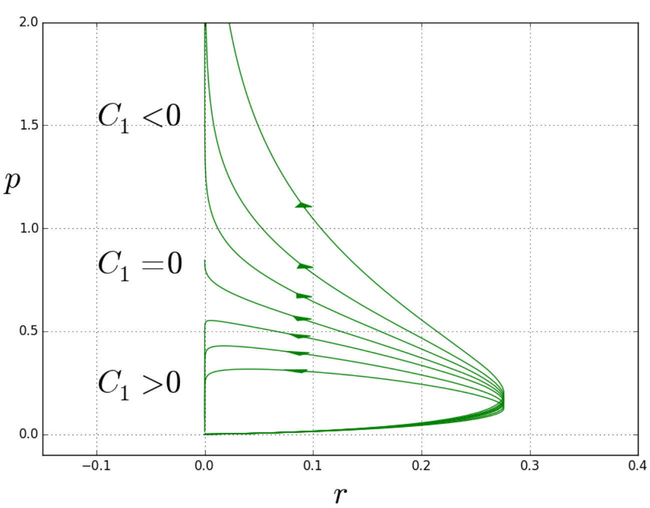


Fig. 6. Numerical results in the (r, p) plane, showing a bifurcation when the evolution parameter C_1 passes through 0. For $C_1 > 0$, the solution converges to the trivial equilibrium. For $C_1 = 0$, a persistence of panic suddenly occurs. For $C_1 < 0$, the solution leaves the compact set $[0, 1]^5$ and diverges to infinity.

manifold [Carr, 2012; Perko, 2001; Kuznetsov, 2004; Dang-Vu & Delcarte, 2000; Faria & Magalhaes, 1995a, 1995b]. We will momentarily consider positive or negative values of the parameters, in order to establish a complete mathematical analysis.

As mentioned in our introduction, we reduce the analysis to the case of constant imitation functions, that is:

$$F(r, c) = k_1, \quad G(r, p) = k_2, \quad H(c, p) = k_3,$$

where $k_1, k_2, k_3 \in [-1, 1]$. Furthermore, as we study the asymptotic behavior of the PCR system, we assume

$$\gamma(t) = \varphi(t) = 1, \quad \forall\, t \geq T,$$

for a given $T > 0$. Substituting $t - T$ by t, we can without loss of generality replace the study in $[t_0, +\infty[$.

Finally, we consider the following system

$$\begin{cases} \dot{r} = -(B_1 + B_2)r + q(1 - r) + k_1 rc + k_2 rp \\ \dot{c} = B_1 r - c(r + c + p + q) - C_2 c \\ \qquad + C_1 p - k_1 rc + k_3 cp \\ \dot{p} = B_2 r + C_2 c - C_1 p - k_2 rp - k_3 cp \\ \dot{q} = -q(1 - r). \end{cases} \tag{15}$$

The Jacobian matrix J, evaluated in $(0, 0, 0, 0)$ and $C_1 = 0$, reads

$$J = \begin{pmatrix} -B_1 - B_2 & 0 & 0 & 1 \\ B_1 & -C_2 & 0 & 0 \\ B_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{16}$$

Its characteristic polynomial is

$$\chi_J(\lambda) = \lambda(1 + \lambda)(C_2 + \lambda)(B_1 + B_2 + \lambda).$$

Assuming $C_2 \neq 1$, $B_1 + B_2 \neq 1$ and $C_2 \neq B_1 + B_2$, we can conclude that J admits four eigenvalues 0, -1, $-C_2$ and $-B_1 - B_2$. As J has one zero eigenvalue, we are going to search for a local equation of the center manifold in a neighborhood of $C_1 = 0$. The next proposition gives an equation of the center manifold in a neighborhood of $C_1 = 0$.

Proposition 5.1. *Assume $C_2 \neq 1$, $B_1 + B_2 \neq 1$ and $C_2 \neq B_1 + B_2$. Then the Jacobian matrix J of system (15) can be written in a diagonal form with* *eigenvalues*

$$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 = 0.$$

Moreover, in a neighborhood of $C_1 = 0$, there exist new coordinates (x, y, z, w) such that the PCR system (15) is given by

$$\begin{cases} \dot{x} = \lambda_1 x + \cdots \\ \dot{y} = \lambda_2 y + \cdots \\ \dot{z} = \lambda_3 z + \cdots \\ \dot{w} = \dfrac{C_1}{\lambda_3} w^2 (1 + O(w)). \end{cases} \tag{17}$$

We refer to the Appendix for the complete proof of this proposition. The change of coordinates involved in the diagonalization of the matrix J highlights the particular role played by the total population involved in the disaster, that is

$$w = r + c + p + q,$$

that we have previously considered in order to build a Lyapunov function. The next proposition states for the local stability of the equilibrium points $\mathscr{P}_{\bar{p}}$.

Proposition 5.2. *Assume $C_1 = 0$. Then the equilibrium points $\mathscr{P}_{\bar{p}}$ of the PCR system (15) are locally stable, but not asymptotically stable.*

Proof. The local stability follows from the form of the last equation in system (17), in which all the terms vanish when $C_1 = 0$. ■

The equation of the center manifold shows that in a neighborhood of $C_1 = 0$, the PCR system is topologically equivalent to the following differential system:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ \dot{z} = -z \\ \dot{w} = \varepsilon w^2, \end{cases} \tag{18}$$

where $\varepsilon = -C_1$.

Remark 5.1. As there is an infinite number of critical points in the case $C_1 = 0$, a natural question is to find which one is attempted by the solution, considering a fixed initial condition. Table 2 shows numerical results for the persistence of panic $\bar{p}$ for different values of the parameters B_1, B_2 and C_2. The other parameters ($\alpha_i, \delta_i, \mu_i, i \in \{1, 2\}$ and C_1) are supposed to be null. It seems that an increase of B_1 induces a decrease of $\bar{p}$, while an increase of C_2 or B_2 exacerbates this persistence. In the next section, we will study in detail the inhibition effect of the imitation parameter μ_1.

Table 2. Numerical results for the persistence of panic $\bar{p}$ under a variation of the parameters B_1, B_2 and C_2.

$B_1 = 0.5$			$B_1 = 0.7$		
B_2	C_2	$\bar{p}$	B_2	C_2	$\bar{p}$
0.2	0.1	0.70550	0.2	0.1	0.69018
0.2	0.2	0.84365	0.2	0.2	0.83984
0.3	0.1	0.74474	0.3	0.1	0.72252
0.3	0.2	0.86674	0.3	0.2	0.85823
0.4	0.1	0.77494	0.4	0.1	0.74885
0.4	0.2	0.88409	0.4	0.2	0.87294

We are now going to study the dynamic of the normal form exhibited in the local equation of the center manifold (17). To that aim, we consider the following two-dimensional dynamical system

$$\begin{cases} \dot{w} = \alpha + \varepsilon w^2 \\ \dot{v} = -v \end{cases} \tag{19}$$

with parameters α and ε.

The first parameter is naturally introduced to avoid a systematic degeneracy [Kuznetsov, 2004], while the second parameter ε corresponds to the parameter $-C_1$ in the PCR system. Figure 7 shows different phase portraits for the system (19) according to both parameters α and ε. The gray zone in the diagram corresponds to the region of the parameter C_1 in the PCR system. To understand the bifurcation that suddenly causes the emergence of an infinite number of critical points, we have to find the equilibrium points in system (19), by solving

$$\begin{cases} \alpha + \varepsilon w^2 = 0 \\ -v = 0. \end{cases}$$

The only critical points are for $v = 0$. If $\alpha \times \varepsilon > 0$, there is not any critical point. If $\alpha \times \varepsilon < 0$, there are two critical points given by

$$\overline{w} = \pm\sqrt{\frac{-\alpha}{\varepsilon}}, \quad \overline{v} = 0.$$

One is a saddle, the other one being a node. They collapse if ε is fixed and α tends to 0, forming a classical saddle-node bifurcation. If α is fixed and ε tends to 0, they are pushed to infinity and

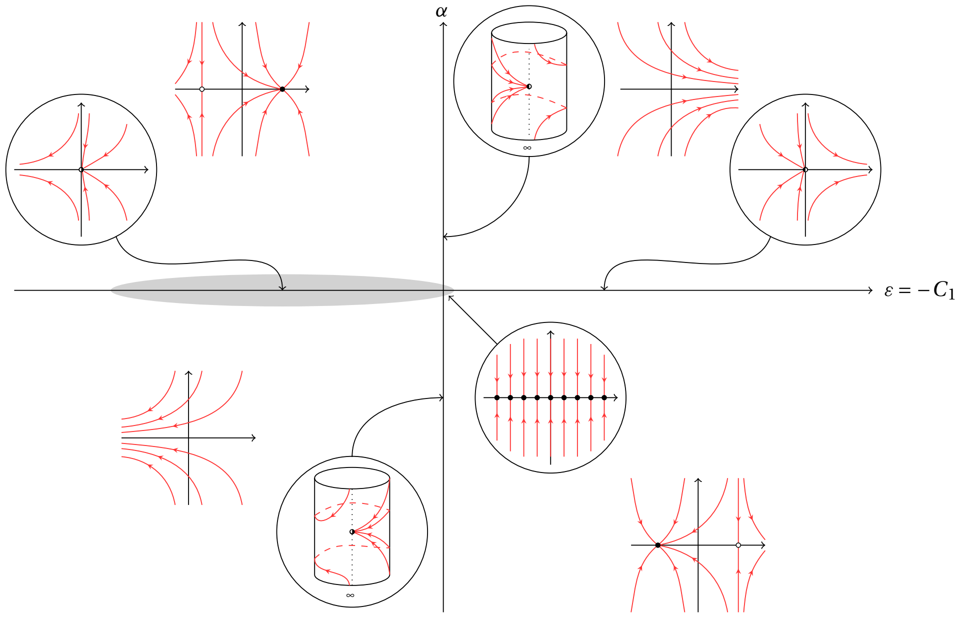
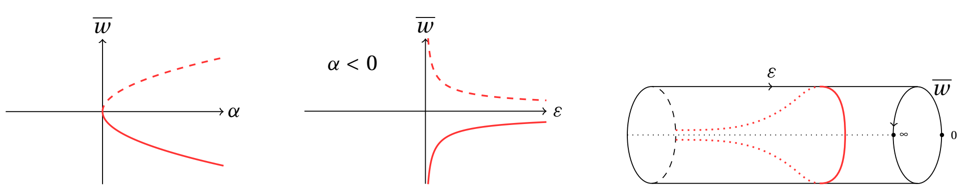


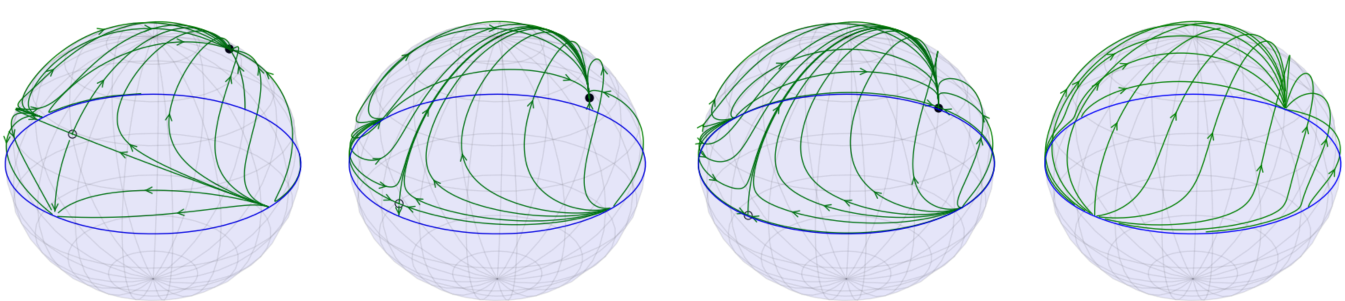
Fig. 7. Bifurcation diagram for system $\dot{w} = \alpha + \varepsilon w^2$, $\dot{v} = -v$ showing phase portraits in the (w, v) plane. The gray zone, on the left of ε axis, corresponds to the possible values of parameter C_1 in the PCR system. The infinite number of equilibrium points occurring for $\alpha = \varepsilon = 0$ accounts for the persistence of panic in the model.

merge in a degenerate way. This bifurcation can be seen on a cylinder, by rolling the $(\varepsilon, \overline{w})$ plane (see Fig. 8). It can also be studied on the Poincaré sphere S^2 (see Fig. 9), where the points at the infinity in the (w, v) plane are projected on the equator, on which antipodal points are naturally identified [Perko, 2001; Blows & Rousseau, 1993]. The codimension-2 degeneracy caused by $\alpha = \varepsilon = 0$ accounts for the persistence of panic exhibited in the PCR system (see Fig. 10).



(a) (b) (c)

Fig. 8. Bifurcation diagrams. (a) Classical saddle-node bifurcation diagram, (b) saddle-node bifurcation at infinity and (c) saddle-node bifurcation at the infinity, seen on a cylinder, by rolling the $(\varepsilon, \overline{w})$ plane.



(a) (b) (c) (d)

Fig. 9. Phase portraits of system $\dot{w} = \alpha + \varepsilon w^2$, $\dot{v} = -v$, on the Poincaré sphere S^2, with $\alpha = 1$, showing the saddle-node bifurcation at infinity. When ε passes through 0, the saddle and the node are pushed to antipodal points of the equator and finally vanish for $\varepsilon > 0$. (a) $\varepsilon = -0.5$, (b) $\varepsilon = -0.01$, (c) $\varepsilon = -0.0001$ and (d) $\varepsilon = +0.5$.

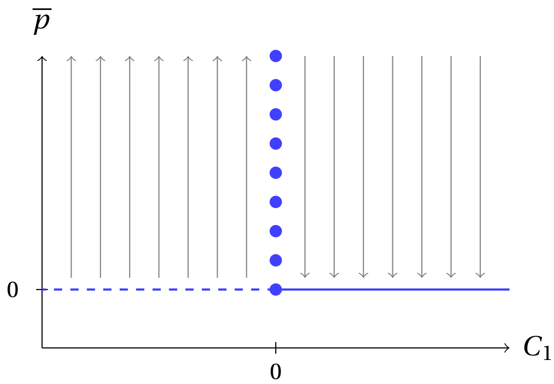


Fig. 10. Bifurcation diagram for the PCR system. The infinite number of equilibrium points appearing for $C_1 = 0$ corresponds to a saddle-node degeneracy.

Remark 5.2. The computation of the Lyapunov exponents [Chen & Lai, 1996; Dang-Vu & Delcarte, 2000], well known as a measure of sensibility of the solution to the initial condition confirms this behavior (see Fig. 11). Indeed, for $C_1 > 0$, the Lyapunov exponents are negative. If $C_1 = 0$, the maximum Lyapunov exponent is null. If $C_1 < 0$, it is even positive, which does not mean chaos, since the orbits of the PCR system (15) leave the compact set $[0, 1]^5$ (see Fig. 6).

It also shows that the solution of the PCR system lies in a larger context of saddle-node bifurcations, that could lead to instability. Indeed, the solution evolves on a fragile ridge, and a small perturbation of the system, caused by an external phenomenon, or an inherent variation of one parameter, could on one side provoke an unexpected asymptotic behavior of the solution.

Remark 5.3. The persistence of panic $\bar{p}$ has to be apprehended with precaution. The geographical observations can record in some specific situations a difficult return to a daily behavior, with a longer panic behavior duration. But a *stricto sensu* persistence is not an established, observed phenomenon, except in the cases with a large mortality [Crocq, 1994]. Indeed, it does not mean that the parameter C_1 cannot be chosen with a zero value. The interaction parameters that act in parallel with C_1 also play a decisive role, as we are going to show in the next section.

Another point of view is to write system (19) as a gradient system

$$\begin{cases} \dot{v} = -\dfrac{\partial V}{\partial v} \\[2ex] \dot{w} = -\dfrac{\partial V}{\partial w}, \end{cases}$$

with potential

$$V(w, v) = -\alpha w - \frac{\varepsilon}{3} w^3 + \frac{1}{2} v^2.$$

Figure 12 shows the corresponding bifurcation surface, which is a fold with a degeneracy around 0. The section of this surface by a vertical plane of equation $\varepsilon = \varepsilon_0$ ($\varepsilon_0 \neq 0$) corresponds to a saddle-node bifurcation shown in Fig. 8(a), while

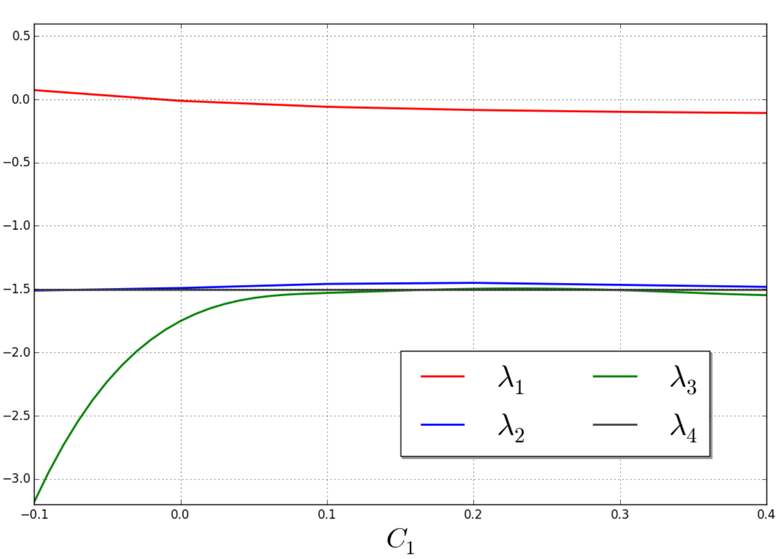


Fig. 11. Numerical results for the Lyapunov exponents of the PCR system under a variation of the parameter C_1.

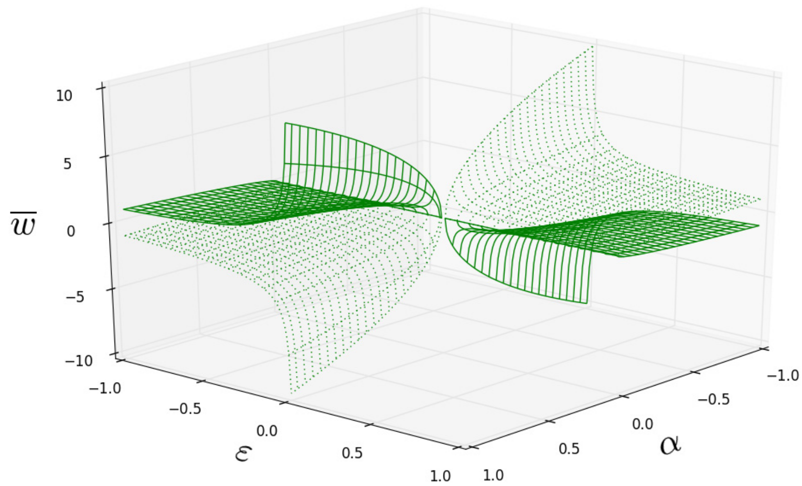


Fig. 12. Surface of codimension-2 bifurcation in system $\dot{w} = \alpha + \varepsilon w^2$, $\dot{v} = -v$. The tear in the origin corresponds to the degeneracy exhibited in the PCR system. The dotted parts of the surface indicate the unstable equilibrium points. The section of this surface by a vertical plane of equation $\varepsilon = \varepsilon_0$ corresponds to a classical saddle-node bifurcation, while the section by a plane of equation $\alpha = \alpha_0$ is related to a saddle-node bifurcation at infinity.

the section by a plane of equation $\alpha = \alpha_0$ ($\alpha_0 \neq 0$) is related to Fig. 8(b).

5.2. *Inhibition of panic*

In this section, we shall study the effect of the imitation process between panic and control behaviors, that acts in parallel with the evolution process. We have previously mentioned (see Table 2) that the persistence of panic, that occurs when C_1 is null, could change when other evolution parameters vary. Table 3 shows numerical results for the persistence of panic $\bar{p}$ for various increasing values of μ_1.

Here, we focus our attention on the variation of the persistence of panic under a perturbation of the imitation parameter μ_1, which means that we only consider imitation from panic behavior to control behavior. Thus, we take

$$\alpha_i = \delta_i = 0, \quad i \in \{1, 2\}, \quad \mu_2 = 0$$

Table 3. Numerical results for the persistence of panic under a variation of the imitation parameter μ_1. The values of the other parameters are: $B_2 = 0.3$, $C_2 = 0.1$. An increase of μ_1 inhibits the persistence of panic.

$B_1 = 0.4$		$B_1 = 0.7$	
μ_1	$\bar{p}$	μ_1	$\bar{p}$
0.0	0.76178	0.0	0.72252
0.1	0.64451	0.1	0.61713
0.2	0.49784	0.2	0.48676
0.3	0.37016	0.3	0.36852
0.4	0.28922	0.4	0.28856

and we just write μ instead of μ_1, in order to lighten our notations. As in the previous section, we study the asymptotic behavior of the solution, thus

$$\gamma(t) = \varphi(t) = 1,$$

for all $t \geq t_0$. We recall that the function h_1 involved in the imitation term between panic and control behaviors satisfies the property (2) presented in Sec. 2:

$$0 \leq h_1(s) \leq 1, \quad \forall\, s \in \mathbb{R}.$$

Indeed, we reduce the analysis to the case $h_1(s) = k$ for all $s \in \mathbb{R}$, where k denotes a real constant between 0 and 1. Finally, we consider the following dynamical system, in which μ is seen as a perturbation parameter:

$$\begin{cases} \dot{r} = -(B_1 + B_2)r + q\left(1 - \dfrac{r}{r_m}\right) \\ \dot{c} = B_1 r - C_2 c - c(1 - b) + \mu k c p \\ \dot{p} = B_2 r + C_2 c - \mu k c p \\ \dot{q} = -q\left(1 - \dfrac{r}{r_m}\right) \\ \dot{b} = c(1 - b), \end{cases} \tag{20}$$

where $(r, c, p, q, b) \in \mathbb{R}^5$ and $t \geq t_0$. We are interested in the solutions passing through $(0, 0, 0, 1, 0)$ at $t = t_0$, and we would like to compare the panic components p_μ and p_{μ^*} of two solutions obtained for two different values $\mu > \mu^*$ of the imitation

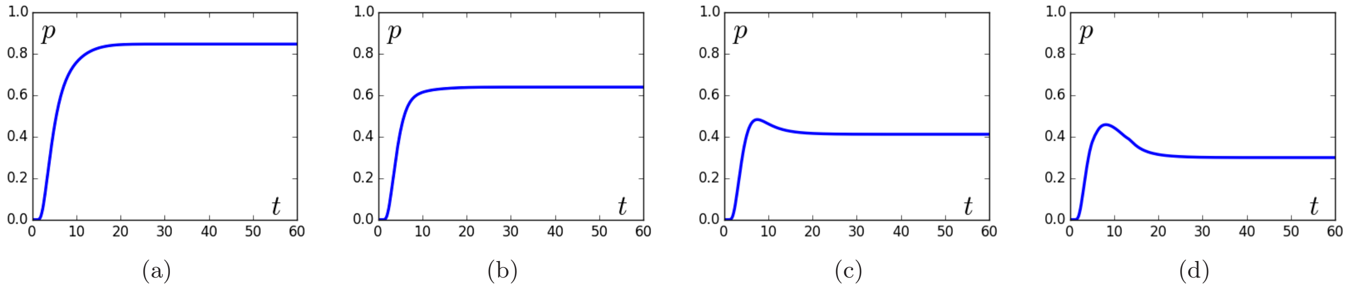


Fig. 13. Numerical results showing the inhibition of the persistence of the panic behavior under an increase of the imitation parameter μ_1. The values of the other parameters are: $C_1 = 0$, $B_1 = B_2 = C_2 = 0.2$, $\alpha_i = \delta_i = 0.1$, $i \in \{1, 2\}$, $s_1 = s_2 = 0$, $\mu_2 = 0.1$. (a) $\mu_1 = 0.1$, (b) $\mu_1 = 0.3$, (c) $\mu_1 = 0.5$ and (d) $\mu_1 = 0.7$.

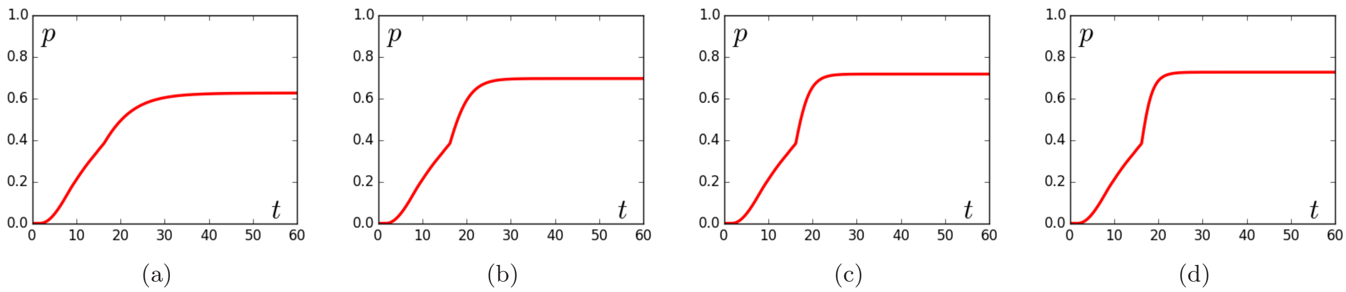


Fig. 14. Numerical results showing the increase of the persistence of the panic behavior under an increase of the imitation parameter δ_1. The values of the other parameters are: $C_1 = 0$, $B_1 = 0.1$, $B_2 = 0.01$, $C_2 = 0.2$, $\alpha_i = \mu_i = 0.1$, $i \in \{1, 2\}$, $s_1 = s_2 = 0$, $\delta_2 = 0.1$. (a) $\delta_1 = 0.1$, (b) $\delta_1 = 0.4$, (c) $\delta_1 = 0.7$ and (d) $\delta_1 = 0.95$.

parameter. To that aim, we fix μ^* and introduce $\nu > 0$ such that $\mu = \mu^* + \nu$.

Proposition 5.3. *Assume $\nu > 0$ is sufficiently small and $\mu^* \in [0, 1]$. Let p_μ and p_{μ^*} denote respectively the panic components of the solutions of system (20) obtained for μ and μ^*. Then*

$$p_{\mu^*}(t) \geq p_\mu(t), \quad \forall\, t \geq t_0.$$

Once again, we refer the reader to the Appendix for the complete proof, that is based on an expansion method. This expansion method can be used to study the effect of other parameters on the persistence of panic. For instance, an increase of the parameter δ_1, which models the imitation process from reflex behavior to panic behavior, accentuates the persistence of panic. At the opposite, a change of the parameters α_1, α_2, which model imitation between reflex and control, does not affect this persistence. Figure 13 shows the control exerted by an increase of the imitation parameter μ_1, while Fig. 14 illustrates the effect of the imitation parameter δ_1. In the case $C_1 > 0$, when panic does not persist, this inhibition effect can be used to accelerate the convergence of p to 0. Finally, Fig. 15 shows a bifurcation diagram for the PCR system, taking into account the bifurcation effect of parameter C_1, and the inhibition effect of parameter μ_1.

Remark 5.4. The effect of the imitation parameters α_i, δ_i, μ_i, $i \in \{1, 2\}$, satisfies the initial choices made in the modeling. The only consideration of

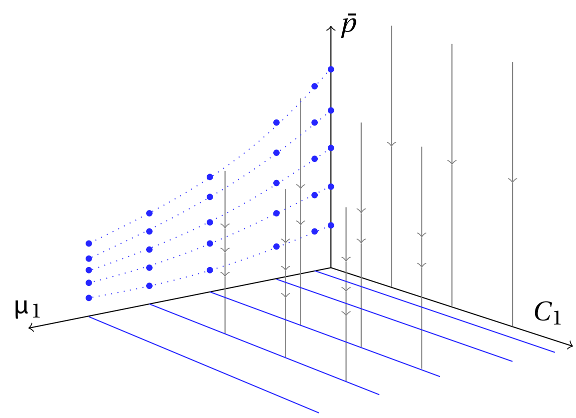


Fig. 15. Bifurcation diagram for the PCR system. The varying parameters are C_1 and μ_1; $\bar{p}$ denotes the persistence of panic. The section of this diagram by the plane of equation $\mu_1 = 0$ is related to Fig. 10. When $C_1 = 0$, there is an infinite number of equilibrium points. The height of each point decreases under an increase of the inhibition parameter μ_1. For $C_1 > 0$, the greater μ_1 is, the faster the trajectories converge to 0.

evolution process from panic behavior to control behavior is not sufficient. The emotion contagion phenomena have to be taken into account, in a non-neglected proportion.

6. Conclusion

The mathematical results presented in this paper represent a new step in the qualitative validation of the PCR system, as a model of human behavior during catastrophic events. The positiveness and the boundedness are the first properties required for a population dynamic model. They are now rigorously proved.

The study of the equilibrium points shows the particular role played by the evolution parameter C_1 from panic behavior to control behavior. When the evolution acts in a fluidity context, that is $C_1 > 0$, the return of all individuals to a daily behavior is guaranteed. When at the opposite this evolution is blocked ($C_1 = 0$), a persistence of panic occurs, that can fortunately be inhibited by an increase of the imitation parameter μ_1 that acts in parallel. Roughly speaking, the model can evolve towards two possible states. The first one is a successful state, with a global return of the population to tranquility, whereas the second one is a problematic state, with a plug in the panic behavior subgroup.

The transitional dynamic highlights the decisive emptying role of the function $\varphi(t)$: before its action, which models the return to daily behavior, a structurally stable equilibrium takes place among the three behavior subgroups, whatever the asymptotic equilibrium the solution converges to. During that transitional phase, periodic phenomena can occur if a succession of catastrophic events is also taken into account in the model.

Finally, the analysis of the bifurcation provoked when the evolution parameter C_1 passes through 0, made by stating a local equation of the center manifold, highlights the potential role of the total population involved in the catastrophe mechanism, and shows that the solution of the PCR system lies in a larger context of saddle-node bifurcations, near a degeneracy isolated case that accounts for the possible persistence of panic. Interaction process can affect this persistence in a positive way, when individuals in a panic behavior imitate individuals in a reflex or control behavior, or in a negative way, when at the opposite, imitation increases the flow towards panic behavior.

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Appendix

Proof [Proof of Proposition 5.1]. J admits four eigenvalues 0, -1, $-C_2$ and $-B_1 - B_2$. The associated eigenvectors are given by

$$
\begin{aligned}
X_0 &= (0,0,1,0)^T\\
X_{-1} &= \left(\frac{1}{B_1+B_2-1}, \frac{B_1}{(B_1+B_2-1)(C_2-1)}, \frac{-B_2}{B_1+B_2-1} - \frac{C_2B_1}{(B_1+B_2-1)(C_2-1)}, 1\right)^T\\
X_{-C_2} &= (0,-1,1,0)^T\\
X_{-B_1-B_2} &= \left(\frac{(B_1+B_2)(B_1+B_2-C_2)}{B_1C_2-B_2(B_1+B_2-C_2)}, \frac{-B_1(B_1+B_2)}{B_1C_2-B_2(B_1+B_2-C_2)}, 1, 0\right)^T
\end{aligned}
$$

so J can be written in a diagonal form. We write $J = PDP^{-1}$ with

$$
P = \begin{pmatrix} g_1 & g_2 & 0 & 0\\ g_3 & g_4 & -1 & 0\\ 1 & g_5 & 1 & 1\\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -B_1-B_2 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -C_2 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

where the coefficients g_i, $1 \le i \le 5$ are given by

$$
\begin{cases}
g_1 = \dfrac{(B_1+B_2)(B_1+B_2-C_2)}{B_1C_2-B_2(B_1+B_2-C_2)}\\[2mm]
g_2 = \dfrac{1}{B_1+B_2-1}\\[2mm]
g_3 = \dfrac{-B_1(B_1+B_2)}{B_1C_2-B_2(B_1+B_2-C_2)}\\[2mm]
g_4 = \dfrac{B_1}{(B_1+B_2-1)(C_2-1)}\\[2mm]
g_5 = \dfrac{-B_2}{B_1+B_2-1} - \dfrac{C_2B_1}{(B_1+B_2-1)(C_2-1)}
\end{cases}
$$

and satisfy

$$
\begin{cases}
g_1 + g_3 + 1 = 0\\
g_2 + g_4 + g_5 + 1 = 0.
\end{cases}
$$

We compute the inverse of P and obtain

$$
P^{-1} = \begin{pmatrix} \dfrac{1}{g_1} & 0 & 0 & \dfrac{-g_2}{g_1}\\ 0 & 0 & 0 & 1\\ \dfrac{-(1+g_1)}{g_1} & -1 & 0 & g_2 + g_4 + \dfrac{g_2}{g_1}\\ 1 & 1 & 1 & 1 \end{pmatrix}.
$$

Let $R = (r,c,p,q)^T$ and $X = (x,y,z,w)^T$. We have $R = PX$ and $X = P^{-1}R$. After some basic computations, we get the following system:

$$\begin{cases} \dot{x} = \lambda_1 x + P_1(x,y,z) + wQ_1(x,y,z) \\ \dot{y} = \lambda_2 y + P_2(x,\ y,\ z) \\ \dot{z} = \lambda_3 z - C_1(x-(1+g_2+g_4)y+z+w) \\ \qquad + w(-(1+g_1)x + g_4 y - z) + P_3(x,y,z) \\ \qquad + wQ_2(x,y,z) \\ \dot{w} = -w(-(1+g_1)x + g_4 y - z), \end{cases}$$

where P_1, P_2, P_3 are homogeneous polynomials of degree 2 in x, y, z, and Q_1, Q_2 homogeneous polynomials of degree 1 in x, y, z. We then look for a Taylor expansion of (x, y, z) in w and C_1, around $(w, C_1) = (0, 0)$. So we write

$$\begin{cases} x = h_1, \quad y = h_2, \quad z = h_3 \\ h_i = a_i C_1^2 + b_i C_1 w + c_i w^2 + \cdots, \quad i \in \{1,2,3\}. \end{cases}$$

A local equation of the center manifold is given by

$$\begin{aligned} &Dh(w, C_1)[\lambda_4 w + f(w, h(w, C_1), C_1)] \\ &\quad = Bh(w, C_1) + g(w, h(w, C_1), C_1), \end{aligned}$$

where

$$\lambda_4 = 0, \quad h = (h_1, h_2, h_3)^T$$

$$B = \mathrm{diag}\left(-2, -1, \frac{-1}{2}\right)$$

$$f(w, (x,y,z), C_1) = -w(-(1+g_1)x + g_4 y - z)$$

and

$$g(w,(x,y,z),C_1) = \begin{pmatrix} P_1(x,y,z) + wQ_1(x,y,z) \\ P_2(x,y,z) \\ -C_1(x-(1+g_2+g_4)y+z+w) + w(-(1+g_1)x+g_4y-z) + P_3(x,y,z) + wQ_2(x,y,z) \end{pmatrix}.$$

We obtain

$$\begin{aligned} &(b_1 C_1 + 2c_1 w + \cdots)(-w(-(1+g_1)h_1 - g_4 h_2 - h_3)) \\ &\quad = \lambda_1 h_1 + P_1(h_1,h_2,h_3) + wQ_1(h_1,h_2,h_3) + \cdots \\ &(b_2 C_1 + 2c_2 w + \cdots)(-w(-(1+g_1)h_1 - g_4 h_2 - h_3)) \\ &\quad = \lambda_2 h_2 + P_2(h_1,h_2,h_3) + \cdots \\ &(b_3 C_1 + 2c_3 w + \cdots)(-w(-(1+g_1)h_1 - g_4 h_2 - h_3)) \\ &\quad = \lambda_3 h_3 - C_1(w + h_1 - (1+g_2+g_4)h_2 + h_3) \\ &\qquad + w(-(1+g_1)h_1 + g_4 h_2 - h_3) \\ &\qquad + P_3(h_1,h_2,h_3) + wQ_2(h_1,h_2,h_3) + \cdots \end{aligned}$$

where the dots indicate terms of order higher than 3. An identification of the terms in C_1^2, $C_1 w$ and w^2 produces

$$\begin{cases} a_1 = b_1 = c_1 = 0 \\ a_2 = b_2 = c_2 = 0 \\ a_3 = c_3 = 0 \\ b_3 = \dfrac{1}{\lambda_3}, \end{cases}$$

thus $h_1 = h_2 = 0$ and $h_3 = \frac{C_1}{\lambda_3} w + \cdots$. An induction reasoning allows us to compute higher order terms of the form $d_n w^n$ with $n \geq 3$, and to prove that their coefficients d_n are null. Thus, we write

$$\dot{w} = \frac{C_1}{\lambda_3} w^2 (1 + \cdots).$$

The center manifold is consequently given in a neighborhood of $C_1 = 0$ by

$$\begin{cases} \dot{x} = \lambda_1 x + \cdots \\ \dot{y} = \lambda_2 y + \cdots \\ \dot{z} = \lambda_3 z + \cdots \\ \dot{w} = \dfrac{C_1}{\lambda_3} w^2 (1 + O(w)) \end{cases}$$

and this achieves the proof. ■

Proof [Proof of Proposition 5.3]. The main idea of the demonstration is to find an expansion of the solution of system (20), in a Taylor series according to the parameter $\nu = \mu - \mu^*$. We divide system (20) into two subsystems

$$\begin{cases} \dot{r} = -(B_1 + B_2) r + q\left(1 - \dfrac{r}{r_m}\right) \\ \dot{q} = -q\left(1 - \dfrac{r}{r_m}\right), \end{cases} \tag{A.1}$$

$$\begin{cases} \dot{c} = B_1 r - C_2 c - c(1-b) + \mu^* kcp + \nu kcp \\ \dot{p} = B_2 r + C_2 c - \mu^* kcp - \nu kcp \\ \dot{b} = c(1-b). \end{cases}$$

The Poincaré expansion theorem [Verhulst, 1996] guarantees that the solution can be written as a Taylor series in ν. We are then looking for the first terms in that expansion. For more convenience, let (r, c, p, q, b), $(r_0, c_0, p_0, q_0, b_0)$ denote respectively the solutions corresponding to the parameter values μ and μ^*. We remark that the components r and q do not depend on ν, and write

$$\begin{cases} c = c_0 + \nu c_1 + \nu^2 c_2 + \cdots \\ p = p_0 + \nu p_1 + \nu^2 p_2 + \cdots \\ b = b_0 + \nu b_1 + \nu^2 b_2 + \cdots. \end{cases}$$

A necessary condition for $(c,\ p,\ b)$ to be a solution of system (A.1) is

$$\begin{aligned} &\dot c_0 + \nu \dot c_1 + \nu^2 \dot c_2 + \cdots \\ &\quad = B_1 r - C_2(c_0 + \nu c_1 + \nu^2 c_2 + \cdots) \\ &\qquad - (c_0 + \nu c_1 + \nu^2 c_2 + \cdots) \\ &\qquad \times (1 - b_0 - \nu b_1 - \nu^2 b_2 - \cdots) \\ &\qquad + \mu^* k(c_0 + \nu c_1 + \cdots)(p_0 + \nu p_1 + \cdots) \\ &\qquad + \nu k(c_0 + \nu c_1 + \cdots)(p_0 + \nu p_1 + \cdots) \\ &\dot p_0 + \nu \dot p_1 + \nu^2 \dot p_2 + \cdots \\ &\quad = B_2 r + C_2(c_0 + \nu c_1 + \nu^2 c_2 + \cdots) \\ &\qquad - \mu^* k(c_0 + \nu c_1 + \cdots)(p_0 + \nu p_1 + \cdots) \\ &\qquad - \nu k(c_0 + \nu c_1 + \cdots)(p_0 + \nu p_1 + \cdots) \\ &\dot b_0 + \nu \dot b_1 + \nu^2 \dot b_2 + \cdots \\ &\quad = (c_0 + \nu c_1 + \nu^2 c_2 + \cdots) \\ &\qquad \times (1 - b_0 - \nu b_1 - \nu^2 b_2 - \cdots). \end{aligned}$$

An identification of the terms of order 0 in ν yields the following differential system, whose unknown functions are c_0, p_0 and b_0:

$$\begin{cases} \dot c_0 = B_1 r - C_2 c_0 - c_0(1 - b_0) + \mu^* k c_0 p_0 \\ \dot p_0 = B_2 r + C_2 c_0 - \mu^* k c_0 p_0 \\ \dot b_0 = c_0(1 - b_0). \end{cases}$$

The same goes for terms of order 1:

$$\begin{cases} \dot c_1 = -C_2 c_1 + c_0 b_1 - c_1(1 - b_0) \\ \qquad\quad + \mu^* k(c_0 p_1 + c_1 p_0) + k c_0 p_0 \\ \dot p_1 = C_2 c_1 - \mu^*(c_0 p_1 + c_1 p_0) - k c_0 p_0 \\ \dot b_1 = -c_0 b_1 + c_1(1 - b_0). \end{cases}$$

We recall that the initial condition is fixed to $(0, 0, 0, 1, 0)$. Thus $c_0(0) = p_0(0) = b_0(0) = 0$ and $c_1(0) = p_1(0) = b_1(0) = 0$. The initial condition also affects the derivatives as follows:

$$\begin{cases} \dot r(0) = 1, \quad \dot q(0) = -1, \\ \dot c_0(0) = \dot p_0(0) = \dot b_0(0) = 0 \\ \dot c_1(0) = \dot p_1(0) = \dot b_1(0) = 0. \end{cases}$$

We compute the second derivatives of c_0, p_0, b_0, which produces

$$\begin{cases} \ddot c_0 = B_1 \dot r - C_2 \dot c_0 - \dot c_0(1 - b_0) \\ \qquad\quad + c_0 \dot b_0 + \mu^* k(\dot c_0 p_0 + c_0 \dot p_0) \\ \ddot p_0 = B_2 \dot r + C_2 \dot c_0 - \mu^* k(\dot c_0 p_0 + c_0 \dot p_0) \\ \ddot b_0 = \dot c_0(1 - b_0) - c_0 \dot b_0, \end{cases}$$

thus $\ddot c_0(0) = B_1$, $\ddot p_0(0) = B_2$ and $\ddot b_0(0) = 0$. Similarly, we compute the derivatives of c_1, p_1, b_1:

$$\begin{aligned} \ddot c_1 &= -C_2 \dot c_1 + \dot c_0 b_1 + c_0 \dot b_1 - \dot c_1(1 - b_0) \\ &\quad + c_1 \dot b_0 + k(\dot c_0 p_0 + c_0 \dot p_0) \\ &\quad + \mu^* k(\dot c_0 p_1 + c_0 \dot p_1 + \dot c_1 p_0 + c_1 \dot p_0) \\ \ddot p_1 &= C_2 \dot c_1 - k(\dot c_0 p_0 + c_0 \dot p_0) \\ &\quad - \mu^* k(\dot c_0 p_1 + c_0 \dot p_1 + \dot c_1 p_0 + c_1 \dot p_0) \\ \ddot b_1 &= -\dot c_0 b_1 - c_0 \dot b_1 + \dot c_1(1 - b_0) - c_1 \dot b_0 \end{aligned}$$

thus $\ddot c_1(0) = \ddot p_1(0) = \ddot b_1(0) = 0$. After some more computations, we finally obtain

$$\begin{cases} c^{(3)}(0) = p^{(3)}(0) = b^{(3)}(0) = 0 \\ c^{(4)}(0) = p^{(4)}(0) = b^{(4)}(0) = 0 \\ c^{(5)}(0) = 6B_1 B_2 > 0, \\ p^{(5)}(0) = -6B_1 B_2 < 0, \\ b^{(5)}(0) = 0 \\ b^{(6)}(0) = 6B_1 B_2 > 0. \end{cases}$$

Consequently, we have $c_1(t) > 0$, $p_1(t) < 0$ and $b_1(t) > 0$ on a time interval $[0, \eta[$, with $\eta > 0$. To complete the proof, we are going to show that $\eta = +\infty$. To that aim, we examine three cases for $(b_1(\eta), c_1(\eta), p_1(\eta))$ to leave the domain $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^-$. We first assume $b_1(\eta) = 0$, $c_1(\eta) > 0$, $p_1(\eta) < 0$, and obtain

$$\dot b_1(\eta) = c_1(\eta)(1 - b_0(\eta)) > 0,$$

which leads to a contradiction. Thus $b_1(t) > 0$ for all $t \geq t_0$. Let us now suppose that $c_1(\eta) = 0$, $b_1(\eta) > 0$

and $p_1(\eta) < 0$. As $c_1 + p_1 + b_1 = 0$ and $\mu^* \in [0,\ 1]$, we obtain

$$\dot{c}_1(\eta) = c_0(\eta)(b_1(\eta)(1 - \mu^* k) + p_0(\eta)) > 0,$$

which yields one more time a contradiction. Thus $c_1(t) > 0$ for all $t \geq t_0$. Finally, $p_1(\eta) = 0$ implies $c_1(\eta) + p_1(\eta) = 0$, which is also excluded. Thus $p_1(t) < 0$, for all $t \geq t_0$. Consequently, for ν sufficiently small, the panic component p_{μ^*} is greater than p_μ, and this achieves the proof. ■