

# DETECTION OF SYNCHRONIZATION PHENOMENA IN NETWORKS OF HINDMARSH-ROSE NEURONAL MODELS

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**Abstract.** This paper addresses networks of chemically coupled Hindmarsh-Rose neurons. After a brief presentation of the model, two synchronisation phenomena are studied. We first show that the minimal coupling strength needed to obtain complete synchronisation only depends on the in-degree of the network nodes and follows a power law. Then we propose an algorithm of numerical detection of burst synchronization in networks of bursting oscillators. Finally, this algorithm is applied to networks of different topologies. Numerical results show that the coupling strength needed to observe burst synchronization is roughly a linear function of the network diameter.

**Keywords.** Neuron models, Dynamical systems, Synchronization, Complex networks

## 1 Introduction

Synchronization motion of coupled chaotic oscillators is of main interest in different scientific domains. During the last two decades, synchronization in chaotic dynamical systems has attracted a great deal of attention [14, 5, 1] since the seminal papers by Yamada and Fujisaka [18] and Pecora and Carrol [15]. Let us recall that synchronization is a phenomenon characteristic of many processes in natural systems and nonlinear science. It remains the subject of intensive research and is today considered as one of the basic nonlinear phenomena studied in mathematics, physics, engineering or life science. This word has a Greek root, *syn* = *common* and *chronos* = *time*, which means to share common time or to occur at the same time, that is correlation or agreement in time of different processes.

Synchronization of two dynamical systems generally means that one system somehow follows the motion of another. A lot of research has been carried out and, as a result, showed that even chaotic systems could synchronize when they are coupled. Many researchers have discussed the theory, the design or applications of synchronized motion in coupled chaotic systems. A broad variety of applications have emerged, for example increasing the power of lasers, synchronizing the output of electronic circuits, controlling oscillations in chemical reactions or encoding electronic messages for secure communications.

There are different synchronization regimes, some of

which are: identical (or complete) synchronization, which is defined as the coincidence of states of interacting systems; generalized synchronization, which extends the identical synchronization phenomenon and implies the presence of some functional relation between two coupled systems; phase synchronization, which means driving of phases of chaotic oscillators, whereas their amplitudes remain; or the burst synchronization, which means that all the oscillators fire the same number of bursts starting at the same moment.

The conditions on the network to make burst synchronization appear are weaker than the ones needed to observe a complete synchronization phenomena [2, 7, 16, 17, 19]. In particular, complete synchronization in nonlinearly coupled networks of Hindmarsh-Rose neurons necessitates equal in-degrees of all network nodes, which is biologically unrealistic. That is why in this paper we are also interested in burst synchronization.

The rest of this paper is organized as follows. In section 2 we introduce the networks of chemically coupled Hindmarsh-Rose systems. Section 3 presents our results on complete synchronization, while in section 4 we develop a new algorithm aiming at detecting burst synchronization in networks of oscillators. Finally, in section 5 we apply this algorithm to networks of different topologies and sizes.

## 2 Hindmarsh-Rose neuron networks

In 1952, a mathematical model that describes neuron activity has been proposed by two neurophysiologists, A. L. Hodgkin and A. F. Huxley [10]. Different neuron models have been then developed and studied, see for example [11, 13] and references therein cited. In this paper, we consider on one of them, the Hindmarsh-Rose model (HR), which results from a simplification and a generalization of the Hodgkin-Huxley model, see [8, 9]. As observed in various biological systems, neuron activity presents different time scales. This can be explicitly observed in the HR model, which is a *slow-fast* autonomous system of three ordinary differential equations. The two first equations control the fast dynamics while the third one controls the slow dynamics. The HR model reads as

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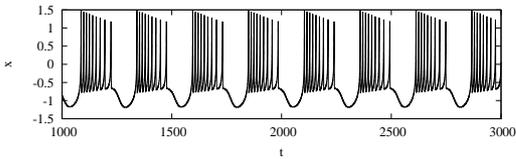


Figure 1: Time series  $x(t)$  of (2) when parameters are fixed as in (5). For this set of parameters values, a HR system exhibit a periodic bursting behaviour.

follows,

$$(HR) \begin{cases} \dot{x} &= y + ax^2 - x^3 - z + I \\ \dot{y} &= 1 - dx^2 - y \\ \dot{z} &= \epsilon(b(x - c_x) - z) \end{cases} \quad (1)$$

Parameters  $a$ ,  $b$  and  $d$  are experimentally determined,  $c_x$  is the equilibrium  $x$ -coordinate of the two-dimensional system given by the first two equations of (1) when  $I = 0$  and  $z = 0$  and parameter  $I$  corresponds to the applied current. Finally, parameter  $\epsilon$  represents the ratio of time scales between fast and slow fluxes across the membrane of a neuron. This HR neuron model can exhibit most of biological neuron behaviour, such as *spiking* or *bursting*. With appropriate parameter settings, the HR model exhibits periodic behavior characterized by fast periods of spiking called bursts, followed by slow quiescent inter-burst periods, as shown in Fig. 1.

Hereafter, for all numerical experiments, we use HR system with the following coordinate changes, see [3],  $y = 1 - y$ ,  $z = 1 + I + z$ ,  $d = a + \alpha$ ,  $c = -1 - I - bx_c$ . Applying this transformation, we obtain,

$$\begin{cases} \dot{x} &= ax^2 - x^3 - y - z \\ \dot{y} &= (a + \alpha)x^2 - y \\ \dot{z} &= \epsilon(bx + c - z) \end{cases} \quad (2)$$

Nerve cells interact via synapses. A synapse is the functional contact part which exists between two neurons or between a neuron and another cell (muscular cell, sensory receptor, etc.). It operates the conversion of an action potential fired by the presynaptic neuron into a signal in the postsynaptic cell. Usually, two different types of synapses are discerned. *Chemical synapses* need some neurotransmitters to transmit information while through *electrical synapses*, the signal is transmitted electrically through gap-junctions. The size of the synaptic cleft is characteristic of one or the other kind of synapse. In the case of electrical synapses, this synaptic cleft is about two nanometers, while it can reach from ten to forty nanometers in the case of chemical synapses. Since the large majority of synapses in the nervous system are chemical, we focus our study on them. They can be modeled by nonlinear coupling functions.

Let us consider a network composed by  $n$  HR neurons. These neurons are coupled by their first variable  $x_i$ . A

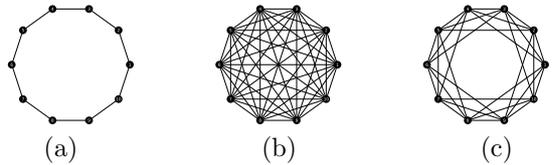


Figure 2: (a) *Ring topology network* : each neuron is connected to its two closest neighbours. (b) *Full topology network* : each neuron is connected to all the others. (c) *k-regular topology network* : each neuron is connected to its  $k$  nearest left and right neighbours.

model of this network is given by

$$\begin{cases} \dot{x}_i &= ax_i^2 - x_i^3 + y_i - z_i - \sum_{j=1}^n c_{ij}h(x_i, x_j) \\ \dot{y}_i &= (a + \alpha)x_i^2 - y_i \\ \dot{z}_i &= \epsilon(bx_i + c - z_i) \end{cases} \quad (3)$$

for  $i = 1, \dots, n$ , where  $h$  is the coupling function and  $\{c_{ij}\}$  is the network adjacency matrix. When the neurons are chemically coupled, the coupling function  $h$  is given by [3] and reads as

$$h(x_i, x_j) = g_{syn} \frac{(x_i - V)}{1 + \exp(-\lambda(x_j - \Theta))} \quad (4)$$

where  $g_{syn}$  is the coupling strength,  $\Theta$  is the threshold reached by every action potential for a neuron. Parameter  $V$  is the reversal potential and must be larger than  $x_i(t)$  for all  $i$  and all  $t$  since synapses are supposed excitatory. The parameters are fixed as follows throughout this paper,

$$a = 2.8, \alpha = 1.6, c = 5, b = 9, \epsilon = 0.001 \quad (5)$$

$$V = 2, \lambda = 10, \Theta = -0.25 \quad (6)$$

### 3 Complete synchronization

We first focus on the complete synchronization phenomenon within networks defined as in (3) with the coupling function given in (4). Due to the shape of this nonlinear coupling function, the complete synchronization phenomenon can only arise when all the neurons of the network receive the same number of signals from the other neurons [3]. That is why we limit our study to regular topology networks such as ring networks, full networks or  $k$ -regular networks in the bidirectional or unidirectional case (see Fig. 2). In each case, we consider the evolution of the coupling strength needed to observe the complete synchronization phenomena within a network, according to the in-degree of all the neurons of the network. This study points out that this coupling strength follows a law given by equation (7).

$$g_{syn}^d = \frac{g_{syn}^*}{d} \quad (7)$$

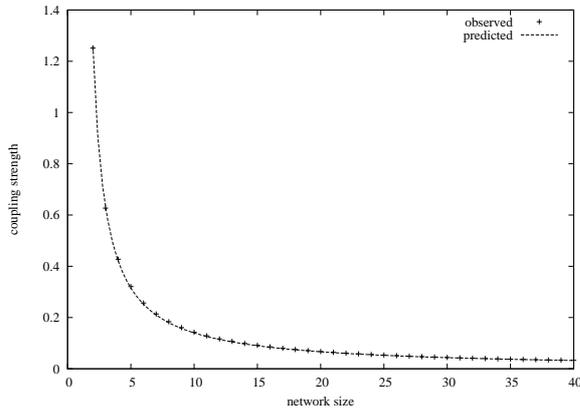


Figure 3: Observed and predicted coupling strength for complete synchronisation in a full network as a function of the network size. The results are similar for the other network topologies studied.

where  $g_{syn}^*$  is the coupling strength needed to make two bidirectionally coupled neurons synchronize and  $d$  is the in-degree of the neurons (see also [4]). This law presents the classical shape of power laws, as one can see on Fig. 3.

## 4 Algorithm of burst synchronization detection

We consider that a network of coupled neurons presents a burst synchronization behavior if the neurons fire bursts starting all at the same time. Unlike complete synchronization, burst synchronization is not easy to detect numerically. Even the distinction between fast and slow periods could be difficult, since the dynamics of single neurons could change in unpredictable way when the coupling force is slightly modified.

In this section we propose a general algorithm of burst synchronization detection in networks of coupled oscillators. Its application is not restricted to HR neurons, it can be used to detect burst synchronization in networks composed of any oscillators displaying burst behavior.

Our algorithm can be decomposed in four main steps. In order to detect burst synchronization, bursts of different neurons must be matched. To do this, one needs to determine the start time of each burst, and before detecting bursts, spikes must be detected first.

### 4.1 Spike detection

The first step of our algorithm is the detection of spikes. For our needs it suffices to find the local maxima of  $x_i(t)$  for each neuron  $i$ . Thus each spike is associated to the time when the corresponding local maximum occurs. Fig. 4 shows an example of time pattern of bursts in a network of 5 neurons. Each spike is represented as a

point on the horizontal line corresponding to the neuron in which the spike occurs.

### 4.2 Burst start detection

Once all spikes localized, we need to determine the first spike of each burst. Since each burst is preceded by a quiescent period, the idea is to consider the inter-spike distances. When the distance between two consecutive spikes is large enough, the second spike is considered as the first of a new burst. It is not easy to determine a threshold beyond which an inter-spike distance could be considered as “large enough”. Indeed, the coupling force applied on the in-degrees of the network nodes change significantly the form of the attractors of individual neurons and a threshold applicable to a given combination of these parameters could be completely inappropriate for another combination. To overcome this difficulty, we use the  $k$ -means clustering method [12] to separate the inter-spike distances in two groups. When applied to classify one-dimensional observations into two groups, the  $k$ -means clustering method is very simple. Suppose that  $d_1, \dots, d_n$  are the distances between all pairs of consecutive spikes of a given neuron sorted in non-decreasing order. For fixed  $i$ , let  $\mu_1(i)$  be the mean of  $d_1, \dots, d_i$  and let  $\mu_2(i)$  be the mean of  $d_{i+1}, \dots, d_n$ . Let

$$i^* = \operatorname{argmin} \left\{ \sum_{j=1}^i (d_j - \mu_1(i))^2 + \sum_{j=i+1}^n (d_j - \mu_2(i))^2 \right\}$$

Then  $d_{i^*+1}, \dots, d_n$  are considered as inter-burst distances. Fig. 4 shows the bursts detected by our algorithm as segments below the spike patterns.

For certain coupling force values and network topologies, the behavior of individual neurons can change from bursting to spiking. An indicator allowing to distinguish between these two behaviors is the ratio between the smallest inter-burst distance and the largest inter-spike distance,  $\frac{d_{i^*+1}}{d_{i^*}}$ . This ratio close to one indicates spiking behavior. With our parameter settings, this ratio is about 4.2 for a single non-coupled neuron.

### 4.3 Burst matching

The next step of our algorithm is to match the bursts fired by different neurons. We define the distance between two bursts as the absolute value of the difference of their starting times determined at the previous step. Let  $b_1$  be a burst emitted by a neuron and  $b_2$  be a burst emitted by another one. We say that  $b_1$  and  $b_2$  match if  $b_1$  is the burst of the first neuron closest to  $b_2$  and  $b_2$  is the burst of the second neuron closest to  $b_1$ . We say that bursts  $b_1, \dots, b_n$  fired by  $n$  different neurons match if each pair of them match. Fig. 4 represents groups of matching bursts by dots at the beginning of the burst segments.

A necessary condition for burst synchronisation is that almost all bursts belong to  $n$ -tuples of matching bursts.

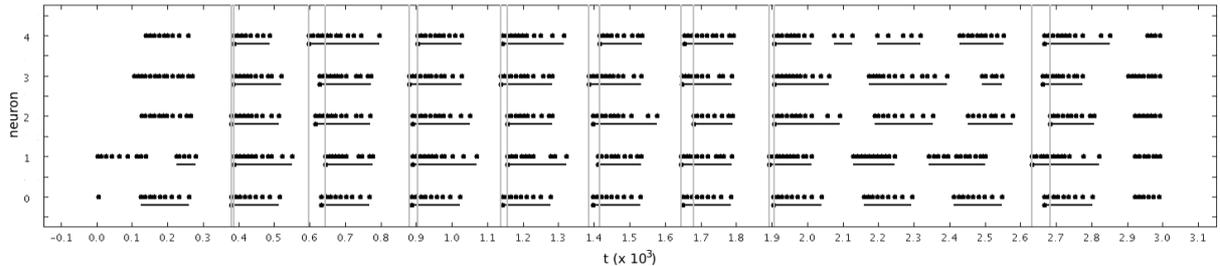


Figure 4: Example of burst pattern in a network of five neurons numbered from 0 to 4. Each spike is represented by a dot on the horizontal line corresponding to the neuron in which the spike occurs. Bursts detected by our algorithm are represented by horizontal segments under the corresponding spikes. When a burst belongs to a group of matching bursts, there is a dot at the beginning of its segment. The gray vertical lines delimit the largest distance within each group of matching bursts.

This condition is measured by the ratio of the number of bursts belonging to matching  $n$ -tuples and the total number of bursts in the network, over a long time period.

#### 4.4 Burst synchronization

Another condition for burst synchronization is that all the bursts within a matching group start in a small time interval. We measure the largest distance between two bursts in each  $n$ -tuple of matching bursts and we take the mean of this distance over all  $n$ -tuples in the observed period. The distances are visualized by grey vertical lines in Fig. 4. If this average is below a given threshold, then the oscillators are said to exhibit burst synchronization. The thresholds used in the numerical experiments are discussed in the next section.

To recapitulate, there are three conditions for burst synchronization:

- The oscillators must have bursting behavior, or the ratio  $\frac{d_{i^*} + 1}{d_{i^*}}$  must be above a given threshold for all oscillators in the network.
- The ratio between the number of matching bursts and the total number of bursts must be close to one.
- The average span of the bursts within each matching group must be below a given threshold.

## 5 Simulations

In this section, the algorithm presented in the previous section is applied to different kinds of networks.

### 5.1 Network topologies

A necessary condition for synchronization of two network nodes is that either one of them must be influenced by the other one or both of them must be influenced by a third node. At network level this implies the existence

of at least one “root” node from which all nodes can be reached.

We impose a second condition, the absence of cycles in the network. The reason for this condition is that cycles could significantly modify the individual neuron behavior. In the presence of cycles, the bursting phenomenon could even disappear for certain coupling strength values. To illustrate this fact, let us consider the simplest cycle case, two neurons with bidirectional coupling. Fig. 5 shows that when the coupling strength grows the bursting is progressively transformed in spiking. From the moment when complete synchronization is observed, bursting behavior comes back but in different form and disappears again for very big coupling forces. In the presence of longer cycles, the individual behavior could be even more perturbed. Our experiments show that in acyclic networks bursting is more stable.

We have chosen two topologies which satisfy the above conditions and which allow to build easily networks of different size sharing the same characteristics. The first type of network is a simple string in which each neuron receives signal from the previous one. The second type is a complete oriented network in which neuron  $i$  receives signal from neurons  $1, \dots, i - 1$  as shown in Fig. 6.

### 5.2 Experimental results

In this section we are interested in the values of the synaptic coupling strength  $g_{\text{syn}}$  for which burst synchronization appears. The numerical results are obtained using a Java code implementing the Runge-Kutta 4 integration method. GraphStream library [6] is used for generation and manipulation of networks.

The three burst synchronization indicators identified in Section 4 are shown in Fig. 7.

Fig. 7(a) shows the evolution of the ratio between the shortest inter-burst distance and the longest inter-spike distance. For the sake of clarity, only the last neuron of a 20 node network is presented. The ratio is stable from a certain coupling strength value. For the other neurons the

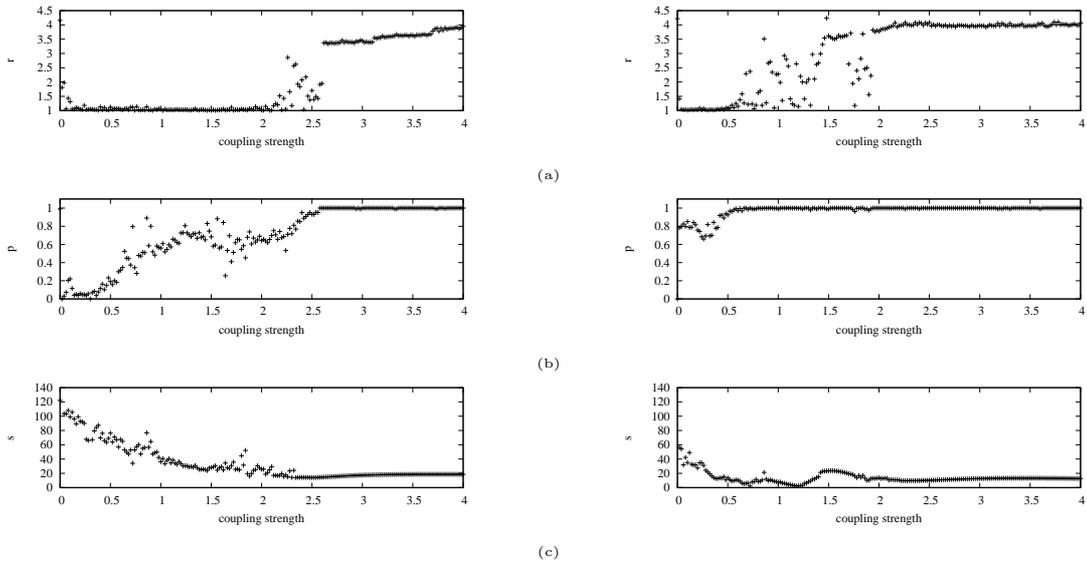


Figure 7: Three burst synchronization indicators as a function of the coupling strength. (a)  $r$ : ratio between the shortest inter-burst and the longest inter-spike distances. (b)  $p$ : proportion of matching bursts. (c)  $s$ : average span of burst starts within matching groups. All results are for networks of size 20 with string topology (left) and complete oriented topology (right).

curve is similar, but the stabilization point is shifted to the left. The coupling force needed in complete network is less than the force needed in string network because of the greater in-degrees. Indeed, the in-degree of node  $i$  is  $i - 1$  in a complete network, while it is only 1 in a string network.

The second indicator, the ratio of matching bursts, is shown in Fig. 7(b). Once again, there is a threshold beyond which this ratio becomes practically 1, meaning that all bursts belong to matching groups. The last indicator is the average span of burst starts within each matching group. Fig. 7(c) shows that as the other indicators, this one stabilizes from certain coupling force value.

Taking into account the three indicators, one can determine the values of the coupling strength for which burst synchronization is observed. Fig. 8 summarizes these results. We can see that for string networks the needed coupling strength grows linearly with the network size, while for complete networks it does not depend on the network size. It is interesting to note that in the last case a desynchronization phenomenon is observed.

## 6 Conclusion

In this paper we are interested in the minimal coupling force needed to synchronize a network of chemically coupled Hindmarsh-Rose oscillators. The study of the complete synchronization phenomenon pointed out that this strength follows a power law depending only on the in-degree of the network nodes.

In order to study burst synchronization in networks of

coupled oscillators, we develop an algorithm to numerically detect this phenomenon. The application of this algorithm to networks of different topologies and sizes led us to the hypothesis that the burst synchronization threshold of the coupling strength could be a linear function of the network diameter. This assumption fits well with our preliminary experimental results as seen in Fig. 8. Indeed, the diameter of a full oriented network of size  $n$  is 1, while this diameter is  $n - 1$  in the case of a string network. Our hypothesis is to be confirmed (or disproved) experimentally on other types of networks and justified and completed theoretically.

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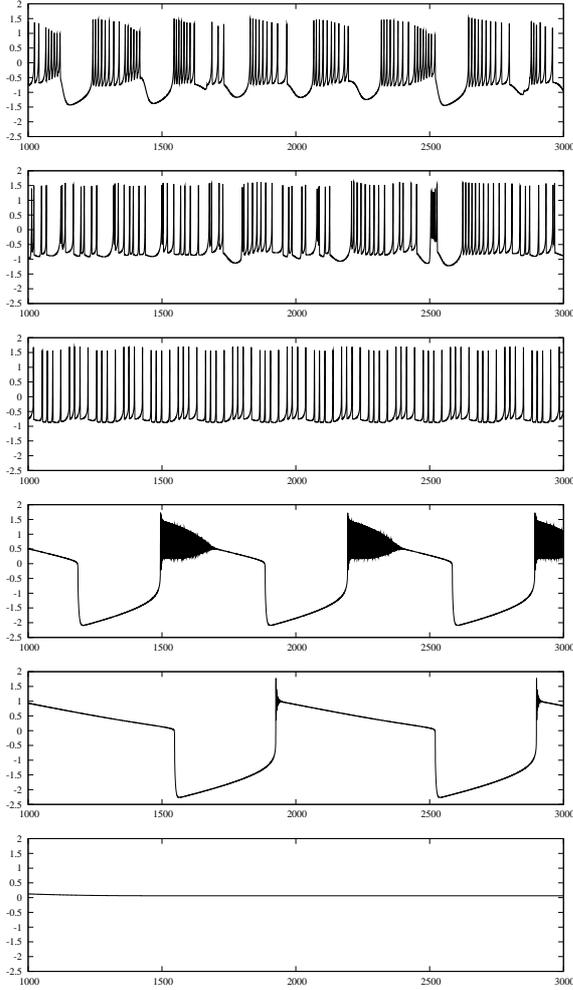


Figure 5: Times series  $x(t)$  of a neuron 1 within a network with cycles for an increasing coupling strength (from top to bottom) 0.1, 0.5, 1.0, 1.3 and 3.0. Beyond a given value of the coupling strength, there is no more bursting behavior exhibited by the neuron.

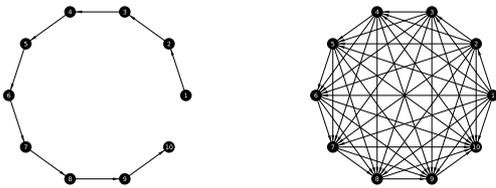


Figure 6: Examples of string network (left) and complete oriented network (right) of size 10

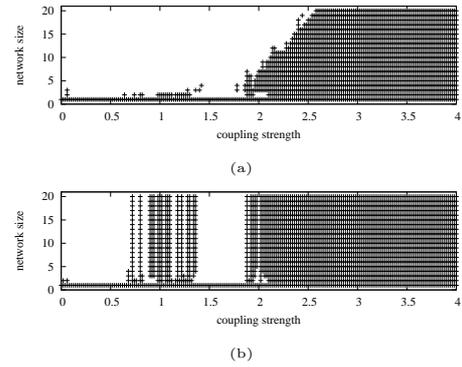


Figure 8: The  $x$  axis presents the coupling strength and the  $y$  axis represents the network size. Each dot indicates the presence of burst synchronization for the corresponding coupling strength and network size. (a) string network, (b) complete oriented network.

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