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BURST SYNCHRONIZATION OF COUPLED OSCILLATORS: TOWARDS UNDERSTANDING THE INFLUENCE OF THE NETWORK TOPOLOGY

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Abstract.
This paper addresses the question of burst synchronization in networks of chemically coupled Hindmarsh-Rose neurons. After a brief description of the model and of an algorithm of numerical detection of burst synchronization, we present numerical experiments designed to give an insight on the influence of the network topology on the minimal coupling strength needed to obtain burst synchronization in the network. Two topological characteristics are studied: the network diameter and the in-degrees of the nodes. Our numerical simulations show that when the diameter grows, the network becomes more difficult to synchronize, while networks with bigger in-degrees of the nodes synchronize more easily.

Keywords. Neuron models, Dynamical systems, Synchronization, Complex networks

1 Introduction

Synchronization of two dynamical systems generally means that one system somehow follows the motion of another. A lot of research has been carried out and, as a result, showed that even chaotic systems could synchronize when they are coupled. Many researchers have discussed the theory, the design or applications of synchronized motion in coupled chaotic systems [1, 9, 12]. There are different synchronization regimes. Oscillators firing bursts exhibit *burst synchronization*, when they all fire the same number of bursts starting at the same moment.

The conditions on the network to make burst synchronization appear are weaker than the ones needed to observe a complete synchronization phenomena [2, 6, 10, 11, 13]. In particular, complete synchronization in nonlinearly coupled networks of Hindmarsh-Rose neurons necessitates equal in-degrees of all network nodes, which is biologically unrealistic. That is why in this paper we are interested in burst synchronization.

We focus on networks composed of Hindmarsh-Rose neuon models, given by (1).

$$(HR)\begin{cases} \dot{x} &= y + ax^2 - x^3 - z + I \\ \dot{y} &= 1 - dx^2 - y \\ \dot{z} &= \epsilon(b(x - c_x) - z) \end{cases} \quad (1)$$

Parameters a, b and d are experimentally determined, c_x is the equilibrium x-coordinate of the two-dimensional system given by the first two equations of (1) when $I = 0$ and $z = 0$ and parameter I corresponds to the applied current. Finally, parameter ϵ represents the ratio of time scales between fast and slow fluxes across the membrane of a neuron. This HR neuron model can exhibit most of biological neuron behavior, such as *spiking* or *bursting*. With appropriate parameter settings, the HR model exhibits periodic behavior characterized by fast periods of spiking called bursts, followed by slow quiescent inter-burst periods, as shown in Fig. 1.

Hereafter, for all numerical experiments, we use HR system with the following coordinate changes, see [3], $y = 1-y$, $z = 1+I+z$, $d = a+\alpha$, $c = -1-I-bx_c$. Applying this transformation, we obtain,

$$\begin{cases} \dot{x} &= ax^2 - x^3 - y - z \\ \dot{y} &= (a+\alpha)x^2 - y \\ \dot{z} &= \epsilon(bx + c - z) \end{cases} \quad (2)$$

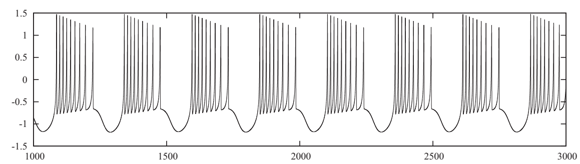

Figure 1: Time series (t, x) of (2) when parameters are fixed as in (5). For this set of parameters values, a HR system exhibit a periodic bursting behavior.

Let us consider a network composed by n HR neurons. These neurons are coupled by their first variable x_i with a coupling function modeling chemical synapses. A model of this network is given by

$$\begin{cases} \dot{x_i} &= ax_i^2 - x_i^3 + y_i - z_i - \sum_{j=1}^{n} c_{ji} h(x_i, x_j) \\ \dot{y_i} &= (a+\alpha)x_i^2 - y_i \\ \dot{z_i} &= \epsilon(bx_i + c - z_i) \end{cases} \quad (3)$$

for $i = 1, \ldots, n$, where h is the coupling function and $\{c_{ij}\}$ is the network adjacency matrix. When the neurons are

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chemically coupled, the coupling function h is given by [3] and reads as

$$h(x_i, x_j) = g_{\text{syn}} \frac{(x_i - V)}{1 + \exp(-\lambda(x_j - \Theta))} \tag{4}$$

where g_{syn} is the coupling strength, Θ is the threshold reached by every action potential for a neuron. Parameter V is the reversal potential and must be larger than $x_i(t)$ for all i and all t since synapses are supposed excitatory. The parameters are fixed as follows throughout this paper,

$$a = 2.8,\ \alpha = 1.6,\ c = 5,\ b = 9,\ \epsilon = 0.001 \tag{5}$$

$$V = 2,\ \lambda = 10,\ \Theta = -0.25 \tag{6}$$

These parameter values are commonly used in the literature (see for example [3].

The rest of this paper is organized as follows. In section 2, we recall the algorithm we developed to detect burst synchronization phenomena in network of coupled oscillators, and in section 3 we apply this algorithm to networks of different topologies and sizes.

2 Algorithm of burst synchronization detection

We consider that a network of coupled neurons presents a burst synchronization behavior if the neurons fire bursts starting all at the same time. Unlike complete synchronization, burst synchronization is not easy to detect numerically. Even the distinction between fast and slow periods could be difficult, since the dynamics of single neurons could change in unpredictable way when the coupling force is slightly modified.

In this section we recall the main steps of a general algorithm of burst synchronization detection in networks of coupled oscillators [4]. Its application is not restricted to HR neurons, it can be used to detect burst synchronization in networks composed of any oscillators displaying burst behavior.

Our algorithm can be decomposed in four main steps. In order to detect burst synchronization, bursts of different neurons must be matched. To do this, one needs to determine the start time of each burst, and before detecting bursts, spikes must be detected first.

The first step of our algorithm is the detection of spikes. For our needs it suffices to find the local maxima of $x_i(t)$ for each neuron i. Thus each spike is associated to the time when the corresponding local maximum occurs. Fig. 2 shows an example of time pattern of bursts in a network of 5 neurons. Each spike is represented as a point on the horizontal line corresponding to the neuron in which the spike occurs.

Once all spikes localized, we need to determine the first spike of each burst. Since each burst is preceded by a quiescent period, the idea is to consider the inter-spike distances. When the distance between two consecutive spikes is large enough, the second spike is considered as the first of a new burst.

For certain coupling force values and network topologies, the behavior of individual neurons can change from bursting to spiking. An indicator allowing to distinguish between these two behaviors is the ratio between the smallest inter-burst distance and the largest inter-spike distance. This ratio close to one indicates spiking behavior. With our parameter settings, this ratio is about 4.2 for a single non-coupled neuron.

The next step of our algorithm is to match the bursts fired by different neurons. We define the distance between two bursts as the absolute value of the difference of their starting times determined at the previous step.

A necessary condition for burst synchronisation is that almost all bursts belong to n-tuples of matching bursts. This condition is measured by the ratio of the number of bursts belonging to matching n-tuples and the total number of bursts in the network, over a long time period.

Another condition for burst synchronization is that all the bursts within a matching group start in a small time interval. We measure the largest distance between two bursts in each n-tuple of matching bursts and we take the mean of this distance over all n-tuples in the observed period.

To recapitulate, there are three conditions for burst synchronization:

- The oscillators must have bursting behavior.
- The ratio between the number of matching bursts and the total number of bursts must be close to one.
- The average span of the bursts within each matching group must be below a given threshold.

3 Numerical Simulations

In this section, the algorithm presented in the previous section is applied to different kinds of networks. The numerical results are obtained using a code implementing Runge-Kutta 4 integration method and developed in CUDA C [8]. The code was executed on NVIDIA Tesla C2050 GPUs. This implementation allows us to reduce the running times two orders of magnitude compared to the equivalent CPU implementation and thus to simulate networks containing thousands of nodes in reasonable time. The post-processing (including the algorithm from the previous section) is implemented in Java. GraphStream library [5] is used to generate and to manipulate networks.

Our numerical experiments aim to answer the following question : What is the coupling strength g_{syn} (see (4)) needed to synchronize the bursts of all the neurons of a given network? The answer is not obvious in the general case and that is why we start by studying particular

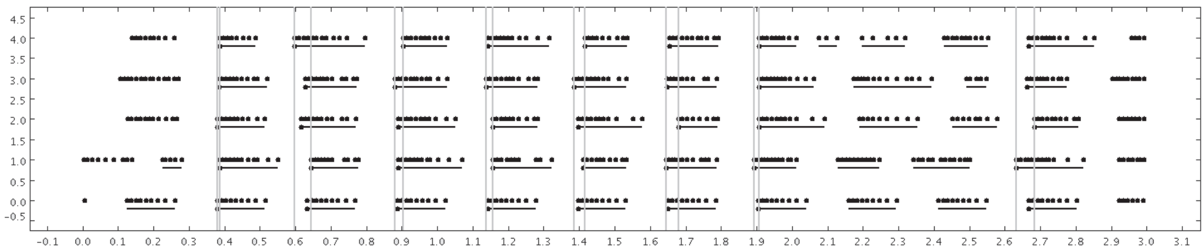

Figure 2: Illustration of the burst synchronization detection algorithm. The four steps of the algorithm are represented on this figure.

network topologies. These particular cases are chosen to give us an insight on the influence of different network parameters on the synchronization threshold.

3.1 Network topology

A necessary condition for synchronization of two nodes of a network is that either one of them must be influenced by the other one or both of them must be influenced by a third node. At network level this implies the existence of at least one "root" node from which all nodes can be reached.

We impose a second condition, which is the absence of cycles in the network. The reason for this restriction is that cycles could significantly modify the individual neuron behavior. In the presence of cycles, the bursting phenomenon could even disappear for certain coupling strength values. To illustrate this fact, let us consider the simplest cycle case, two neurons with bidirectional coupling. Fig. 3 shows that when the coupling strength grows the bursting is progressively transformed in spiking. From the moment when complete synchronization is observed, bursting behavior comes back but in different form and disappears again for very big coupling forces. In the presence of longer cycles, the individual behavior could be even more perturbed. Our experiments show that in acyclic networks, the bursting motion is more stable.

It is easy to see that in acyclic networks the root node is unique. Our first observation was that the coupling strength needed to synchronize a given node to the root node depends on the distance between them. In other words, networks with smaller diameter require less coupling strength to synchronize. To study the influence of the network diameter, we use networks constructed by levels. Level 0 contains the root node. Nodes of level l receive signal only from nodes of level $l-1$. Thus the distance between the root and all the nodes of level l is exactly l. An example of such a network is given in Fig. 4.

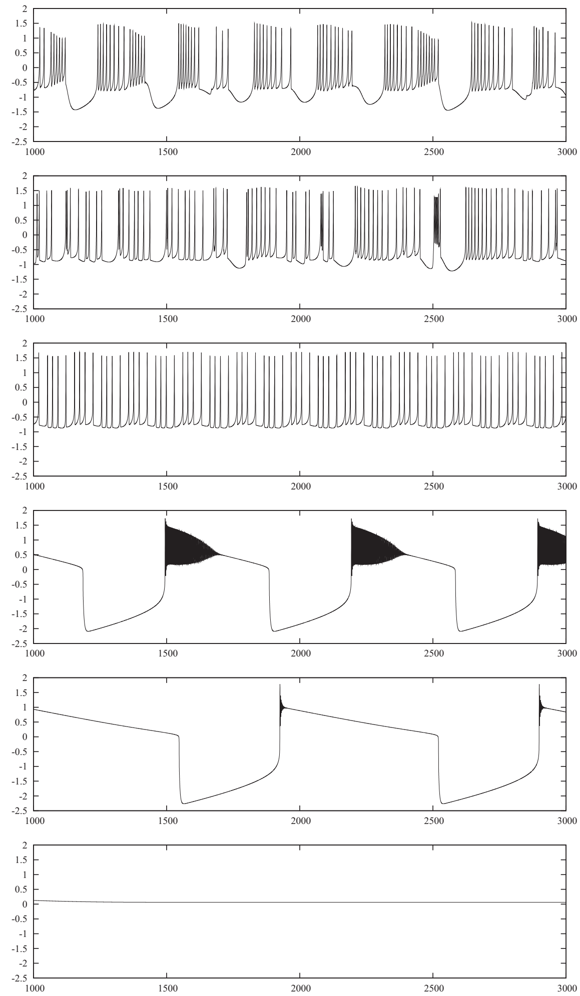

Figure 3: Times series (t, x) of a neuron (1) within a network with cycles for an increasing coupling strength (from top to bottom) 0.1, 0.5, 1.0, 1.3 and 3.0. Beyond a given value of the coupling strength, there is no more bursting behavior exhibited by the neuron.

3.2 Regular and quasi-regular networks

The simplest case of network constructed by levels is a chain network in which each level contains a single node connected to the node from the previous level. The results obtained for chain networks are summarized on Fig. 5. Fig. 5 (a)-(c) show the values of the three burst synchro-

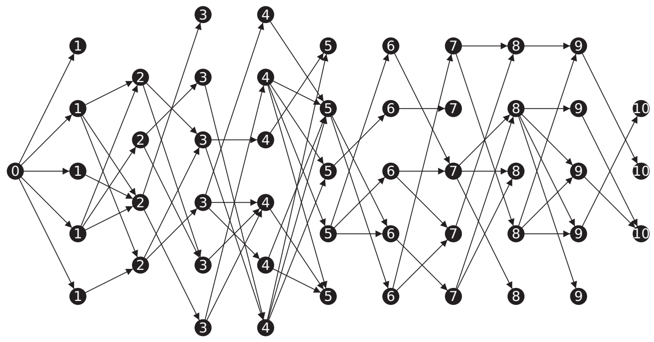

Figure 4: Example of a network constructed by levels containing 50 nodes distributed in 11 levels. Each node receives signal only from nodes of the previous level.

nization indicators defined in section 2. For each indicator we can see the presence of a threshold value of the coupling strength, above which the value of the indicator changes abruptly. These thresholds are shown as green lines on the figure. Fig. 5 (d) shows the three threshold lines drawn together. We can see that the first indicator, the ratio between the shortest inter-burst and the longest inter-spike distances, is predominant. In other words, when all the neurons have clearly expressed burst behavior, their bursts are synchronized. The same observation is valid for all the other network topologies we tested.

The experimental results show that the minimal coupling strength needed to synchronize a chain network scales linearly with the network diameter. In other words,

$$g(l) = Al + B \tag{7}$$

where $g(l)$ is the minimal coupling strength needed to synchronize the network up to level l. The values of coefficients A and B fitted by the least squares method to the simulation data are $A = 0.044$ and $B = 1.906$. Fig. 5 (d) shows also this linear approximation.

The same results are obtained for tree networks in which each node receives a signal from exactly one node from the previous level. Indeed, in these networks the in-degree of each node, except the root, is one and the behavior on each path from the root to another node is exactly the same as in a chain network.

The network diameter is not the only topological characteristics influencing the burst synchronization. The law observed for chain and tree networks is not preserved when we introduce nodes of different in-degrees. To illustrate the influence of the in-degree, consider a modified chain network as shown in Fig. 6(a). At a given level l we introduce k nodes each of them receiving signal from the node of level $l-1$ and sending signal to the node of level $k+1$. Thus, the in-degree of the root is 0, the in-degree of the node on level $l+1$ is k and the in-degrees of all the other nodes are 1.

Fig. 7 shows the burst synchronization thresholds for modified chain networks in which $k = 2$ or 5 nodes are introduced on level 20. When the network is synchronized up to level 20, there is no need to increase the coupling strength to synchronize the next several levels. In fact the behavior of the node on level 21 is roughly the same as if it was coupled with one node but with k times greater coupling strength. Nevertheless, this influence is only local, some more levels away the modification is progressively "forgotten" and the coupling strength joins again the linear law (7).

To study further the influence of the nodes in-degree, we consider a regular network shown in Fig. 6(b). In this type of networks all levels except level 0 contain the same number of nodes k and each node is connected to all the nodes from the previous level. In order to ensure the same in-degree of all nodes, the nodes of level 1 are connected to the root with k links. Fig. 8 shows the experimental results for this kind of networks. For the sake of simplicity, only the cases $k = 2$ and $k = 3$ are shown, but the results are similar for greater values of k. We can see that the coupling strength needed for burst synchronization grows linearly with the network diameter also in this case. Moreover, the simulation data fits to the lines

$$g_k(l) = \frac{Al + B}{k} \tag{8}$$

where $g_k(l)$ is the minimal coupling strength needed to synchronize a regular network with k nodes per level up to level l and A and B are the same coefficients as in the case of chain network. In particular, for $k = 1$, we obtain exactly (7). This result is not surprising, because a neuron coupled with k identical neurons with coupling strength g behaves as if it was coupled with a single neuron but with coupling strength kg.

3.3 Random networks

We have seen that the burst synchronization is influenced by two main characteristics of the network: the network diameter and the in-degree of the nodes. For regular networks in which the nodes in-degrees are the same, the coupling strength needed for burst synchronization grows linearly with the network diameter. In this section, we consider networks with heterogeneous in-degrees. They are always constructed by levels as described previously but each level contains a random number of nodes randomly connected to the nodes of the previous level. For our experiment we fixed 384 nodes randomly distributed on 64 levels. We then generated different random subsets of all possible links between adjacent levels in order to obtain different average in-degrees. Fig. 9 shows the experimental results for average in-degrees 1, 1.5, 2 and 3. The case of in-degree 1 corresponds to a tree and logically we observe a linear growth of the coupling strength with the network diameter. For bigger in-degrees, the coupling strength grows stepwisely with the network diameter. This can be explained by looking again at Fig. 7. As we have seen, when a neuron receives more strength than it needs to synchronize, it synchronizes his successors several levels away. Fig. 9 shows also that the growth of the coupling force becomes slower when the in-degree is bigger. The places and the heights of the jumps in the lines are difficult to predict. In order to do this, an aggregate parameter as the average in-degree is not sufficient. The coupling strength needed to synchronize a neuron seems to depend not only on its in-degree, but also on the in-degrees of its direct and indirect predecessors.

4 Conclusion and perspectives

In this paper, we are interested in the minimal coupling force needed to obtain burst synchronization within a network of chemically coupled Hindmarsh-Rose oscillators according to the topology of networks. We study the influence of two topological network characteristics: the network diameter and the in-degrees of the nodes. We performed numerical tests on different network topologies in order to highlight the role of these characteristics.

Our experiments show that the coupling force needed to synchronize a network grows with its diameter. Inversely, when the in-degree of the nodes grows, the network becomes easier to synchronize. In the case of (quasi-)regular networks where (almost) all the nodes have the same in-degree, the coupling force grows linearly with the network diameter. In irregular networks with heterogeneous in-degrees we observe stepwise growth of the coupling strength due to the fact that some nodes receive stronger signal than others which they propagate to their successors.

In this paper, we present preliminary results which need to be confirmed by other numerical simulations. To make the general trends more evident, we have started by considering a restricted class of network topologies (acyclic networks with specific level structure). In future numerical simulations we will progressively remove these restrictions. It will be interesting to make a more detailed study of the influence of the in-degrees and to understand the evolution of the coupling force needed to synchronize a node as a function of its in-degree, but also of the in-degrees of its predecessors, the in-degrees of their predecessors, etc. Finally, a mathematical formalism describing the burst synchronization needs to be developed and used to justify theoretically our experimental observations.

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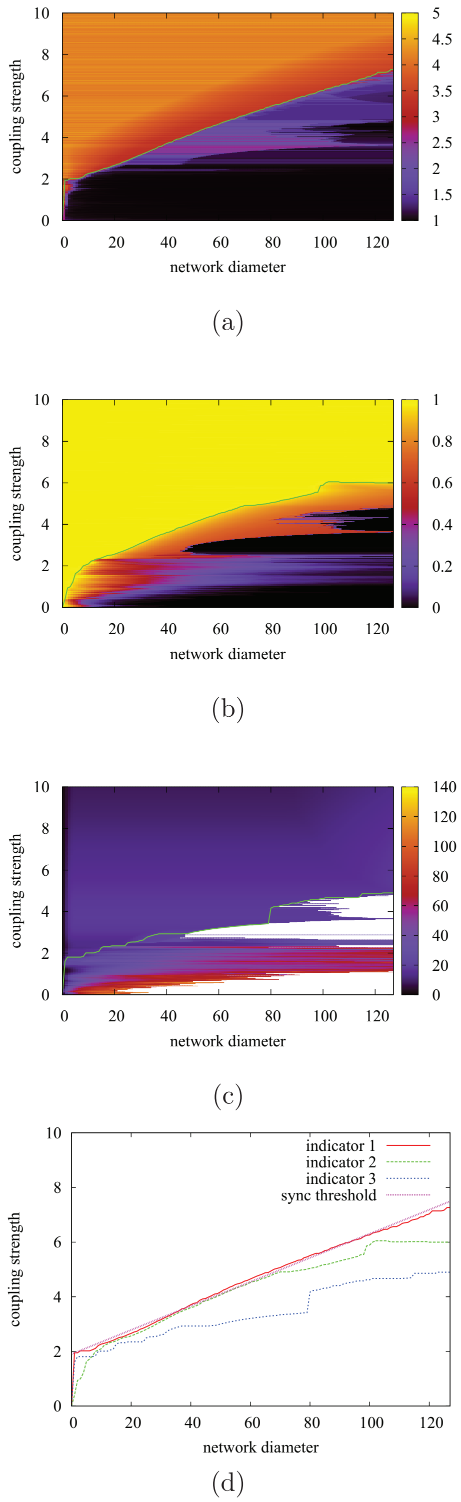

Figure 5: Burst synchronization indicators for a chain network as a function of the coupling strength and the network diameter. (a) Ratio between the shortest inter-burst and the longest inter-spike distances. (b) Proportion of matching bursts. (c) Average span of burst starts within matching groups. (d) Superposition of the three indicators and linear approximation of the synchronization threshold.

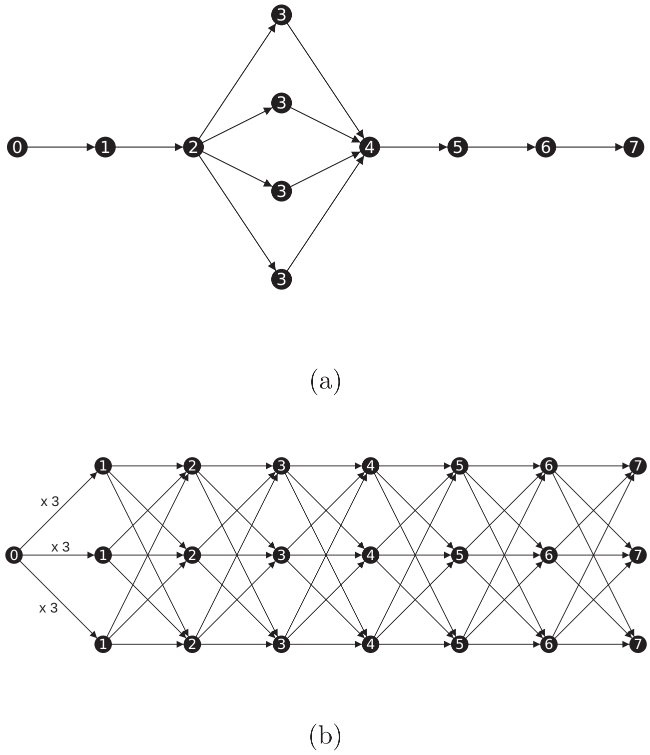

Figure 6: (a) Modified chain network. Level 3 contains four nodes. (b) Regular level network. Each level contains 3 nodes connected to all the nodes from the previous level.

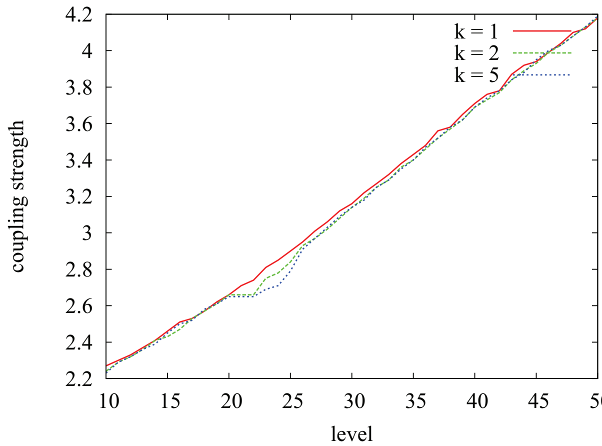

Figure 7: Coupling strength needed to synchronize modified chain networks. k nodes are introduced on level 20. Results for $k = 1, 2$ and 5.

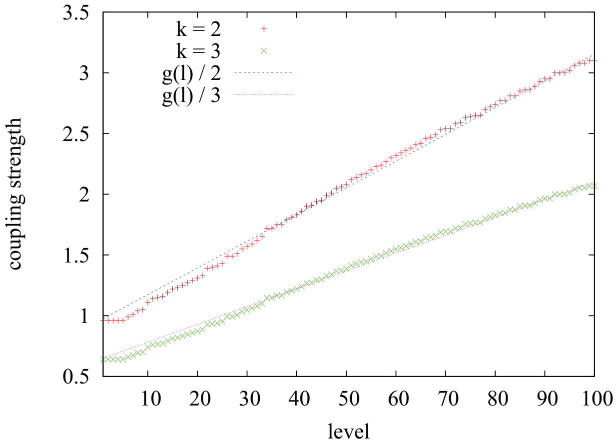

Figure 8: Coupling strength needed to synchronize regular network with k nodes per level. Results for $k = 2$ and 3.

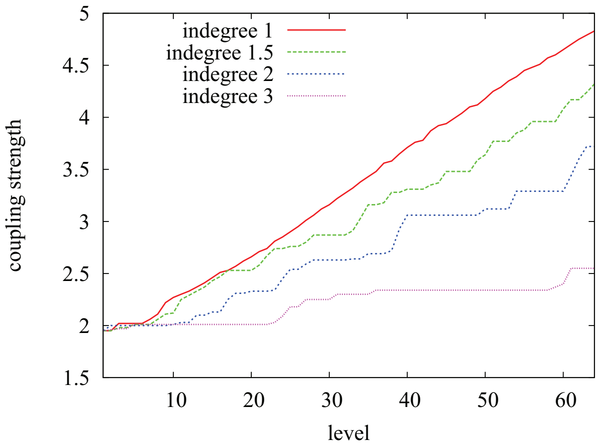

Figure 9: Coupling strength needed to synchronize random networks with different average in-degrees.