Mathematical Biosciences 229 (2011) 50–63

Contents lists available at ScienceDirect

Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

The chikungunya disease: Modeling, vector and transmission global dynamics

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ARTICLE INFO

Article history:
Received 16 March 2010
Received in revised form 23 October 2010
Accepted 29 October 2010
Available online 9 November 2010

Keywords:
Chikungunya
Modeling
Competitive systems
Global stability

ABSTRACT

Models for the transmission of the chikungunya virus to human population are discussed. The chikungunya virus is an alpha arbovirus, first identified in 1953. It is transmitted by *Aedes* mosquitoes and is responsible for a little documented uncommon acute tropical disease. Models describing the mosquito population dynamics and the virus transmission to the human population are discussed. Global analysis of equilibria are given, which use on the one hand Lyapunov functions and on the other hand results of the theory of competitive systems and stability of periodic orbits.

1. Introduction

An unprecedented chikungunya epidemic has appeared on the Reunion Island (775,000 inhabitants) with over 244,000 reported and 205 deaths (directly or indirectly linked) as of April 2006. *Aedes albopictus* [1], long present on the island, is the main vector of this disease. *Aedes aegypti* [2], which is also known to transmit dengue fever, is the other vector of chikungunya. After the Grande Comore Island epidemic, first cases were reported in the Reunion Island in March 2005. It was the first time that a chikungunya epidemic was described in this part of the world.

The Asian tiger mosquito or forest day mosquito (*Aedes albopictus*), from the mosquito family Culicidae, is characterized by its black and white striped legs, small black and white body. It is native of the tropical and subtropical areas of Southeast Asia. In the past couple of decades this specie has invaded many countries throughout the world, through the increasing transport of goods and international travels. It has recently appeared in Europe, like in France [3], in the USA and in Australia.

The chikungunya is an arthropod-borne viral disease (arbovirus). The name is derived from the Makonde word meaning "that which bends up" in reference to the stooped posture developed as a result of the arthritic symptoms of the disease. It was first described by Marion Robinson and Lumsden [4,5], following an outbreak in 1952 on the Makonde Plateau, in Tanzania [5].

Some arboviruses are able to cause emergent diseases and transmit the virus upon biting, allowing it to enter the bloodstream which can cause viremia. The dynamics of arboviral diseases like dengue or chikungunya are influenced by many factors such as humans, the mosquito vector, the virus itself, as well as the environment which affects all the present mechanisms directly or indirectly.

Through the 20th century, mathematical models have been established as major tools for epidemiological models (see [6–8] and references therein), and in particular the study of vector-borne infections [9–12].

In [13,14], the authors study a SI-SIR model in which the total human population size is constant. Models with a variable human population are studied in [15,16]. In [15] the authors use a non classical contact rate among humans that depend on the total vector population size. Such models do not take into account the dynamics of the vector.

In [17,18], mathematical models describing the dynamics of vector-borne diseases taking into account the controlling mechanisms applied on the vector population are developed. In [17], the human population is supposed to be constant and the incidence rate among humans depends on the total vector population size. Moreover, only the local stability of equilibria was studied analytically.

This paper deals with two models involving differential equations for the mosquito population and virus transmission to the human population. Following [15], we consider a model which takes into account the dynamics of the vector with a non-constant population size and a contact rate that depends on the vector population size.

The local and global stability of equilibria are studied. This work can be seen as a complement to the study of [17,18]. The conclusion discusses the use of a contact rate among humans that depends on the vector population size.

This paper is organized as follows. In the second section we give the biological explanation of the problem and address the vector

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doi:10.1016/j.mbs.2010.10.008

life cycle (embryonic, larvae, pupae and adult stages), the reproduction and oviposition habitat selection, and finally the transmission of the virus phenomenon.

The third section deals with the formulation of dynamical models, first of all for the population growth that is the *Aedes albopictus* mosquito, and secondly for the virus transmission to the human population. The first model uses a stage structure model; the second uses SI and SIR type models.

The fourth section is devoted to the mathematical analysis of both models, focusing on the boundedness and the positivity of the solutions, and on local or global stability of equilibria.

For the first system we use the Lyapunov theory to establish the global stability of the endemic equilibrium, while for the second we use the general theory of competitive systems and compound matrices. As usual in mathematical epidemiological studies, we also found two thresholds parameters that determine the global dynamical behavior.

2. Biological problem

2.1. Vector life cycle

The life of the vector, consists of four stages: embryonic stage (eggs), larvae stage, pupae stage and adult stage. The first three stages need water for their development while the last one needs only air. In this paper we will not distinguish the larvae stage and the pupae stage: these two stages are called the immature stage. The lifespan of each stage depends on several factors, such as temperature or the availability of food and water [19].

2.2. Oviposition habitat selection

Mosquitoes have developed various approaches to avoid predation. Many species breed in areas where predators are rare or absent. The oviposition habitat selection is made to make sure that the mosquito larvae can develop relatively unmolested. That is the case of the Asian tiger mosquito or *Aedes albopictus* mosquito which is a container-inhabiting species and which lays its eggs in any water-containing receptacle in urban, suburban, rural and forest areas. The primary immature habitats of this species are artificial containers [20,21].

These habitats are largely devoid of predators, thus offering a relatively secure refuge for larval development. Most female can lay their eggs not only on moist substrate but on dry substrate that is subject to flooding with rains or tides. They breed in small, often ephemeral, pools such as those in tree holes, bamboo pots and leaf axils. This strategy does have a negative side for mosquito: the development depends on the availability of water and they must be able to develop quickly before the water dries. Mosquitoes lay their eggs on the water or any moist surface, but they can also breed in natural habitats like vegetation or near rivers ...

2.3. The embryonic stage

The eggs need 48–72 hours to become mature [22]. They need water to hatch, but they are desiccation-resistant and cold-resistant and they have the capacity to cling to the inner side of any potential containers. Moreover, eggs are capable of winter diapause and mature eggs can wait until two years to hatch if the hydration conditions are not sufficient for the development of larvae [23,24].

2.4. The larvae and pupae stages

Depending on temperature and availability of food, *Aedes albopictus* can complete larval development (four stages) between five to ten days; the pupae stage needs two days to develop [25]. An increase in larval density or a decrease in food (for example, a decrease in water due to evaporation) can cause more mortality and reduction in the number of adult subjects. Though limited food is the primary cause of death, parasites and predators may exert substantial influence on the population size. The amount of water in the containers also plays an important role in determining mosquito density. Although the overall population in containers appears greater, it is actually decreasing as resources decline and intra-specific competition increases [26,27], resulting in greater larval mortality and the production of small-sized adults. Moreover, it has been observed that the larvae of *Aedes albopictus* are cannibal, they are able to eat earlier-stage larva under certain conditions.

2.5. The adult stage

The exit of the pupae stage normally happens very early in the morning, perhaps in order to escape numerous predators whose main activity takes place during daytime [21,28]. Before any activity, male and female need to have a meal of sugar and water [29]. The flight range of adults is limited (from one to two hundred meters), they have not been observed to fly in strong winds [30,31]. Its major means of dispersal is through the transport of used and waste tires. The move of other water-holding containers could also play a role in expanding its range. The life expectancy for males is fifteen days whereas it is from two to four weeks for females [22] and can reach ten weeks under labs conditions.

2.6. Reproduction

Aedes albopictus is very aggressive during the daytime [32] with biting generally occurring during early morning and late afternoon [30]. Females require blood to produce their eggs. They feed on a number of hosts including human (indoors and outdoors), domestic and wild animals and birds. Their generalized feeding behavior contributes to their being potential vectors.

Females lay eggs one by one above the water level or on the sides of a variety of containers which serve as breeding habitat. They rely on rainfalls to raise water level in containers and inundate the eggs for hatching. Females lay from one hundred to three hundred eggs per oviposition and have from one to four ovipositions during their life.

2.7. Transmission of the virus

A vector is infected after biting an infected human. There is a delay (from seven to twelve days) or incubation period when mosquitoes are incurring the disease but still unable to transmit it. Recent research have shown, that a genetic mutation in the chikungunya virus identified in Réunion Island, has facilitated the transmission by *Aedes albopictus*. Indeed, the extrinsic incubation period was reduced to two days [33]. Mosquitoes remain infective until death. Vertical transmission in the vector has not yet been observed until today.

A human is infected after being bitten by an infected vector, after a delay of four to seven days (incubation period), the human is able to transmit the virus. This period, which can go on five to seven days, is the viraemia period. After this time, human recovers.

3. Vector population and virus transmission modeling

3.1. Formulation of a dynamical model for vector population growth

To describe the *Aedes albopictus* population dynamics we use a stage structured model, which consists of three main stages (see

Fig. 1): embryonic (E), larvae (L, which consists here of the larvae and pupae populations) and adult (A, which consists only of adults females). Even if eggs and immature stages are both aquatic, we dissociate them because these two populations respond differently to control measures. Indeed, eggs can cling and are desiccation-resistant and hence, drying the breeding sites does not kill eggs, but only larvae and pupae. Moreover, chemical interventions on the breeding sites has impact on the larvae population, but not on the eggs.

We assume that the number of laid eggs is proportional to the number of females.

The above hypotheses lead to the following equations.

$$\begin{cases} \frac{dE}{dt}(t) = bA(t) - sE(t) - dE(t), \\ \frac{dL}{dt}(t) = sE(t) - s_L L(t) - d_L L(t), \\ \frac{dA}{dt}(t) = s_L L(t) - d_m A(t). \end{cases} \tag{1}$$

Moreover, as we said in Sections 2.2 and 2.4, it has been observed that mosquitoes are able to detect the best breeding site for the eggs development. Indeed if there are too much eggs in the oviposition habitat or too few nutrients and water resources, then females laid less eggs or choose another site. It seems reasonable to express this biological phenomenon with a mathematical model which explicitly incorporates the idea of limited carrying capacity resources. This model should take into account the availability of nutrients and the occupation by eggs or larvae of the available breeder sites. That is why we assume that,

- *per capita* oviposition rate is given by,

$$b\left(1 - \frac{E(t)}{K_E}\right)A(t),$$

where K_E is the availability of nutrients and space, b represents the rate at which the population would grow if they were unencumbered by environmental degradation,
- the transition rate from class E to class L is s but when the availability of food is not sufficient for the class L, then the larvae can eat the young larvae to complete its developement and we suppose that the death rate due to the lack of food is proportional to the young larvae sE and to the coefficent L/K_L that represent the availability of food for each larvae. At the end, the number of eggs that hatch and survive is given by,

$$s\left(1 - \frac{L(t)}{K_L}\right)E(t).$$

Then system (1) reads as follows,

$$\begin{cases} \frac{dE}{dt}(t) = b\left(1 - \frac{E(t)}{K_E}\right)A(t) - sE(t) - dE(t), \\ \frac{dL}{dt}(t) = s\left(1 - \frac{L(t)}{K_L}\right)E(t) - s_L L(t) - d_L L(t), \\ \frac{dA}{dt}(t) = s_L L(t) - d_m A(t). \end{cases} \tag{2}$$

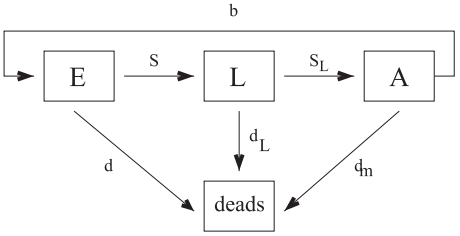

Fig. 1. A stage structured model for *Aedes albopictus* population dynamics. E states for eggs, L for larvae and pupae, A for female adult. s, s_L, b, d, d_L, d_m are nonnegative system parameters. In the diagram, b = eggs laying rate, s = E to L transfer rate, s_L = L to A transfer rates, d, d_L, d_m = mortality rates of eggs, larvae and adult population.

This system is mathematically well defined over the whole $\mathbb{R}^3$. Nevertheless, the region of biological interest is Δ which is given by,

$$\Delta = \left\{ \begin{matrix} 0 \leqslant E \leqslant K_E \\ (E,L,A) | \, 0 \leqslant L \leqslant K_L \\ 0 \leqslant A \leqslant \frac{s_L}{d_m} K_L \end{matrix} \right\}, \tag{3}$$

which its interior, denoted $int(\Delta)$, is given by

$$int(\Delta) = \left\{ \begin{matrix} 0 < E < K_E \\ (E,L,A) | \, 0 < L < K_L \\ 0 < A < \frac{s_L}{d_m} K_L \end{matrix} \right\}. \tag{4}$$

We will see in Lemma 4.3 that Δ is a positive invariant set for system (2).

3.2. A compartmental model for the virus transmission to human population

Let us denote by N_H the human population size for which we assume an exponential growth. Then, its dynamics is described by,

$$\frac{dN_H}{dt}(t) = (b_H - d_H)N_H(t), \tag{5}$$

where b_H and d_H are, respectively, the human birth and natural death rates.

Let $\overline{S}_H, \overline{I}_H$ and $\overline{R}_H$ denote the total number of respectively susceptible, infective, and immune in the human population and $\overline{S}_m, \overline{I}_m$ be the total number of susceptible and infective mosquitoes. The immune class in the vector population does not exist, since mosquitoes carry the infection throughout their life. The model is schematically represented in Fig. 2.

The effective contact rate β_H is the average number of contacts per day which would result in infection if the vector is infectious and, as in [15,34], we can assume that it is constant. The effective contact rate β_m is the average number of contacts per day that effectively transmit the infection to vectors. These hypotheses lead to the following equations,

$$\begin{cases} \frac{d\overline{S}_H}{dt}(t) = b_H(\overline{S}_H(t) + \overline{I}_H(t) + \overline{R}_H(t)) - \beta_H \frac{\overline{I}_m(t)}{A(t)} \overline{S}_H(t) - d_H \overline{S}_H(t), \\ \frac{d\overline{I}_H}{dt}(t) = \beta_H \frac{\overline{I}_m(t)}{A(t)} \overline{S}_H(t) - \gamma \overline{I}_H(t) - d_H \overline{I}_H(t), \\ \frac{d\overline{R}_H}{dt}(t) = \gamma \overline{I}_H(t) - d_H \overline{R}_H(t), \\ \frac{d\overline{S}_m}{dt}(t) = s_L L(t) - d_m \overline{S}_m(t) - \beta_m \frac{\overline{I}_H(t)}{N_H(t)} \overline{S}_m(t), \\ \frac{d\overline{I}_m}{dt}(t) = \beta_m \frac{\overline{I}_H(t)}{N_H(t)} \overline{S}_m(t) - d_m \overline{I}_m(t). \end{cases} \tag{6}$$

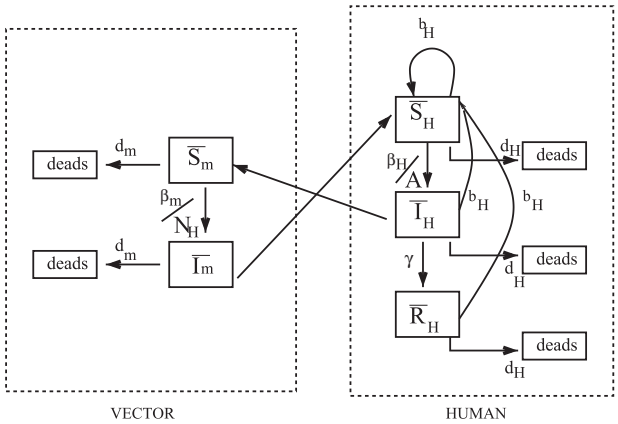

Fig. 2. A compartmental model for the chikungunya virus transmission with the nonnegative parameters: β_m = effective contact rate between susceptible vectors and humans, β_H = effective contact rate between susceptible humans and vectors, γ = recovery rate of infected humans, d_H = mortality rate of human population, d_m = mortality rate of vector population.

All parameters in this model are positive.

Introducing the proportions $S_H = \overline{S}_H/N_H, I_H = \overline{I}_H/N_H, R_H = \overline{R}_H/N_H, S_m = \overline{S}_m/A, I_m = \overline{I}_m/A$ in system (6) and by using relations $\overline{S}_H + \overline{I}_H + \overline{R}_H = N_H$ and $\overline{S}_m + \overline{I}_m = A$, the adult mosquito population is assumed to be described by the third equation of system (2), we obtain the following system that describes the dynamics of the proportion of individuals in each class, with the notation $u'(t) = \frac{du}{dt}(t)$,

$$\begin{cases} \begin{cases} E'(t) = bA(t)\left(1 - \frac{E(t)}{K_E}\right) - (s+d)E(t), \\ L'(t) = sE(t)\left(1 - \frac{L(t)}{K_L}\right) - (s_L + d_L)L(t), \\ A'(t) = s_L L(t) - d_m A(t), \end{cases} & (a) \\ \begin{cases} S'_H(t) = -(b_H + \beta_H I_m(t))S_H(t) + b_H, \\ I'_H(t) = \beta_H I_m(t) S_H(t) - (\gamma + b_H) I_H(t), \\ I'_m(t) = -\left(s_L \frac{L(t)}{A(t)} + \beta_m I_H(t)\right) I_m(t) + \beta_m I_H(t). \end{cases} & (b) \end{cases} \tag{7}$$

Remark 1. Obviously, this system has two different time scales, the one of mosquitoes of the order of weeks and the human lifespan of order of decades. Moreover, whether $b_H \neq d_H$ or $b_H = d_H$, when we consider the proportions, just above given, simple computations lead to the same system (7). Indeed, it suffices to note that $S'_H = (1/N_H^2)(\overline{S}'_H N_H - \overline{S}_H N'_H), I'_H = (1/N_H^2)(\overline{I}'_H N_H - \overline{I}_H N'_H)$ and $S_H + I_H + R_H = 1$, similarly $I'_m = (1/A^2)(\overline{I}'_m A - \overline{I}_m A')$ and $S_m + I_m = 1$, where N'_H is given by (5) and $\overline{S}'_H, \overline{I}'_H, \overline{I}_m$ are given in system (6).

This system, as we will see in Section 5.2, is defined on the bounded subset of $\mathbb{R}^6$, which is the region of biological interest, $\Delta \times \Omega$, where Δ is given by Eq. 3 and,

$$\Omega = \left\{ (S_H, I_H, I_m) \in \mathbb{R}^3_+ \middle| \begin{matrix} 0 \leqslant S_H + I_H \leqslant 1 \\ 0 \leqslant I_m \leqslant 1 \end{matrix} \right\} \tag{8}$$

and which its interior, denoted $int(\Omega)$, is given by

$$int(\Omega) = \left\{ (S_H, I_H, I_m) \in \mathbb{R}^3_+ \middle| \begin{matrix} 0 < S_H + I_H < 1 \\ 0 < I_m < 1 \end{matrix} \right\}. \tag{9}$$

Obviously, this model may be enhanced by taking into account the delay between the transfer to mosquitoes and the transmission to humans (from five to six days), see [35]. One can also use a SEI type model for the vector, see [12], although, if we consider a huge mosquito population, the number of mosquitoes in state E (exposed) can be neglected in comparison to the whole population.

4. Analysis of the population dynamics models

We investigate the asymptotic behavior of orbits starting in the non-negative cone,

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R} / x \geqslant 0, y \geqslant 0, z \geqslant 0\}.$$

Let us also denote,

$$\mathbb{R}^{*3}_+ = \{(x, y, z) \in \mathbb{R} / x > 0, y > 0, z > 0\}.$$

Obviously, system (2) which is a $\mathcal{C}^\infty$ differential system, admits a unique maximal solution for any associated Cauchy problem. We shall use the following threshold parameter,

$$r = \left(\frac{b}{s+d}\right)\left(\frac{s}{s_L + d_L}\right)\left(\frac{s_L}{d_m}\right), \tag{10}$$

which arises in an obvious manner, when computing equilibria. It is easy to prove the following result.

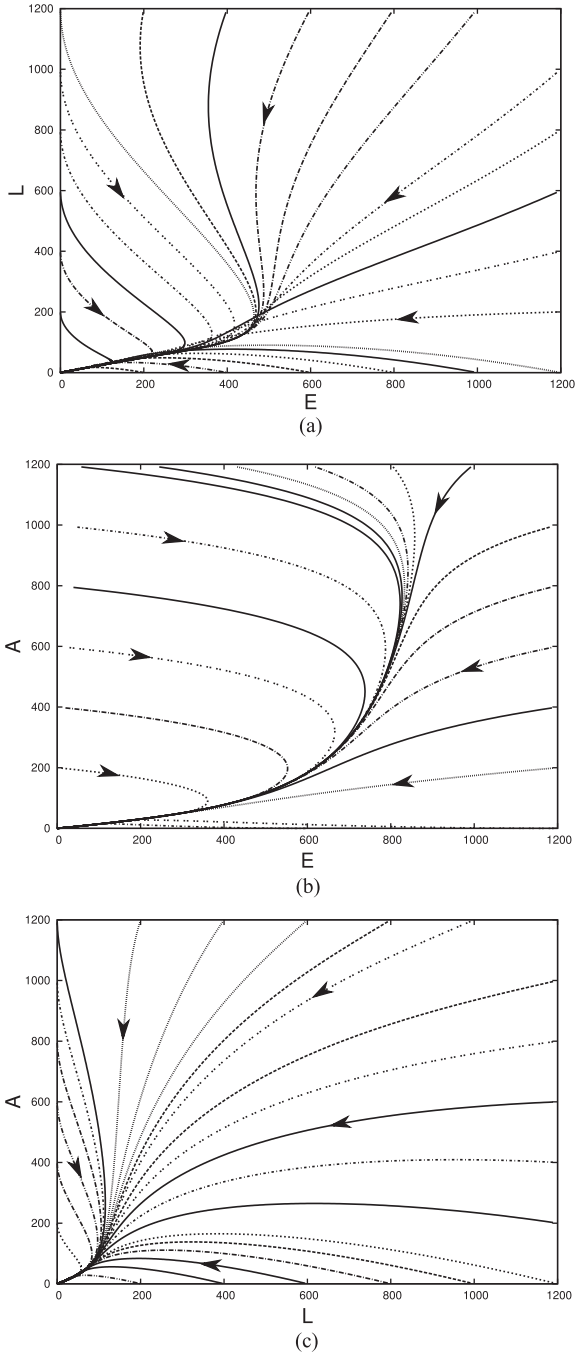

Fig. 3. Phase portraits of system (2) with parameters: $b = 5$, $s = 0.2$, $d = 0.6$, $K_E = 1000$, $s_L = 0.3$, $d_L = 0.6$, $K_L = 500$, $d_m = 0.7$. In this case $r = 0.595238$, then all trajectories tend to the mosquito-free equilibrium X_0^*.

Proposition 4.1. *System (2) always has the mosquito-free equilibrium* $X_0^* = (0, 0, 0)$.

- *If* $r \leqslant 1$, *then system (2) has no other equilibrium.*
- *If* $r > 1$, *there is an unique endemic equilibrium,*

$$X^* = \left(1 - \frac{1}{r}\right)\left(\frac{K_E}{\gamma_E}, \frac{K_L}{\gamma_L}, \frac{s_L}{d_m}\frac{K_L}{\gamma_L}\right) = (E^*, L^*, A^*),$$

where,

$$\gamma_E = 1 + \frac{(s+d)d_m K_E}{b s_L K_L} \quad \text{and} \quad \gamma_L = 1 + \frac{(s_L + d_L)K_L}{s K_E}.$$

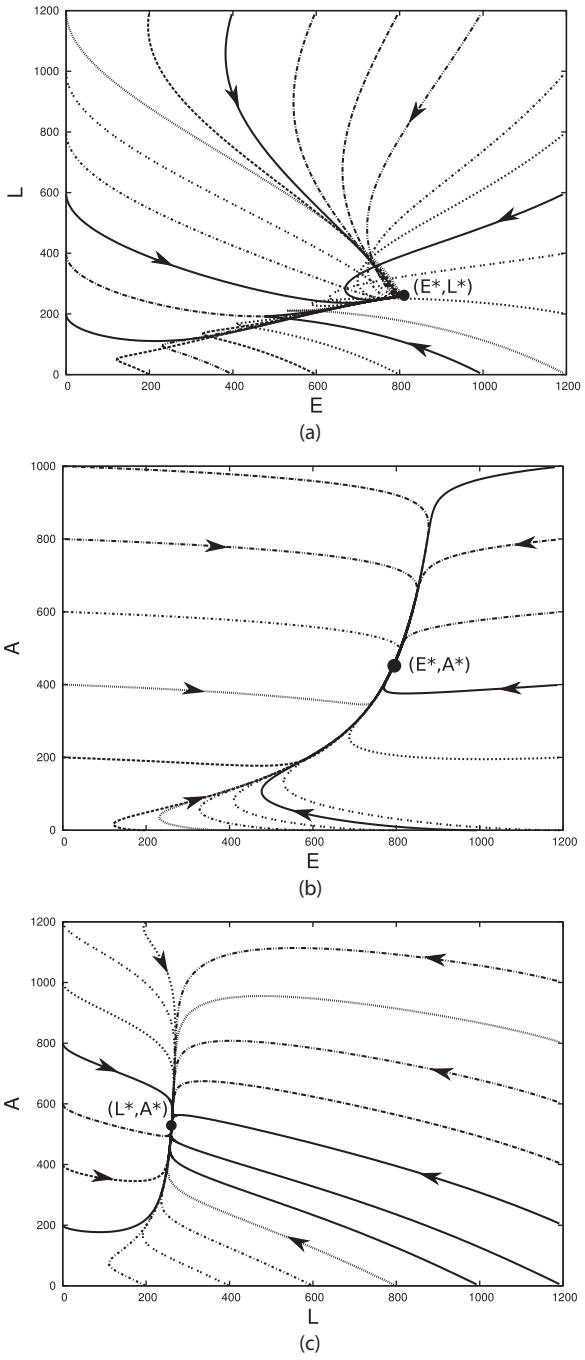

Fig. 4. Phase portraits of system (2) with parameters: $b = 6$, $s = 0.5$, $d = 0.2$, $K_E = 1000$, $s_L = 0.5$, $d_L = 0.25$, $K_L = 500$, $d_m = 0.25$. In this case $r = 11.428571$, then all trajectories tend to the endemic equilibrium X^*.

4.1. Non-negativity and boundedness of solutions

Lemma 4.2. *Let $(t_0, X_0 = (E_0, L_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}^3_+$ and $([t_0, T[, X = (E, L, A))(T \in]t_0, +\infty])$ be the maximal solution of the Cauchy problem associated to (2) with the initial condition (t_0, X_0). Then,*

$$\forall t \geqslant t_0, X(t) \in \mathbb{R}^3_+.$$

Proof. Let $(t_0, X_0 = (E_0, L_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}^3_+$ and $([t_0, T[, X = (E, L, A))$ be the maximal solution of the Cauchy problem associated to (2) with the initial condition (t_0, X_0). Let us assume that this solution becomes negative, i.e. $\tilde{t}_1 > t_0$ exists such that $X(\tilde{t}_1) \notin \mathbb{R}^3_+$. Let us define, $t_1 = \inf\{t, X(t) \notin \mathbb{R}^3_+\}$, i.e.

$$t_0 \leqslant t < t_1, X(t) \in \mathbb{R}^3_+$$

and $\varepsilon > 0$ exists such that

$$\forall t_1 < t \leqslant t_1 + \varepsilon, X(t) \notin \mathbb{R}^3_+. \tag{11}$$

Since $N_0^* = (0,0,0)$ is an equilibrium, the uniqueness of the solutions implies $X(t_1) \neq (0,0,0)$. For $t = t_1$, six cases are possible.

1. $X(t_1) = (0, L(t_1), A(t_1))$ with $(L(t_1), A(t_1)) \in (\mathbb{R}_+^*)^2$. Then, taking t_1 as an initial time, $E'(t_1) = bA(t_1) > 0$. Since,
$$\begin{aligned} E(t) &= E'(t_1)(t - t_1) + \underset{t \to t_1}{\circ}(t - t_1) \\ &= bA(t_1)(t - t_1) + \underset{t \to t_1}{\circ}(t - t_1), \end{aligned}$$
thus, $\tilde{\varepsilon} > 0$ exists such that, $\forall t_1 < t \leqslant t_1 + \tilde{\varepsilon}$ we have $E(t) > 0$. Besides $L(t_1)$ and $A(t_1)$ are positive for all $t \in [t_1, t_1 + \tilde{\tilde{\varepsilon}}]$, therefore $\forall t \in [t_1, t_1 + min\{\tilde{\varepsilon}, \tilde{\tilde{\varepsilon}}\}]$,
$$X(t) \in \mathbb{R}^3_+,$$
which is a contradiction.
2. Let $X(t_1) = (0,0,A(t_1))$ with $A(t_1) > 0$. We can show as above that $\tilde{\varepsilon} > 0$ exists such that $\forall t_1 < t \leqslant t_1 + \tilde{\varepsilon}, E(t) > 0$. Now from the second equation of (2), since $L(t_1) = 0$, $L'(t_1) = 0$ and $L''(t_1) = sE'(t_1) = sbA(t_1) > 0$, then
$$\begin{aligned} L(t) &= L''(t_1)\frac{(t - t_1)^2}{2} + \underset{t \to t_1}{\circ}((t - t_1)^2) \\ &= sbA(t_1)\frac{(t - t_1)^2}{2} + \underset{t \to t_1}{\circ}((t - t_1)^2), \end{aligned}$$
thus, $\tilde{\tilde{\varepsilon}} > 0$ exists such that $\forall t_1 < t \leqslant t_1 + \tilde{\tilde{\varepsilon}}$, we have $L(t) > 0$. Besides $A(t_1) > 0$, is positive forall $t \in [t_1, t_1 + \tilde{\tilde{\varepsilon}}]$, therefore $\forall t \in [t_1, t_1 + min\{\tilde{\varepsilon}, \tilde{\tilde{\varepsilon}}\}]$
$$X(t) \in \mathbb{R}^3_+,$$
which is a contradiction.

Similar proof can easily be done for the other cases that are $X(t) = (E(t_1), 0, A(t_1))$ or $X(t) = (E(t_1), L(t_1), 0)$ or $X(t) = (E(t_1), 0, 0)$ or $X(t) = (0, L(t_1), 0)$. □

Lemma 4.3. *The set*

$$\Delta = \left\{ (E,L,A) \middle| \begin{array}{c} 0 \leqslant E \leqslant K_E \\ 0 \leqslant L \leqslant K_L \\ 0 \leqslant A \leqslant \frac{s_L}{d_m} K_L \end{array} \right\}$$

is an invariant region under the flow induced by (2).

Proof. Let $(t_0, X_0 = (E_0, L_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}^3_+$ and $([t_0, T, [X = (E, L, A))$ be the maximal solution of the Cauchy problem associated to system (2) with the initial condition $(t_0, X_0), T \in]t_0, +\infty]$. Let $t_1 \in [t_0, T[$. We only have to show that,

1. if $E(t_1) \leqslant K_E$ then $\forall\ t_1 \leqslant t < T$, $E(t) \leqslant K_E$
2. if $L(t_1) \leqslant K_L$ then $\forall\ t_1 \leqslant t < T, L(t) \leqslant K_L$
3. if $A(t_1) \leqslant \frac{s_L}{d_m} K_L$ then $\forall t_1 \leqslant t < T, A(t) \leqslant \frac{s_L}{d_m} K_L$

since we have already shown that solutions are nonnegative, Lemma 4.2.

1. Assume that $\varepsilon_1 > 0$ exists such that $E(t_1 + \varepsilon_1) > K_E$. Let,
$$t_1^* = \inf\{t \geqslant t_1 |\ E(t) > K_E\}.$$

Since, $E(t_1^*) = K_E$, then,

$$E(t) = K_E + E'(t_1^*)(t - t_1^*) + \underset{t \to t_1^*}{o}(t - t_1^*).$$

Moreover, from the first equation of system (2), $E\prime(t_1^*) = -(s+d)K_E < 0$, then there exists $\tilde{\varepsilon} > 0$ such that $\forall t_1^* \leqslant t < t_1^* + \tilde{\varepsilon}, E(t) < K_E$ which is a contradiction. As a result, $\forall t \in [t_0, T[, E(t) \leqslant K_E$.

2. Assume that $\varepsilon_1 > 0$ exists such that $L(t_1 + \varepsilon_1) > K_L$. Let,

$$t_1^* = \inf\{t \geqslant t_1 |\ L(t) > K_L\}.$$

Since $L(t_1^*) = K_L$, then,

$$L(t) = K_L + L'(t_1^*)(t - t_1^*) + \underset{t \to t_1^*}{o}(t - t_1^*).$$

Moreover from the second equation of system (2), $L'(t_1^*) = -(s_L + d_L)K_L < 0, \tilde{\varepsilon} > 0$ exists such that $\forall t_1^* \leqslant t < t_1^* + \tilde{\varepsilon}, L(t) < K_L$ which is a contradiction. As a result, $\forall t \in [t_0, T[, L(t) \leqslant K_L$.

3. Assume that $\varepsilon_1 > 0$ exists such that $A(t_1 + \varepsilon_1) > \frac{s_L}{d_m} K_L$. Let,

$$t_1^* = \inf\{t \geqslant t_1 |\ A(t) > \frac{s_L}{d_m} K_L\}.$$

Since $A(t_1^*) = \frac{s_L}{d_m} K_L$, then,

- If $L(t_1^*) < K_L$, then, $A'(t_1^*) < 0$.

$$A(t) = \frac{s_L}{d_m} K_L + A'(t_1^*)(t - t_1^*) + \underset{t \to t_1^*}{o}(t - t_1^*)$$

with

$$A'(t_1^*) = s_L\big(L(t_1^*) - K_L\big).$$

- If $L(t_1^*) = K_L$ then, $A'(t_1^*) = 0$ and

$$A(t) = \frac{s_L}{d_m} K_L + A''(t_1^*)\frac{(t - t_1^*)^2}{2} + \underset{t \to t_1^*}{o}\left((t - t_1^*)^2\right)$$

with $A''(t_1^*) = s_L L'(t_1^*) = -s_L(s_L + d_L)K_L < 0$.

In both cases, there exists $\tilde{\varepsilon} > 0$ such that $\forall t_1^* < t \leqslant t_1^* + \tilde{\varepsilon}, A(t) < \frac{s_L}{d_m} K_L$ which is a contradiction. As a result $\forall t \in [t_0, T[, A(t) \leqslant \frac{s_L}{d_m} K_L$. To conclude, Δ is invariant under the flow induced by (2). □

Proposition 4.4. *All non-negative solutions (i.e. solutions initiating in $\mathbb{R}^3_+$) eventually enter the set Δ.*

Proof. Let $(t_0, X_0 = (E_0, L_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}^3_+$ such that $(E_0, L_0, A_0) \notin \Delta$ (since Δ is invariant) and $([t_0, T[, X = (E, L, A))$ be the maximal solution of the Cauchy problem associated to system (2) with the initial condition (t_0, X_0).

We know that Δ is an invariant region (Lemma 4.3).

It is then sufficient to show that there exists $t \geqslant t_0$ such that $X(t) \in \Delta$.

- Assume that for all $t \in [t_0, +\infty[, E(t) > K_E$. Then, due to the first equation of system (2), for all $t \in [t_0, +\infty[, E'(t) < -(s+d)K_E$. Then, by comparison, $\forall t \in [t_0, +\infty[$, we have,

$$E(t) \leqslant E_0 - (s+d)K_E(t - t_0).$$

For $t_1 = t_0 + \frac{E_0 - K_E}{(s+d)K_E}$, we obtain $E(t) \leqslant K_E$ which is a contradiction. Therefore, for all $t > t_1, E(t) \leqslant K_E$.

- If $L(t_1) \leqslant K_L$, then the solution $L(t)$ belongs to Δ which is invariant. Otherwise, let assume on the contrary that for all $t \in [t_1, +\infty[$, t_1 given above, $L(t) > K_L$. Then $\forall t \in [t_1, +\infty[$, due to the second equation of system (2), $L'(t) < -(s_L + d_L)K_L$. Then, by comparison $\forall t \in [t_1, +\infty[$ we have,

$$L(t) \leqslant L(t_1) - (s_L + d_L)K_L(t - t_1).$$

For $t_2 = t_1 + \frac{L(t_1) - K_L}{(s_L + d_L)K_L}$, we obtain $L(t_2) \leqslant K_L$ which is a contradiction. Thus, there exists $t_2 > t_1$ such that $L(t_2) \leqslant K_L$.

$$L(t) \leqslant K_L.$$

- If $A(t_2) \leqslant \frac{s_L}{d_m} K_L$, the solution $A(t)$ is within Δ which is invariant. Otherwise, let assume on the contrary that for all $t \in [t_2, +\infty[, A(t) > \frac{s_L}{d_m} K_L$. Then, due to the third equation of system (2), $\forall\ t \in [t_2, +\infty[$,

$$A'(t) < s_L(L(t) - K_L) < 0.$$

Then, there exists $c > 0$ such that $A'(t) \leqslant c$, since $L(t)$ is now bounded. By comparison, we have $\forall\ t \in [t_2, T[$,

$$A(t) \leqslant A(t_2) - c(t - t_2).$$

For $t_3 = t_2 + \frac{A(t_2) - \frac{s_L}{d_m} K_L}{c}$, we have $A(t) \leqslant \frac{s_L}{d_m} K_L$ which is a contradiction. To conclude, for $t \geqslant max(t_1, t_2, t_3)$, $(E(t), L(t), A(t)) \in \Delta$. □

4.2. Stability of the equilibria

Proposition 4.5. *The mosquito-free equilibrium $X_0^* = (0, 0, 0)$ is locally asymptotically stable iff $r < 1$.*

Proof. The local stability of the mosquito-free equilibrium X_0^* is given by the Jacobian matrix of the system (2) evaluated at this point, $DF(X_0^*)$,

$$D_F(X_0^*) = \begin{pmatrix} -(s+d) & 0 & b \\ s & -(s_L + d_L) & 0 \\ 0 & s_L & -d_m \end{pmatrix}. \tag{12}$$

The characteristic equation of (12) is given by,

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3,$$

where,

$$\alpha_1 = (s+d) + (s_L + d_L) + d_m,$$
$$\alpha_2 = (s+d)(s_L + d_L) + (s+d)d_m + (s_L + d_L)d_m,$$
$$\alpha_3 = d_m(s+d)(s_L + d_L)(1 - r).$$

We apply the Routh-Hurwitz criterion. Clearly $\alpha_1 > 0$, $\alpha_2 > 0$ and $D_1 = \alpha_1\alpha_2 - \alpha_3 > 0$ since,

$$\begin{aligned} D_1 &= \alpha_1\alpha_2 - \alpha_3 \\ &= ((s+d) + (s_L + d_L) + d_m)(s+d)((s_L + d_L) + d_m) \\ &\quad + d_m(s_L + d_L)(r(s+d) + (s_L + d_L) + d_m) > 0. \end{aligned}$$

If $r < 1$ then $\alpha_3 > 0$, thus using the Routh-Hurwitz criterion all eigenvalues of $D_F(X_0^*)$ have negative real part, thus X_0^* is locally asymptotically stable for (2). If $r \geqslant 1$ then $\alpha_3 < 0$ and we show that $D_F(X_0^*)$ has at least one eigenvalue with non-negative real part consequently X_0^* is not asymptotically stable. □

Remark 2. Moreover, we can easily prove that X_0^* is globally asymptotically stable for $r \leqslant 1$ using quadratic Lyapunov function (the proof is similar to the one given below in Proposition 4.7 for X^*, see also Fig. 3).

Proposition 4.6. *If $r > 1, X^*$ is locally asymptotically stable.*

Proof. The local stability of the endemic equilibrium X^* is given by the Jacobian matrix of the system (2) evaluated at this point,

$$D_F(X^*) = \begin{pmatrix} -d_1 & 0 & d_2 \\ d_3 & -d_4 & 0 \\ 0 & d_5 & -d_6 \end{pmatrix} \tag{13}$$

with

$$d_1 = \frac{bs_L}{d_m\gamma_L}\frac{K_L}{K_E}\left(1-\frac{1}{r}\right)+(s+d),$$
$$d_2 = (s+d)\frac{d_m\gamma_L K_E}{s_L\gamma_E K_L},$$
$$d_3 = \frac{(s_L+d_L)\gamma_E K_L}{\gamma_L K_E},$$
$$d_4 = \frac{sK_E}{\gamma_E K_L}\left(1-\frac{1}{r}\right)+(s_L+d_L),$$
$$d_5 = s_L,$$
$$d_6 = d_m.$$

The characteristic equation of (12) is given by,

$$\chi_{X^*}(\lambda) = \lambda^3+\alpha_1\lambda^2+\alpha_2\lambda+\alpha_3,$$

where

$$\begin{aligned}\alpha_1 &= d_1+d_4+d_6,\\ \alpha_2 &= d_1d_4+d_1d_6+d_6d_4,\\ \alpha_3 &= d_1d_4d_6-d_2d_3d_5\\ &= d_m\left(1-\frac{1}{r}\right)\left(\frac{bss_L}{d_m\gamma_L\gamma_E}\left(1-\frac{1}{r}\right)+(s+d)\frac{sK_E}{\gamma_E K_L}\right.\\ &\quad\left.+(s_L+d_L)\frac{bs_L}{d_m\gamma_L}\frac{K_L}{K_E}\right).\end{aligned}$$

If $r>1$, then $\alpha_1>0$,$\alpha_2>0$ and $\alpha_3>0$ and,

$$\begin{aligned}D_1 &= \alpha_1\alpha_2-\alpha_3\\ &= \alpha_1\times\left(\frac{bs_L}{d_m\gamma_L}\frac{K_L}{K_E}\left(1-\frac{1}{r}\right)+(s+d)\right)\times\left(\frac{sK_E}{\gamma_E K_L}\left(1-\frac{1}{r}\right)\right.\\ &\quad+(s_L+d_L)+d_m)+d_m\left(\frac{sK_E}{\gamma_E K_L}\left(1-\frac{1}{r}\right)\right)\left(\frac{sK_E}{\gamma_E K_L}\left(1-\frac{1}{r}\right)\right.\\ &\quad+(s_L+d_L)+d_m)+d_m(s_L+d_L)\left(\frac{sK_E}{\gamma_E K_L}\left(1-\frac{1}{r}\right)\right.\\ &\quad\left.\left.+(s+d)+(s_L+d_L)+d_m\right)>0.\right.\end{aligned}$$

Then, thanks to the Routh-Hurwitz criterion all eigenvalues of $D_F(X^*)$ have negative real part. Consequently X^* is locally asymptotically stable (see also Fig. 4). □

Proposition 4.7. *If $r>1$ the endemic equilibrium X^* is globally asymptotically stable in int(Δ) (Δ is given by* Lemma 4.3*).*

Proof. Assume $r>1$. Let $X^*(E^*,L^*,A^*)=(x^*,y^*,z^*)$. To prove the global stability of X^*, we use the Lyapunov function $V_1:\mathbb{R}^3\to\mathbb{R}$ defined by,

$$V_1(x,y,z) = \frac{1}{2}\left(a_1(x-x^*)^2+a_2(y-y^*)^2+a_3(z-z^*)^2\right),$$

where $a=(a_1,a_2,a_3)^T\in(\mathbb{R}_+^*)^3$ is a positive constant vector. Note that since $r>1$, then x^*,y^* and z^* are positive. We have,

$$V_1(X^*)=0 \quad\text{and}\quad \forall(x,y,z)\in\mathbb{R}_3+\setminus\{X^*\}, V_1(x,y,z)>0.$$

Hence, V_1 is well defined. The orbital derivative, that is the derivative of V_1 along solutions of system (2) is,

$$\begin{aligned}\dot V_1(x,y,z) &= a_1(x-x^*)\left(bz\left(1-\frac{x}{K_E}\right)-(s+d)x\right)\\ &\quad+a_2(y-y^*)\left(sx\left(1-\frac{y}{K_L}\right)-(s_L+d_L)y\right)\\ &\quad+a_3(z-z^*)(s_Ly-d_mz).\end{aligned}\tag{14}$$

Let $\tilde x=x-x^*,\tilde y=y-y^*,\tilde z=z-z^*$ and $\tilde X=(\tilde x,\tilde y,\tilde z)^T$.

Then

$$\dot V_1(x,y,z) = \tilde X^T\begin{pmatrix}-a_1(s+d) & 0 & a_1b\left(1-\frac{x^*}{K_E}\right)\\ a_2s\left(1-\frac{y^*}{K_L}\right) & -a_2(s_L+d_L) & 0\\ 0 & a_3s_L & -a_3d_m\end{pmatrix}\tilde X - \frac{a_1b}{K_E}\tilde x^2z-\frac{a_2s}{K_L}\tilde y^2x.$$

Let $A_1 = -D + R_1$ with,

$$D=\begin{pmatrix}a_1(s+d) & 0 & 0\\ 0 & a_2(s_L+d_L) & 0\\ 0 & 0 & a_3d_m\end{pmatrix}$$

and

$$R_1=\begin{pmatrix}0 & 0 & a_1b\left(1-\frac{x^*}{K_E}\right)\\ a_2s\left(1-\frac{y^*}{K_L}\right) & 0 & 0\\ 0 & a_3s_L & 0\end{pmatrix}.$$

Let us denote by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^3$. Then the orbital derivative reads as,

$$\dot V_1(x,y,z) = \left\langle A_1\tilde X,\tilde X\right\rangle - \frac{a_1b}{K_E}\tilde x^2z-\frac{a_2s}{K_L}\tilde y^2x.$$

The *symetric* matrix S_1 defined by,

$$S_1 = -D+\frac{1}{2}(R_1^T+R_1)$$

is given using simple algebraic computations by

$$S_1=\begin{pmatrix}-a_1(s+d) & \frac{a_2}{2}(s_L+d_L)\frac{y^*}{x^*} & \frac{a_1}{2}(s+d)\frac{x^*}{z^*}\\ \frac{a_2}{2}(s_L+d_L)\frac{y^*}{x^*} & -a_2(s_L+d_L) & \frac{a_3s_L}{2}\\ \frac{a_1}{2}(s+d)\frac{x^*}{z^*} & \frac{a_3s_L}{2} & -a_3d_m\end{pmatrix}.$$

Therefore, we have,

$$\left\langle A_1\tilde X,\tilde X\right\rangle = \left\langle S_1\tilde X,\tilde X\right\rangle.$$

The characteristic polynomial of S_1 is,

$$\chi_{S_1}(\lambda) = \lambda^3+\alpha_1\lambda^2+\alpha_2\lambda+\alpha_3,$$

where,

$$\alpha_1 = a_1(s+d)+a_2(s_L+d_L)+a_3d_m,$$

$$\begin{aligned}\alpha_2 &= \frac{1}{4}a_1(s+d)\beta_1+\frac{1}{4}a_2(s_L+d_L)\beta_2+\frac{1}{4}a_3\beta_3+\frac{3}{4}(a_1a_3d_m(s+d)\\ &\quad+a_1a_2(s+d)(s_L+d_L)+a_2a_3d_m(s_L+d_L))\end{aligned}$$

with

$$\beta_1 = \left(a_3d_m-a_1(s+d)\left(\frac{x^*}{z^*}\right)^2\right),$$
$$\beta_2 = \left(a_1(s+d)-a_2(s_L+d_L)\left(\frac{y^*}{x^*}\right)^2\right),$$
$$\beta_3 = (a_2d_m(s_L+d_L)-a_3s_L^2)$$

and

$$\alpha_3 = \frac{1}{4}(a_1a_2(s+d)(s_L+d_L)\beta_1+a_2a_3d_m(s_L+d_L)\beta_2+a_1a_3(s+d)\beta_3).$$

Let us choose a_1, a_2, and a_3 satisfying,

$$a_1 = \frac{1}{s+d}\left(\frac{y^*}{x^*}\right)^2 > 0,$$
$$a_2 = \frac{s+d}{s_L+d_L}\left(\frac{x^*}{y^*}\right)^2 a_1,$$
$$a_3 = \frac{d_m(s+d)}{s_L^2}\left(\frac{x^*}{y^*}\right)^2 a_1 = a_1\frac{s+d}{d_m}\left(\frac{x^*}{z^*}\right)^2$$

with such a choice, one can easily verify that $\beta_1 = \beta_2 = \beta_3 = 0$. Thus $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_3 = 0$, therefore the characteristic polynomial reads as,

$$\chi_{S_1}(\lambda) = \lambda(\lambda^2 + \alpha_1\lambda + \alpha_2).$$

Then,

$$S_1 = \begin{pmatrix} -\left(\frac{y^*}{x^*}\right)^2 & \frac{y^*}{2x^*} & \frac{d_m y^*}{2s_L x^*} \\ \frac{y^*}{2x^*} & -1 & \frac{d_m}{2s_L} \\ \frac{d_m y^*}{2s_L x^*} & \frac{d_m}{2s_L} & -\frac{d_m^2}{s_L^2}, \end{pmatrix}$$

since S_1 has one zero eigenvalue and two negatives eigenvalues. The matrix S_1 satisfies $\forall \tilde{X} \in \mathbb{R}^3_+$,

$$\langle S_1\tilde{X}, \tilde{X}\rangle \leqslant 0.$$

Note that if $\tilde{X} \notin \mathrm{Ker}(S_1)$ then $\langle S_1\tilde{X}, \tilde{X}\rangle < 0$.

Then $\forall (x,y,z) \in \Delta\backslash\{X^*\}$,

$$\tilde{X} \notin \mathrm{Ker}(S_1) \Rightarrow \dot{V}_1(x,y,z) = \left\langle A_1\tilde{X}, \tilde{X}\right\rangle - \left(\frac{a_1 b}{K_E}\tilde{x}^2 z + \frac{a_2 s}{K_L}\tilde{y}^2 x\right)$$
$$\leqslant \left\langle A_1\tilde{X}, \tilde{X}\right\rangle < 0.$$

If $\tilde{X} \in \mathrm{Ker}(S_1)$, it is sufficient to verify that

$$\left(\frac{a_1 b}{K_E}\tilde{x}^2 z + \frac{a_2 s}{K_L}\tilde{y}^2 x\right) > 0.$$

Note that

$$\left(\frac{a_1 b}{K_E}\tilde{x}^2 z + \frac{a_2 s}{K_L}\tilde{y}^2 x\right) = 0$$
$$\iff \begin{cases} x = 0 \quad \text{and} \quad z = 0 \\ \text{or} \quad x = x^* \quad \text{and} \quad y = y^* \\ \text{or} \quad z = 0 \quad \text{and} \quad y = y^* \end{cases}$$
$$\iff \begin{cases} \tilde{x} = -x^* \quad \text{and} \quad \tilde{z} = -z^* \\ \text{or} \quad \tilde{x} = 0 \quad \text{and} \quad \tilde{y} = 0 \\ \text{or} \quad \tilde{z} = -z^* \quad \text{and} \quad \tilde{y} = 0. \end{cases}$$

Let

$$a_1 = \frac{(s_L+d_L)^2}{s+d}\left(\frac{y^*}{x^*}\right)^2,$$
$$a_2 = s_L + d_L,$$
$$a_3 = \left(\frac{s_L+d_L}{s_L}\right)^2 d_m,$$

we have

$$S_1 = (s_L+d_L)^2\begin{pmatrix} -\left(\frac{y^*}{x^*}\right)^2 & \frac{y^*}{2x^*} & \frac{d_m y^*}{2s_L x^*} \\ \frac{y^*}{2x^*} & -1 & \frac{d_m}{2s_L} \\ \frac{d_m y^*}{2s_L x^*} & \frac{d_m}{2s_L} & -\frac{d_m^2}{s_L^2} \end{pmatrix}$$

and then

$$\mathrm{Ker}(S_1) = \left\{ z\begin{pmatrix} \frac{x^*}{z^*} \\ \frac{y^*}{z^*} \\ 1 \end{pmatrix}, z \in \mathbb{R} \right\}.$$

Finally, we easily see that

$$\left\{ (x,y,z) \in \mathbb{R}^3 \middle/ \begin{array}{l} (\tilde{x},\tilde{y},\tilde{z}) \in \mathrm{Ker}(S_1) \\ \text{and } \left(\frac{a_1 b}{K_E}\tilde{x}^2 z + \frac{a_2 s}{K_L}\tilde{y}^2 x\right) = 0 \end{array} \right\} = \{X_0^*; X^*\}.$$

Consequently $\forall (x,y,z) \in \Delta\backslash\{X^*\}$,

$$\dot{V}_1(x,y,z) < 0,$$

i.e. V_1 is a strict Lyapunov function and X^* is globally asymptotically stable in Δ. □

5. Dynamics analysis of the virus transmission model

This section addresses the existence and global stability of equilibrium points of (7) by showing the persistence of the system and using the theory of competitive systems. We will focus on the case $r > 1$,r given by (10) which is the condition of survival of all populations as we studied in the previous section.

For this aim, we shall use the following reproduction number [7,8], which is defined as the average number of secondary infections produced by an infected individual in a completely susceptible population

$$R_0 = \frac{\beta_m\beta_H}{d_m(\gamma + b_H)} = \frac{\beta_m\beta_H}{s_L\frac{L^*}{A^*}(\gamma + b_H)}, \tag{15}$$

which arises by computing the steady state.

5.1. Existence of equilibria

Proposition 5.1. *We assume that $r > 1$. System (7) always has the disease free equilibrium $N_0^* = (E^*, L^*, A^*, 1, 0, 0)$. Moreover, if $R_0 > 1$, it has an unique endemic equilibrium with disease $N^* = (E^*, L^*, A^*, S_H^*, I_H^*, I_m^*)$ defined on $\Delta \times \Omega$, the last are done by (3) and (8), where*

$$S_H^* = \frac{b_H}{\beta_H + b_H} + \frac{\beta_H}{(\beta_H + b_H)R_0},$$
$$I_H^* = \frac{d_m b_H}{\beta_m(\beta_H + b_H)}(R_0 - 1), \tag{16}$$
$$I_m^* = \frac{b_H}{\beta_H + b_H R_0}(R_0 - 1)$$

and (E^*, L^*, A^*) is the endemic equilibrium of the independent system (2) given by Proposition 4.1.

Proof. Obviously, as $R_0 > 1$, both equilibria are non-negative. Besides, one can easily see that $(1,0,0)$ is an equilibrium of the subsystem of (7) given by the three last equations, then it is clear that N_0^* is an equilibrium of (7), belonging to $\Delta \times \Omega$.

It is also easy to check that N^* is an equilibrium of (7), thus we only have to show that (S_H^*, I_H^*, I_m^*) belongs to Ω, since we know that (E^*, L^*, A^*) is in Δ.

Since $R_0 > 1$, then,

$$I_m^* = \frac{R_0 - 1}{R_0 + \frac{\beta_H}{b_H}} \leqslant 1$$

and

$$I_H^* = \frac{R_0 - 1}{R_0}\frac{1}{\left(1 + \frac{\gamma}{b_H}\right)\left(1 + \frac{b_H}{\beta_H}\right)} \leqslant 1.$$

Moreover,

$$S_H^* + I_H^* = \frac{b_H}{\beta_H + b_H} + \frac{\beta_H}{(\beta_H + b_H)R_0} + \frac{R_0 - 1}{R_0}\frac{1}{\left(1+\frac{\gamma}{b_H}\right)\left(1+\frac{b_H}{\beta_H}\right)}$$
$$= \frac{1}{\beta_H + b_H}\left(b_H\left(1+\frac{\beta_H}{b_H+\gamma}\right) + \frac{\beta_H}{R_0}\left(1-\frac{b_H}{b_H+\gamma}\right)\right),$$

which is less than 1, since $R_0 > 1$ and hence,

$$b_H + \frac{\gamma}{R_0} < b_H + \gamma \iff \frac{b_H}{b_H+\gamma} + \frac{1}{R_0}\left(1-\frac{b_H}{b_H+\gamma}\right) < 1$$
$$\iff \frac{b_H\beta_H}{b_H+\gamma} + \frac{\beta_H}{R_0}\left(1-\frac{b_H}{b_H+\gamma}\right) < \beta_H$$
$$\iff b_H\left(1+\frac{\beta_H}{b_H+\gamma}\right) + \frac{\beta_H}{R_0}\left(1-\frac{b_H}{b_H+\gamma}\right) < \beta_H + b_H$$
$$\iff \frac{1}{\beta_H+b_H}\left(b_H\left(1+\frac{\beta_H}{b_H+\gamma}\right) + \frac{\beta_H}{R_0}\left(1-\frac{b_H}{b_H+\gamma}\right)\right) < 1. \quad \square$$

Proposition 5.2. *The equilibrium $N_0^* = (E^*, L^*, A^*, 1, 0, 0)$ is globally asymptotically stable in Ω iff $R_0 \leqslant 1$.*

Proof. Similar to proof of Proposition 4.7, using the Lyapunov function $V_1 : \mathbb{R}^6 \to \mathbb{R}$ defined by,

$$V_1(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{1}{2}\left(a_1(x-E^*)^2 + a_2(y-L^*)^2 + a_3(z-A^*)^2\right)$$
$$+ \frac{1}{2}(a_4(x_4-1)^2 + a_5x_5^2 + a_6x_6^2),$$

where,

$$a_1 = \frac{(s_L+d_L)^2}{s+d}\left(\frac{L^*}{E^*}\right)^2,\ a_2 = s_L + d_L,$$
$$a_3 = \left(\frac{s_L+d_L}{s_L}\right)^2 d_m,\ a_4 = \frac{b_H}{(\gamma+b_H)},$$
$$a_5 = \frac{\beta_H^2}{d_m(\gamma+b_H)},\ a_6 = R_0^2. \quad \square$$

System (7) is the coupling of the two subsystems (7a) and (7b), for which the coupling term is the function $s_L\frac{L(t)}{A(t)}I_m(t)$, that is system (7a) drives system (7b). Therefore, since the previous section was devoted to the study of the subsystem (7a) corresponding to the population dynamics we only have to analyze the subsystem (7b),

$$\begin{cases} S_H'(t) = -(b_H + \beta_H I_m(t))S_H(t) + b_H, \\ I_H'(t) = \beta_H I_m(t)S_H(t) - (\gamma + b_H)I_H(t), \\ I_m'(t) = -\left(s_L\frac{L(t)}{A(t)} + \beta_m I_H(t)\right)I_m(t) + \beta_m I_H(t). \end{cases}$$

5.2. Global stability of the endemic equilibrium with disease

First of all, for the reader convenience, let us recall some useful preliminaries, see [36] and [37] in which a similar analysis has been done.

Definition 5.3. Consider the following systems,

$$x' = f(t,x), \tag{17}$$

$$y' = g(y), \tag{18}$$

where f and g are continuous and locally Lipschitz in $x \in \mathbb{R}^n$, thus the solutions exist for all positive time. System (17) is called asymptotically autonomous with limit system (18) if $f(t,x) \to g(x)$ as $t \to \infty$ uniformly for $x \in \mathbb{R}^n$.

Lemma 5.4 [36]. *Let e be a locally asymptotically stable equilibrium of (18) and ω be the ω-limit set of a forward bounded solution $x(t)$ of (17). If ω contains a point y_0 such that the solution of (18), with $y(0) = y_0$ converges to e as $t \to \infty$, then $\omega = e$, i.e., $x(t) \to e$ as $t \to \infty$.*

Corollary 1 [36]. *If the solutions of system (17) are bounded and the equilibrium e of the limit system (18) is globally asymptotically stable, then every solution $x(t)$ of the system (17) satisfies $x(t) \to e$ as $t \to \infty$.*

Let us apply this result to our subsystem (7b).

Since $X^* = (E^*, L^*, A^*)$, the endemic equilibrium of subsystem (7a), is globally asymptotically stable for $r > 1$ (Proposition 4.7), then $\frac{L(t)}{A(t)} \to \frac{L^*}{A^*}$ as $t \to +\infty$ uniformly. Therefore, thanks to the results above, system (7b) is a three-dimensional asymptotically autonomous differential system with limit system,

$$\begin{cases} S_H'(t) = -(b_H + \beta_H I_m(t))S_H(t) + b_H, \\ I_H'(t) = \beta_H I_m(t)S_H(t) - (\gamma + b_H)I_H(t), \\ I_m'(t) = -\left(s_L\frac{L^*}{A^*} + \beta_m I_H(t)\right)I_m(t) + \beta_m I_H(t). \end{cases} \tag{19}$$

The equilibrium of which are $\mathcal{L}_0^* = (1,0,0)$ and $\mathcal{L}^* = (S_H^*, I_H^*, I_m^*)$ if $R_0 > 1$.

First of all, note that the region of biological interest Ω given by (8) is positively invariant under the flow induced by (19), as the vector field on the boundary does not point to the outside of Ω which is obvious (similar proof is done for Lemma 4.3).

Theorem 5.5. *If $R_0 > 1$, the endemic equilibrium with disease $\mathcal{L}^*$ of system (19) is globally asymptotically stable in int(Ω).*

To prove this theorem we shall use some preliminary results about competitive systems, see [38–40] and stability of periodic orbits, which we recall here for the reader convenience.

Let $D \subset \mathbb{R}^n$ be an open set, and $x \mapsto f(x) \in \mathbb{R}^n$ be a C^1 function defined in D. We consider the autonomous system in $\mathbb{R}^n$ given by,

$$x' = f(x). \tag{20}$$

System (20) is *competitive* in D, if, for some diagonal matrix $H = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, where ε_i, $(i = 1 \ldots, n)$, is either 1 or -1, the matrix $H(DF(x))H$ has non-positive off-diagonal elements for $x \in D$, where $DF(x)$ is the Jacobian matrix of (20), see [39,40]. It is shown in [39] that, if D is convex, the flow of such a system preserves, for $t < 0$, the partial order in $\mathbb{R}^n$ defined by the orthant

$K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \varepsilon_i x_i \geqslant 0\}$.

Looking at the Jacobian matrix of system (19) and choosing the matrix H such as

$$H = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we can see that system (19) is competitive in Ω, with respect to the partial order defined by the orthant

$K = \{(S_H, I_H, I_m) \in \mathbb{R}^3 : S_H \leqslant 0, I_H \geqslant 0, I_m \leqslant 0\}$.

We recall additional definitions that we will use later [41] and also [38,13,15]. Suppose that (20) has a periodic solution $x = p(t)$ with minimal period $\omega > 0$ and orbit $\gamma = \{p(t) : 0 \leqslant t \leqslant \omega\}$. This orbit is *orbitally stable* iff, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $x(t)$, for which the distance of $x(0)$ from γ is less than δ, remains at a distance less than ε from γ, for all $t \geqslant 0$. It is *asymptotically orbitally stable*, if the distance of $x(t)$ from γ also tends to zero as t goes to infinity. This orbit γ is *asymptotically orbitally stable with asymptotic phase* if it is asymptotically orbitally stable and there is $b > 0$ such that, any solution x (t), for which the distance of $x(0)$ from γ is less than b, satisfies $|x(t) - p(t-\tau)| \to 0$ as $t \to +\infty$ for some τ which may depend on $x(0)$ [41].

System (20) is persistent in the sense described in [42], i.e. iff each solution x (t) starting in $int(\Omega)$, has the property that lim $inf_{t\to+\infty} x(t)$ is at a positive distance from the boundary of Ω.

Definition 5.6. We say that system (20) has the *property of stability of periodic orbits*, iff the orbit of any periodic solution $\gamma(t)$, if it exists, is asymptotically orbitally stable.

The following theorem is the main tool to prove the global stability of the endemic equilibrium with disease.

Theorem 5.7 (*[13,41]*). *Assume that n = 3, D convex and bounded. Suppose that (20) is competitive, persistent and has the property of stability of periodic orbits. If x_0 is the only equilibrium in int (D), and if it is locally asymptotically stable, then it is globally asymptotically stable in int (D).*

Now, let us go back and apply those results to the global asymptotic stability study of (S_H^*, I_H^*, I_m^*). The proof of this theorem is similar to the one in [38]. In order to prove the persistence of system (19), we shall prove the following proposition.

Proposition 5.8. *On the boundary of Ω, system (19) has only one ω – limit point which is the equilibrium $\mathcal{L}_0^*$. Moreover for $R_0 > 1$, $\mathcal{L}_0^*$ cannot be the ω-limit of any orbit in int(Ω).*

Proof. The vector field is transversal to the boundary of Ω, except in the S_H-axis which is invariant with respect to (19). On the S_H-axis we have

$$S_H' = b_H(1 - S_H).$$

which implies that $S_H(t) \to 1$ as $t \to \infty$. Therefore, $\mathcal{L}_0^*$ is the only ω – *limit* point on the boundary of Ω.

To prove the second part of the proposition, we consider the functional

$$V = I_m + d_m \frac{1+R_0}{2}\frac{1}{\beta_H} I_H, \tag{21}$$

the derivative of which along solutions is given by,

$$\begin{aligned}
\dot{V} &= \dot{I_m} + d_m\left(\frac{1+R_0}{2}\right)\frac{1}{\beta_H}\dot{I_H} \\
&= -s_L\frac{L^*}{A^*} I_m + \beta_m(1-I_m)I_H \\
&\quad + d_m\left(\frac{1+R_0}{2}\right)\frac{1}{\beta_H}[\beta_H I_m(t)S_H(t) - (\gamma+b_H)I_H(t)] \\
&= -d_m I_m + \beta_m(1-I_m)I_H + d_m\left(\frac{1+R_0}{2}\right) I_m S_H \\
&\quad - d_m\left(\frac{1+R_0}{2}\right)\frac{(\gamma+b_H)}{\beta_H} I_H \\
&= \left[\beta_m(1-I_m) - d_m\frac{1+R_0}{2}\frac{(\gamma+b_H)}{\beta_H}\right] I_H + \left[d_m\frac{1+R_0}{2}S_H - d_m\right] I_m \\
&= \left[(1-I_m) - \left(\frac{1+R_0}{2}\right)\frac{d_m(\gamma+b_H)}{\beta_H\beta_m}\right]\beta_m I_H \\
&\quad + \left[\left(\frac{1+R_0}{2}\right)S_H - 1\right] d_m I_m \\
&= \left[(1-I_m) - \left(\frac{1+R_0}{2}\right)\frac{1}{R_0}\right]\beta_m I_H + \left[\left(\frac{1+R_0}{2}\right)S_H - 1\right] d_m I_m \\
&= \left[(1-I_m) - \frac{1}{2}\left(\frac{1}{R_0}+1\right)\right]\beta_m I_H \\
&\quad + \left[S_H - \left(\frac{2}{1+R_0}\right)\right] d_m\left(\frac{1+R_0}{2}\right) I_m.
\end{aligned}$$

Since $R_0 > 1$, then $\frac{1}{2}(1/R_0 + 1) < 1$ and $2/(1 + R_0) < 1$. Therefore, there exists a neighborhood U of $\mathcal{L}_0^*$ such that for $(S_H, I_H, I_m) \in U \cup int(\Omega)$ the expression inside of the brackets are positives. In this neighborhood, we have $\dot{V} > 0$ unless $I_H = I_m = 0$. Moreover, the level sets of V are the planes

$$I_m + d_m\left(\frac{1+R_0}{2}\right)\frac{1}{\beta_H} I_H = c,$$

which go away from the S_H-axis as c increases. Since V increases along the orbits starting in $U \cup int(\Omega)$, we conclude that they go away from $\mathcal{L}_0^*$.

This proves the proposition, and therefore, the persistence of system, (19) when $R_0 > 1$. □

Theorem 5.9. *The trajectory of any nonconstant periodic solution to (19), if it exists, is asymptotically orbitally stable with asymptotic phase.*

To prove this we used the following results.

Theorem 5.10 [43]. *A sufficient condition for a periodic orbit $\gamma = \{p(t): 0 \leqslant t \leqslant \omega\}$ of (20) to be asymptotically orbitally stable with asymptotic phase is that the linear non-autonomous system,*

$$y'(t) = \frac{\partial f^{[2]}}{\partial x}(p(t))y(t) \text{ is asymptotically stable.} \tag{22}$$

Eq. (22) is called the *second compound equation* of (20) and $\partial f^{[2]}/\partial x$ is the *second compound matrix* [38,43] of the Jacobian matrix $\partial f^{[2]}/\partial x$ of f. Generally speaking, for a $n \times n$ matrix A and an integer $1 \leqslant k \leqslant n$, the kth additive compound matrix of A is denoted by $A^{[k]}$. This is a $N \times N$ matrix, $N = \binom{n}{k}$, defined by

$$A^{[k]} = D_+(I + hA)^{(k)}|_{h=0},$$

where $B^{(k)}$ is the kth exterior power of a $n \times n$ matrix B and D_+ denotes the right-hand derivative. For example, if $n = 3$ with the notations,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

We have,

$$A^{[1]} = A,$$

$$A^{[2]} = \begin{pmatrix} a_{1,1}+a_{2,2} & a_{2,3} & -a_{1,3} \\ a_{3,2} & a_{1,1}+a_{3,3} & a_{1,2} \\ -a_{3,1} & a_{2,1} & a_{2,2}+a_{3,3} \end{pmatrix},$$

$$A^{[3]} = tr(A).$$

Proof. of Theorem 5.10 The Jacobian matrix of (19) is given by

$$DF = \begin{pmatrix} -(b_H + \beta_H I_m) & 0 & -\beta_H S_H \\ \beta_H I_m & -(\gamma + b_H) & \beta_H S_H \\ 0 & \beta_m(1-I_m) & -(s_L\frac{L^*}{A^*} + \beta_m I_H) \end{pmatrix}.$$

For the solution $\gamma(t)$, (22) becomes

$$\begin{cases} X' = -(2b_H + \beta_H I_m + \gamma)X + \beta_H S_H Y + \beta_H S_H Z, \\ Y' = \beta_m(1-I_m)X - (b_H + \beta_H I_m + s_L\frac{L^*}{A^*} + \beta_m I_H)Y, \\ Z' = \beta_H I_m Y - (s_L\frac{L^*}{A^*} + \beta_m I_H + \gamma + b_H)Z. \end{cases} \tag{23}$$

In order to prove that (23) is asymptotically stable, we consider the following Lyapunov function, where $\|.\|$is the norm in $\mathbb{R}^3$ by

$$\|(X,Y,Z)\| = \sup\{|X|, |Y| + |Z|\}$$

with

$$V(t) = V(X(t), Y(t), Z(t); S_H(t), I_H(t), I_m(t)) = \left\| \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{I_H(t)}{I_m(t)} & 0 \\ 0 & 0 & \frac{I_H(t)}{I_m(t)} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \right\| = \sup\left(|X|, \frac{I_H}{I_m}(|Y| + |Z|)\right).$$

Suppose that the solution $p(t) = (S_H(t), I_H(t), I_m(t))$ is periodic of minimal period ω. Then Proposition 5.8 implies that the orbit γ of $p(t)$ remains at a positive distance of the boundary of Ω. Therefore

$I_H(t) \geqslant \varepsilon$ and $I_m(t) \geqslant \varepsilon$ with $0 \leqslant \varepsilon \leqslant 1$.

Hence, the function V is well defined along $p(t)$ and there exists a constant $c > 0$ such that,

$$V(X, Y, Z; S_H, I_H, I_m) \geqslant c|(X, Y, Z)| \tag{24}$$

for all $(X, Y, Z) \in \mathbb{R}^3$ and $(S_H, I_H, I_m) \in \gamma$.

The right-hand derivative of $V(t)$ exists and its calculation is described in [44] and [45]. In fact direct computation yields,

$$D_+|X(t)| \leqslant -(2b_H + \beta_H I_m + \gamma)|X(t)| + \beta_H S_H(|Y(t)| + |Z(t)|) \leqslant -(2b_H + \beta_H I_m + \gamma)|X(t)| + \beta_H S_H \frac{I_m}{I_H}\left(\frac{I_H}{I_m}(|Y(t)| + |Z(t)|)\right)$$

and

$$D_+|Y(t)| \leqslant \beta_m(1 - I_m)|X(t)| - \left(b_H + \beta_H I_m + s_L\frac{L^*}{A^*} + \beta_m I_H\right)|Y(t)|, \tag{25}$$

$$D_+|Z(t)| \leqslant \beta_H I_m|Y(t)| - \left(s_L\frac{L^*}{A^*} + \beta_m I_H + \gamma + b_H\right)|Z(t)|. \tag{26}$$

Thus,

$$\begin{aligned} D_+\left[\frac{I_H}{I_m}(|Y(t)| + |Z(t)|)\right] &= \left(\frac{I_H'}{I_H} - \frac{I_m'}{I_m}\right)\frac{I_H}{I_m}(|Y(t)| + |Z(t)|) + \frac{I_H}{I_m}D_+(|Y(t)| \\ &\quad + |Z(t)|) \leqslant \left(\frac{I_H'}{I_H} - \frac{I_m'}{I_m}\right)\frac{I_H}{I_m}(|Y(t)| + |Z(t)|) \\ &\quad + \frac{I_H}{I_m}\left[\beta_m(1 - I_m)|X(t)| - (b_H + s_L\frac{L^*}{A^*} \right. \\ &\quad \left. + \beta_m I_H)(|Y(t)| + |Z(t)|)\right] \\ &\leqslant \left(\frac{I_H'}{I_H} - \frac{I_m'}{I_m} - b_H - s_L\frac{L^*}{A^*} - \beta_m I_H\right)(|Y(t)| \\ &\quad + |Z(t)|)\frac{I_H}{I_m} + \frac{I_H}{I_m}\beta_m(1 - I_m)|X(t)|. \end{aligned}$$

Then we can obtain

$$D_+V(t) \leqslant \sup\{g_1(t), g_2(t)\}V(t), \tag{27}$$

where

$$g_1(t) = -(2b_H + \beta_H I_m + \gamma) + \beta_H S_H(t)\frac{I_m(t)}{I_H(t)},$$

$$g_2(t) = \frac{I_H}{I_m}\beta_m(1 - I_m) + \frac{I_H'}{I_H} - \frac{I_m'}{I_m} - b_H - s_L\frac{L^*}{A^*} - \beta_m I_H.$$

Rewriting the last two equations of (19) as:

$$\frac{I_H'}{I_H} + (\gamma + b_H) = \beta_H\frac{I_m}{I_H}S_H,$$

$$\frac{I_m'}{I_m} + s_L\frac{L^*}{A^*} = \beta_m\frac{I_H}{I_m}(1 - I_m),$$

then

$$g_1(t) = -(2b_H + \beta_H I_m + \gamma) + \frac{I_H'}{I_H} + (\gamma + b_H) = \frac{I_H'}{I_H} - (b_H + \beta_H I_m),$$

$$\begin{aligned} g_2(t) &= \frac{I_H}{I_m}\beta_m(1 - I_m) + \frac{I_H'}{I_H} - \frac{I_m'}{I_m} - b_H - \beta_m\frac{I_H}{I_m}(1 - I_m) - \beta_m I_H \\ &= \frac{I_H'}{I_H} - b_H - \beta_m I_H, \end{aligned}$$

we obtain

$$\begin{aligned} \sup\{g_1(t), g_2(t)\} &= \sup\left\{\frac{I_H'}{I_H} - (b_H + \beta_H I_m), \frac{I_H'}{I_H} - b_H - \beta_m I_H\right\} \\ &\leqslant -b_H + \frac{I_H'}{I_H} \end{aligned}$$

and thus, from Eq. (27) and Gronwall's inequality, we obtain

$$V(t) \leqslant V(0)I_H(t)e^{-b_H t} \leqslant V(0)e^{-b_H t},$$

since $0 < I_H < 1$ in $int(\Omega)$, which implies that $V(t) \to 0$ as $t \to \infty$. By (24), it turns out that

$(X(t), Y(t), Z(t)) \to 0$ as $t \to \infty$.

Therefore, system (23) is asymptotically stable and Theorem 5.9 holds. □

Theorem 5.11. *Consider system (19). If $R_0 > 1$, then $\Omega - \{(S_H, 0, 0): 0 \leqslant S_H \leqslant 1\}$ is an asymptotic stability region for the endemic equilibrium with disease $\mathcal{L}^*$. Moreover all trajectories starting in the $S_H -$ axis approach the disease-free equilibrium $\mathcal{L}_0^*$.*

Proof. The first part of the theorem follows from the transversality of the vector field of (19) on $\Omega - \{(S_H, 0, 0): 0 \leqslant S_H \leqslant 1\}$ and theorem (5.5).The second part is proved by Proposition 5.8. □

The graphs shown, Fig. 5(a)-(b)–Fig. 6(a)-(b), were obtained after the numerical integration of system (7). In the numerical simulations b_H happen to be very small with respect to the other parameters, since the average expected life in humans is about 60 years, whereas the length of the infected period is a few days and the vector life expectancy is about 4–10 weeks.

Theorem 5.12. *Assume $R_0 > 1$ an $b_H \ll 1$, then the solutions of system (19) oscillate to the endemic equilibrium with disease.*

Proof. The existence of oscillations around the equilibrium $\mathcal{L}^*$ depends on whether the characteristic equation, defined by the Jacobian matrix (5.2),

$$P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C,$$

where

$$\begin{aligned} A &= (b_H + \beta_H I_m^*) + (\gamma + b_H) + (d_m + \beta_m I_H^*), \\ B &= (b_H + \beta_H I_m^*)(\gamma + b_H) + (b_H + \beta_H I_m^*)(d_m + \beta_m I_H^*) \\ &\quad + (\gamma + b_H)(d_m + \beta_m I_H^*) - \beta_H S_H^*\beta_m(1 - I_m^*) \\ &= (b_H + \beta_H I_m^*)(\gamma + b_H) + (b_H + \beta_H I_m^*)(d_m + \beta_m I_H^*) \\ &\quad + (\gamma + b_H)\beta_m I_H^*, \\ C &= (b_H + \beta_H I_m^*)(\gamma + b_H)(d_m + \beta_m I_H^*) + \beta_H S_H^*\beta_H I_m^*\beta_m(1 - I_m^*) \\ &\quad - (b_H + \beta_H I_m^*)\beta_H S_H^*\beta_m(1 - I_m^*) \\ &= (\gamma + b_H)\left((b_H + \beta_H I_m^*)\beta_m I_H^* + \beta_H d_m I_m^*\right). \end{aligned}$$

has eigenvalues with imaginary part different from zero. Recall that a polynomial of degree three has eigenvalues with imaginary part different from zero if the discriminant

$$\Delta = \frac{1}{4}q^2 + \frac{1}{27}p^3 \tag{28}$$

is bigger than zero, where

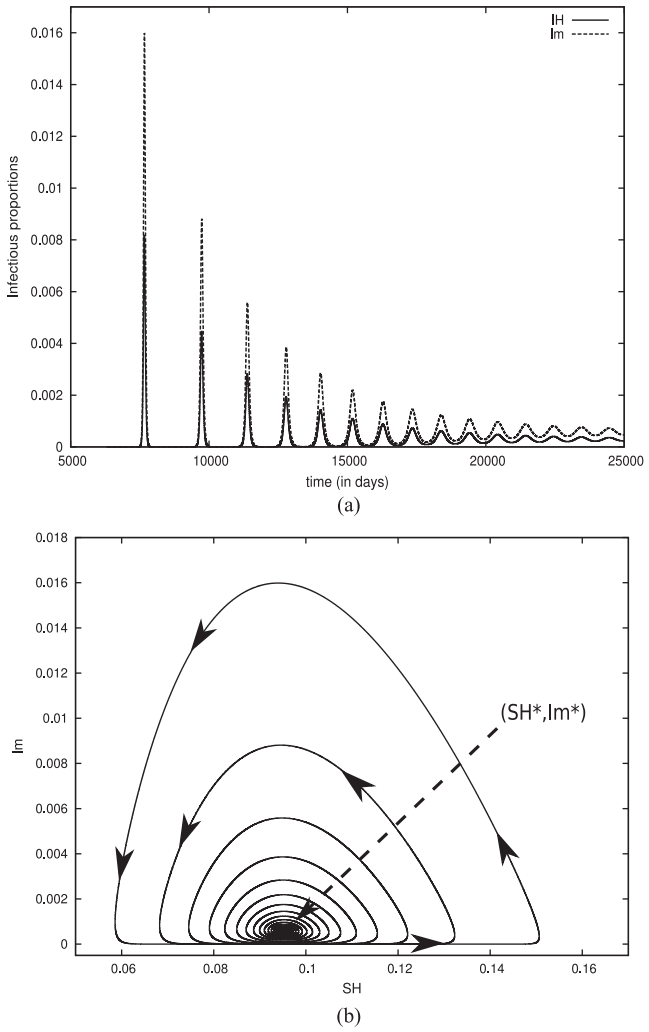

Fig. 5. Numerical solutions of model (7) (b)). The graphs (a) ans (b) show the proportion of infective humans and infective vectors versus time and the phase portrait in the $(S_H,\ I_m)$ plan. The parameters in the simulation are: $d_m = 0.25$, $b_H = 0.0000457$, $\beta_M = 0.5$, $\beta_H = 0.75$, $\gamma_H = 0.1428$ and then $R_0 = 10.500841$.

$$q = \frac{2}{27}A^3 - \frac{AB}{3} + C,\ p = B - \frac{A^2}{3}.$$

We substitute I_H^*, I_m^* and R_0 in the coefficients A, B and C and expand them in Taylor series around $b_H = 0$. After some computations we obtain the following approximations,

$$A = \gamma + d_m + \left(1 + \frac{\beta_m\beta_H}{d_m\gamma} + \frac{\beta_m}{\gamma} - \frac{d_m}{\beta_H}\right)b_H + \mathcal{O}(b_H^2),$$
$$B = \left((\gamma + d_m)\frac{\beta_m\beta_H}{d_m\gamma} + \beta_m - \frac{d_m\gamma}{\beta_H}\right)b_H + \mathcal{O}(b_H^2),$$
$$C = (\beta_H\beta_m - d_m\gamma))b_H + \mathcal{O}(b_H^2).$$

On the other hand, in terms of the coefficients A, B and C in Eq. (28), and collecting terms $\mathcal{O}(b_H^2)$, we get,

$$\Delta = \frac{1}{27}(\gamma + d_m)^3(\beta_H\beta_m - \gamma d_m)b_H + \mathcal{O}(b_H^2).$$

The term $\beta_H\beta_m - \gamma d_m$ is positive since $R_0 > 1$, therefore

$$\lim_{b_H \to 0} \frac{\Delta}{b_H} > 0,$$

which implies that for b_H sufficiently small and positive, $\Delta > 0$. This prove the theorem (this result is similar to that one given in [13]). □

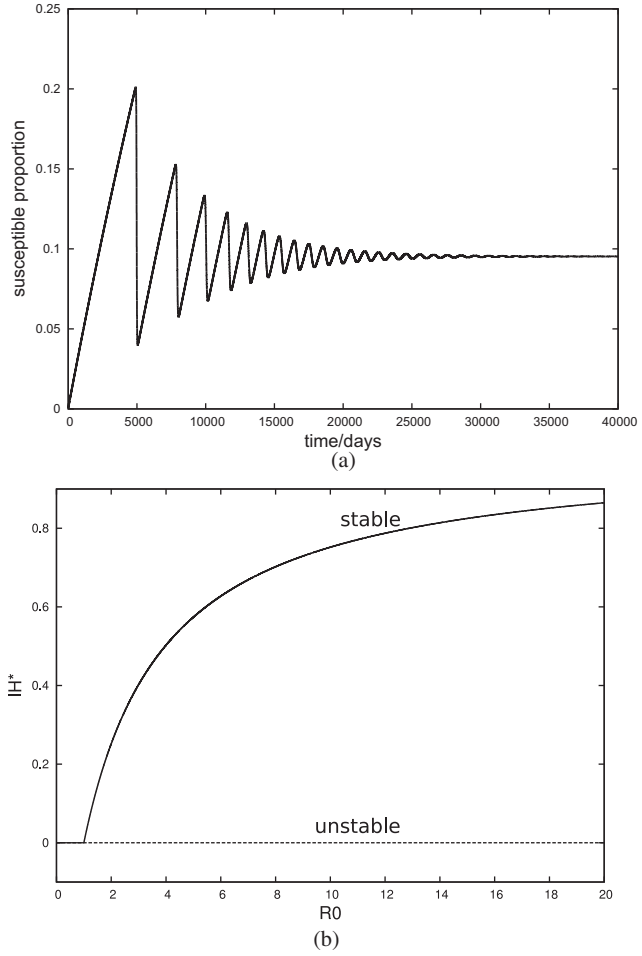

Fig. 6. (a) proportion of susceptible humans versus time (in days). (b) bifurcation diagram for equilibria of model (19) with respect to R_0. For $R_0 > 1$ we plot the proportion of infective humans I_H^* given in Eq. (16), and we fix the parameters $b_H = 0.0000457$, $\beta_M = 0.9$, $\beta_H = 0.9$, $\gamma_H = 0.1428$, $d_m = 0.25$.

Next, we analyze the asymptotic behavior of the total population $N_H(t)$, and the total number of individuals in the epidemiological classes $\overline{S}_H, \overline{I}_H$ and $\overline{R}_H$. For this we introduce the following parameters

$$R = \frac{\beta_H\beta_m}{d_m(\gamma + d_H)}$$

and

$$R_1 = R.S_H^*(1 - I_m^*) = \begin{cases} R, & \text{if } R_0 \leqslant 1, \\ \frac{\gamma + b_H}{\gamma + d_H}, & \text{if } R_0 > 1. \end{cases}$$

First we study the dynamics of solutions whose initial conditions are outside the subspace $\overline{I}_H = \overline{I}_m = 0$. For $R_0 \neq 1$ we have the following results.

Proposition 5.13. *For $b_H > d_H$, $(\overline{S}_H(t), \overline{I}_H(t), \overline{R}_H(t))$ tend, as $t \to \infty$, to $(\infty, 0, 0)$ if $R < 1$ and to (∞, ∞, ∞) if $(R_0 \leqslant 1$ and $R > 1)$ or $(R_0 > 1)$.*

Proof. Since $I_m' \to 0$ and $s_L \frac{L}{A} \to d_m$ as $t \to \infty$, in the limit, the proportion of infectious mosquitoes is related to the proportion of infected humans as

$$I_m = \frac{\beta_m}{d_m} I_H(1 - I_m),$$

thus, the limit form of the equation for $\bar{I}_H(t)$ is given in system (6) by

$$\bar{I}'_H = \left(\frac{\beta_m\beta_H}{d_m}S_H(1-I_m) - (\gamma+d_H)\right)\bar{I}_H = (\gamma+d_H)(R_1-1)\bar{I}_H,$$

which implies that $\bar{I}_H(t)$ declines exponentially if $R_1 < 1$, and grows exponentially if $R_1 > 1$. Moreover,

$$R_1 < 1 \Longleftrightarrow R < 1$$

and

$$R_1 > 1 \Longleftrightarrow (R_0 \leqslant 1 \quad \text{and} \quad R > 1) \quad \text{or} \quad (R_0 > 1).$$

The solution $\bar{R}_H(t)$ is given by

$$\bar{R}_H(t) = \bar{R}_{H_0}(t)e^{-d_H t} + e^{-d_H t}\gamma\int_0^t \bar{I}_H(s)e^{d_H s}ds.$$

From the exponential nature of $\bar{I}_H(t)$, it follows that $\bar{R}_H(t)$ declines exponentially if $R < 1$, and grows exponentially if $R_1 > 1$. □

6. Conclusion

We have proposed models to describe the vector (*Aedes albopictus* mosquito) population dynamics and the chikungunya virus transmission to human population.

First of all, we have proposed model (2) to describe the vector population dynamics which takes into account auto-regulation phenomena of eggs and larvae stages. We have shown that this model is well defined. For this model we found that,

$$r = \left(\frac{b}{s+d}\right)\left(\frac{s}{s_L+d_L}\right)\left(\frac{s_L}{d_m}\right)$$

is the threshold condition for the existence of the endemic state, where $(\frac{b}{s+d}), (\frac{s}{s_L+d_L})$ and $(\frac{s_L}{d_m})$ are respectively eggs, larvae and adults growth rates. For $r > 1$, we proved, using a Lyapunov functional, that the endemic equilibrium is globally asymptotically stable.

Moreover, following [15], we have proposed model (7) to describe the virus transmission to the human population. This is a model with variable human population and the contact rate among humans depends on the vector population.

In the case $r > 1$ (the biological interesting case) we found the following threshold parameters:

$$R_0 = \frac{\beta_m\beta_H}{d_m(\gamma+b_H)},$$
$$R = \frac{\beta_m\beta_H}{d_m(\gamma+d_H)}$$

and

$$R_1 = \begin{cases} R, & \text{if} \quad R_0 \leqslant 1, \\ \frac{\gamma+b_H}{\gamma+d_H}, & \text{if} \quad R_0 > 1. \end{cases}$$

On the one hand, parameter R_0 is the threshold condition for the existence of endemic proportions of infected humans and infected mosquitoes. On the other hand, the basic reproduction number R_1 controls the asymptotic behavior of the number of infected humans.

For $b_H = d_H$, we have $R_0 = R$ and hence, $(\bar{S}_H(t), \bar{I}_H(t), \bar{R}_H(t))$ tend, as $t \to +\infty$, to $(N_H(0), 0, 0)$ if $R < 1$ and to $N_H.(S^*_H, I^*_H, R^*_H)$ if $R > 1$, and this work complete [17] (global stability of the disease-free equilibrium and the endemic equilibrium with disease).

For $b_H > d_H$,$(\bar{S}_H(t), \bar{I}_H(t), \bar{R}_H(t))$ tend, as $t \to +\infty$, to $(+\infty, 0, 0)$ if $R < 1$ and to (∞, ∞, ∞) if $R_1 > 1$.

Before concluding about the possible actions to take to eradicate the disease, we formulate some remarks about proposed models.

- The infective proportion I_H and the total number of infective humans $\bar{I}_H$ may have different behaviors. Thus, I_H may tend to zero and $\bar{I}_H$ would grow exponentially (case $R_0 < 1 < R$).
- For proposed models, the dynamics of the vector does not depend directly on parameters K_E and K_L. We have already pointed out that drying the breeding sites, and then reducing the carrying capacity K_L, has an impact on the parameters d_L and s. However, the size of the vector population depends on the carrying capacity, but the threshold parameter r does not and the proportion $\frac{I^*}{A^*}$ (whose expression is reduced to d_m) does not either. Otherwise, it would be the same for threshold parameters R_0 and R_1.
- The use of the non classical incidence rate among humans leads to some simplifications on the threshold parameter R_0 (we obtain $\frac{A^*}{A^*}$ instead of $\frac{A^*}{N^*_H}$). That is another reason of its non-dependence on r, K_L and K_E.
- Our proposed models include models considering classical incidence rate with a constant human population. More precisely, for $b_H = d_H$, the model with a classical incidence rate among human population $(\frac{\beta_H\bar{S}_H\bar{I}_m}{N_H})$ leads to the system (7) where β_H have to be replaced by $\beta_H\frac{A(t)}{N_H}$. As $A(t)$ tends to A^* for $t \to +\infty$, exactly the same calculations can be made (by substituting β_H by $\beta_H\frac{A^*}{N_H}$) and we obtain the following threshold parameter

$$R_0 = \frac{\beta_m\beta_H}{d_m(\gamma+b_H)}\frac{A^*}{N_H}(=R)$$
$$= \frac{\beta_m\beta_H}{d_m(\gamma+b_H)}\frac{1}{N_h}\left(1-\frac{1}{r}\right)\frac{sK_Es_LK_L}{d_m(sK_E+(s_L+d_L)K_L)},$$

  which controls the global stability of equilibria. The same approach can be applied to show the global stability of the endemic equilibrium with disease for models in [18] if we consider no influence of the disease on the death rate for vector population.

Following these models and the previous remarks, the eradication of the disease can be achieved if the mosquito population is eradicated or if parameter R is lowered below unity.

Threshold parameter r may provide conditions in order to control the proliferation of the mosquito population. Indeed, even if there is no chikungunya epidemic, we need to be vigilant since in less than 20 years *Aedes albopictus* has developed capabilities to adapt to non tropical regions. That is why several measures have been applied in Europe in order to control the mosquito proliferation. One of the possible interventions to reduce the impact of the epidemic would be to reduce the number of mosquitoes. In this aim we have to focus on some parameters that human intervention can easily control. For instance chemical intervention can reduce the threshold parameter r: chemical alduticide increases the mortality rate of mosquitoes d_m and chemical larvicide increases the death rate of the larvae d_L. Nevertheless, such intervention has fatal consequences, in particular on the environment. Moreover, the vector can develop a resistance to some insecticides that reduce the impact of chemical interventions. That is why, other method of control must be considered. For instance, the reduction of the number of breeding sites, by drying them, has a double impact. First, it reduces the transfer rate s between the eggs compartment and the immature stage: if there is no water, eggs cannot hatch. Second, it increases the death rate d_L of the immature stage: water is necessary for the development of the immature stage. Note that drying the breeding sites has little impact on d, the death rate of eggs, because they are resistant to desiccation. Another example is the introduction of sterile male mosquitoes in the population (see [46]) that has an incidence on the egg-laying. These interventions allow reducing the birth rate b and then the threshold r, and thus control the mosquitoes proliferation.

In case of an epidemic, we have to reduce the parameter R. Clearly, chemical adulticide can be used to increase the death rate d_m of the vector with its bad consequences on the environment. Moreover, all efforts made by the human population, like the use of mosquito nets or repulsive against mosquitoes bites, wearing appropriate clothes, or isolating infected patients in hospitals, will reduce the contact rates β_h and β_m without the disadvantages of the chemical alduticide. All these efforts and interventions can be formulated in terms of an optimal control problem [47].

However, if we want to eradicate the disease just by controlling the vector population, models with non-classical incidence rate lead necessary to eradicate the vector population while models with a classical incidence rate require only to reduce it. In the case of a classical incidence rate, all measures cited previously (drying the breeding sites, larvicide) impact on K_E, K_L,s,d_L and hence R.

Acknowledgments

This research is supported by region "Haute-Normandie", France. The authors would like to thank the editors and the anonymous referees for their careful reading and very valuable comments that improved the presentation of this paper.

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