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The Effect of non-Selective Harvesting in Predator-Prey Model with Modified Leslie-Gower and Holling Type II Schemes

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Abstract

In this paper, we study the effect of harvesting on the qualitative properties of predator-prey model with modified Leslie-Gower and Holling Type II functional responses. The model is given by a system of two ordinary differential equations with non-selective constants harvesting terms. We investigate the impact of harvesting terms on the boundedness of solutions, on the existence of the attraction set, on the stability of different equilibrium points. A Lyapunov function is used to prove the global stability of the interior equilibrium. We also, discussed the policy of optimal harvest and we got the solution for the interior equilibrium by the Pontryagin maximum criterion. Finally, our theoretical results are illustrated by a numerical simulations.

1 Introduction

It has long been apparent that the predator-prey models have contributed in understanding this path to greater dynamical complexity of ecological systems. The first predator-prey model has been proposed by Lotka (see, [8]), in the same context Volterra has developed a predator -prey competition model (see, [11]). In [9], Odum combined the two last models and present a new model which called Lotka-Volterra model. In the last decades, various dynamics of Lotka-Volterra model has attracted several authors from many fields, biology, economics... with many different functional response.

The functional response of predators describes the rate of a predator consumes prey [1,4,5,15] and represent a main element in the research of predator-prey models, the most used one is the Holling type II which is characterized by a decelerating intake rate which comes from the assumption that the consumer is limited by its capacity to process food. In this way the predator-prey model proposed by Leslie-Gower describes the properties of a stochastic model interaction between two species (see, [7]). In the study of predator-prey interaction, some studies that treat population can be extended by considering harvesting. In [6], Clark introduce a mathematical model with harvesting of renewable resources and studied the harvesting problem with a combined ecologically independent and logistically growing fish species. Brauer and Soudack [3] developed a predator-prey model with

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constant rate harvesting in prey and studied its dynamical behavior. They also discussed the relation between the presence of harvesting and the region of asymptotic stability in the predator-prey plane. Zhang et al. [14] treated the problem of combined harvesting of prey-predator model with prey reserve by considering different harvesting effort. Rojas-Palma and Gonzalez-Olivares [10] considered an open access fishery of the predator-prey type where the functional response of the predators takes the form of Holling type-*III* and the prey growth is affected by the Allee-effect in which both prey and predator species are subjected to non-selective harvesting based on the catch per unit effort.

In this work, we consider a predators prey model with non-selective harvesting. The model given by a system of two-species food chain which describes a prey population u which serves as food for a predator v with modified Leslie-Gower and Holling type-*II* functional responses. The equations of the two populations in the presence of harvesting can be written as follows:

$$\begin{cases} \dfrac{du}{dt} = (r_1 - b_1 u - \dfrac{a_1 v}{u+k_1})u - E_1 u, \\ \dfrac{dv}{dt} = (r_2 - \dfrac{a_2 v}{u+k_2})v - E_2 v, \end{cases} \tag{1}$$

with $u(0) \geqslant 0$ and $v(0) \geqslant 0$, where u and v represent the population densities at time t; r_1, r_2, a_1, a_2 and b_1, are model parameters assuming only positive values. These parameters are defined as follows: r_1 is the growth rate of prey u, b_1 measures the strength of competition among individuals of species u, al is the maximum value which per capita reduction rate of u can attain, k_l (respectively, k_2) measures the extent to which environment provides protection to prey u (respectively, to predator v), r_2 describes the growth rate of v, and a_2 has a similar meaning to a_1, E_1 the effort applied to harvest the prey, E_2 the effort applied to harvest the predator.

The model without harvesting is studied in [2] and the model without harvesting and with delay is studied in [12,13].

The paper is organized as follows: Section 2 is devoted to the positivity and the boundedness of solutions. In section 3, we prove the local stability of the possible equilibrium points. Global stability of the positive equilibrium is studied in section 4. In section 5, we maximize the objective functional form of the studied model. In the last section, we give some some discussions and some numerical simulations in order to illustrate our theoretical results.

2 Positivity and boundedness

In this section, we will prove that, under some assumptions the solutions of system (1) which start in $\mathbb{R}^2_+$ are ultimately bounded. We denote by $\mathbb{R}^2_+$ the nonnegative quadrant and by $int(\mathbb{R}^2_+)$ the positive quadrant.

Lemma 1. *The positive quadrant $int(\mathbb{R}^2_+)$ is invariant for system (1).*

Proof. Let

$$h_1(u,v) = (r_1 - b_1 u - \frac{a_1 v}{u+k_1}) - E_1$$

and

$$h_2(u,v) = (r_2 - \frac{a_2 v}{u+k_2}) - E_2.$$

Integrating equation (1) with initial conditions $u(0) > 0$ and $v(0) > 0$, we have

$$u(t) = u_0 \exp(\int_0^t h_1(u(s), v(s)ds) > 0,$$

$$v(t) = v_0 \exp(\int_0^t h_2(u(s), v(s)ds) > 0.$$

Hence all solutions starting in $\mathbb{R}^2_+$ remain in $\mathbb{R}^2_+$ for all $t > 0$.

Next, we give the following result:

Lemma 2. *Let ϕ be an absolutely-continuous function satisfying the differential inequality $\frac{d\phi(t)}{dt} + \alpha_1\phi(t) \leq \alpha_2$, where $t \geq 0$, $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\alpha_1 \neq 0$. Then there exists T such that for all $t \geq T \geq 0$,*

$$\phi(t) \leq \frac{\alpha_2}{\alpha_1} - (\frac{\alpha_2}{\alpha_1} - \phi(T))e^{-\alpha_1(t-T)}.$$

Proof. The proof of this Lemma is based on the Gronwall Lemma.

Definition 1. A solution $\phi(t,t_0,x_0,y_0)$ of system (1) is said to be ultimately bounded with respect to $\mathbb{R}^2_+$ if there exists a compact region $B \in \mathbb{R}^2_+$ and a finite time $T(T = T(t_0,x_0,y_0))$ such that, for any $(t_0,x_0,y_0) \in \mathbb{R} \times \mathbb{R}^2_+$, $\phi(t,t_0,x_0,y_0) \in B$ for all $t \geq T$.

Theorem 3. *Let $r_1 > E_1$ and B the set defined by:*

$$B = \{(u,v) \in \mathbb{R}^2_+ : 0 \leq u \leq \frac{r_1 - E_1}{b_1}, 0 \leq u + v \leq L\},$$

where

$$L = \frac{1}{4a_2b_1}(a_2(r_1 - E_1)(r_1 - E_1 + 4) + (r_2 - E_2 + 1)^2(r_1 - E_1 + b_1k_2)).$$

Then

1. *B is positively invariant,*
2. *all solutions of (1) initiating in $\mathbb{R}^2_+$ are ultimately bounded with respect to $\mathbb{R}^2_+$.*

Proof.

1. If $(u(0),v(0)) \in B$, then from Lemma (1) we have $(u(t),v(t))$ remain in $\mathbb{R}^2_+$ for all $t \geq 0$.
   Now, we have to show that $u(t) \leq \frac{r_1-E_1}{b_1}$ and $u(t) + v(t) \leq L$ for all $t > 0$.
   Since $u > 0$ and $v > 0$ in $Int(\mathbb{R}^2_+)$, every solution $\phi(t) = (u(t),v(t))$ of (1) which starts in $Int(\mathbb{R}^2_+)$. Then, $u(t)$ satisfies the following inequality

$$0 \leq \frac{du(t)}{dt} \leq (u(t))(r_1 - E_1 - b_1u(t)).$$

   Then, we have $u(t) \leq y(t)$ where $y(t)$ is the solution of the following equation:

$$\frac{dy(t)}{dt} = y(t)(r_1 - b_1y(t)) - E_1y(t)$$

   $y(0) = u(0) > 0$, and

$$y(t) = \frac{1}{\frac{b_1}{r_1 - E_1} + ce^{-r_1t}}$$

   where $c = \frac{1}{u(0)} - \frac{b_1}{r_1-E_1}$.
   Therefore

$$u(t) \leq \frac{r_1 - E_1}{b_1} \tag{2}$$

for all $t \geq 0$.
Now we prove that $u(t)+v(t) \leq L$ for all $t \geq 0$.
Define the function $x(t)=u(t)+v(t)$, then we have

$$\frac{dx}{dt}=\frac{du}{dt}+\frac{dv}{dt}=(r_1-b_1u-\frac{a_1v}{u+k_1})u-E_1u+(r_2-\frac{a_2v}{u+k_2})v-E_2v. \tag{3}$$

As all parameters are positive and all solutions initiating in $\mathbb{R}^2_+$ remain in $\mathbb{R}^2_+$, then

$$\frac{dx}{dt} \leq (r_1-b_1u)u-E_1u+(r_2-\frac{a_2v}{u+k_2})v-E_2v.$$

Therefore

$$\max_{u\in\mathbb{R}_+}(r_1-E_1-b_1u)u=\frac{(r_1-E_1)^2}{4b_1}.$$

Thus, we have

$$\frac{dx(t)}{dt} \leq \frac{(r_1-E_1)^2}{4b_1}-x(t)+u+v+(r_2-E_2-\frac{a_2v}{u+k_2})v$$

and

$$\frac{dx(t)}{dt}+x(t) \leq \frac{(r_1-E_1)^2}{4b_1}+u+(1+r_2-E_2-\frac{a_2v}{u+k_2})v.$$

From inequality 2, we get:

$$\frac{dx(t)}{dt}+x(t) \leq \frac{(r_1-E_1)^2}{4b_1}+\frac{r_1-E_1}{b_1}+(1+r_2-E_2-\frac{a_2b_1v}{r_1+b_1k_2})v.$$

and

$$\max_{v\in\mathbb{R}_+}(1+r_2-E_2-\frac{a_2b_1v}{r_1-E_1+b_1k_2})v=\frac{(r_2-E_2+1)^2(r_1-E_1+b_1k_2)}{4a_2b_1}$$

Consequently

$$\frac{dx(t)}{dt}+x(t) \leq L.$$

Using Lemma 2, (with $\alpha_1=1$ and $\alpha_2=L$), we find

$$x(t) \leq L-(L-x(T))e^{-(t-T)} \text{ for all } t \leq T \leq 0.$$

Then

$$x(t) \leq L-(L-x(T))e^{-t} \text{ if } T=0$$

Since $(u(0),v(0)) \in B$, we have $x(t)=u(t)+v(t) \leq L$ for all $t \geq 0$.

2. Secondly we prove that

$$\overline{\lim_{t\longrightarrow\infty}}\, u(t) \leq \frac{r_1-E_1}{b_1}$$

and

$$\overline{\lim_{t\longrightarrow\infty}}\, u(t)+v(t) \leq L.$$

From (2) and Lemma 2, we obtain

$$\overline{\lim_{t\longrightarrow\infty}}\, u(t) \leq \frac{r_1-E_1}{b_1},$$

since the solutions of the following initial value problem

$$\begin{aligned} \frac{du}{dt} &= (r_1 - b_1 u)u - E_1 u, \\ u(0) &\geq 0, \end{aligned} \tag{4}$$

satisfies

$$\overline{\lim_{t\longrightarrow\infty}} u(t) \leq \frac{r_1 - E_1}{b_1}.$$

For the second inequality. Let $\varepsilon > 0$, then there exists $T_1 > 0$ such that for all $t \leq T_1$ we have $u(t) < 1 + \frac{\varepsilon}{2}$. From equation (19) with $T = T_1$, we get

$$x(t) = u(t) + v(t) \leq L - (L - x(T_1))e^{(T_1 - t)} \tag{5}$$

and for all $t \geq T_1 \geq 0$, we have

$$x(t) \leq L - (Le^{T_1} - (u(T_1) + v(T_1)e^{T_1})e^{-t} \leq L - (L - (u(T_1) + v(T_1)e^{T_1})e^{-t}. \tag{6}$$

Then

$$x(t) \leq L + \frac{\varepsilon}{2} - (L + \frac{\varepsilon}{2} - (u(T_1) + v(T_1)e^{T_1})e^{-t} \ \forall t \geq T_1 \geq 0. \tag{7}$$

Let $T_2 \geq T_1$ such that

$$\mid (L + \frac{\varepsilon}{2} - (u(T_1) + v(T_1)e^{T_1})e^{-t} \mid \leq \frac{\varepsilon}{2}.$$

Then

$$u(t) + v(t) \leq L + \frac{\varepsilon}{2} \ \forall t > T_2.$$

Hence $\overline{\lim_{t\longrightarrow\infty}} x(t) \leq L$, we deduce the result.

3 Local stability

In this section, we prove the local stability of the possible equilibrium points of system (1).

It's easy to show that system (1) has three trivial equilibria (belonging to the boundary of $\mathbb{R}^2_+$; i.e. at which one or more of populations has zero density) $P_0 = (0,0)$, $P_1(\tilde{u},0)$ and $P_2(0,\tilde{v})$ where $\tilde{u} = \frac{r_1 - E_1}{b_1}$ and $\tilde{v} = \frac{(r_2 - E_2)k_2}{a_2}$.

The following result gives the existence of the non trivial equilibrium point.

Proposition 4. *Let us assume the following condition:*

$$a_1((r_2 - E_2)k_2) < a_2((r_1 - E_1)k_1) \text{ and } r_2 > E_2.$$

Then system (1) *has a unique interior equilibrium* $P_3(u^*, v^*)$.

Proof. From system (1), a steady state satisfies

$$(r_1 - E_1 - b_1 u^*)(u^* + k_1) = a_1 v^* \tag{8}$$

and

$$v^* = \frac{r_2(u^* + k_2) - E_2(u^* + k_2)}{a_2}. \tag{9}$$

We get

$$a_2 b_1 u^{*2} + (a_1(r_2 - E_2) - a_2(r_1 - E_1) + a_2 b_1 k_1)u^* + a_1(r_2 - E_2)k_2 - a_2(r_1 - E_1)k_1 = 0.$$

Thus

$$u^* = \frac{1}{2a_2b_1}(-(a_1(r_2-E_2)-a_2(r_1-E_1+a_2b_1k_1))\pm\Delta^{1/2}),$$

$$v^* = \frac{(r_2-E_2)(u^*+k_2)}{a_2}.$$

Where

$$\Delta = (a_1(r_2-E_2)-a_2(r_1-E_1)+a_2b_1k_1)^2-4a_2b_1(a_1(r_2-E_2)k_2-a_2(r_1-E_1)k_1).$$

If

$$a_1((r_2-E_2)k_2) < a_2((r_1-E_1)k_1)$$

holds, then $\Delta > 0$ and we also obtain that $u^*_+ > 0$ and $u^*_- < 0$.

Therefore, system (1) possesses a unique positive interior equilibrium $P_3(u^*, v^*)$ given by

$$u^* = \frac{1}{2a_2b_1}(-(a_1(r_2-E_2)-a_2(r_1-E_1+a_2b_1k_1))+\Delta^{1/2}),$$

$$v^* = \frac{(r_2-E_2)(u^*+k_2)}{a_2}.$$

Next, we prove the local stability for different equilibrium points under some parameter assumptions. To do that,one need to compute the Jacobian matrix M at an equilibrium point (u, v)

$$M = \begin{bmatrix} r_1-E_1-2b_1u-\dfrac{a_1v(u+k_1)-a_1uv}{(u+k_1)^2} & \dfrac{-a_1u}{u+k_1} \\ \dfrac{a_2v^2}{(u+k_2)^2} & r_2-E_2-\dfrac{2a_2v}{u+k_2} \end{bmatrix} \quad (10)$$

Then, we have the following results:

Proposition 5. *i) The equilibrium point P_0 is stable only if $E_1 > r_1$ and $E_2 > r_2$.*
ii) The equilibrium point P_1 is stable if

$$r_1 > E_1 \text{ and } E_2 > r_2.$$

iii)The equilibrium point P_2 is stable if is stable if

$$E_2 < r_2 \text{ and } r_1 - \frac{a_1k_2}{a_2k_1}(r_2-E_2) < E_1.$$

iv) The equilibrium point P_3 is stable if

$$r_1 - r_2 < E_1 - E_2 \text{ and } k_2 < k_1$$

Proof. i), ii) and iii) an easy computation gives the results.
iv) From equations (18) and (19), the Jacobian matrix M becomes:

$$M = \begin{bmatrix} -b_1u^* + \dfrac{a_1u^*v^*}{(u^*+k_1)^2} & \dfrac{-a_1u^*}{u^*+k_1} \\ \dfrac{a_2v^2}{(u^*+k_2)^2} & -\dfrac{a_2v^*}{u^*+k_2} \end{bmatrix} \quad (11)$$

and

$$det(M) = b_1u^*\frac{a_2v^*}{u^*+k_2} - \frac{a_2a_1u^*v^{*2}}{(u^*+k_2)(u^*+k_1)^2} + \frac{a_2a_1u^*v^{*2}}{(u^*+k_1)(u^*+k_2)^2}.$$

If $k_2 < k_1$, then $det(M) > 0$.

$$tr(M) = -b_1u^* + \frac{a_1u^*v^*}{(u^*+k_1)^2} - \frac{a_2v^*}{u^*+k_2}$$

using (18) and (19)

$$tr(M) = r_1 - E_1 - 2b_1u^* - \frac{a_1v^*(u^*+k_1) - a_1u^*v^*}{(u^*+k_1)^2} - (r_2 - E_2)$$

Therefore, $tr(M) < 0$ if $(r_1 - E_1) < (r_2 - E_2)$.

4 Global stability

In this section, we prove the global stability of the interior equilibrium by using the Lyapunov function.

Lemma 6. *If*

$$r_1 > E_1 \tag{12}$$

and

$$2a_1L_1 < (r_1 - E_1)k_1. \tag{13}$$

Then $f_1(u,v) < 0$, *where* $f_1(u,v) = -b_1(u^*+k_1) + \frac{a_1v}{u+k_1}$.

Proof. Let

$$f_1(u,v) = -b_1(u^*+k_1) + \frac{a_1v}{u+k_1},$$

from (22), we have

$$f_1(u,v) = E_1 - r_1 + \frac{a_1v^*}{u^*+k_1} + \frac{a_1v}{u+k_1}.$$

As B is an attracting positively invariant set and all solutions satisfy $0 \le u \le \frac{r_1-E_1}{b_1}$ and $0 \le u+v \le L_1$. Then

$$f_1(u,v) \le E_1 - r_1 + \frac{a_1}{k_1}(v+v^*) \le E_1 - r_1 + \frac{2a_1L_1}{k_1}.$$

Therefore if (12) and (13), we get $f_1(u,v) < 0, \forall (u,v) \in B$ and $t \ge 0$.

Lemma 7. *If*

$$k_1 < 2k_2 \tag{14}$$

and

$$4(r_1 - E_1 + b_1k_1) < a_1. \tag{15}$$

Then, $g(u,v) = -a_1f_1(u,v) - f_2^2(u,v) < 0$.
Where

$$f_1(u,v) = -b_1(u^*+k_1) + \frac{a_1v}{u+k_1},$$

$$f_2(u,v) = \frac{1}{2}(-a_1 + \frac{a_1v}{u+k_2}).$$

Proof. Let

$$g(u,v) = -a_1(-b_1(u^*+k_1 + \frac{a_1v}{u+k_1})) - \frac{1}{4}(-a_1 + \frac{a_1v}{u+k_2})^2 < 0.$$

Then

$$\frac{\partial g(u,v)}{\partial v} = \frac{-a_1^2}{u+k_1} - \frac{1}{2}(\frac{-a_1^2}{u+k_2} + \frac{a_1^2v}{(u+k_2)^2})$$

and

$$\frac{\partial^2 g(u,v)}{\partial v^2} = -\frac{a_1^2}{2(u+k_2)^2} < 0.$$

Hence $\frac{\partial g}{\partial v}$ is strictly decreasing.

As

$$\begin{aligned}\frac{\partial g}{\partial v} &= -\frac{a_1^2}{u+k_1} + \frac{a_1^2}{2(u+k_2)} \\ &= \frac{a_1^2(-u-2k_2+k_1)}{2(u+k_2)(u+k_1)}\end{aligned}$$

and if (14) satisfies, we have $\frac{\partial g}{\partial v} < 0$ on $\mathbb{R}^+$.

So $g(u,v)$ is strictly decreasing on $\mathbb{R}^+$, this yields to $g(u,v) < g(u,0)$ for $(u,v) \in B$.

Therefore

$$g(u,v) < a_1 b_1 (u^* + k_2) - \frac{1}{4} a_1^2.$$

As $0 < u^* \leq \frac{r_1 - E_1}{b_1}$, then $g(u,v) < a_1(r_1 - E_1 + \frac{b_1}{k_1} - \frac{1}{4} a_1)$ and due to (15), we have $\forall (u,v) \in B,\ g(u,v) < 0$.

Theorem 8. *Assume (12), (13), (14) and (15) then the interior equilibrium P_3 is globally asymptotically stable.*

Proof. The proof is based on a positive definite Lyapunov function.

Let

$$L(u,v) = \frac{a_1}{a_2}(u^* + k_2)(v - v^* - v^* \ln(\frac{v}{v^*})) + (u^* + k_1)(u - u^* - u^* \ln(\frac{u}{u^*})).$$

This function is defined and continuous on $int(\mathbb{R}_+^2)$, $L(u^*,v^*) = 0$, $L(u,v) > 0\ \forall u,v > 0$ and $P^*(u^*,v^*)$ is the global minimum of L.

Since the solutions of the system are bounded and the set B which is positively invariant, we restrict our study on this set.

Computing the time derivative of L along the solution of the system (1) and using (8) and 9, we have

$$\begin{aligned}\frac{dL}{dt} &= (u^*+k_1)(u-u^*)(-b_1(u-u^*) + \frac{a_1 u^*}{u^*+k_1} - \frac{a_1 v}{u+k_1}) + \frac{a_1(u^*+k_2)(v-v^*)}{a_2}(\frac{a_2 v^*}{u^*+k_2} - \frac{a_2 v}{u+k_2}) \\ &= (u^*+k_1)(u-u^*)(-b_1(u-u^*) + \frac{a_1 v^*(u+k_1) - a_1 v(u^*+k_1)}{(u+k_1)(u^*+k_1)} \\ &\quad + a_1(u^*+k_2)(v-v^*)\frac{v^*(u+k_2) - v(u^*+k_2)}{(u^*+k_2)(u+k2)} \\ &= (u^*+k_1)(u-u^*)(-b_1(u-u^*) + \frac{-a_1 k_1(v-v^*) - a_1 u(v-v^*) + a_1 v(u-u^*)}{(u+k1)(u^*+k_1)} \\ &\quad + a_1(u^*+k_2)(v-v^*)\frac{-k_2(v-v^*) - u(v-v^*) + v(u-u^*)}{(u^*+k_2)(u+k_2)}.\end{aligned}$$

From expression of $\frac{dL}{dt}$, we have

$$\frac{dL}{dt} = (-b_1(u^*+k_1) + \frac{a_1 v}{u+k_1})(u-u^*)^2 + (-a_1 + \frac{a_1 v}{u+k_2})(u-u^*)(v-v^*) - a_1(v-v^*)^2. \tag{16}$$

The above equation can be written as

$$\frac{dL}{dt} = -(u-u^*, v-v^*)\ \ H\ \begin{pmatrix} u-u^* \\ v-v^* \end{pmatrix}. \tag{17}$$

Where

$$H = \begin{pmatrix} -f_1(u,v) & -f_2(u,v) \\ -f_2(u,v) & a_1 \end{pmatrix},$$

$$f_1(u,v) = -b_1(u^* + k_1) + \frac{a_1 v}{u + k_1},$$

$$f_2(u,v) = \frac{1}{2}(-a_1 + \frac{a_1 v}{u + k_2}).$$

Using Lemma 6 and Lemma 7, we get

1. $f_1(u,v) < 0$,
2. $g(u,v) = -a_1 f_1(u,v) - f_2^2(u,v) < 0$.

Since $a_1 > 0$, we have H is a positive-definite matrix by Sylvester's criterion. Then, $\frac{\partial L}{\partial t} < 0$ along all trajectories in the first quadrant except (u^*, v^*).

So P^* is globally asymptotically stable.

5 Optimal harvesting policy

Let us define the following objective functional:

$$J(E_1, E_2) = \int_0^\infty e^{-\delta t}(p_1 u - c_1)E_1(t) + (p_2 v - c_2)E_2(t)dt,$$

where

- c_1 fishing cost per unit effort for prey species,
- c_2 fishing cost per unit effort for predator species,
- p_1 price per unit biomass of the prey,
- p_2 price per unit biomass of the predator.

Our objective is to maximize the form of the harvesting model with the instantaneous annual rate of discount δ subject to the state constraints of the system (1) and the control constraints $0 \le E_i \le E_i^{max} (i = 1,2)$.

Hamiltonian for the problem is given by

$$\begin{aligned} H =& e^{-\delta t}[(p_1 u - c_1)E_1(t) + (p_2 v - c_2)E_2(t)] \\ &+ \lambda_1[(r_1 - b_1 u - \frac{a_1 v}{u + k_1})u - E_1 u] \\ &+ \lambda_2[(r_2 - \frac{a_2 v}{u + k_2})v - E_2 v]. \end{aligned} \tag{18}$$

where λ_1 and λ_2 are adjoint variables.

We have

$$\frac{\partial H}{\partial E_1} = e^{-\delta t}(p_1 u - c_1) - \lambda_1 u = s_1(t), \tag{19}$$

$$\frac{\partial H}{\partial E_2} = e^{-\delta t}(p_2 v - c_2) - \lambda_2 v = s_2(t). \tag{20}$$

Hence the optimal control policy is

$$E_i(t) = \begin{cases} E_i^{max} & \text{if } s_i(t) > 0, \\ 0 & \text{if } s_i(t) < 0, \\ E_i^* & \text{if } s_i(t) = 0. \end{cases}$$

for $i = 1,2$ and singular control $s_i(t)$ $(i = 1,2)$.

From (19) and (20) we get

$$\lambda_1 = e^{-\delta t}(p_1 - \frac{c_1}{u}), \tag{21}$$

$$\lambda_2 = e^{-\delta t}(p_2 - \frac{c_2}{v}). \tag{22}$$

By Pontryagin's maximum principle, the adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial u}, \tag{23}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial v}. \tag{24}$$

From (18) and (23), we have

$$\frac{d\lambda_1}{dt} = -[e^{-\delta t}E_1 p_1 + \lambda_1(r_1 - 2b_1 u - \frac{a_1 v(u+k_1) - a_1 vu}{(u+k_1)^2}) + \lambda_2(\frac{a_2 v^2}{(u+k_2)^2})] \tag{25}$$

From (18) and (24), we get

$$\frac{d\lambda_2}{dt} = -[e^{-\delta t}E_2 p_2 + \lambda_1(-\frac{a_1 u}{(u+k_1)}) + \lambda_2(r_2 - 2\frac{a_2 v}{(u+k_2)})]. \tag{26}$$

Using equilibrium conditions (25) and (26), we find

$$\frac{d\lambda_1}{dt} = -[e^{-\delta t}E_1 p_1 + \lambda_1(-b_1 u + \frac{a_1 vu}{(u+k_1)^2}) + \lambda_2(\frac{a_2 v^2}{(u+k_2)^2})], \tag{27}$$

$$\frac{d\lambda_2}{dt} = -[e^{-\delta t}E_2 p_2 + \lambda_1(-\frac{a_1 u}{(u+k_1)}) + \lambda_2(-\frac{a_2 v}{(u+k_2)})]. \tag{28}$$

From (23) and (24) and by integrating (27) and (28)

$$\lambda_1 = \frac{1}{\delta}e^{-\delta t}[E_1 p_1 + (p_1 - \frac{c_1}{u})(-b_1 u + \frac{a_1 vu}{(u+k_1)^2}) + (p_2 - \frac{c_2}{v})(\frac{a_2 v^2}{(u+k_2)^2})], \tag{29}$$

$$\lambda_2 = \frac{1}{\delta}e^{-\delta t}[E_2 p_2 + (p_1 - \frac{c_1}{u})(-\frac{a_1 u}{(u+k_1)}) + (p_2 - \frac{c_2}{v})(-\frac{a_2 v}{(u+k_2)})]. \tag{30}$$

From (21) and (22), we get

$$e^{-\delta t}(p_1 - \frac{c_1}{u}) = \frac{1}{\delta}e^{-\delta t}[E_1 p_1 + (p_1 - \frac{c_1}{u})(-b_1 u + \frac{a_1 vu}{(u+k_1)^2}) + (p_2 - \frac{c_2}{v})(\frac{a_2 v^2}{(u+k_2)^2})], \tag{31}$$

$$e^{-\delta t}(p_2 - \frac{c_2}{v}) = \frac{1}{\delta}e^{-\delta t}[E_2 p_2 + (p_1 - \frac{c_1}{u})(-\frac{a_1 u}{(u+k_1)}) + (p_2 - \frac{c_2}{v})(-\frac{a_2 v}{(u+k_2)})]. \tag{32}$$

Equations (31) and (32) give the optimal harvesting efforts as

$$E_1 = \frac{1}{p_1}[\delta p_1 - \frac{c_1\delta}{u} - (p_1 - \frac{c_1}{u})(-b_1 u + \frac{a_1 vu}{(u+k_1)^2}) - (p_2 - \frac{c_2}{v})(\frac{a_2 v^2}{(u+k_2)^2})], \tag{33}$$

$$E_2 = \frac{1}{p_2}[\delta p_2 - \frac{\delta c_2}{v} - (p_1 - \frac{c_1}{u})(-\frac{a_1 u}{(u+k_1)}) - (p_2 - \frac{c_2}{v})(-\frac{a_2 v}{(u+k_2)})]. \tag{34}$$

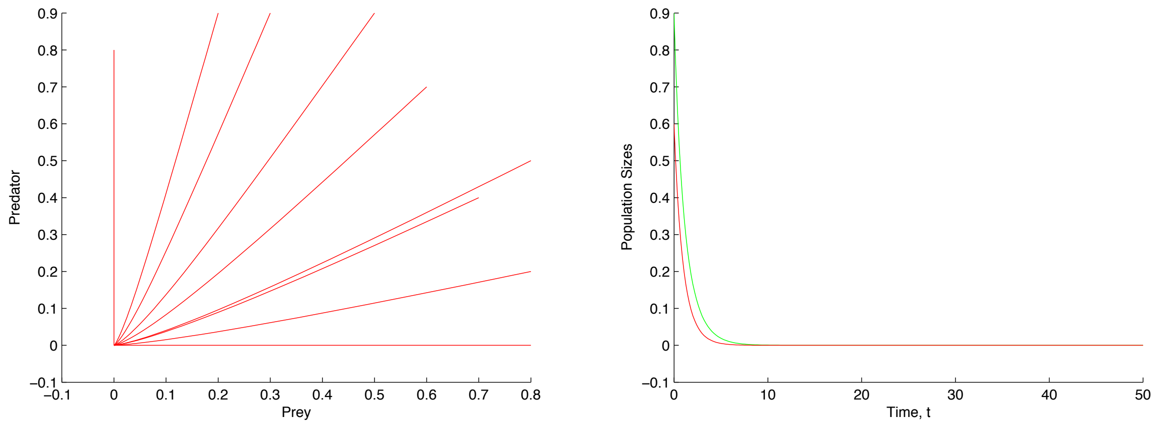

Fig. 1 Local stability for P_0 for $E_1 < E_2, r_1 = 1.8; r_2 = 2.1; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_1 = 2.5; E_2 = 3$.

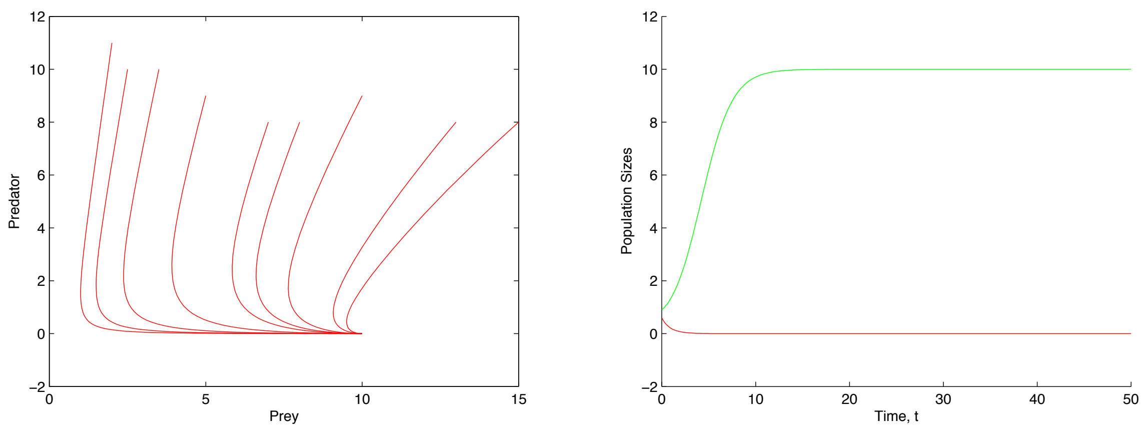

Fig. 2 Stability for P_1 for $E_1 < E_2, r_1 = 1.8; r_2 = 2; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_1 = 1.2; E_2 = 3$.

6 Numerical simulations and discussions

In this section, some numerical simulations are given to illustrate the results presented in this paper, we examine the three cases of harvesting $E_1 < E_2, E_1 > E_2$ and $E_1 = E_2$ (in left we plot u vs v and in the right (t,u) and (t,v)) spaces.

In this paper, we have investigated the effect of harvesting for a modified Leslie-Gower predator-prey model. A qualitative analysis show the impact of harvesting in terms of boundedness of solutions and the existence of an attraction set. According to different parameters of the model, the local stability of different equilibrium point is studied. We prove the global stability for the interior equilibrium using a Lyapunov function. Finally, by the Hamiltonian and the Pontryagin's maximum principle, the optimal harvesting are established and some numerical simulations are given to illustarte our results.

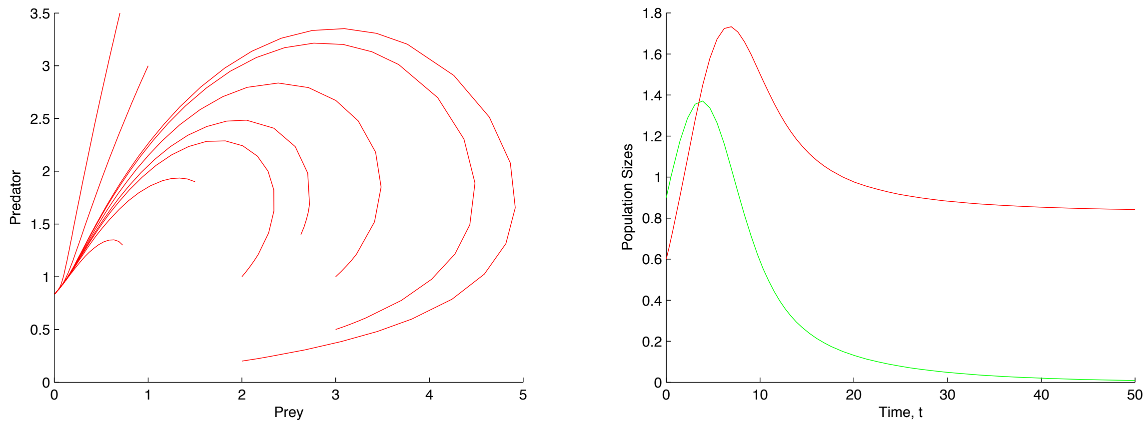

Fig. 3 Stability for P_2 for $E_1 < E_2, r_1 = 2; r_2 = 2.5; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_1 = 1.5; E_2 = 2$.

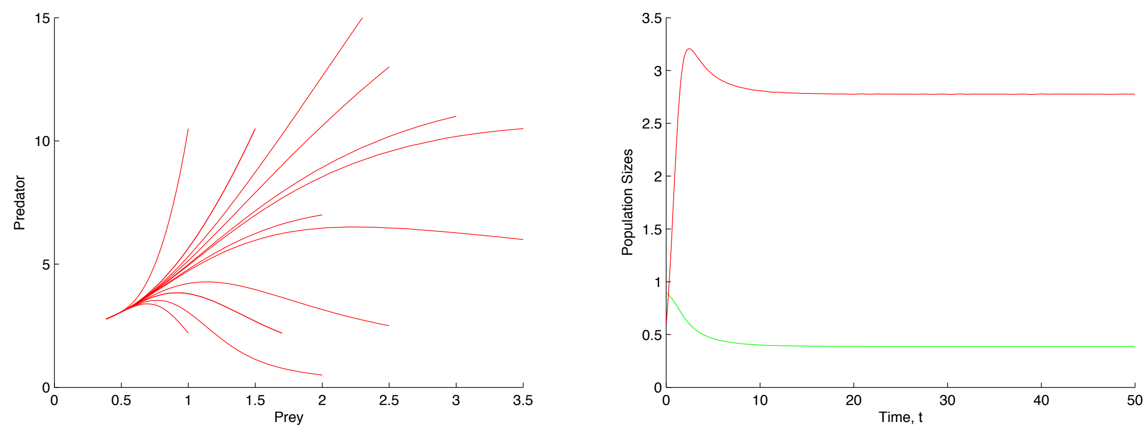

Fig. 4 Stability for P_3 for $E_1 < E_2, r_1 = 2; r_2 = 4; b_1 = 0.6; a_1 = 1; a_2 = 1; k_1 = 10; k_2 = 1; E_1 = 1.5; E_2 = 2$.

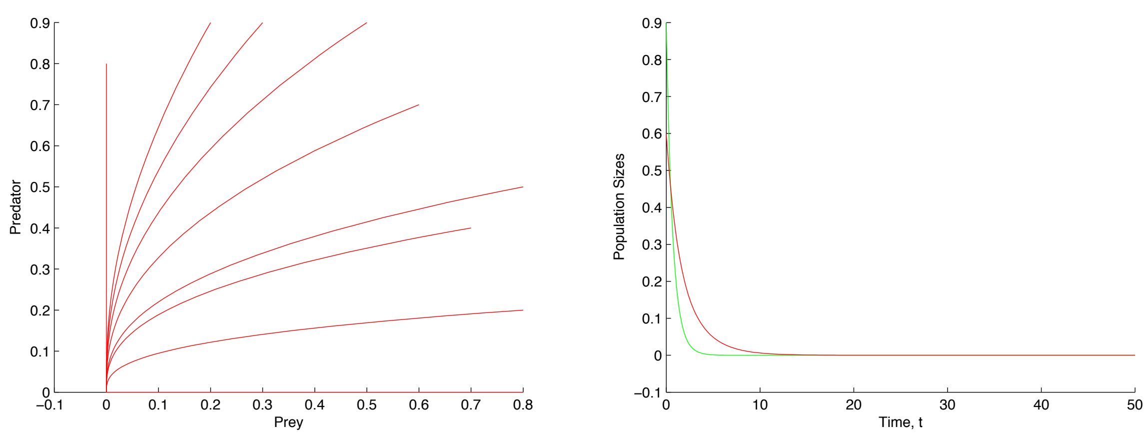

Fig. 5 Stability for P_0 for $E_1 > E_2, r_1 = 1.8; r_2 = 2.1; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_2 = 2.5; E_1 = 3$.

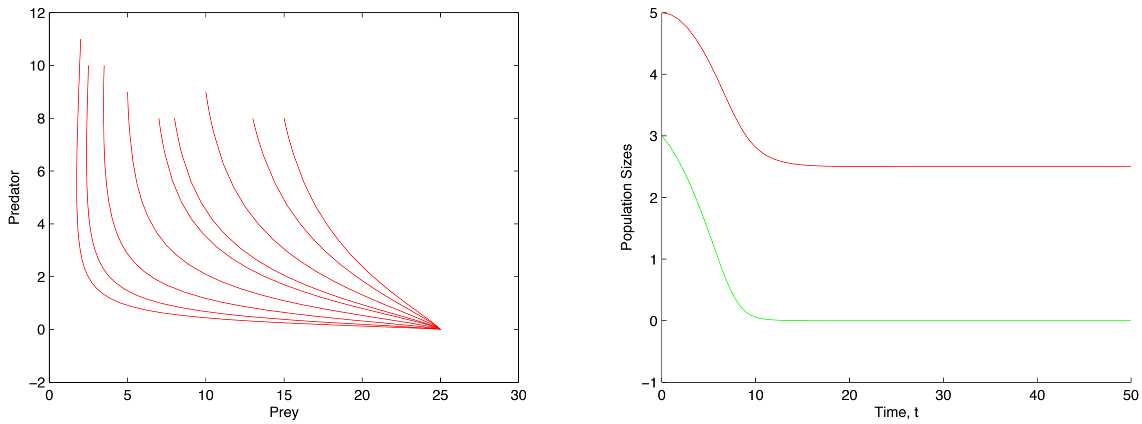

Fig. 6 Stability for P_1 for $E_1 > E_2, r_1 = 3; r_2 = 1; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_1 = 1.5; E_2 = 2$.

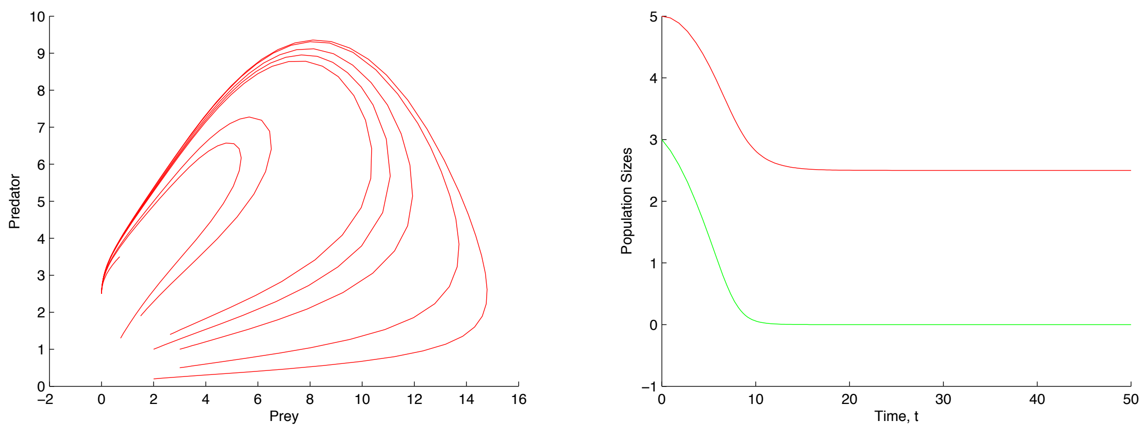

Fig. 7 Stability for P_2 for $E_1 > E_2, r_1 = 3; r_2 = 1.5; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 3; E_1 = 2; E_2 = 1$.

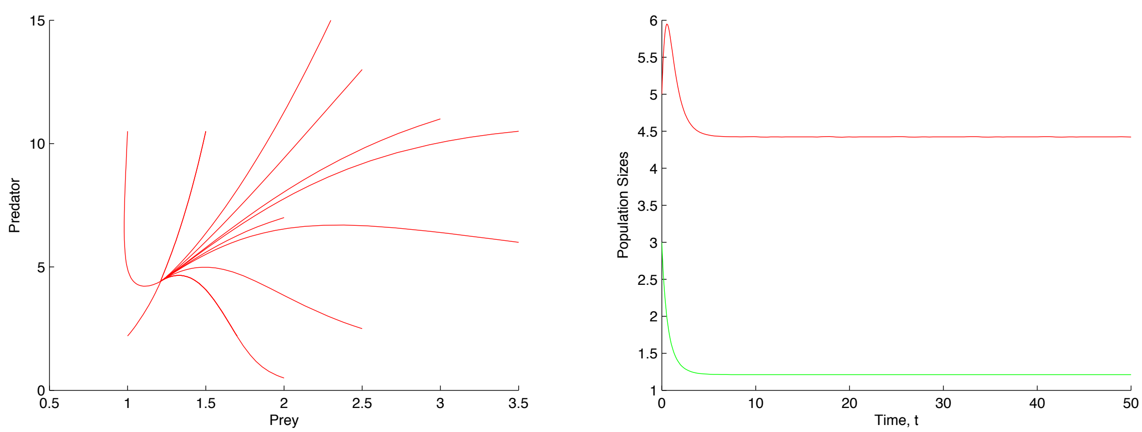

Fig. 8 Stability for P_3 for $E_1 > E_2, r_1 = 4; r_2 = 4; b_1 = 0.6; a_1 = 1; a_2 = 1; k_1 = 15; k_2 = 1; E_1 = 3; E_2 = 2$.

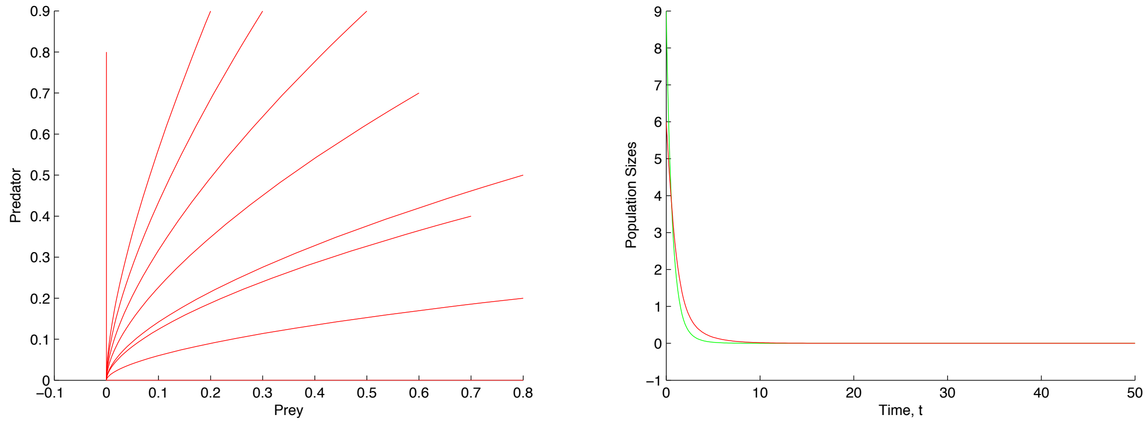

Fig. 9 Stability for P_0 for $E_1 = E_2, r_1 = 1.8; r_2 = 2.1; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_1 = 2.5; E_2 = 2.5.$

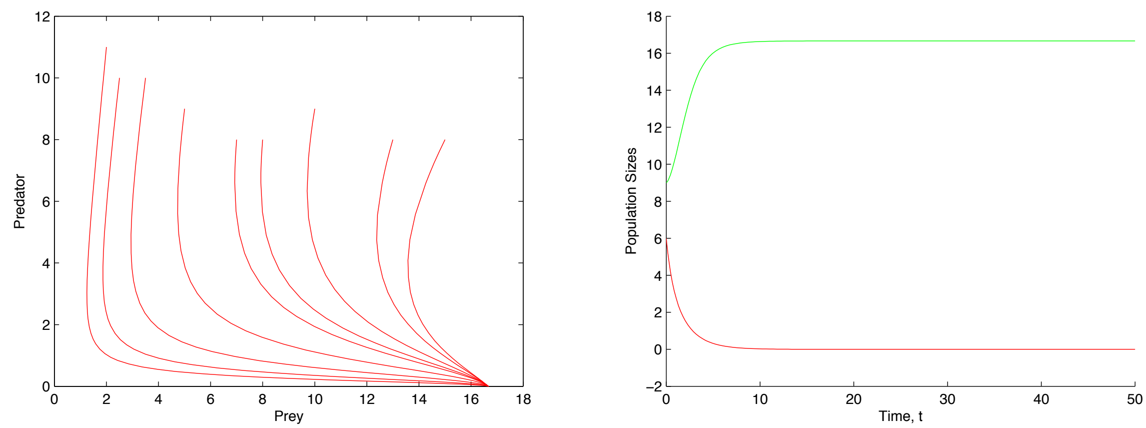

Fig. 10 Stability for P_1 for $E_1 = E_2, r_1 = 4; r_2 = 2.5; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 1; E_1 = 3; E_2 = 3.$

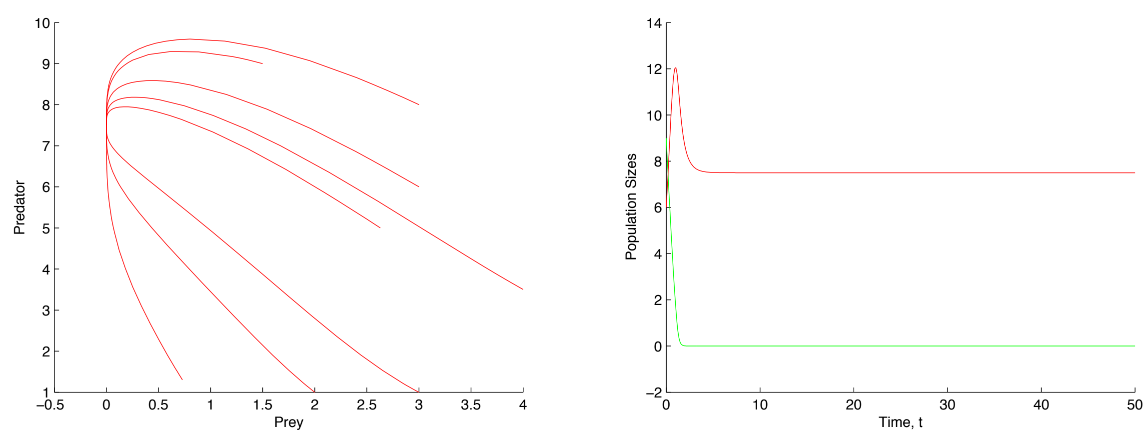

Fig. 11 Stability for P_2 for $E_1 = E_2, r_1 = 1; r_2 = 2.5; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 1; k_2 = 3; E_1 = 1; E_2 = 1.$

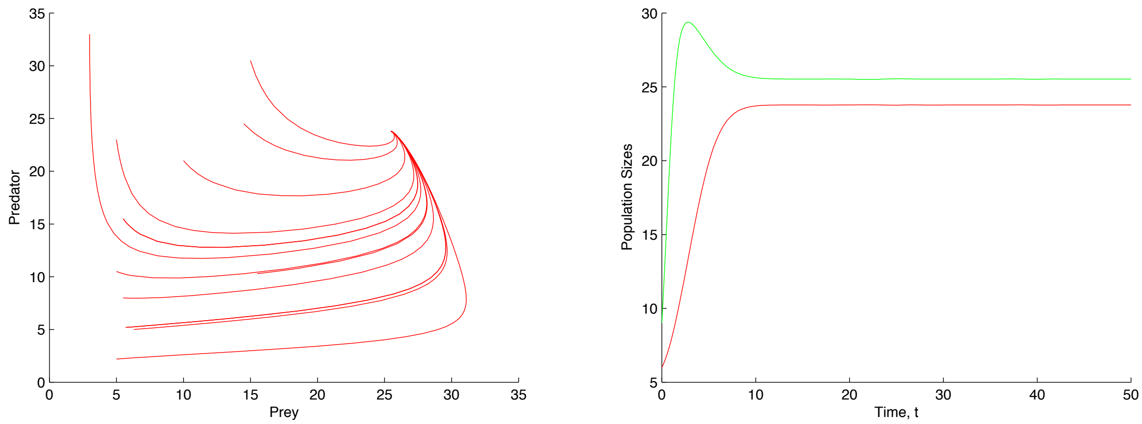

Fig. 12 Stability for P_3 for $E_1 = E_2$ $r_1 = 3; r_2 = 1.5; b_1 = 0.06; a_1 = 0.7; a_2 = 0.6; k_1 = 10; k_2 = 3; E_1 = 1; E_2 = 1$.

References

[1] M.A. Aziz-Alaoui, Study of a Leslie-Gower-type tritrophic population, Chaos Sol. and Fractals 14 (8), 1275- 1293, (2002).
[2] M.A. Aziz-Alaoui, M. Daher Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type IIschemes, Appl. Math. Lett. 16 (2003) 1069-1075.
[3] Brauer, F., Soudack, A.C.: Stability regions in predator-prey systems with constant-rate prey harvesting. J. Math. Biol. 8, 55-71 (1979).
[4] L. Chen, F. Chen, L. Chen, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge, Nonlinear Anal. Real World Appl. 11 (1) (2010).
[5] S. Chen, J. Shi, J. Wei, Global stability and Hopf bifurcation in a delayed diffusive Leslie-Gower predator-prey system, Internat. J. Bifur. Chaos 22(03) (2012). 246-252.
[6] C.W. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources. Wiley, New York (1976).
[7] P.H. Leslie, J.C. Gower, The properties of a stochastic model for the predator-prey type of interaction between two species, Biometrika 47 (1960) 219-231.
[8] A.J. Lotka, Elements of Physical Biology, Waverly Press, Williams Wilkins Company, Baltimore, MD,USA, 1925.
[9] E.P. Odum, Fundamentals of Ecology, Saunders, Philadelphia, 1971.
[10] A. Rojas-Palma, E. Gonzalez-Olivares, Optimal harvesting in a predator-prey model with Allee effect and sigmoid functional response. Appl. Math. Model. 36, 1864-1874 (2012).
[11] V. Volterra, Variations and fluctuations of the number of individuals in animal species living together, ICES J. Mar. Sci. 3 (1928) 3-51.
[12] R. Yafia, F. El Adnani and H. Talibi Limit cycle and numerical similations for small and large delays in a predator-prey model with modified Leslie-Gower and Holling-type II scheme, Nonlinear Analysis: Real World Applications Vol.9, (2008), pp: 2055-2067.
[13] R. Yafia, F. El Adnani and H. Talibi, Stability of limit cycle in a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay. Applied Mathematical Sciences, Vol. 1, no. 3, (2007), pp: 119-131.
[14] R. Zhang, J. Sun, H. Yang, Analysis of a prey-predator fishery model with prey researve. Appl. Math. Sci. 1, 2481-2492 (2007).
[15] Z. Zeng, M. Fan, Study on a non-autonomous predator-prey system with Beddington-DeAngelis functional response, Math. Comput. Modelling 48(11) (2008) 1755-1764.