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# Multispiral chaos

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**Abstract**

In this paper, we give piecewise linear (PWL) versions of dynamical systems such as Duffing equations. Then, by modifying the piecewise-linear function, various 'multispiral' strange attractors are shown. These attractors appear as a result of the combination of several 'one-spiral' attractors similar to Rossler's or similar to Chua's double scroll. This is demonstrated by some numerical results describing how the dynamic changes and gives rise to 'multispiral-attractor' as the number of segments of the piecewise-linearity increases. Bifurcation phenomena and transition from order to 'multispiral-chaos' are studied.

## 1 Introduction

In [1]–[3], systems of nonautonomous or autonomous differential equations having the piecewise-linear function, characteristic of Chua's resistor (see Fig. 1(a)),

$g(v_C) =$

$$G_A v_C + \frac{1}{2}(G_B - G_A)[|v_C + B_p| - |v_C - B_p|], \quad (1)$$

for which the corresponding dimensionless form is [4]:

$$f_2(x) = m_1 x + \frac{1}{2}(m_0 - m_1)[|x+1| - |x-1|], \quad (2)$$

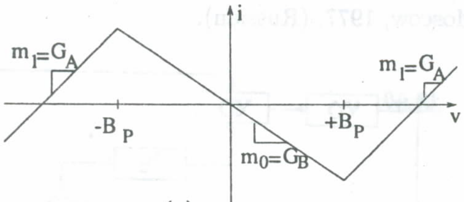


**Figure 1(a):** v-i characteristic of Chua's nonlinear resistor. The slopes of the inner and outer regions are $G_B$ and $G_A$, respectively, while $B_P$ indicate the breakpoints.

are extended to obtain systems showing 'multispiral' strange attractors (i.e. strange attractors with more than two spirals). This is achieved by modifying Chua's piecewise linearity in order to have additional segments (see Fig. 1(b)). That is $f_2(.)$ is replaced by the function $f_N(.)$. This 'extended' characteristic nonlinear element has additional segments compared to $f_2(.)$, allowing for multi-spiral strange attractors. It is given by

$$f_N(x) = \begin{cases} m_k x + sgn(x)\xi_k & \text{if } s_{k-1} \le |x| \le s_k, \\ & k \in \mathcal{I}_{N-2} \\ m_{N-1}x + sgn(x)\xi_{N-1} & \text{if } |x| \ge s_{N-2}, \end{cases} \quad (3)$$

where $N \in I\!N$, $N \ge 2$, and

- $\mathcal{I}_N = \{0, \cdots, N\}$, $\mathcal{I}_N^* = \{1, \cdots, N\}$;
- $(m_k)_{k \in \mathcal{I}_{N-1}}$ and $(\xi_k)_{k \in \mathcal{I}_{N-1}}$ are two finite and real sequences;
- $(s_k)_{k \in \mathcal{I}_{N-2}}$, is a finite and positive real sequence that is strictly increasing.

Furthermore, we will set

$$s_{-1} = 0 \text{ and } s_N = +\infty. \quad (4)$$

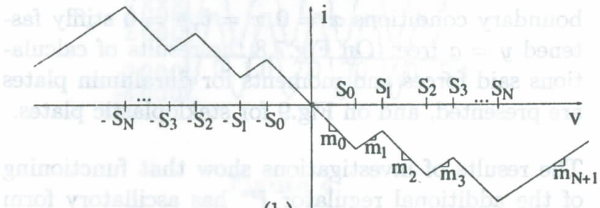


**Figure 1(b):** The $f_N$-characteristic of the nonlinear resistor in the multispiral case. $(s_k)_{k \in \mathcal{I}_{N-2}}$ indicate the breakpoints, while $(m_k)_{k \in \mathcal{I}_{N-1}}$ indicate the slopes of the segments.

The parameters $(m_k), k \in \mathcal{I}_{N-1}$, are the slopes of $f_N$ in each of the linear segments $[-s_0, s_0]$, $([s_{k-1}, s_k])_{k \in \mathcal{I}^*_{N-1}}$ respectively. Throughout the paper we will assume that

$$\xi_0 = 0 \text{ and } s_0 = 1. \tag{5}$$

**Remark** The function $f_N$ is continuous if the parameters $(\xi_k)_{k \in \mathcal{I}_{N-1}}$ satisfy :

$$\begin{aligned} &\forall \xi_0 \in I\!R,\ \forall k \in \mathcal{I}^*_{N-2}, \\ &\xi_{k+1} = (m_k - m_{k+1})s_k + \xi_k. \end{aligned} \tag{6}$$

As a result,

$$\begin{aligned} &\forall \xi_0 \in I\!R,\ \ \forall k \in \mathcal{I}^*_{N-1}, \\ &\xi_k = \xi_0 + \sum_{j=1}^{k} (m_{j-1} - m_j)s_{j-1}. \end{aligned} \tag{7}$$

See the proof in [1].

In the following sections we will give the dynamical multispiral systems, (different from those in [1]) that is, the systems which allow for the emergence of chaotic attractors with $N$ spirals ($N \geq 2$). In addition to some specific parameters, all these systems will also depend on the parameters of the set

$$\mathcal{E}_N = \{(s_k)_{k \in \mathcal{I}_{N-2}}, (m_k)_{k \in \mathcal{I}_{N-1}}\} \subset I\!R^{2N-1}. \tag{8}$$

However, in order to reduce the number of parameters needed to obtain multi-spiral attractors, the parameters $(m_k)_{k \in \mathcal{I}_{N-1}}$ are chosen to satisfy:

$$m_{2j} = m_0 \ \text{ and } \ m_{2j+1} = m_1, \ \ j = 1, 2, 3 \cdots, \tag{9}$$

and we will determine the parameters $(s_k)_{k \in \mathcal{I}^*_{N-2}}$ such that the following relationship is satisfied:

$$\mathcal{E}_{2j-2} \subset \mathcal{E}_{2j} \ \text{ and } \ \mathcal{E}_{2j-1} \subset \mathcal{E}_{2j+1}, \ j = 2, 3, 4 \cdots. \tag{10}$$

We will see that a large number of equilibrium states may exist in such systems (when $N$ increases), and this allows for the emergence of different types of strange attractors, especially multispiral ones. We will also study the transition to chaotic behaviour via sequences of period-doubling bifurcations of limit cycles.

All the phase portrait figures presented in this paper are done in the $x-y$ plane. They have been obtained by integrating the systems of differential equations using the most common fourth-order Runge-Kutta's method. In order to obtain reliable numerical results, the step size has been chosen to be equal to $10^{-3}$ (or $10^{-4}$ ), and the first $10^6$ (or $10^7$ ) steps are discarded to avoid the transient regime.

## 2 PWL-Duffing System

The classical Duffing equations are given by

$$\left\{ \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^3 - \varepsilon y + \gamma \cos(\omega t), \end{aligned} \right. \tag{11}$$

where $x$ and $y$ are functions of $t$. For the values of the parameters given by

$$\varepsilon = 0.25,\ \gamma = 0.3,\ \omega = 1.0, \tag{12}$$

the phase portrait is given by Fig. 2(a).

The simple corresponding PWL version of this system is obtained by repalcing $x^3 - x$ by $f_2(x)$ given by Eq. (2), that is

$$\left\{ \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -f_2(x) - \varepsilon y + \gamma \cos(\omega t), \end{aligned} \right. \tag{13}$$

Figure 2(b) shows the phase portrait of this equation for the parameters given by (12) and for

$$m_0 = -0.0845,\ m_1 = 0.66, \tag{14}$$

(we have used the initial condition $X_0 = (-0.3, 0.02)$).

When the set of parameters $\mathcal{D}_2 = \mathcal{E}_2 \cup \{\varepsilon, \gamma, \omega\}$ is determined, (it corresponds to system (13) and allows for 2-spiral attractors), the new set of parameters allowing 4-spiral attractors will be $\mathcal{D}_4 = \mathcal{D}_2 \cup \{s_1, s_2\}$, where we only have to look for two new parameters $s_1$ and $s_2$. Then the new set allowing 6-spiral attractors will be $\mathcal{D}_6 = \mathcal{D}_4 \cup \{s_3, s_4\}$ and so on .

Figures 2 (c) and 2(d) present respectively four-spiral attractor and six-spiral attractor for PWL-Duffing equations.

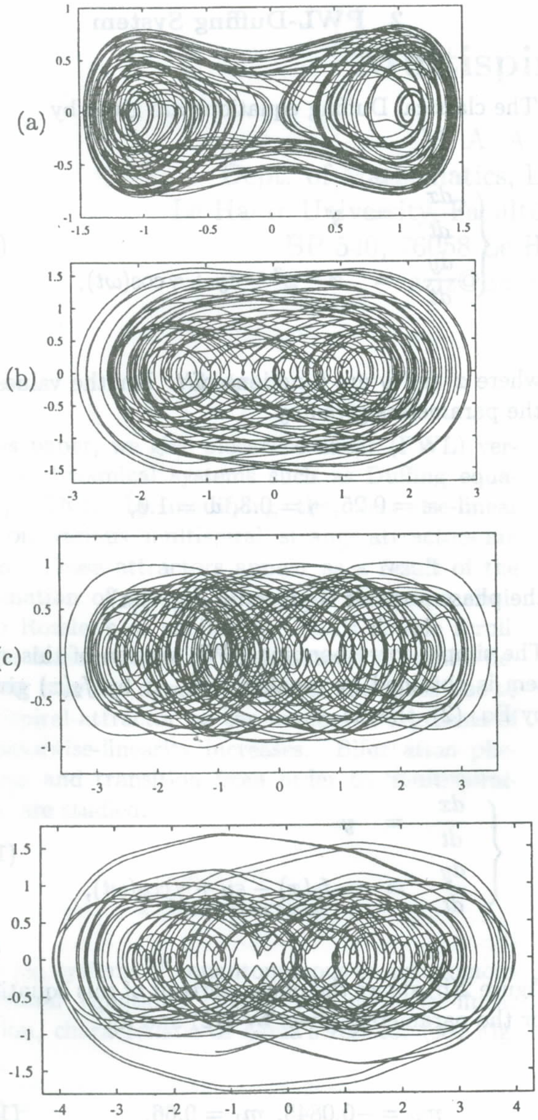


**Figure 2:** Phase portraits of (a) simple Duffing equation, and the PWL version of Duffing system with (b) 2 spirals, (c) 4 spirals ($s_1 = 1.4, s_2 = 2.7$) (d) 6 spirals ($s_3 = 3.67, s_4 = 4.5$). The other parameters are given by Eqs.(9, 12, 14).

## 3 Chua's system

In this section, we deal with Chua's system for which the dimensionless state equations are given by

$$\left\{\begin{array}{rcl} \dfrac{dx}{dt} & = & \alpha(y - x - f_2(x)) \\ \dfrac{dy}{dt} & = & x - y + z \\ \dfrac{dz}{dt} & = & -\beta y - \gamma z, \end{array}\right. \tag{15}$$

where $f_2(.)$ is given by Eq. (2), and $x, y$ and $z$ are functions of $t$. To obtain strange chaotic attractors with $N$ spirals, $N \geq 2$, we integrate the differential system

$$\left\{\begin{array}{rcl} \dfrac{dx}{dt} & = & \alpha(y - x - f_N(x)) \\ \dfrac{dy}{dt} & = & x - y + z \\ \dfrac{dz}{dt} & = & -\beta y - \gamma z, \end{array}\right. \tag{16}$$

where $f_N(.)$ is given by Eq. (3). This system depends on the parameters of the set

$$\mathcal{C}_N = \{\alpha, \beta, \gamma\} \cup \mathcal{E}_N \subset I\!R^{2N+2}, \tag{17}$$

where $\mathcal{E}_N$ is given by Eq. (8).

To obtain the 10-spiral strange attractor given by Fig. 3, we have employed the parameters

$$\left\{\begin{array}{l} \beta = 11.79,\ \gamma = 0.04, \\ m_0 = m_2 = m_4 = m_6 = m_8 = -1.4, \\ m_1 = m_3 = m_5 = m_7 = m_9 = -0.6, \\ s_0 = 1,\ s_1 = 1.9,\ s_2 = 2.6, \\ s_3 = 3.75,\ s_4 = 4.75,\ s_5 = 5.85, \\ s_6 = 6.46,\ s_7 = 7.5,\ s_8 = 8.55. \end{array}\right. \tag{18}$$

The value of parameter $\alpha$ is given in the caption of Fig. 3.

(a)
(b)
(c)
(d)

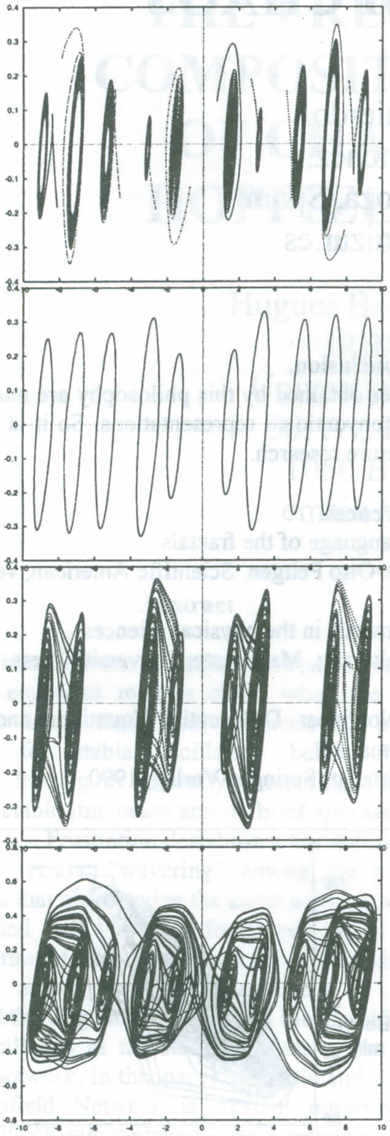

**Figure 3:** Phase portraits of coexisting attractors, for the same parameters and various initial conditions, illustrating the birth of a 10-spiral strange atractor for Chua's system. The parameters are given by Eq.(18), and by (a) $\alpha = 6.1$, (b) $\alpha = 6.4$, (c) $\alpha = 8.0$, (d) $\alpha = 9.35$.

## 4 Conclusion

In this paper we give numerical results on multi-spiral attractors for piecewise linear dynamical systems different from those given in the references we cite. These attractors appear as a result of the combination of several one-spiral attractors similar to Rossler's or to Chua's double scroll. Figures displaying this phenomenon are done.

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