Computers and Mathematics with Applications 59 (2010) 1462–1472

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A multi-step differential transform method and application to non-chaotic or chaotic systems

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ARTICLE INFO

Article history:
Received 27 March 2009
Received in revised form 1 November 2009
Accepted 2 November 2009

Keywords:
Differential transform method
Multi-step differential transform method
Lotka–Volterra system
Chen system
Lorenz system

ABSTRACT

The differential transform method (DTM) is an analytical and numerical method for solving a wide variety of differential equations and usually gets the solution in a series form. In this paper, we propose a reliable new algorithm of DTM, namely multi-step DTM, which will increase the interval of convergence for the series solution. The multi-step DTM is treated as an algorithm in a sequence of intervals for finding accurate approximate solutions for systems of differential equations. This new algorithm is applied to Lotka–Volterra, Chen and Lorenz systems. Then, a comparative study between the new algorithm, multi-step DTM, classical DTM and the classical Runge–Kutta method is presented. The results demonstrate reliability and efficiency of the algorithm developed.

1. Introduction

The differential transform method (DTM) is a numerical as well as analytical method for solving integral equations, ordinary and partial differential equations. The method provides the solution in terms of convergent series with easily computable components. The concept of the differential transform was first proposed by Zhou [1] and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the nth derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computations of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found in [2–23].

The DTM introduces a promising approach for many applications in various domains of science. However, DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. It is the purpose of this paper is to propose a reliable algorithm of the DTM. The new algorithm, multi-step DTM, presented in this paper, accelerates the convergence of the series solution over a large region and improve the accuracy of the DTM. The validity of the modified technique is verified through illustrative examples of Lotka–Volterra, Chen and Lorenz systems.

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doi:10.1016/j.camwa.2009.11.005

Table 1
Operations of differential transformation.

Original function	Transformed function
$f(t) = u(t) \pm v(t)$	$F(k) = U(k) \pm V(k)$
$f(t) = \alpha u(t)$	$F(k) = \alpha U(k)$
$f(t) = u(t)v(t)$	$F(k) = \sum_{l=0}^{k} V(l)U(k-l)$
$f(t) = \frac{du(t)}{dt}$	$F(k) = (k+1)U(k+1)$
$f(t) = \frac{d^m u(t)}{dt^m}$	$F(k) = (k+1)(k+2)\cdots(k+m)U(k+m)$
$f(t) = \int_{t_0}^{t} u(x)dx$	$F(k) = \frac{U(k-1)}{k}, k \geq 1$
$f(t) = t^m$	$F(k) = \delta(k-m)$
$f(t) = \exp(\lambda t)$	$F(k) = \frac{\lambda^k}{k!}$
$f(t) = \sin(wt + \alpha)$	$F(k) = \frac{w^k}{k!} \sin(\pi k/2 + \alpha)$
$f(t) = \cos(wt + \alpha)$	$F(k) = \frac{w^k}{k!} \cos(\pi k/2 + \alpha)$

2. Differential transform method

The differential transform technique is one of the semi-numerical analytical methods for ordinary and partial differential equations that uses the form of polynomials as approximations of the exact solutions that are sufficiently differentiable. The basic definition and the fundamental theorems of the DTM and its applicability for various kinds of differential equations are given in [2–5]. For convenience of the reader, we present a review of the DTM. The differential transform of the kth derivative of function $f(t)$ is defined as follows,

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0}, \tag{1}$$

where $f(t)$ is the original function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$ is defined as,

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^k. \tag{2}$$

From Eqs. (1) and (2), we get,

$$f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{d^k f(t)}{dt^k}\bigg|_{t=t_0}, \tag{3}$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. For implementation purposes, the function $f(t)$ is expressed by a finite series and Eq. (2) can be written as,

$$f(t) \approx \sum_{k=0}^{N} F(k)(t-t_0)^k, \tag{4}$$

here N is decided by the convergence of natural frequency. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1. The main steps of the DTM, as a tool for solving different classes of nonlinear problems, are the following. First, we apply the differential transform (1) to the given problem (integral equation, ordinary differential equation or partial differential equations), then the result is a recurrence relation. Second, solving this relation and using the differential inverse transform (2) we can obtain the solution of the problem.

3. Multi-step differential transform method

Although the DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks: the series solution always converges in a very small region and it has slow convergent rate or completely divergent in the wider region [5–8]. To overcome the shortcoming, we present in this section the multi-step DTM that we have developed for the numerical solution of differential equations. For this purpose, we consider the following nonlinear initial value problem,

$$f(t, u, u', \ldots, u^{(p)}) = 0, \tag{5}$$

subject to the initial conditions $u^{(k)}(0) = c_k$, for $k = 0, 1, \ldots, p-1$.

Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (5). In actual applications of the DTM, the approximate solution of the initial value problem (5) can be expressed by the finite series,

$$u(t) = \sum_{n=0}^{N} a_n t^n \quad t \in [0, T]. \tag{6}$$

The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[0, T]$ is divided into M subintervals $[t_{m-1}, t_m]$, $m = 1, 2, \dots, M$ of equal step size $h = T/M$ by using the nodes $t_m = mh$. The main ideas of the multi-step DTM are as follows. First, we apply the DTM to Eq. (5) over the interval $[0, t_1]$, we will obtain the following approximate solution,

$$u_1(t) = \sum_{n=0}^{K} a_{1n} t^n, \quad t \in [0, t_1], \tag{7}$$

using the initial conditions $u_1^{(k)}(0) = c_k$. For $m \geq 2$ and at each subinterval $[t_{m-1}, t_m]$ we will use the initial conditions $u_m^{(k)}(t_{m-1}) = u_{m-1}^{(k)}(t_{m-1})$ and apply the DTM to Eq. (5) over the interval $[t_{m-1}, t_m]$, where t_0 in Eq. (1) is replaced by t_{m-1}. The process is repeated and generates a sequence of approximate solutions $u_m(t)$, $m = 1, 2, \dots, M$, for the solution $u(t)$,

$$u_m(t) = \sum_{n=0}^{K} a_{mn} (t - t_{m-1})^n, \quad t \in [t_m, t_{m+1}], \tag{8}$$

where $N = K \cdot M$. In fact, the multi-step DTM assumes the following solution,

$$u(t) = \begin{cases} u_1(t), & t \in [0, t_1] \\ u_2(t), & t \in [t_1, t_2] \\ \vdots & \\ u_M(t), & t \in [t_{M-1}, t_M]. \end{cases} \tag{9}$$

The new algorithm, multi-step DTM, is simple for computational performance for all values of h. It is easily observed that if the step size $h = T$, then the multi-step DTM reduces to the classical DTM. As we will see in the next section, the main advantage of the new algorithm is that the obtained series solution converges for wide time regions and can approximate non-chaotic or chaotic solutions.

4. Numerical experiments

To demonstrate the effectiveness of the proposed algorithm as an approximate tool for solving nonlinear problems of differential equations, we apply the proposed algorithm, the multi-step DTM, to Lotka–Volterra, Chen and Lorenz systems.

4.1. Lotka–Volterra system

Lotka–Volterra equations are a pair of first order, nonlinear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, a predator and its prey. They were proposed independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926,

$$\begin{cases} \dfrac{dx}{dt} = x(t)(a - by(t)), \\ \dfrac{dy}{dt} = -y(t)(c - dx(t)), \end{cases} \tag{10}$$

subject to the initial conditions $x(0) = c_x$ and $y(0) = c_y$, where t represents the time, y is the predator density, x is the prey density, a, b, c and d are positive parameters representing the interaction of the two species. In view of the differential transform, given in Eq. (1), and the operations of differential transformation given in Table 1, applying the differential transform to Lotka–Volterra system (10), we obtain,

$$\begin{cases} X(k+1) = \dfrac{1}{k+1} \left\{ aX(k) - b \displaystyle\sum_{l=0}^{k} X(l)Y(k-l) \right\}, \\ Y(k+1) = \dfrac{1}{k+1} \left\{ -cY(k) + d \displaystyle\sum_{l=0}^{k} X(l)Y(k-l) \right\}, \end{cases} \tag{11}$$

where $X(k)$ and $Y(k)$ are the differential transformation of $x(t)$ and $y(t)$, respectively. The differential transform of the initial conditions are given by $X(0) = c_x$ and $Y(0) = c_y$. In view of the differential inverse transform, given in Eq. (2), the DTM

Table 2
Approximate solutions for Lotka–Volterra system (10), when $a = 0.95$, $b = 0.25$, $c = 2.45$, $d = 0.25$, $c_x = 15$ and $c_y = 8$, obtained using multi-step DTM and RK4 method.

t	Multi-step DTM		RK4	
	$x(t)$	$y(t)$	$x(t)$	$y(t)$
0	15	8	15	8
1	4.72786232	5.60596008	4.72786231	5.60596103
2	5.59022815	1.58484276	5.59022854	1.58484276
3	10.90426318	0.97587480	10.90426461	0.97587449
4	18.33290002	3.61672098	18.33289906	3.61672217
5	6.79300603	8.81062122	6.79300590	8.81062070
6	4.63976338	2.62171097	4.63976356	2.62171077
7	8.07809733	1.01629787	8.07809832	1.01629760
8	15.85821888	1.62121047	15.85822082	1.62121043
9	12.45957157	9.29003141	12.45957037	9.29003037
10	4.50205906	4.60359686	4.50205829	4.60359718

series solution for Lotka–Volterra system (10) can be obtained as,

$$\begin{cases} x(t) = \displaystyle\sum_{n=0}^{N} X(n)t^n, \\ y(t) = \displaystyle\sum_{n=0}^{N} Y(n)t^n. \end{cases} \tag{12}$$

Now, according to the multi-step DTM, taking $N = K \cdot M$, the series solution for Lotka–Volterra system (10) is given by,

$$x(t) = \begin{cases} \displaystyle\sum_{n=0}^{K} X_1(n)t^n, & t \in [0, t_1] \\ \displaystyle\sum_{n=0}^{K} X_2(n)(t-t_1)^n, & t \in [t_1, t_2] \\ \vdots \\ \displaystyle\sum_{n=0}^{K} X_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{13}$$

$$y(t) = \begin{cases} \displaystyle\sum_{n=0}^{K} Y_1(n)t^n, & t \in [0, t_1] \\ \displaystyle\sum_{n=0}^{K} Y_2(n)(t-t_1)^n, & t \in [t_1, t_2] \\ \vdots \\ \displaystyle\sum_{n=0}^{K} Y_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{14}$$

where $X_i(n)$ and $Y_i(n)$, for $i = 1, 2, \ldots, M$, satisfy the following recurrence relations,

$$\begin{cases} X_i(k+1) = \dfrac{1}{k+1}\left\{aX_i(k) - b\displaystyle\sum_{l=0}^{k} X_i(l)Y_i(k-l)\right\}, \\ Y_i(k+1) = \dfrac{1}{k+1}\left\{-cY_i(k) + d\displaystyle\sum_{l=0}^{k} X_i(l)Y_i(k-l)\right\}, \end{cases} \tag{15}$$

such that $X_i(0) = X_{i-1}(0)$ and $Y_i(0) = Y_{i-1}(0)$. Finally, if we start with $X_0(0) = c_x$ and $Y_0(0) = c_y$, using the recurrence relations given in (15), then we can obtain the multi-step solution given in Eqs. (13) and (14). Fig. 1 shows the approximate solutions for Lotka–Volterra system (10) obtained using multi-step DTM and RK4 method. In (a) and (b) we take the parameters $a = 0.95$, $b = 0.25$, $c = 2.45$, $d = 0.25$, $c_x = 15$ and $c_y = 8$ and in (c) and (d) we take the parameters $a = 0.95$, $b = 0.2$, $c = 2.5$, $d = 0.35$, $c_x = 10$ and $c_y = 3$. We can observe that the phase plane trajectories obtained using the multi-step DTM are in high agreement with the phase plane trajectories obtained using RK4 method.

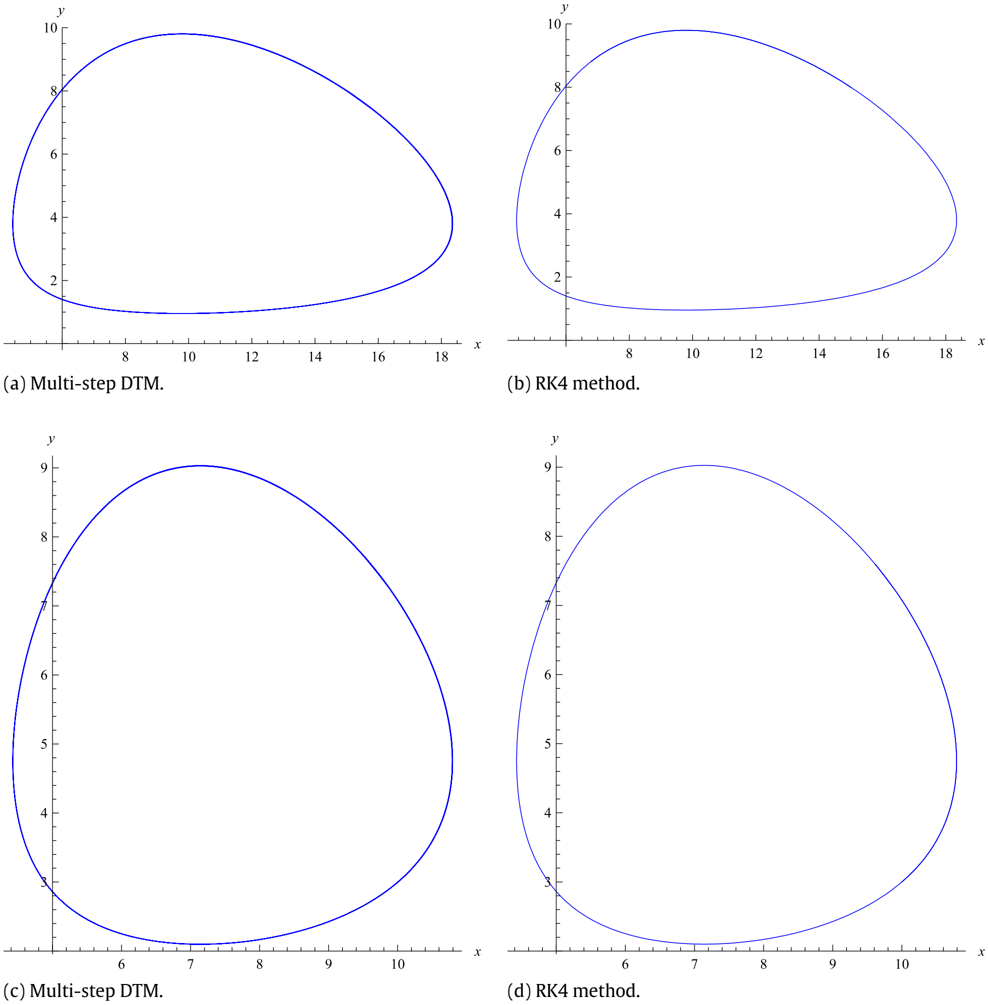


Fig. 1. Comparison between multi-step DTM and RK4 method. Phase plane trajectories for Lotka–Volterra system (10): (a) and (b) using the parameters $a = 0.95$, $b = 0.25$, $c = 2.45$, $d = 0.25$, $c_x = 15$ and $c_y = 8$; (c) and (d) using the parameters $a = 0.95$, $b = 0.2$, $c = 2.5$, $d = 0.35$, $c_x = 10$ and $c_y = 3$.

Table 2 shows the approximate solutions for Lotka–Volterra system (10), when $a = 0.95$, $b = 0.25$, $c = 2.45$, $d = 0.25$, $c_x = 15$ and $c_y = 8$, obtained using multi-step DTM and RK4 method. The high agreement of the approximate solutions emphasizes the efficiency of the multi-step DTM in handling systems of differential equations.

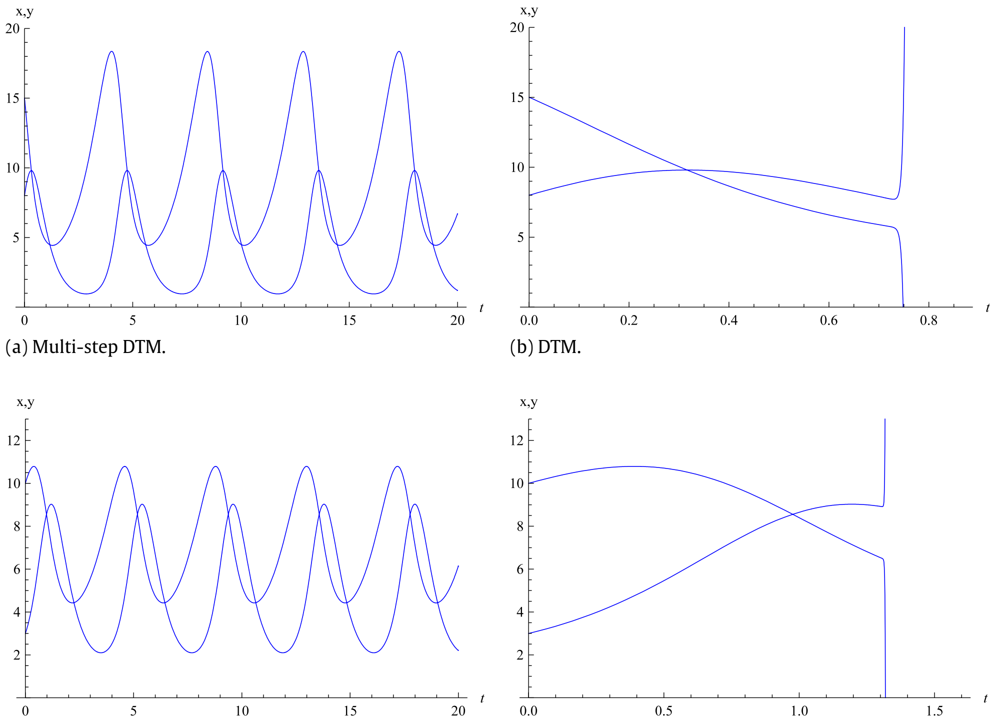
Fig. 2 shows the approximate solutions for Lotka–Volterra system (10) obtained using multi-step DTM and DTM using the same parameters as in Fig. 1. One can see that the multi-step DTM gives the solution over large time intervals while the classical DTM solution diverges for $t > 0.75$ in (b) and $t > 1.35$ in (d). It is to be noted that the multi-step DTM results are obtained when $K = 10$, $M = 100$, $T = 20$ and the DTM results are obtained when $N = 1000$.

4.2. Chen system

Now, we take Chen system into consideration. In 1999, the Chen system is found by Chen and Ueta [24]. The nonlinear differential equations that describe the Chen system are,

$$\begin{cases} \dfrac{dx}{dt} = a(y - x), \\ \dfrac{dy}{dt} = (c - a)x - xz + cy, \\ \dfrac{dz}{dt} = xy - bz, \end{cases} \tag{16}$$

where a, b and c are positive real numbers. Bifurcation studies [25,26] show that Chen system has chaotic behavior for the parameters $(a, b, c) = (35, 3, 28)$ and $(a, b, c) = (35, 12, 28)$. According to the multi-step DTM, the series solution for



(a) Multi-step DTM. (b) DTM.

(c) Multi-step DTM. (d) DTM.

Fig. 2. Time series for Lotka–Volterra system (10), comparison between multi-step DTM and DTM. Obviously the DTM solution diverges for $t > 0.75$ in (b) and $t > 1.35$ in (d).

Chen system (16) is given by,

$$(x(t), y(t), z(t)) = \begin{cases} \sum_{n=0}^{K} (X_1(n), Y_1(n), Z_1(n)) t^n, & t \in [0, t_1] \\ \sum_{n=0}^{K} (X_2(n), Y_2(n), Z_2(n)) (t - t_1)^n, & t \in [t_1, t_2] \\ \vdots \\ \sum_{n=0}^{K} (X_M(n), Y_M(n), Z_M(n)) (t - t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{17}$$

where $X_i(n)$, $Y_i(n)$ and $Z_i(n)$, for $i = 1, 2, \dots, M$, satisfy the following recurrence relations,

$$\begin{cases} X_i(k+1) = \dfrac{a}{k+1} \left\{ Y_i(k) - X_i(k) \right\}, \\ Y_i(k+1) = \dfrac{1}{k+1} \left\{ (c-a) X_i(k) - \sum_{l=0}^{k} X_i(l) Z_i(k-l) + c Y_i(k) \right\}, \\ Z_i(k+1) = \dfrac{1}{k+1} \left\{ \sum_{l=0}^{k} X_i(l) Y_i(k-l) - b Z_i(k) \right\}, \end{cases} \tag{18}$$

such that $X_1(0) = x(0)$, $Y_1(0) = y(0)$, $Z_1(0) = z(0)$ and $X_i(0) = X_{i-1}(0)$, $Y_i(0) = Y_{i-1}(0)$, $Z_i(0) = Z_{i-1}(0)$, for $i = 2, 3, \dots, M$. Figs. 3 and 4 show chaotic attractors for Chen system (16) when $(a, b, c) = (35, 3, 28)$ and $(a, b, c) = (35, 12, 28)$, respectively, using the multi-step DTM solution, at step size $h = 0.01$ and $K = 20$, and RK4 method. We can observe the high agreement between the chaotic attractors obtained using the multi-step DTM and the one obtained using RK4 method for the chaotic Chen system (16). This is much more obvious from Fig. 4.

Fig. 5 shows time series for the chaotic Chen system (16) when $(a, b, c) = (35, 12, 28)$ using the multi-step DTM solution, at step size $h = 0.01$ and $K = 15$, and the DTM, when $N = 1000$. Also in this figure we can see that the multi-step DTM gives the solution over a large interval while the classical DTM diverges for $t > 0.135$.

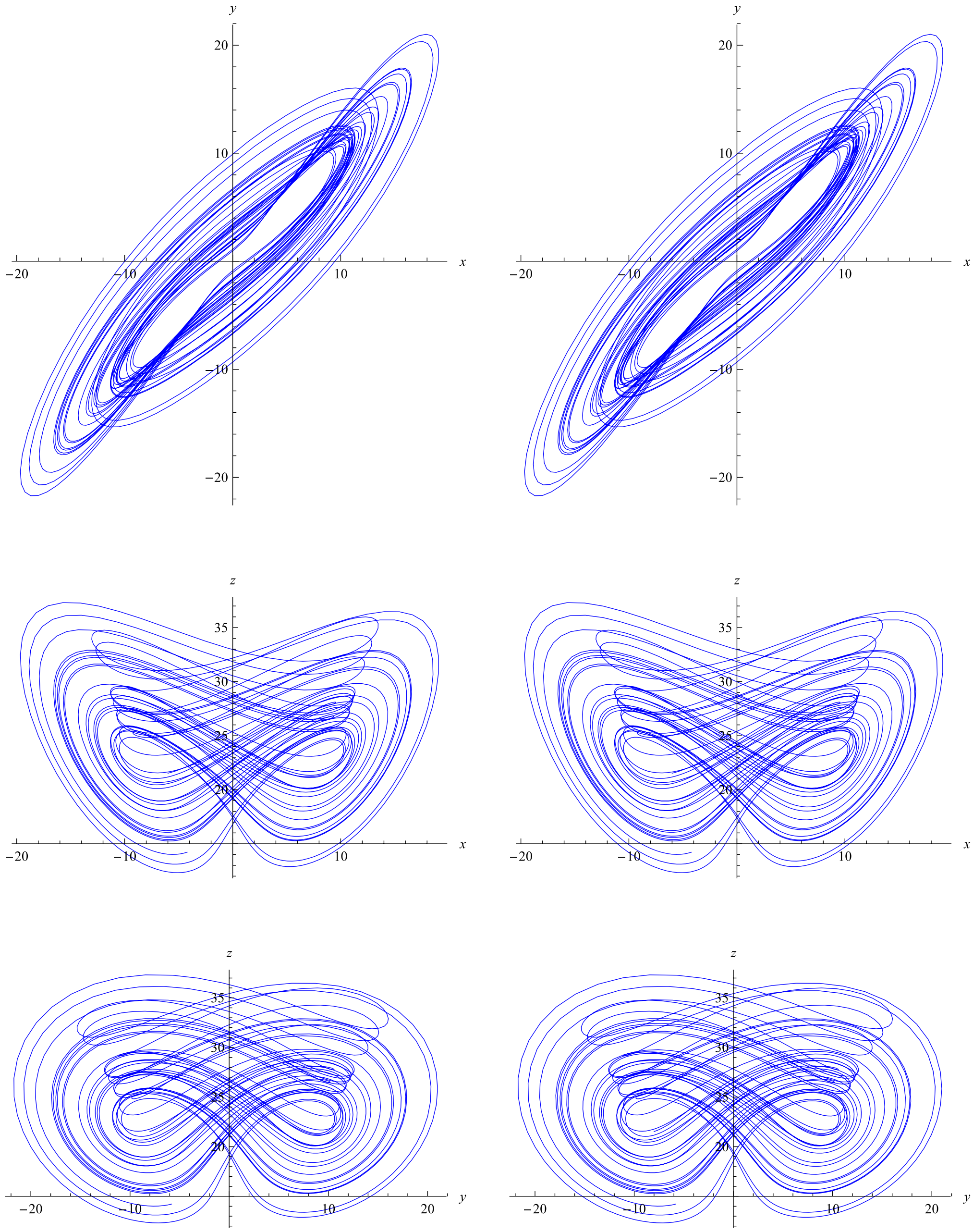


Fig. 3. Chaotic attractors for the chaotic Chen system (16), when $(a, b, c) = (35, 3, 28)$, comparison between multi-step DTM (Left) and RK4 method (Right).

4.3. Lorenz system

Here, we take Lorenz system into consideration. In 1963, the Lorenz system is introduced by Edward Lorenz [27]. The nonlinear differential equations that describe the Lorenz system are,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a(y - x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = cx - xz - y, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = xy - bz, \end{cases} \tag{19}$$

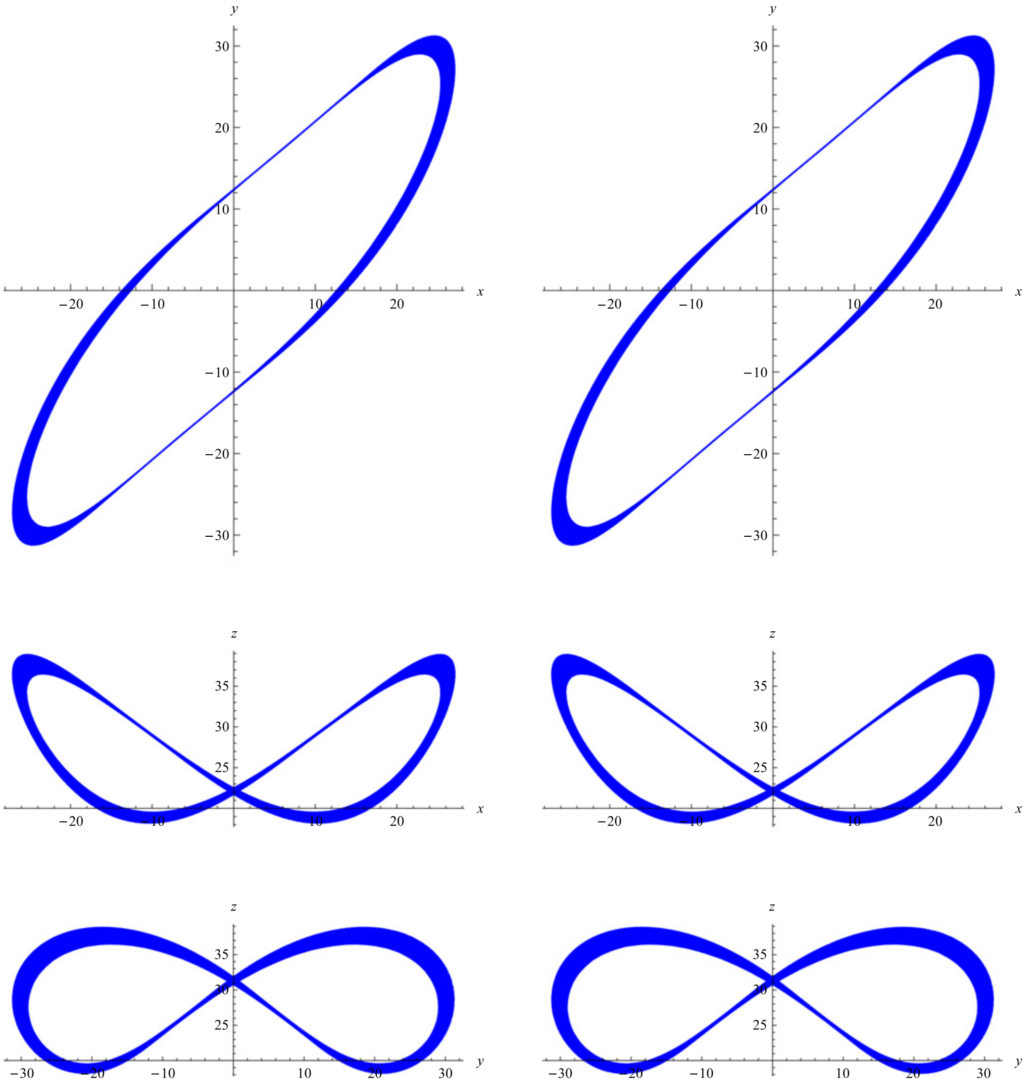
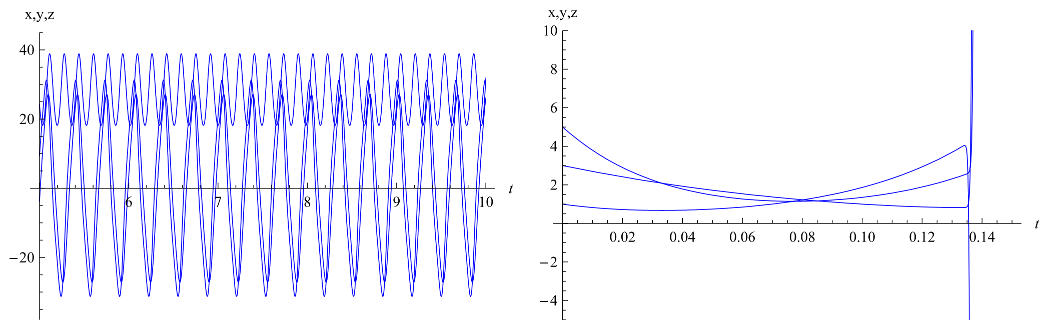


Fig. 4. Chaotic attractors for the chaotic Chen system (16), when $(a, b, c) = (35, 12, 28)$, comparison between multi-step DTM (Left) and RK4 method (Right).



(a) Multi-step DTM.

(b) DTM.

Fig. 5. Time series for the chaotic Chen system (16), comparison between multi-step DTM and DTM. Obviously the DTM solution diverges for $t > 0.135$.

where a, b and c are positive real numbers. For the parameters $(a, b, c) = (10, 8/3, 28)$ Lorenz system can display chaotic attractors. According to the multi-step DTM, the series solution for Lorenz system (19) is given in Eq. (17) where $X_i(n)$, $Y_i(n)$

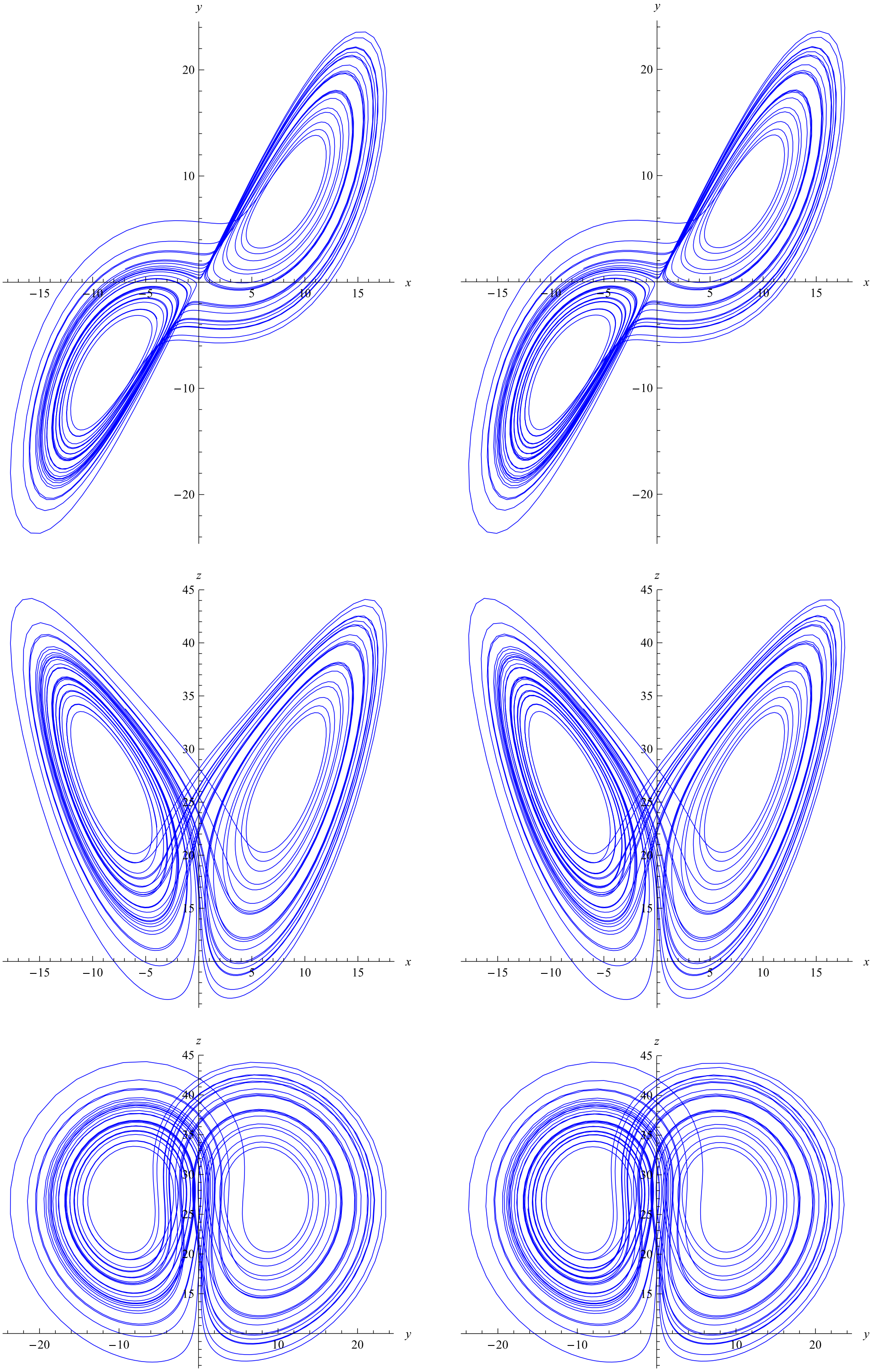
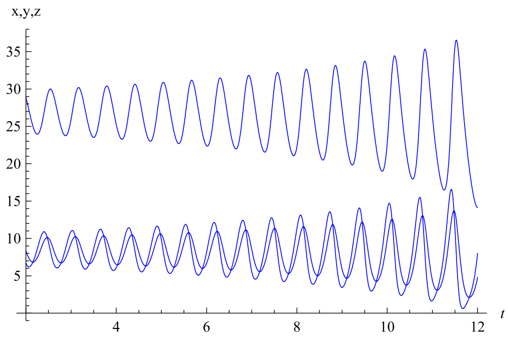
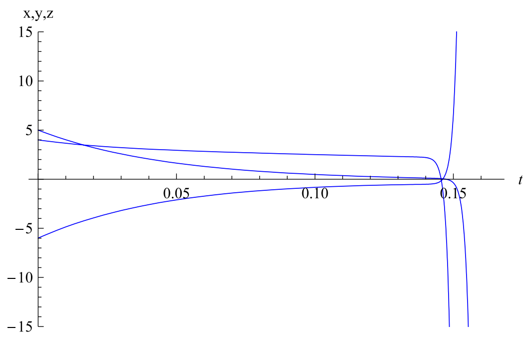


Fig. 6. Chaotic attractors for the chaotic Lorenz system (19), when $(a, b, c) = (10, 8/3, 28)$, comparison between multi-step DTM (Left) and RK4 method (Right).



(a) Multi-step DTM.



(b) DTM.

Fig. 7. Time series for the chaotic Lorenz system (19), comparison between multi-step DTM and DTM. Obviously the DTM solution diverges for $t > 0.15$.

and $Z_i(n)$, for $i = 1, 2, \ldots, M$, satisfy the following recurrence relations,

$$\begin{cases} X_i(k+1) = \dfrac{a}{k+1}\left\{Y_i(k) - X_i(k)\right\}, \\ Y_i(k+1) = \dfrac{1}{k+1}\left\{cX_i(k) - \displaystyle\sum_{l=0}^{k} X_i(l)Z_i(k-l) - Y_i(k)\right\}, \\ Z_i(k+1) = \dfrac{1}{k+1}\left\{\displaystyle\sum_{l=0}^{k} X_i(l)Y_i(k-l) - bZ_i(k)\right\}, \end{cases} \tag{20}$$

such that $X_1(0) = x(0)$, $Y_1(0) = y(0)$, $Z_1(0) = z(0)$ and $X_i(0) = X_{i-1}(0)$, $Y_i(0) = Y_{i-1}(0)$, $Z_i(0) = Z_{i-1}(0)$, for $i = 2, 3, \ldots, M$. Fig. 6 shows the chaotic attractors for the chaotic Lorenz system (19) when $(a, b, c) = (10, 8/3, 28)$ using the multi-step DTM solution, at step size $h = 0.01$ and $K = 20$, and RK4 method. As also shown for Chen system, we can observe the high agreement between the chaotic attractors obtained using the multi-step DTM and the one obtained using RK4 method for the chaotic Lorenz system (19). Fig. 7 shows the time series for the chaotic Lorenz system (19) when $(a, b, c) = (10, 8/3, 28)$ using the multi-step DTM solution, at step size $h = 0.01$ and $K = 15$, and the DTM, when $N = 1000$. Also in this figure we can see that the DTM solution diverges when $t > 0.15$.

5. Discussion

In this work, we carefully propose the multi-step DTM, a reliable modification of the DTM, that improves the convergence of the series solution. The method provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. The validity of the proposed method has been successful by applying it for Lotka–Volterra, Chen and Lorenz systems. The method were used in a direct way without using linearization, perturbation or restrictive assumptions. It provides the solutions in terms of convergent series with easily computable components and the results have shown remarkable performance.

Figs. 1, 3, 4 and 6 show that the multi-step DTM approximate solutions for Lotka–Volterra, Chen and Lorenz systems, respectively, are very closed to the Runge–Kutta approximate solutions. Therefore, the proposed method is very efficient and accurate method that can be used to provide analytical solutions for nonlinear systems of differential equations. Figs. 2, 5 and 7 show that the solutions obtained using DTM have a small interval of convergence while those obtained using multi-step DTM have wide intervals of convergence. This confirms that the new algorithm of the DTM increases the interval of convergence for the series solution. Of course the accuracy can be improved when the step size h becomes smaller and the number of terms in each subinterval K becomes larger.

We have shown that the proposed algorithm is a very accurate and efficient method compared with RK4 method for both non-chaotic or chaotic systems. The method works successfully in handling systems of differential equations directly with a minimum size of computations and a wide interval of convergence for the series solution. This emphasizes the fact that the multi-step DTM is applicable to many other nonlinear models and its reliable and promising when compared with the existing methods.

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