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Nonlinear Analysis: Real World Applications 7 (2006) 1104 – 1118

Nonlinear Analysis
Real World Applications

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Analysis of a predator–prey model with modified Leslie–Gower and Holling-type II schemes with time delay

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Received 17 July 2005; accepted 7 October 2005

Abstract

Two-dimensional delayed continuous time dynamical system modeling a predator–prey food chain, and based on a modified version of Holling type-II scheme is investigated. By constructing a Liapunov function, we obtain a sufficient condition for global stability of the positive equilibrium. We also present some related qualitative results for this system.

Keywords: Time delay; Boundedness; Permanent; Local stability; Global stability; Liapunov functional

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of dominant themes in both ecology and mathematical ecology due to its universal existence and importance. A major trend in theoretical work on prey–predator dynamics has been to derive more realistic models, trying to keep to maximum the unavoidable increase in complexity of their mathematics. In this optic, recently [2], see also [1,5,6] has proposed a first study of two-dimensional system of autonomous differential equation modeling a predator prey system. This model incorporates a modified version of Leslie–Gower functional response as well as that of the Holling-type II.

They consider the following model

$$\begin{cases} \dot{x} = \left(a_1 - bx - \dfrac{c_1 y}{x + k_1}\right) x, \\ \dot{y} = \left(a_2 - \dfrac{c_2 y}{x + k_2}\right) y \end{cases} \tag{1}$$

with the initial conditions $x(0) > 0$ and $y(0) > 0$.

This two species food chain model describes a prey population x which serves as food for a predator y.

The model parameters a_1, a_2, b, c_1, c_2, k_1 and k_2 are assuming only positive values. These parameters are defined as follows: a_1 is the growth rate of prey x, b measures the strength of competition among individuals of species x, c_1

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doi:10.1016/j.nonrwa.2005.10.003

is the maximum value of the per capita reduction rate of x due to y, k_1 (respectively, k_2) measures the extent to which environment provides protection to prey x (respectively, to the predator y), a_2 describes the growth rate of y, and c_2 has a similar meaning to c_1.

It was first motivated more by the mathematics analysis interest than by its realism as a model of any particular natural dynamical system. However, there may be situations in which the interaction between species is modelized by systems with such a functional response. It may, for example, be considered as a representation of an insect pest–spider food chain. Furthermore, it is a first step towards a predator–prey model (of Holling–Tanner type) with inverse trophic relation and time delay, that is where the prey eaten by the mature predator can consume the immature predators.

Let us mention that the first equation of system (1) is standard. By contrast, the second equation is absolutely not standard. This intactness model contains a modified Leslie–Gower term, the second term on the right-hand side in the second equation of (1). The last depicts the loss in the predator population.

The Leslie–Gower formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Indeed, Leslie introduced a predator prey model where the carrying capacity of the predator environment is proportional to the number of prey. He stresses the fact that there are upper limits to the rates of increase of both prey x and predator y, which are not recognized in the Lotka–Volterra model. In case of continuous time, the considerations lead to the following:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a_2 y \left(1 - \frac{y}{\alpha x}\right),$$

in which the growth of the predator population is of logistic form, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a_2 y \left(1 - \frac{y}{C}\right).$$

Here, "C" measures the carry capacity set by the environmental resources and is proportional to prey abundance, $C = \alpha x$, where α is the conversion factor of prey into predators. The term $y/\alpha x$ of this equation is called the Leslie–Gower term. It measures the loss in the predator population due to the rarity (per capita y/x) of its favorite food. In the case of severe scarcity, y can switch over to other population, but its growth will be limited by the fact that its most favorite food, the prey x, is not available in abundance. The situation can be taken care of by adding a positive constant to the denominator, hence the equation above becomes,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a_2 y \left(1 - \frac{y}{\alpha x + d}\right)$$

and thus,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y \left(a_2 - \frac{a_2}{\alpha} . \frac{y}{x + \frac{d}{\alpha}} \right)$$

that is the second equation of system (1).

In this paper, we introduce time delays in model (1), which is a more realistic approach to the understanding of predator–prey dynamics. Time delay plays an important role in many biological dynamical systems, being particularly relevant in ecology, where time delays have been recognized to contribute critically to the stable or unstable outcome of prey densities due to predation. Therefore, it is interesting and important to study the following delayed modified Leslie–Gower and Holling-Type-II schemes:

$$\begin{cases} \dot{x}(t) = \left(a_1 - bx(t) - \dfrac{c_1 y(t)}{x(t) + k_1} \right) x(t), \\ \dot{y}(t) = \left(a_2 - \dfrac{c_2 y(t-r)}{x(t-r) + k_2} \right) y(t) \end{cases} \tag{2}$$

for all $t > 0$. Here, we incorporate a single discrete delay $r > 0$ in the negative feedback of the predator's density.

Let us denote by $\mathbb{R}_+^2$ the nonnegative quadrant and by $int(\mathbb{R}_+^2)$ the positive quadrant. For $\theta \in [-r, 0]$, we use the following conventional notation:

$$x_t(\theta) = x(\theta + t).$$

Then the initial conditions for this system take the form

$$\begin{cases} x_0(\theta) = \phi_1(\theta), \\ y_0(\theta) = \phi_2(\theta) \end{cases} \tag{3}$$

for all $\theta \in [-r, 0]$, where $(\phi_1, \phi_2) \in C([-r, 0], \mathbb{R}_+^2)$, $x(0) = \phi_1(0) > 0$ and $y(0) = \phi_2(0) > 0$.

It is well known that the question of global stability of the positive steady state in a predator–prey system, with a single discrete delay in the predator equation without instantaneous negative feedback, remains a challenge, see [3,5,7]. Our main purpose is to present some results about the global stability analysis on a system with delay containing modified Leslie–Gower and Holling-Type-II terms.

This paper is organized as follows. In the next section, we present some preliminary results on the boundedness of solutions for system (2)–(3). Next, we study some equilibria properties for this system and give a permanence result. In Section 5, the analysis of the global stability is made for a boundary equilibrium and sufficient conditions are provided for the positive equilibrium of both instantaneous system (1) and system with delay (2)–(3) to be globally asymptotically stable. Finally, a discussion which includes local stability results for system (2)–(3) is given.

2. Preliminaries

In this section, we present some preliminary results on the boundedness of solutions for system (2)–(3). We consider (x, y) a noncontinuable solution, see [4], of system (2)–(3), defined on $[-r, A[$, where $A \in]0, +\infty]$.

Lemma 1. *The positive quadrant* $int(\mathbb{R}_+^2)$ *is invariant for system* (2).

Proof. We have to show that for all $t \in [0, A[$, $x(t) > 0$ and $y(t) > 0$. Suppose that is not true. Then, there exists $0 < T < A$ such that for all $t \in [0, T[$, $x(t) > 0$ and $y(t) > 0$, and either $x(T) = 0$ or $y(T) = 0$. For all $t \in [0, T[$, we have

$$x(t) = x(0) \exp\left(\int_0^t a_1 - bx(s) - \frac{c_1 y(s)}{x(s) + k_1}\, \mathrm{d}s\right) \tag{4}$$

and

$$y(t) = y(0) \exp\left(\int_0^t a_2 - \frac{c_2 y(s - r)}{x(s - r) + k_2}\, \mathrm{d}s\right). \tag{5}$$

As (x, y) is defined and continuous on $[-r, T]$, there is a $M \geqslant 0$ such that for all $t \in [-r, T[$,

$$x(t) = x(0) \exp\left(\int_0^t a_1 - bx(s) - \frac{c_1 y(s)}{x(s) + k_1}\, \mathrm{d}s\right) \geqslant x(0) \exp(-TM)$$

and

$$y(t) = y(0) \exp\left(\int_0^t a_2 - \frac{c_2 y(s - r)}{x(s - r) + k_2}\, \mathrm{d}s\right) \geqslant y(0) \exp(-TM).$$

Taking the limit, as $t \to T$, we obtain

$$x(T) \geqslant x(0) \exp(-TM) > 0$$

and

$$y(T) \geqslant y(0) \exp(-TM) > 0,$$

which contradicts the fact that either $x(T) = 0$ or $y(T) = 0$. So, for all $t \in [0, A[$, $x(t) > 0$ and $y(t) > 0$. □

Lemma 2. *For system* (2)–(3), $A = +\infty$ *and*

$$\limsup_{t\to+\infty} x(t) \leqslant K \tag{6}$$

and

$$\limsup_{t\to+\infty} y(t) \leqslant L \tag{7}$$

where $K = a_1/b$ *and* $L = a_2/c_2(K + k_2)e^{a_2 r}$.

Proof. From the first equation of system (2)–(3), we have for all $t \in [0, A[$,

$$\dot{x}(t) < x(t)(a_1 - bx(t)).$$

A standard comparison argument shows that for all $t \in [0, A[$, $x(t) \leqslant \tilde{x}(t)$ where $\tilde{x}$ is the solution of the following ordinary differential equation

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{x}(t)(a_1 - b\tilde{x}(t)), \\ \tilde{x}(0) = x(0) > 0. \end{cases}$$

As $\lim_{t\to+\infty} \tilde{x}(t) = a_1/b$, then $\tilde{x}$ and thus x is bounded on $[0, A[$. Moreover, from Eq. (5), we can define y on all interval $[kr, (k+1)r]$, with $k \in \mathbb{N}$, and it is easy to see that y is bounded on $[0, A[$ if $A < +\infty$. Then $A = +\infty$, see [4, Theorem 2.4].

Now, as for all $t \geqslant 0$, $x(t) \leqslant \tilde{x}(t)$, then

$$\limsup_{t\to+\infty} x(t) \leqslant \limsup_{t\to+\infty} \tilde{x}(t) = K.$$

From the predator equation, we have

$$\dot{y}(t) < a_2 y(t),$$

hence, for $t > r$,

$$y(t) \leqslant y(t-r)e^{a_2 r},$$

which is equivalent, for $t > r$, to

$$y(t-r) \geqslant y(t)\mathrm{e}^{-a_2 r}. \tag{8}$$

Moreover, for any $\mu > 1$, there exists positive T_μ, such that for $t > T_\mu$, $x(t) < \mu K$. According to (8), we have, for $t > T_\mu + r$,

$$\dot{y}(t) < y(t)\left(a_2 - \frac{c_2 e^{-a_2 r}}{\mu K + k_2} y(t)\right),$$

which implies by the same arguments use for x that,

$$\limsup_{t\to+\infty} y(t) \leqslant L_\mu,$$

where $L_\mu = a_2/c_2(\mu K + k_2)e^{a_2 r}$. Conclusion of this lemma holds by letting $\mu \to 1$. □

3. Equilibria

In this section we study some equilibria properties of system (2)–(3). These steady states are determined analytically by setting $\dot{x} = \dot{y} = 0$. They are independent of the delay r. It is easy to verify that this system has three trivial boundary equilibria, $E_0 = (0, 0)$, $E_1 = (a_1/b, 0)$ and $E_2(0, a_2 k_2/c_2)$.

Proposition 3. *System* (2)–(3) *has a unique interior equilibrium* $E^* = (x^*, y^*)$ (*i.e.* $x^* > 0$ *and* $y^* > 0$) *if the following condition holds*

$$\frac{a_2 k_2}{c_2} < \frac{a_1 k_1}{c_1}. \tag{9}$$

Proof. From system (2)–(3), such a point satisfies

$$(a_1 - bx^*)(x^* + k_1) = c_1 y^* \tag{10}$$

and

$$y^* = \frac{a_2(x^* + k_2)}{c_2}. \tag{11}$$

If (9) holds, this system has two solutions (x^*_+, y^*_+) and (x^*_-, y^*_-) given by

$$\begin{cases} x^*_\pm = \dfrac{1}{2c_2 b}(-(c_1 a_2 - a_1 c_2 + c_2 b k_1) \pm \Delta^{1/2}), \\ y^*_\pm = \dfrac{a_2(x^*_\pm + k_2)}{c_2}, \end{cases}$$

where $\Delta = (c_1 a_2 - a_1 c_2 + c_2 b k_1)^2 - 4c_2 b(c_1 a_2 k_2 - c_2 a_1 k_1) > 0$. Moreover, it is easy to see that, $x^*_+ > 0$ and $x^*_- < 0$. □

Linear analysis of system (2)–(3) shows that point E_0 is unstable (it repels in both x and y directions) and point E_1 is also unstable (it attracts in the x-direction but repels in the y-direction).

For $r > 0$, the characteristic equation of the linearized system at E_2 takes the form

$$P_2(\lambda) + Q_2(\lambda)e^{-\lambda r} = 0,$$

where

$$P_2(\lambda) = \lambda^2 - A\lambda,$$

$$Q_2(\lambda) = a_2(\lambda - A),$$

and

$$A = a_1 - \frac{c_1 a_2 k_2}{c_2 k_1}.$$

Let us define

$$F_2(y) = |P_2(iy)|^2 - |Q_2(iy)|^2.$$

It is easy to verify that the equation $F_2(y) = 0$ has one positive root. Therefore, if E_2 is unstable for $r = 0$, it will remain so for all $r > 0$, and if it is stable for $r = 0$, there is a positive constant r_2, such that for $r > r_2$, E_2 becomes unstable.

It is easy to verify that for $r = 0$, E_2 is asymptotically stable if $a_2 k_2/c_2 > a_1 k_1/c_1$, stable (but not asymptotically) if $a_2 k_2/c_2 = a_1 k_1/c_1$ and unstable if $a_2 k_2/c_2 < a_1 k_1/c_1$.

4. Permanence results

Definition 4. System (2)–(3) is said to be permanent, see [4], if there exist α, β, $0 < \alpha < \beta$, independent of the initial condition, such that for all solutions of this system,

$$\min\left\{\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right\} \geqslant \alpha$$

and

$$\max\left\{\limsup_{t\to+\infty} x(t), \limsup_{t\to+\infty} y(t)\right\} \leqslant \beta.$$

Theorem 5. *System* (2)–(3) *is permanent if*

$$L < \frac{a_1 k_1}{c_1}. \tag{12}$$

Proof. By Lemma (2), there is a $\beta = \max\{K, L\} > 0$ independent of the initial condition such that

$$\max\left\{\limsup_{t\to+\infty} x(t), \ \limsup_{t\to+\infty} y(t)\right\} \leqslant \beta.$$

We only need to show that there is a $\alpha > 0$, independent of the initial data, such that

$$\min\left\{\liminf_{t\to+\infty} x(t), \liminf_{t\to+\infty} y(t)\right\} \geqslant \alpha.$$

It is easy to see that, for system (2)–(3), for any $\mu > 1$ and for t large enough, we have $y(t) < \mu L$. Thus, we obtain

$$\dot{x} > x\left(a_1 - bx - \frac{c_1 \mu L}{k_1}\right).$$

By standard comparison arguments, it follows that

$$\liminf_{t\to+\infty} x(t) \geqslant \frac{1}{b}\left(a_1 - \frac{c_1 \mu L}{k_1}\right)$$

and letting $\mu \to 1$, we obtain

$$\liminf_{t\to+\infty} x(t) \geqslant \frac{1}{b}\left(a_1 - \frac{c_1 L}{k_1}\right). \tag{13}$$

Let us denote by $N_1 = 1/b(a_1 - c_1 L/k_1)$. If (12) is satisfied, $N_1 > 0$. From (13) and Lemma 2 and for any $\mu > 1$, there exists a positive constant, T_μ, such that for $t > T_\mu$, $x(t) > N_1/\mu$ and $y(t) < \mu L$. Then, for $t > T_\mu + r$, we have

$$\dot{y}(t) > y(t)\left(a_2 - \frac{\mu c_2}{N_1 + \mu k_2} y(t-r)\right). \tag{14}$$

On the one hand, for $t > T_\mu + r$, these inequalities lead to

$$\dot{y}(t) > -\frac{\mu^2 c_2 L}{N_1 + \mu k_2} y(t),$$

which involves, for $t > T_\mu + r$,

$$y(t-r) < y(t) \exp\left(\frac{\mu^2 c_2 L}{N_1 + \mu k_2} r\right). \tag{15}$$

On the other hand, from (14) and (15), we have for $t > T_\mu + r$,

$$\dot{y}(t) > y(t)\left(a_2 - \frac{\mu c_2}{N_1 + \mu k_2} \exp\left(\frac{\mu^2 c_2 L}{N_1 + \mu k_2} r\right) y(t)\right)$$

which yields

$$\liminf_{t\to+\infty} y(t) \geqslant \frac{a_2(N_1 + \mu k_2)}{\mu c_2} \exp\left(-\frac{\mu^2 c_2 L}{N_1 + \mu k_2} r\right) = y_\mu.$$

Letting $\mu \to 1$, we get

$$\liminf_{t\to+\infty} y(t) \geqslant \frac{a_2(N_1+k_2)}{c_2} \exp\left(-\frac{c_2 L}{N_1+k_2} r\right) = y_1.$$

Let be $\alpha = \min\{N_1, y_1\} > 0$. Then we have shown that system (2)–(3) is permanent. □

5. Global stability analysis

5.1. Stability of E_2

Theorem 6. *If*

$$\left(a_1 - \frac{c_1 y_1}{(k_1+K)}\right) < 0,$$

then E_2 is globally asymptotically stable for system (2)–(3).

Proof. As $\liminf_{t\to+\infty} y(t) \geqslant y_1$ and $\limsup_{t\to+\infty} x(t) \leqslant K$, from the prey's equation we obtain: $\forall \mu \geqslant 1, \exists T_\mu > 0, \forall t > T_\mu$,

$$\dot{x}(t) < x(t)\left(a_1 - bx - \frac{c_1 y_1}{(k_1+\mu K)\mu}\right).$$

As

$$\left(a_1 - \frac{c_1 y_1}{(k_1+K)}\right) < 0,$$

there exists $\mu > 1$ such that

$$\left(a_1 - \frac{c_1 y_1}{(k_1+\mu K)\mu}\right) < 0.$$

Then, by standard comparison arguments, it follows that $\limsup_{t\to+\infty} x(t) \leqslant 0$ and thus, $\lim_{t\to+\infty} x(t) = 0$.

The ω-limit set Ω of every solution with positive initial conditions is then contained in $\{(0, y), y \geqslant 0\}$. Now from (7), we obviously obtain

$\Omega \subset \{(0, y), 0 \leqslant y \leqslant L\}$.

As $E_0 \notin \Omega$, (E_0 is unstable is repels in both x and y directions) and as Ω is nonempty closed and invariant set, therefore $\Omega = \{E_2\}$. □

5.2. Stability of E^* without delay

First, we give some sufficient conditions which insure that the steady state in the instantaneous system, i.e. without time delay, is globally asymptotically stable.

Theorem 7. *The interior equilibrium E^* is globally asymptotically stable if*

$$a_1 + c_1 < b(k_1 + x^*), \tag{16}$$

$$a_1 a_2 < b k_2 (c_2 - a_2). \tag{17}$$

Proof. The proof is based on constructing a suitable Lyapunov function. We define

$$V(x, y) = (x - x^*) - x^* \ln\left(\frac{x}{x^*}\right) + \alpha\left((y - y^*) - y^* \ln\left(\frac{y}{y^*}\right)\right),$$

where $\alpha = k_2 c_1 / k_1 a_2$.

This function is defined and continuous on $int(\mathbb{R}^2_+)$. It is obvious that the function V is zero at the equilibrium E^* and is positive for all other values of x and y, and thus, E^* is the global minimum of V. The time derivative of V along a solution of system (2)–(3) is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = (x - x^*)\left(a_1 - bx - \frac{c_1 y}{x + k_1}\right) + \alpha(y - y^*)\left(a_2 - \frac{c_2 y}{x + k_2}\right).$$

Centering $\mathrm{d}V/\mathrm{d}t$ on the positive equilibrium, we get

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \left(-b + \frac{a_1 - bx^*}{x + k_1}\right)(x - x^*)^2 - \frac{c_2\alpha}{x + k_2}(y - y^*)^2 \\ &\quad + \left(\frac{a_2\alpha}{x + k_2} - \frac{c_1}{x + k_1}\right)(x - x^*)(y - y^*) \\ &= \left(-b + \frac{a_1 - bx^*}{x + k_1}\right)(x - x^*)^2 - \frac{c_2\alpha}{x + k_2}(y - y^*)^2 \\ &\quad + \left(\frac{a_2\alpha}{x + k_2} + \frac{c_1}{x + k_1}\right)\frac{(x - x^*)^2 + (y - y^*)^2}{2}. \qquad (18)\\ &\leqslant \left(-b + \frac{a_1 - bx^*}{k_1} + \frac{c_1}{k_1}\right)(x - x^*)^2 \\ &\quad + \left(-\frac{c_2\alpha}{x + k_2} + \frac{c_1}{k_1}\right)(y - y^*)^2 \\ &\leqslant \left(-b + \frac{a_1 - bx^*}{k_1} + \frac{c_1}{k_1}\right)(x - x^*)^2 \\ &\quad + \frac{c_1}{k_1(x + k_2)}\left(x + k_2 - \frac{c_2 k_2}{a_2}\right)(y - y^*)^2. \qquad (19)\end{aligned}$$

From (16), we obtain

$$\left(-b + \frac{a_1 - bx^*}{k_1} + \frac{c_1}{k_1}\right) < 0.$$

From (6) and (17), there exists $\mu > 1$ and $T > 0$, such that

$$\left(\mu K + k_2 - \frac{c_2 k_2}{a_2}\right) < 0,$$

and for $t > T$,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\leqslant \left(-b + \frac{a_1 - bx^*}{k_1} + \frac{c_1}{k_1}\right)(x - x^*)^2 \\ &\quad + \frac{c_1}{k_1(x + k_2)}\left(\mu K + k_2 - \frac{c_2 k_2}{a_2}\right)(y - y^*)^2.\end{aligned}$$

Thus, $\mathrm{d}V/\mathrm{d}t$ is negative definite provided that (16) and (17) holds true. Finally, E^* is globally asymptotically stable. □

If in particular, we suppose that environment provides the same protection to both prey and predator (i.e. $k_1 = k_2$) then Theorem 7 can be simplified as follows.

Corollary 8. *The interior equilibrium E^* is globally asymptotically stable if*

$$k_1 = k_2 \qquad (20)$$

and

$$a_1 < b(k_1 + x^*). \qquad (21)$$

Proof. From (18), we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \left(-b+\frac{a_1-bx^*}{x+k_1}\right)(x-x^*)^2-\frac{c_2\alpha}{x+k_2}(y-y^*)^2\\&\quad+\left(\frac{c_1(k_2-k_1)x}{k_1(x+k_2)(x+k_1)}\right)(x-x^*)(y-y^*)\\&\leqslant\left(-b+\frac{a_1-bx^*}{k_1}\right)(x-x^*)^2-\frac{c_2\alpha}{x+k_2}(y-y^*)^2\\&\quad+\left(\frac{c_1(k_2-k_1)x}{k_1(x+k_2)(x+k_1)}\right)(x-x^*)(y-y^*),\end{aligned}$$

which is negative definite provided that (20) and (21) holds true and thus E^* is globally asymptotically stable. □

5.3. *Stability of E^* with delay*

In this subsection we shall give a result on the global asymptotic stability of the positive equilibrium for the delayed system.

Theorem 9. *Assume that parameters of system* (2)–(3) *satisfy*

$$\frac{a_1k_1}{c_1}>\max\left\{\frac{a_1}{2c_2},\frac{3}{2}\frac{a_2k_2}{c_2}\right\}. \tag{22}$$

Then, for b large enough, there exists $r_0>0$ such that, for $r\in[0,r_0]$, the interior equilibrium E^* is globally asymptotically stable in $\mathbb{R}_+^2$.

Proof. First of all, we rewrite Eqs. (2) to center it on its positive equilibrium. By using the following change of variables,

$$X(t)=\ln\left(\frac{x(t)}{x^*}\right)$$

and

$$Y(t)=\ln\left(\frac{y(t)}{y^*}\right),$$

the system becomes

$$\begin{cases}\dot{X}(t)=x^*\left(-b+\dfrac{a_1-bx^*}{x(t)+k_1}\right)(e^{X(t)}-1)-\dfrac{c_1y^*}{x(t)+k_1}(e^{Y(t)}-1),\\ \dot{Y}(t)=-\dfrac{c_2y^*}{x(t-r)+k_2}(e^{Y(t-r)}-1)+\dfrac{a_2x^*}{x(t-r)+k_2}(e^{X(t-r)}-1).\end{cases} \tag{23}$$

According to the global existence of solutions established in Lemma 2, we can assume that the initial data exists on $[-2r,0]$ (this can be done by changing initial time).

Now, let $\mu>1$ be fixed and let us define the following Liapunov functional $V:\mathscr{C}([-2r;0],\mathbb{R}^2)\to\mathbb{R}$,

$$\begin{aligned}V(\varphi_1,\varphi_2)&=\int_0^{\varphi_1(0)}(e^u-1)\,\mathrm{d}u+\int_0^{\varphi_2(0)}(e^u-1)\,\mathrm{d}u+\frac{a_2x^*}{2k_2}\int_{-r}^0(e^{\varphi_1(u)}-1)^2\,\mathrm{d}u\\&\quad+\frac{c_2y^*}{2k_2}\int_{-r}^0\int_v^0 e^{\varphi_2(s)}\left(\frac{c_2y^*}{k_2}(e^{\varphi_2(s-r)}-1)^2+\frac{a_2x^*}{k_2}(e^{\varphi_1(s-r)}-1)^2\right)\mathrm{d}s\,\mathrm{d}v\\&\quad+\frac{c_2}{2k_2}r\mu L\int_{-r}^0\left(\frac{c_2y^*}{k_2}(e^{\varphi_2(s)}-1)^2+\frac{a_2x^*}{k_2}(e^{\varphi_1(s)}-1)^2\right)\mathrm{d}s.\end{aligned}$$

Let us define the continuous and nondecreasing function $u:\mathbb{R}_+\to\mathbb{R}_+$ by

$$u(x)=\mathrm{e}^x-x-1.$$

We have $u(0)=0$, $u(x)>0$ for $x>0$ and

$$u(|\varphi(0)|)\leqslant u(\varphi_1(0))+u(\varphi_2(0))\leqslant V(\varphi_1,\varphi_2),$$

where $\varphi=(\varphi_1,\varphi_2)$ and $|\cdot|$ denotes the infinity norm in $\mathbb{R}^2$.

Let $v:\mathbb{R}_+\to\mathbb{R}_+$ be define by

$$v(x)=2u(x)+(e^x-1)^2\left(\frac{a_2x^*r}{2k_2}+\left(\frac{c_2y^*}{k_2}+\frac{a_2x^*}{k_2}\right)\left(\frac{c_2y^*r^2}{4k_2}e^x+\frac{c_2}{2k_2}r^2\mu L\right)\right).$$

It is clear that v is a continuous and nondecreasing function and satisfies $v(0)=0$, $v(x)>0$ and

$$V(\varphi_1,\varphi_2)\leqslant v(\|\varphi\|),$$

where $\|\cdot\|$ denotes the infinity norm in $\mathscr{C}([-2r;0],\mathbb{R}^2)$. Now, let (X,Y) be a solution of (23) and let us compute $\dot V_{(23)}$, the time derivative of V along the solutions of (23).

First, we start with the function

$$V_1(\varphi_1,\varphi_2)=\int_0^{\varphi_1(0)}(e^u-1)\,\mathrm{d}u+\int_0^{\varphi_2(0)}(e^u-1)\,\mathrm{d}u+\frac{a_2x^*}{2k_2}\int_{-r}^0(e^{\varphi_1(u)}-1)^2\,\mathrm{d}u.$$

We have

$$V_1(X_t,Y_t)=\int_0^{X(t)}(e^u-1)\,\mathrm{d}u+\int_0^{Y(t)}(e^u-1)\,\mathrm{d}u+\frac{a_2x^*}{2k_2}\int_{t-r}^t(e^{X(u)}-1)^2\,\mathrm{d}u.$$

Then,

$$\dot V_{1|(23)}=(e^{X(t)}-1)\dot X(t)+(e^{Y(t)}-1)\dot Y(t)+\frac{a_2x^*}{2k_2}((e^{X(t)}-1)^2-(e^{X(t-r)}-1)^2).$$

System (23) gives us

$$\begin{aligned}\dot V_{1|(23)}=&\,x^*\left(-b+\frac{a_1-bx^*}{x(t)+k_1}\right)(e^{X(t)}-1)^2-\frac{c_1y^*}{x(t)+k_1}(e^{Y(t)}-1)(e^{X(t)}-1)\\&-\frac{c_2y^*}{x(t-r)+k_2}(e^{Y(t)}-1)(e^{Y(t-r)}-1)\\&+\frac{a_2x^*}{x(t-r)+k_2}(e^{Y(t)}-1)(e^{X(t-r)}-1)\\&+\frac{a_2x^*}{2k_2}((e^{X(t)}-1)^2-(e^{X(t-r)}-1)^2).\end{aligned}$$

By using several times the obvious inequalities of type

$$\begin{aligned}-\frac{c_1y^*}{x(t)+k_1}(e^{Y(t)}-1)(e^{X(t)}-1)&\leqslant\frac{c_1y^*}{x(t)+k_1}\left(\frac{(e^{Y(t)}-1)^2}{2}+\frac{(e^{X(t)}-1)^2}{2}\right)\\&\leqslant\frac{c_1y^*}{2k_1}((e^{Y(t)}-1)^2+(e^{X(t)}-1)^2),\end{aligned}$$

we get

$$\begin{aligned}\dot V_{1|(23)}\leqslant&\left(x^*\left(-b+\frac{a_1-bx^*}{k_1}+\frac{a_2}{2k_2}\right)+\frac{c_1y^*}{2k_1}\right)(e^{X(t)}-1)^2\\&+\left(\frac{c_1y^*}{2k_1}+\frac{a_2x^*}{2k_2}\right)(e^{Y(t)}-1)^2\\&-\frac{c_2y^*}{x(t-r)+k_2}(e^{Y(t)}-1)(e^{Y(t-r)}-1).\end{aligned}$$

As,

$$e^{Y(t-r)} = e^{Y(t)} - \int_{t-r}^{t} e^{Y(s)} \dot{Y}(s)\,\mathrm{d}s,$$

we obtain

$$\begin{aligned}\dot{V}_{1|(23)} \leqslant & \left(x^*\left(-b+\frac{a_1-bx^*}{k_1}+\frac{a_2}{2k_2}\right)+\frac{c_1y^*}{2k_1}\right)(e^{X(t)}-1)^2 \\ & +\left(\frac{c_1y^*}{2k_1}+\frac{a_2x^*}{2k_2}-\frac{c_2y^*}{x(t-r)+k_2}\right)(e^{Y(t)}-1)^2 \\ & +\frac{c_2y^*}{x(t-r)+k_2}(e^{Y(t)}-1)\int_{t-r}^{t} e^{Y(s)}\dot{Y}(s)\,\mathrm{d}s. \end{aligned} \tag{24}$$

From (23), we have

$$\begin{aligned}&(e^{Y(t)}-1)\int_{t-r}^{t} e^{Y(s)}\dot{Y}(s)\,\mathrm{d}s \\ &\quad = \int_{t-r}^{t} e^{Y(s)}\frac{-c_2y^*}{x(s-r)+k_2}(e^{Y(t)}-1)(e^{Y(s-r)}-1)\,\mathrm{d}s \\ &\qquad + \int_{t-r}^{t} e^{Y(s)}\frac{a_2x^*}{x(s-r)+k_2}(e^{Y(t)}-1)(e^{X(s-r)}-1)\,\mathrm{d}s \\ &\quad \leqslant \left(\frac{c_2y^*}{2k_2}+\frac{a_2x^*}{2k_2}\right)\int_{t-r}^{t} e^{Y(s)}\,\mathrm{d}s.(e^{Y(t)}-1)^2 \\ &\qquad + \frac{c_2y^*}{2k_2}\int_{t-r}^{t} e^{Y(s)}(e^{Y(s-r)}-1)^2\,\mathrm{d}s + \frac{a_2x^*}{2k_2}\int_{t-r}^{t} e^{Y(s)}(e^{X(s-r)}-1)^2\,\mathrm{d}s\end{aligned}$$

and as $y^*e^{Y(s)} \leqslant \mu L$ for s large enough, we obtain, for t large enough,

$$\begin{aligned}(e^{Y(t)}-1)\int_{t-r}^{t} e^{Y(s)}\dot{Y}(s)\,\mathrm{d}s &\leqslant \mu Lr\left(\frac{c_2}{2k_2}+\frac{a_2x^*}{2k_2y^*}\right)(e^{Y(t)}-1)^2 \\ &+\int_{t-r}^{t} e^{Y(s)}\left(\frac{c_2y^*}{2k_2}(e^{Y(s-r)}-1)^2+\frac{a_2x^*}{2k_2}(e^{X(s-r)}-1)^2\right)\mathrm{d}s\end{aligned}$$

and as $x(s-r) \leqslant \mu L$ for s large enough, (24) becomes, for t large enough,

$$\begin{aligned}\dot{V}_{1|(23)} \leqslant & \left(x^*\left(-b+\frac{a_1-bx^*}{k_1}+\frac{a_2}{2k_2}\right)+\frac{c_1y^*}{2k_1}\right)(e^{X(t)}-1)^2 \\ & +\left(\frac{c_1y^*}{2k_1}+\frac{a_2x^*}{2k_2}-\frac{c_2y^*}{x(t-r)+k_2}+\frac{c_2\mu Lr}{k_2}\left(\frac{c_2y^*}{2k_2}+\frac{a_2x^*}{2k_2}\right)\right)(e^{Y(t)}-1)^2 \\ & +\frac{c_2y^*}{k_2}\int_{t-r}^{t} e^{Y(s)}\left(\frac{c_2y^*}{2k_2}(e^{Y(s-r)}-1)^2+\frac{a_2x^*}{2k_2}(e^{X(s-r)}-1)^2\right)\mathrm{d}s. \end{aligned} \tag{25}$$

The next step consists in computation of the time derivative along the solution of (23), of the term

$$\begin{aligned}V_2(\varphi_1,\varphi_2) = & \frac{c_2y^*}{2k_2}\int_{-r}^{0}\int_{v}^{0} e^{\varphi_2(s)}\left(\frac{c_2y^*}{k_2}(e^{\varphi_2(s-r)}-1)^2+\frac{a_2x^*}{k_2}(e^{\varphi_1(s-r)}-1)^2\right)\mathrm{d}s\,\mathrm{d}v \\ & +\frac{c_2}{2k_2}r\mu L\int_{-r}^{0}\left(\frac{c_2y^*}{k_2}(e^{\varphi_2(s)}-1)^2+\frac{a_2x^*}{k_2}(e^{\varphi_1(s)}-1)^2\right)\mathrm{d}s.\end{aligned}$$

We have

$$V_2(X_t, Y_t) = \frac{c_2 y^*}{2k_2} \int_{t-r}^{t} \int_{v}^{t} e^{Y(s)} \left(\frac{c_2 y^*}{k_2} (e^{Y(s-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(s-r)} - 1)^2 \right) \mathrm{d}s\, \mathrm{d}v + \frac{c_2}{2k_2} r\mu L \int_{t-r}^{t} \left(\frac{c_2 y^*}{k_2} (e^{Y(s)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(s)} - 1)^2 \right) \mathrm{d}s.$$

Then,

$$\begin{aligned}
\dot{V}_2|_{(23)} = {} & \frac{c_2 y^*}{2k_2} e^{Y(t)} \left(\frac{c_2 y^*}{k_2} (e^{Y(t-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t-r)} - 1)^2 \right) r \\
& - \frac{c_2 y^*}{2k_2} \int_{t-r}^{t} e^{Y(s)} \left(\frac{c_2 y^*}{k_2} (e^{Y(s-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(s-r)} - 1)^2 \right) \mathrm{d}s \\
& + \frac{c_2}{2k_2} r\mu L \left(\frac{c_2 y^*}{k_2} (e^{Y(t)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t)} - 1)^2 \right) \\
& - \frac{c_2}{2k_2} r\mu L \left(\frac{c_2 y^*}{k_2} (e^{Y(t-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t-r)} - 1)^2 \right) \\
= {} & \frac{c_2 r}{2k_2} \left(\frac{c_2 y^*}{k_2} (e^{Y(t-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t-r)} - 1)^2 \right) (y^* e^{Y(t)} - \mu L) \\
& - \frac{c_2 y^*}{2k_2} \int_{t-r}^{t} e^{Y(s)} \left(\frac{c_2 y^*}{k_2} (e^{Y(s-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(s-r)} - 1)^2 \right) \mathrm{d}s \\
& + \frac{c_2}{2k_2} r\mu L \left(\frac{c_2 y^*}{k_2} (e^{Y(t)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t)} - 1)^2 \right).
\end{aligned}$$

For t large enough, we have $y^* e^{Y(t)} - \mu L < 0$ and thus,

$$\begin{aligned}
\dot{V}_2|_{(23)} \leqslant {} & - \frac{c_2 y^*}{2k_2} \int_{t-r}^{t} e^{Y(s)} \left(\frac{c_2 y^*}{k_2} (e^{Y(s-r)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(s-r)} - 1)^2 \right) \mathrm{d}s \\
& + \frac{c_2}{2k_2} r\mu L \left(\frac{c_2 y^*}{k_2} (e^{Y(t)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t)} - 1)^2 \right).
\end{aligned}$$

This inequality and (25) lead to, for t large enough,

$$\begin{aligned}
\dot{V}|_{(23)} \leqslant {} & \left(x^* \left(-b + \frac{a_1 - bx^*}{k_1} + \frac{a_2}{2k_2} \right) + \frac{c_1 y^*}{2k_1} \right) (e^{X(t)} - 1)^2 \\
& + \left(\frac{c_1 y^*}{2k_1} + \frac{a_2 x^*}{2k_2} - \frac{c_2 y^*}{x(t-r) + k_2} + \frac{c_2 \mu L r}{k_2} \left(\frac{c_2 y^*}{2k_2} + \frac{a_2 x^*}{2k_2} \right) \right) (e^{Y(t)} - 1)^2 \\
& + \frac{c_2}{2k_2} r\mu L \left(\frac{c_2 y^*}{k_2} (e^{Y(t)} - 1)^2 + \frac{a_2 x^*}{k_2} (e^{X(t)} - 1)^2 \right) \\
\leqslant {} & \left(-\frac{bx^*(k_1 + x^*)}{k_1} + \frac{a_1 x^*}{k_1} + \frac{a_2 x^*}{2k_2} + \frac{c_1 y^*}{2k_1} + \frac{c_2 a_2 x^* r\mu L}{2k_2^2} \right) (e^{X(t)} - 1)^2 \\
& + \left(\frac{c_1 y^*}{2k_1} + \frac{a_2 x^*}{2k_2} - \frac{c_2 y^*}{x(t-r) + k_2} + \frac{c_2 \mu L r}{k_2} \left(\frac{c_2 y^*}{k_2} + \frac{a_2 x^*}{2k_2} \right) \right) (e^{Y(t)} - 1)^2.
\end{aligned}$$

Now, if

$$-\frac{bx^*(k_1 + x^*)}{k_1} + \frac{a_1 x^*}{k_1} + \frac{a_2 x^*}{2k_2} + \frac{c_1 y^*}{2k_1} < 0 \tag{26}$$

and

$$\frac{c_1 y^*}{2k_1} + \frac{a_2 x^*}{2k_2} - \frac{c_2 y^*}{x(t-r) + k_2} < 0 \tag{27}$$

then, for r small enough, we can conclude that $\dot{V}|_{(23)}$ is negative definite. Hence V satisfies all the assumptions of Corollary 5.2 in [4] and the theorem follows.

We now study when inequalities (26) and (27) hold.

From (10), we have $-bx^*(k_1 + x^*) = c_1(y^* - a_1(k_1 + x^*)/c_1)$ and using (11), (26) becomes

$$\frac{3}{2}\frac{c_1}{k_1}\frac{a_2(x^* + k_2)}{c_2} - a_1 + \frac{a_2 x^*}{2k_2} < 0$$

which is rewritten as

$$x^*\left(\frac{a_2 k_1}{c_1 k_2} + \frac{3a_2}{c_2}\right) < 2\frac{a_1 k_1}{c_1} - 3\frac{a_2 k_2}{c_2}. \tag{28}$$

As $x^* \leqslant \frac{a_1}{b}$, then the following inequality

$$a_1\left(\frac{a_2 k_1}{c_1 k_2} + \frac{3a_2}{c_2}\right) < b\left(2\frac{a_1 k_1}{c_1} - 3\frac{a_2 k_2}{c_2}\right) \tag{29}$$

implies (28). Now, if the following inequality holds

$$\frac{c_1 y^*}{2k_1} + \frac{a_2 x^*}{2k_2} - \frac{c_2 y^*}{\dfrac{a_1}{b} + k_2} < 0, \tag{30}$$

then, for t large enough, (27) holds too. Using (11), (30) is reformulated as

$$\frac{c_1 y^*}{2k_1} + \frac{c_2 y^* - a_2 k_2}{2k_2} < \frac{c_2 y^*}{\dfrac{a_1}{b} + k_2}$$

$$\frac{c_1}{2k_1} + \frac{c_2}{2k_2} < \frac{c_2}{\dfrac{a_1}{b} + k_2} + \frac{a_2}{2y^*}$$

$$\frac{c_1}{2k_1} + \frac{c_2}{2k_2} < \frac{c_2}{\dfrac{a_1}{b} + k_2} + \frac{c_2}{2(x^* + k_2)}$$

$$\frac{c_1}{k_1} < c_2\left(\frac{2}{\dfrac{a_1}{b} + k_2} + \frac{1}{x^* + k_2} - \frac{1}{k_2}\right).$$

As $x^* \leqslant a_1/b$, then the last inequality is satisfied if

$$\frac{c_1}{k_1} < c_2\left(\frac{3}{\dfrac{a_1}{b} + k_2} - \frac{1}{k_2}\right) \tag{31}$$

that is if

$$\frac{a_1 c_1 k_2}{k_1} + a_1 c_2 < b\frac{k_2}{k_1}(2c_2 k_1 - c_1). \tag{32}$$

In conclusion, if

$$\frac{a_1 k_1}{c_1} > \max\left\{\frac{a_1}{2c_2}, \frac{3}{2}\frac{a_2 k_2}{c_2}\right\}$$

and for b large enough, there exists a unique interior equilibrium and for r small enough it is globally asymptotically stable. □

6. Discussion

It is interesting to discuss the effect of time delay r on the stability of the positive equilibrium of system (2)–(3). We assume that positive equilibrium E^* exists for this system.

Linearizing system (2)–(3) at E^*, we obtain

$$\dot{X}(t) = A_{11}X(t) + A_{12}Y(t), \quad \dot{Y}(t) = A_{21}X(t-r) + A_{22}Y(t-r), \tag{33}$$

where

$$A_{11} = -bx^* + \frac{c_1 x^* y^*}{(x^*+k_1)^2},$$

$$A_{12} = -\frac{c_1 x^*}{x^*+k_1},$$

$$A_{21} = \frac{c_2 (y^*)^2}{(x^*+k_2)^2} = \frac{a_2^2}{c_2}$$

and

$$A_{22} = -\frac{c_2 y^*}{x^*+k_2} = -a_2.$$

The characteristic equation for (33) takes the form

$$P(\lambda) + Q(\lambda)e^{-\lambda r} = 0 \tag{34}$$

in which

$$P(\lambda) = \lambda^2 - A_{11}\lambda$$

and

$$Q(\lambda) = -A_{22}\lambda + (A_{11}A_{22} - A_{12}A_{21}).$$

When $r = 0$, we observe that the jacobian matrix of the linearized system is

$$J = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

One can verify that the positive equilibrium E^* is stable for $r = 0$ provided that

$$a_1 < bk_1. \tag{35}$$

Indeed, when (35) holds, then it is easy to verify that

$$Tr(J) = A_{11} + A_{22} < 0$$

and

$$Det(J) = A_{11}A_{22} - A_{12}A_{21} > 0.$$

By denoting

$$F(y) = |P(iy)|^2 - |Q(iy)|^2,$$

we have

$$F(y) = y^4 + (A_{11}^2 - A_{22}^2)y^2 - (A_{11}A_{22} - A_{12}A_{21})^2.$$

If (35) holds, then $\lambda = 0$ is not a solution of (34). It is also easy to verify that if (35) holds, then equation $F(y) = 0$ has at least one positive root. By applying standard theorem on the zeros of transcendental equation, see [4, Theorem

4.1], we see that there is a positive constant r_0 (which can be evaluated explicitly), such that for $r > r_0$, E^* becomes unstable. Then, the global stability of E^* involves restrictions on length of time delay r. Therefore, it is obvious that time delay has a destabilized effect on the positive equilibrium of system (2)–(3).

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