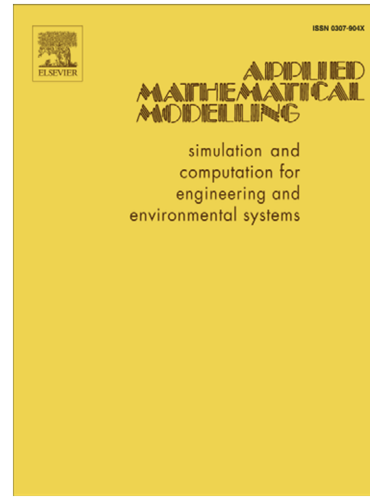
Accepted Manuscript



Existence of periodic travelling waves solutions in predator prey model with diffusion

Radouane Yafia, M.A. Aziz-Alaoui

PII: S0307-904X(12)00459-3
DOI: http://dx.doi.org/10.1016/j.apm.2012.08.003
Reference: APM 9036

To appear in: *Appl. Math. Modelling*

Received Date: 8 June 2011
Revised Date: 19 July 2012
Accepted Date: 15 August 2012

Existence of periodic travelling waves solutions in predator prey model with diffusion

Radouane YAFIA*†, M. A. Aziz-Alaoui‡

July 19, 2012

Abstract

This paper deals with the qualitative analysis of the travelling waves solutions of a reaction diffusion model that refers to the competition between the predator and prey with modified Leslie-Gower and Holling type II schemes. The well posedeness of the problem is proved. We establish sufficient conditions for the asymptotic stability of the unique nontrivial positive steady state of the model by analyzing roots of the forth degree exponential polynomial characteristic equation. We also prove the existence of a Hopf bifurcation which leads to periodic oscillating travelling waves by considering the diffusion coefficient as a parameter of bifurcation. Numerical simulations are given to illustrate the analytical study.

Keywords :
Predator prey model, diffusion, stability, periodic travelling waves, Hopf bifurcation

1 Introduction

The simplest reaction diffusion models for cyclic populations involve two interacting species, with densities u and v:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = D_u \frac{\partial^2}{\partial x^2}u(t,x) + f(u,v) \\ \\ \frac{\partial}{dt}v(t,x) = D_v \frac{\partial^2}{\partial x^2}v(t,x) + g(u,v) \end{cases} \tag{1.1}$$

where $f(u,v)$ and $g(u,v)$ model the local activity (absence of diffusion), the densities u and v may represent the predator and prey, host and parasite, herbivore and grazer, etc. Here x is spatial coordinate and t denotes time. Our focus on cyclic populations means that we assume that the local dynamics f and g are such that the spatially uniform equations $\frac{du}{dt} = f(u,v)$; $\frac{dv}{dt} = g(u,v)$ have a stable periodic solution (limit cycle), which oscillates around the unstable coexistence steady state. The theory of periodic travelling waves is essentially the same for models with three or more interacting species.

The model under study is given by a reaction diffusion system based on the predator prey interaction species with modified Leslie-Gower and Holling type II schemes (see, [1]):

$$\begin{cases} \frac{\partial}{dt}u(t,x) = d_1 \frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\left(r_1 - b_1 u(t,x) - \frac{a_1 v(t,x)}{u(t,x)+k_1}\right), \\ \\ \frac{\partial}{dt}v(t,x) = d_2 \frac{\partial^2}{\partial x^2}v(t,x) + v(t,x)\left(r_2 - \frac{a_2 v(t,x)}{u(t,x)+k_2}\right), \end{cases} \tag{1.2}$$

*Corresponding author: R. Yafia.
†Ibn Zohr University, Polydisciplinary Faculty of Ouarzazate, B.P: 638, Ouarzazate, Morocco. yafia1@yahoo.fr
‡Laboratoire de Mathématiques Appliquées, 25 Rue Ph. Lebon, BP 540, 76058Le Havre Cedex, France. aziz.alaoui@univ-lehavre.fr

where all parameters in (1.2) are positives. The functions $u(t,x)$ and $v(t,x)$ are densities of the prey and predator, respectively, d_1 and d_2 are the diffusion coefficients, r_1 is the growth rate of the prey u, b_1 measures the strength of the competition among individuals of species u, a_1 is the maximal value which per capita reduction rate of u can attain, k_1 (respectively; k_2) measures the extent to which environment provides to prey u (respectively to predator v) r_2 describes the growth rate of u and a_2 has a similar meaning to a_1.
The first model proposed in this optic is given by an ordinary differential equations (see, [1]) and read as follows

$$\begin{cases} \frac{dx}{dt} = \left(r_1 - b_1 x - \frac{a_1 y}{x+k_1}\right) x, \\ \\ \frac{dy}{dt} = \left(r_2 - \frac{a_2 y}{x+k_2}\right) y \end{cases} \tag{1.3}$$

with initial conditions $x(0) > 0$ and $y(0) > 0$.
This two species food chain model describes a prey population x which serves as food for a predator y.
The delayed model of (1.3) (see, [8]) is given by a system of two delayed differential equations as follows:

$$\begin{cases} \frac{dx(t)}{dt} = \left(r_1 - b_1 x(t) - \frac{a_1 y(t)}{x(t)+k_1}\right) x(t), \\ \\ \frac{dy(t)}{dt} = \left(r_2 - \frac{a_2 y(t-\tau)}{x(t-\tau)+k_2}\right) y(t) \end{cases} \tag{1.4}$$

for all $t > 0$. Here, the discrete delay $\tau > 0$ has been incorporated in the negative feedback of the predator's density.
The notion of global stability is studied by many other authors in the predator-prey systems with delay [2, 17]. In [1], authors studied the boundeness and global stability of system (1.3)and in [8] authors studied the global stability and persistence of the delayed system (1.4) by using liapunov functional.
The existence of periodic solutions and their stability are studied in [18, 19], by considering the delay as a parameter of bifurcation.
The spatio-temporal predator prey model without modification is given by (see Huang et al. [6])

$$\begin{cases} \frac{\partial}{dt} u(t,x) = d_1 \frac{\partial^2}{\partial x^2} u(t,x) + u(t,x)\left(r_1 - \frac{1}{K} u(t,x) - B\frac{v(t,x)}{Eu(t,x)+1}\right), \\ \\ \frac{\partial}{dt} v(t,x) = d_2 \frac{\partial^2}{\partial x^2} v(t,x) + v(t,x)\left(r_2 - Cv(t,x) + D\frac{u(t,x)}{Eu(t,x)+1}\right), \end{cases} \tag{1.5}$$

where all parameters in (1.5) are positives, for the meaning of all constants in (1.5) see [6], in which authors studied the existence of travelling waves of system (1.5). The method used to prove the results are shooting argument in $\mathbb{R}^4$ together with a lyapunov function and Lasalle's Invariance Principle.
Travelling waves have been observed in nature in many cyclic animals see [3, 4, 7, 12, 15], other authors are interested in the study of periofic travelling waves see [13, 16] and references therein.

Our goal in this paper is to study the dynamics of the resulting travelling waves system of equation (1.2). We prove the well posedeness of the problem and asymptotic stability of the non trivial steady state with respect to the diffusion coefficient. We establish the existence of limit cycle via Hopf bifurcation theorem, by using the diffusion coefficient as a parameter of bifurcation. The current work is organized as follows: In section 2, the well posedeness of the problem is proved. Section (3) is devoted to the existence and stability of the steady states of the model. In section 4, we prove the existence of limit cycle (periodic travelling waves) of the model. In the end we give some numerical simulations and discussions.

2 Well posedeness of the problem

The reaction diffusion system we focus on here reads as,

$$\begin{cases} \frac{\partial}{dt}u(t,x) = d_1\frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\left(r_1 - b_1 u(t,x) - \frac{a_1 v(t,x)}{u(t,x)+k_1}\right), \\ \frac{\partial}{dt}v(t,x) = d_2\frac{\partial^2}{\partial x^2}v(t,x) + v(t,x)\left(r_2 - \frac{a_2 v(t,x)}{u(t,x)+k_2}\right), \end{cases} \tag{2.1}$$

For further simplification, taking
$u^*(r_1 t, \sqrt{\frac{r_1}{d_2}}x) = \frac{b_1}{r_1}u(t,x)$, $v^*(r_1 t, \sqrt{\frac{r_1}{d_2}}x) = \frac{a_2 b_2}{r_1 r_2}v(t,x)$, $d = \frac{d_1}{d_2}$, $a = \frac{a_1 r_2}{a_2 r_1}$, $b = \frac{r_2}{r_1}$, $e_1 = \frac{b_1 k_1}{r_1}$, $e_2 = \frac{b_1 k_2}{r_1}$, $t' = r_1 t$, $x' = \sqrt{\frac{r_1}{d_2}}x$
and dropping the stars on u, v and the primes on x, t for convenience, we obtain the following system

$$\begin{cases} \frac{\partial}{dt}u(t,x) = d\frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\left(1 - u(t,x)\right) - \frac{au(t,x)v(t,x)}{u(t,x)+e_1}, \\ \frac{\partial}{dt}v(t,x) = \frac{\partial^2}{\partial x^2}v(t,x) + bv(t,x)\left(1 - \frac{v(t,x)}{u(t,x)+e_2}\right). \end{cases} \tag{2.2}$$

We will show that the reaction diffusion system (2.2) generates a dynamical system and it is biologically well posed on suitable Banach space.
Let us set $F = (F_1, F_2)$, $U = (u, v)$ and $D = diag[d, 1]$, where

$$F_1(u,v) = u(1-u) - \frac{auv}{u+e_1}$$

$$F_2(u,v) = bv(1 - \frac{v}{u+e_2})$$

Henceforth, considering also a boundary conditions and system (2.2) can be written as

$$\begin{cases} \frac{\partial}{\partial t}U(t,x) = D\Delta U(t,x) + F(U),\ x\in\Omega,\ t>0 \\ \frac{\partial}{\partial n}U(t,x) = 0,\ x\in\partial\Omega,\ t>0 \\ U(0,x) = \varphi(x),\ x\in\Omega \end{cases} \tag{2.3}$$

Let X be the Banach space $X_1 \times X_2$, where $X_1 = X_2 = \mathcal{C}(\overline{\Omega})$.
The norm is defined by

$$|\varphi| = |\varphi_1| + |\varphi_2|.$$

Let A_u^0 and A_v^0 be the differential operators

$$\begin{cases} A_u^0 u = d_1\Delta u \\ A_v^0 v = d_2\Delta v, \end{cases} \tag{2.4}$$

defined on the domain $\mathcal{D}(A_u^0)$ and $\mathcal{D}(A_v^0)$ respectively, where

$$\mathcal{D}(A_u^0) = \left\{u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\overline{\Omega}) : A_u^0 u \in \mathcal{C}(\overline{\Omega}), \frac{\partial u}{\partial n}(x) = 0, x\in\partial\Omega\right\}$$

$$\mathcal{D}(A_v^0) = \left\{v \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\overline{\Omega}) : A_v^0 v \in \mathcal{C}(\overline{\Omega}), \frac{\partial v}{\partial n}(x) = 0, x\in\partial\Omega\right\}.$$

The closures A_u of A_u^0, and A_v of A_u^0 in X_i $(i = 1, 2)$ generate analytic semigroups of bounded linear operators $T_u(t)$ and $T_v(t)$ for $t \geq 0$ such that

$$\begin{cases} u(t) = T_u(t)\varphi_1 \\ v(t) = T_v(t)\varphi_2 \end{cases} \tag{2.5}$$

are solutions of the following abstract linear differential equations in X_i

$$\begin{cases} u'(t) = A_u u(t) \\ v'(t) = A_v v(t) \end{cases} \tag{2.6}$$

An additional property of the semigroup is that for each $t > 0$, $T_u(t)$ and $T_v(t)$ are compact operators. In the language of partial differential equations

$$\begin{cases} u(t,x) = [T_u(t)\varphi_1](x) \\ v(t,x) = [T_v(t)\varphi_2](x) \end{cases} \tag{2.7}$$

are classical solutions of the initial boundary value problem (2.3) with $F_1 = F_2 = 0$.
Let

$$T(t) : X \longrightarrow X$$

defined by

$$T(t) = T_u(t) \times T_v(t).$$

Then $T(t)$ is a semigroup of operators on X generated by the operator $A = A_u \times A_v$ defined on

$$\mathcal{D}(A) = \mathcal{D}(A_u) \times \mathcal{D}(A_v)$$

and

$$U(t,x) = [T(t)\varphi](x)$$

is the solution of the following linear system

$$\begin{cases} \frac{\partial}{\partial t}U(t,x) = D\Delta U(t,x), \ x \in \Omega, \ t > 0 \\ \frac{\partial}{\partial n}U(t,x) = 0, \ x \in \partial\Omega, \ t > 0 \\ U(0,0) = \varphi(x), \ x \in \Omega \end{cases} \tag{2.8}$$

Observe that the nonlinear term F is twice continuously differentiable on U. Therefore, we can define the map $[F^*(\varphi)](x) = F(\varphi(x))$ which maps X into itself and equation (2.3) can be viewed as an abstract ordinary differential equation in X given by

$$\begin{cases} z'(t) = Az(t) + F^*(z(t)) \\ z(0) = \varphi \end{cases} \tag{2.9}$$

While a solution $z(t)$ of (2.9) can be obtained under restriction that $\varphi \in \mathcal{D}(A)$, a mild solution can be obtained for every $\varphi \in X$ by requiring only that $z(t)$ is a continuous solution of the following integral equation

$$z(t) = T(t)\varphi + \int_0^t T(t-s)F^*(z(s))ds, \ t \in [0, \alpha), \tag{2.10}$$

where $\alpha = \alpha(\varphi) \leq \infty$. Restricting our attention to function φ in the set :

$$X_\Lambda = \left\{\varphi \in X : \varphi(x) \in \Lambda, x \in \overline{\Omega}\right\}$$

where

$$\Lambda = \left\{U = (u, v) \in \mathbb{R}^2 : u \geq 0, v \geq 0\right\},$$

and taking into account the definition of the function F_i $i = 1, 2$, we obtain that $F_1(0, v) = 0$ and $F_2(u, 0) = 0$ for $U \in \Lambda$. Thus, Corollary 3.2, p. 129 in [14] implies that the Nagumo condition for the positive invariance of Λ is satisfied, i.e.,

$$\lim_{h\to 0} h^{-1} dist(\Lambda, U + hF(U)) = 0, \; U \in \Lambda. \tag{2.11}$$

On the other hand, a direct application of the strong parabolic maximum principle can be used to show that the linear semigroup $T(t)$ leaves X_Λ positively invariant, i.e.,

$$T(t)X_\Lambda \subset X_\Lambda, \; t \geq 0. \tag{2.12}$$

Finally, conditions (2.11) and (2.12) together allow us to apply Theorem 3.1, p. 127 in [14], and we obtain the following result.

Theorem 2.1. *For each $\varphi \in X_\Lambda$,* (2.2) *has a unique mild solution $z(t) = z(\varphi, t) \in X_\Lambda$ and a classical solution $U(t, x) = [z(t)](x)$. Moreover, the set X_Λ is positively invariant under flow $\Psi_t(\varphi) = z(\varphi, t)$ induced by* (2.2)

3 steady states and stability

Consider again system (2.2), there are several reasonable parameters restrictions.
First, we require that $0 < d \leq 1$, which indicates that the prey population does not disperse faster than the predator.
We also require that $ae_2 < e_1$, which ensures that system (2.2) has a positive equilibrium point corresponding to constant coexistence of the two species. System (2.2) has forth equilibrium points: $E_0 = (0, 0)$, $E_1 = (1, 0)$, $E_2 = (0, e_2)$, $E_* = (u_*, v_*)$ which are equilibria of the corresponding ordinary differential equation system (2.2) without diffusion, where

$$u_* = \frac{1 - a - e_1 + \sqrt{(a + e_1 - 1)^2 + 4(e_1 - ae_2)}}{2}, \tag{3.1}$$

and

$$v_* = u_* + e_2. \tag{3.2}$$

Without diffusion , the equilibrium point E_0 corresponding to the absence of both species is unstable, E_1 corresponding to the prey at the environment carrying capacity in the absence of predators is a saddle point. If $ae_2 < e_1$, E_2 corresponding to the predator at the environment carrying capacity in the absence of prey is also a saddle point and E_* corresponding to coexistence of the two species is asymptotically stable if $p(u_*) > 0$ and unstable if $p(u_*) < 0$, where

$$p(x) = 2x^2 + (b + e_1 - 1)x + be_1. \tag{3.3}$$

Furthermore, E_* is asymptotically stable if $b + e_1 - 1 \geq 0$ or $0 < u_* < \alpha_1$ or $\alpha_2 < u_* < 1$ and it is unstable if $b + e_1 - 1 > 0$ and $\alpha_1 < u_* < \alpha_2$, where α_1 and α_2 are the roots of the polynomial $p(x)$:

$$\alpha_{1,2} = \frac{1 - b - e_1 \pm \sqrt{(b + e_1 - 1)^2 + 8be_1}}{4}.$$

Particulary, if $e_1 - 1 \geq 0$, E_* is globally asymptotically stable. If $b + e_1 - 1 \geq 0$ the system has no limit cycle.
In order to establish the existence of travelling waves solutions of system (2.2), we assume that the solutions have the spacial form $u(t,x) = u(x+ct)$ and $v(t,x) = v(x+ct)$, where $s = x+ct$, and c is the wave speed. Then the system (2.2) becomes

$$\begin{cases} cu' = du'' + u\,(1-u) - \frac{auv}{u+e_1}, \\ cv' = v'' + bv\left(1 - \frac{v}{u+e_2}\right), \end{cases} \tag{3.4}$$

where $'$ denotes the differentiation with respect to the travelling wave variable s.
Rewrite (3.4) as a system of first order ordinary differential equations in $\mathbb{R}^4$:

$$\begin{cases} u' = w, \\ v' = z, \\ w' = \frac{c}{d}w + \frac{1}{d}u(u-1) + \frac{a}{d}\frac{uv}{u+e_1}, \\ z' = cz - bv + \frac{bv^2}{u+e_2}. \end{cases} \tag{3.5}$$

System (3.5) has forth equilibrium points $E_0 = (0,0,0,0)$, $E_1 = (1,0,0,0)$, $E_2 = (0,e_2,0,0)$ and $E_* = (u_*, v_*, 0, 0)$, where u_* and v_* are given in (3.1) and (3.2) .
The linearized system of the system (3.5) around an equilibrium point $(f, g, 0, 0)$ is given by:

$$\begin{cases} u' = w, \\ v' = z, \\ w' = \frac{c}{d}w + \left(\frac{1}{d}\,(2f-1) + \frac{a}{d}\frac{ge_1}{(f+e_1)^2}\right)u + \frac{a}{d}\frac{f}{f+e_1}v, \\ z' = cz - b\frac{g^2}{(f+e_2)^2}u + \left(-b + 2b\frac{g}{f+e_2}\right)v. \end{cases} \tag{3.6}$$

and the characteristic matrix is reads as

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{d}\,(2f-1) + \frac{a}{d}\frac{ge_1}{(f+e_1)^2} & \frac{a}{d}\frac{f}{f+e_1} & \frac{c}{d} & 0 \\ -b\frac{g^2}{(f+e_2)^2} & -b + 2b\frac{g}{f+e_2} & 0 & c \end{pmatrix}$$

and the associated characteristic equation is given by

$$Q(\lambda, d) = \lambda^4 - (c+\gamma)\lambda^3 + (c\gamma - \alpha - \upsilon)\lambda^2 + (\gamma\upsilon + \alpha c)\lambda + \alpha\upsilon - \beta\mu, \tag{3.7}$$

where

$$\alpha = \frac{1}{d}\,(2f-1) + \frac{a}{d}\frac{ge_1}{(f+e_1)^2},$$

$$\beta = \frac{a}{d}\frac{f}{f+e_1},$$

$$\gamma = \frac{c}{d},$$
$$\mu = -b\frac{g^2}{(f+e_2)^2},$$
$$\upsilon = -b + 2b\frac{g}{f+e_2}.$$

For $\beta\mu = 0$, (3.7) becomes

$$\begin{aligned} Q(\lambda, d) &= \lambda^4 - (c+\gamma)\lambda^3 + (c\gamma - \alpha - \upsilon)\lambda^2 + (\gamma\upsilon + \alpha c)\lambda + \alpha\upsilon \\ &= (\lambda^2 - c\lambda - \upsilon)(\lambda^2 - \gamma\lambda - \alpha). \end{aligned}$$

Remark 3.1. *$\beta\mu = 0$, implies that $\beta = 0$ or $\mu = 0$.*
If $\beta = 0$, we have $f = 0$ and from the equation (3.7) we deduce the stability of the equilibrium point E_2.
If $\mu = 0$, we have $g = 0$ and from the equation (3.7) we deduce the stability of the equilibrium points E_0 and E_1.

At the equilibrium point $(f, g, 0, 0)$, with $(f, g) = (u_*, v_*)$ we have

$$f - 1 + \frac{a}{d}\frac{fg}{f+e_1} = 0 \text{ and } g = f + e_2$$

and we obtain that

$$\alpha = \frac{1}{d}f - \frac{a}{d}\frac{fg}{(f+e_1)^2},$$
$$\beta = \frac{a}{d}\frac{f}{f+e_1},$$
$$\gamma = \frac{c}{d},$$
$$\mu = -b,$$
$$\upsilon = b,$$

and the characteristic equation is as follows

$$Q(\lambda, d) = \lambda^4 - c(1+\frac{1}{d})\lambda^3 + (\frac{c^2}{d} - b - \frac{f}{d}(1 - \frac{ag}{(f+e_1)^2}))\lambda^2 + \frac{c}{d}(b + f(1 - \frac{ag}{(f+e_1)^2}))\lambda + \frac{bf}{d}(\frac{a}{f+e_1} + 1 - \frac{ag}{(f+e_1)^2}). \quad (3.8)$$

Note that

$$\frac{p(f)}{f+e_1} = b + f(1 - \frac{ag}{(f+e_1)^2},$$

with $p(x)$ is the polynomial defined in (3.3).
For more simplification, denote by $\rho = \rho(f) = \frac{af}{f+e_1} - b$, $\theta = \theta(c) = c^2 + b(1-d)$ and $\sigma = \sigma(f) = \frac{p(f)}{f+e_1}$.
Then

$$Q(\lambda, d) = \lambda^4 - c(1+\frac{1}{d})\lambda^3 + \frac{\theta - \sigma}{d}\lambda^2 + \frac{c\sigma}{d}\lambda + \frac{b(\rho+\sigma)}{d} \quad (3.9)$$

Theorem 3.1. *1) If $c > 0$, $\sigma < 0$ and $\rho + \sigma > 0$, the equilibrium point $(f, g, 0, 0)$ is a saddle point.*
2) If $c < 0$, $\sigma < 0$, $-\rho < \sigma < 0$ and $X_1 < 1 + d < X_2$, the equilibrium point $(f, g, 0, 0)$ is asymptotically stable and unstable if $1 + d > X_2$. Where

$$X_1 = \frac{(c^2+2b)\sigma + \sqrt{(c^2+2b)^2\sigma^2 + 4\sigma^2 b\rho}}{-2b\rho} < 0, \quad (3.10)$$

$$X_2 = \frac{(c^2+2b)\sigma - \sqrt{(c^2+2b)^2\sigma^2 + 4\sigma^2 b\rho}}{-2b\rho} > 0. \quad (3.11)$$

Proof. 1) From the Hurwitz criteria, we deduce that the equilibrium point $(f, g, 0, 0)$ is a saddle point.
2) Let a_i, $i = 1, 2, 3, 4$ denote the coefficients of the polynomial Q defined in equation (3.9).
As $c < 0$, $\sigma < 0$, $-\rho < \sigma < 0$, then $a_1 > 0$, $a_2 > 0$ and $a_4 > 0$. To apply the Hurwitz criteria we need that $a_3 > 0$.
Suppose now $X_1 < 1 + d < X_2$, where X_i, $i = 1, 2$ defined in (3.10) and (3.11) are the roots of the following polynomial

$$D(x) = -b\rho x^2 - (c^2 + 2b)\sigma x + \sigma^2. \tag{3.12}$$

Which imply that $D(1 + d) > 0$ and by a simple calculus, we obtain that $a_3 > 0$. From Hurwitz criteria we deduce the result.
To obtain the switch of stability, one needs to find the purely imaginary roots of (3.9).
Let $\lambda = ik$ a root of (3.9), then

$$Q(ik, d) = 0,$$

imply that

$$(ik)^4 - c(1 + \frac{1}{d})\lambda^3 + \frac{\theta - \sigma}{d}(ik)^2 + \frac{c\sigma}{d}(ik) + \frac{b(\rho + \sigma)}{d}.$$

Separating the real and imaginary parts, we obtain

$$k^4 - \frac{\theta - \sigma}{d}k^2 + b\frac{\rho + \sigma}{d} = 0, \tag{3.13}$$

and

$$c(1 + \frac{1}{d})k^3 + \frac{c\sigma}{d}k = 0. \tag{3.14}$$

From equation (3.14), we have

$$k^2 = -\frac{\sigma}{1 + d} > 0.$$

By replacing the last quantity in equation (3.1), we obtain that

$$D(1 + d) = 0$$

and the equilibrium point $(f, g, 0, 0)$ is unstable for $1 + d > X_2$ and (3.9) has a conjugate purely imaginary roots at $d = d_0 = X_2 - 1$. □

4 Existence of limit cycle

In this section, we consider the diffusion coefficient d as a parameter of bifurcation and we prove that the system (3.5) has a limit cycle as d passes trough the critical value d_0 via Hopf bifurcation theorem.

Theorem 4.1. *Assume $c < 0$, $\sigma < 0$, $-\rho < \sigma < 0$ and $X_1 < 1 + d < X_2$. Then, there exists $\varepsilon_0 > 0$ such that, for each $0 \leq \varepsilon < \varepsilon_0$, equation (3.5) has a family of periodic solutions $p_l(\varepsilon)$ with period $T_l = T_l(\varepsilon)$, for the parameter values $d = d(\varepsilon)$ such that $p_l(0) = (f, g, 0, 0)$, $T_l(0) = \frac{2\pi}{k}$ and $d(0) = d_0 = X_2 - 1$, where X_2 is given in equation (3.11).*

Proof. From system (3.5) the nonlinear term is given by

$$H(d, X) = \begin{pmatrix} 0 \\ \\ 0 \\ \\ \frac{1}{d}(u^2 + u_*(u_* - 1)) + \frac{a}{d}\frac{(u+u_*)(v+v_*)}{u+u_*+e_1} - \\ \left(\frac{1}{d}(2f - 1) + \frac{a}{d}\frac{ge_1}{(f+e_1)^2}\right)u - \frac{a}{d}\frac{f}{f+e_1}v \\ \\ -bv_* + \frac{b(v+v_*)^2}{u+u_*+e_2} + b\frac{g^2}{(f+e_2)^2}u - 2b\frac{g}{f+e_2}v \end{pmatrix} \tag{4.1}$$

where $X = (u, v, w, z) \in \mathbb{R}^4$.
Then, we have

$$H(d, 0) = 0 \text{ and } \frac{\partial}{\partial X} H(d, 0) = 0.$$

Now, we need to verify the transversality condition. From equation (3.9), we have

$$Q(ik, d_0) = 0 \text{ and } \frac{\partial}{\partial \lambda} Q(ik, d_0) \neq 0$$

As the function Q is continuous and from the implicit functional Theorem, we have $Q(\lambda(d), d) = 0$ for d in a neighborhood of d_0 and

$$\lambda'(d)_{|d=d_0} = -\frac{-k^4 - ick^3 - bk^2}{-c\sigma - 4d_0 ik^3 + 3cd_0 k^2 + 2(c^2 + 2b - bd_0 - \sigma)ik}. \tag{4.2}$$

Therefore

$$Re(\lambda'(d)_{|d=d_0}) = k^4 \frac{M}{N},$$

where

$$M = -4c(d_0 + 1)(k^2 + b) - 2c((1 - d_0)k^2 + c^2 + b(1 - d_0)),$$
$$N = (3ack^2 - c\sigma)^2 + (4(a - 1)k^3 - 2(c^2 + 2b - b(d_0 + 1) - \sigma)k)^2.$$
$$Re(\lambda'(d)_{|d=d_0}) > 0.$$

From the Hopf bifurcation theorem for ordinary differential equations [9, 11], we deduce the result □

5 Numerical simulations and discussions

With Matlab software we illustrate our result by some numerical simulations. The method used to compute the travelling waves solutions is to use BVP solver with projection conditions and a phase condition with the following parameters values $a = 0.5$, $b = 0.25$, $c = 2$, $e_1 = 2$, $e_2 = 2.5$ and $0 < d < 1$ From figure 1, we observe that for small density prey the density of predator decreases

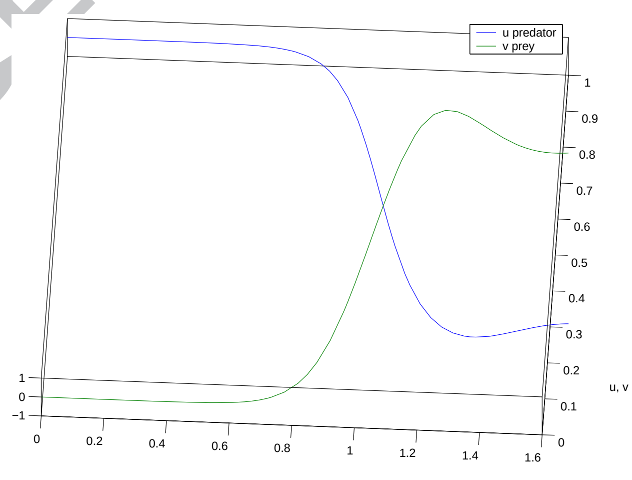


Figure 1: Travelling wave solution connecting the steady states $(1, 0)$ and (u_*, v_*), predator (blue line) and prey (green line)

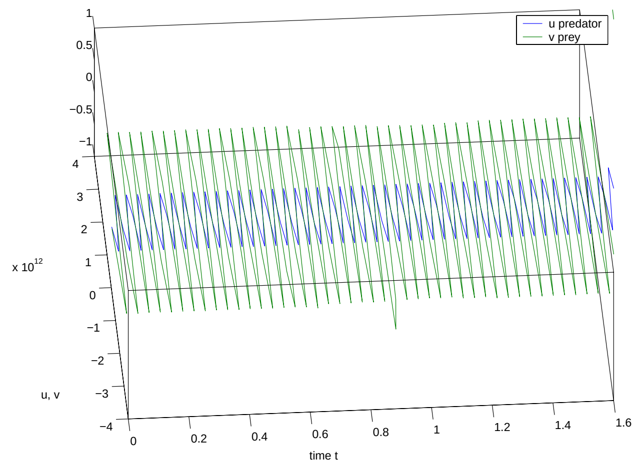


Figure 2: An illustration of the existence of periodic travelling waves connecting the steady states $(1, 0)$ and (u_*, v_*) for $d = d_0$ predator (blue line) and prey (green line)

to reach the equilibrium state u_* and the prey density of prey increases to reach the equilibrium v_* for $d < d_0$. But in figure 2, the two population coexist but they oscillate periodically around the nontrivial steady state (u_*, v_*) for $d = d_0$ and the two population can survive together for long time (because the prey may exists with a high density (10^{12}) and there is no extinction of the two population). In Fig. 3, an illustration of the periodic travelling waves solution with respect to the time and space variables and Neumann boundary conditions. The solutions of the reaction diffusion system are represented by a surface, for each fixed x the solutions are represented by lines which correspond with travelling waves in dimension two. We observe that, the periodicity of the invasion of the predators imply the periodicity of the prey.

In this paper, we show the existence of periodic travelling waves solutions via Hopf bifurcation Theorem by considering the diffusion coefficient as a parameter of bifurcation. A numerical simulations are given to illustrate the theoretical study of the reaction diffusion system modelling the invasion of the prey by the predator species. The predation is an established case of cycling in prey species. Here, the ability of predation to explain periodic travelling waves in prey population (see Fig. 2), which have recently been found in a number of spatiotemporal field studies. The nature of periodic waves in this systems, and the way in which they can be generated by the invasion of predators into a prey population. A theoretical calculation that predicts, as a function of one parameter ratio, whether such an invasion will leads to a periodic travelling waves that would be observed. The result gives an insight into the types of predator prey systems in which one would expects to see periodic travelling waves following an invasion by predators.

5.1 Conclusion

In this work, a spatio-temporal system reaction-diffusion modelling predator-prey population with modified Leslie-Gower and Holling type-II functional response is studied. By using the semigroup theory, we showed the well-posedeness of the problem and the positivity of solutions. In Theorem 3.1, the conditions of change of stability of possible steady states are given. By considering the diffusion coefficient as a parameter of bifurcation and by applying the Hopf bifurcation Theorem, we prove that there exists a critical value of diffusion parameter at which a small amplitude periodic solution in $\mathbb{R}^4$ which corresponds to a small amplitude travelling wave solution connecting the two equilibrium points occurs (Theorem 4.1).

Acknowledgment: We would like to thank the referees and the editor for their careful reading of the original manuscript and their many valuable comments and suggestions that greatly improved

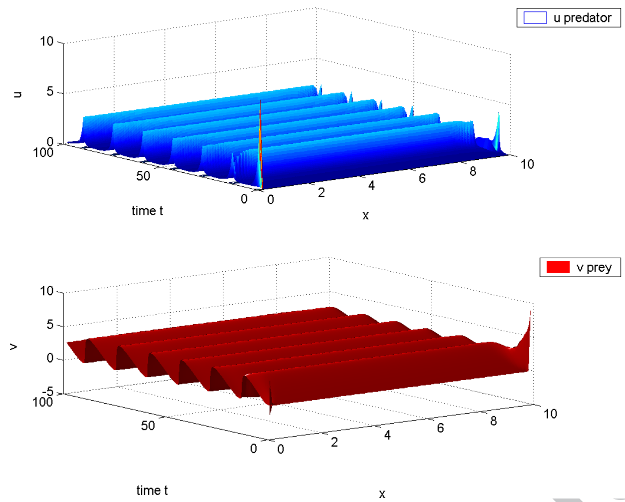


Figure 3: Periodic travelling waves with respect to the time and space variables

the presentation of this work. We also thank the Prof. A. Talibi for their discussions and remarks.

References

[1] M. A. Aziz-Alaoui and M. Daher Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type *II* schemes, Appl. Math. Lett. 16 (2003) 1069-1075.

[2] E. Beretta, Y. Kuang, Global analyses in some delayed ratio-depended predatorŰprey systems, Nonlinear Anal. Theory Methods Appl. 32 (3) (1998) 381-408.

[3] Bierman, S. M., Fairbairn, J. P., Petty, S. J., Elston, D. A., Tidhar, D. and Lambin, X. Changes over time in the spatiotemporal dynamics of cyclic populations of field voles (Microtus agrestis L.). Am. Nat. 167, (2006) 583-590.

[4] Bjornstad, O. N., Peltonen, M., Liebhold, A. M. and Baltensweiler,W. Waves of larch budmoth outbreaks in the European Alps. Science 298, (2002) 1020-1023.

[5] Graham, I. M. and Lambin, X. The impact of weasel predation on cyclic ?eld-vole survival: the specialist predator hypothesis contradicted. J. Anim. Ecol. 71, (2002) 946-956.

[6] J. Huang, G. Lu and S. Ruan, Existence of traveling wave solutions in a diffuse predator prey model, J. Math. Biol., 46, (2003) 132-152.

[7] Mackinnon, J. L., Petty, S. J., Elston, D. A., Thomas, C. J., Sherratt, T. N. and Lambin, X. Scale invariant spatio- temporal patterns of field vole density. J. Anim. Ecol. 70, (2001) 101-111.

[8] A. F. Nindjin and M. A. Aziz-Alaoui and M. Cadivel, Analysis of a a predator-prey model with modified Leslie-Gower and Holling-type *II* schemes with time delay, Nonlinear Analysis, Real World Applications, 7(5), (2006) pp: 1104-1118.

[9] B. D. Hassard, N. D. Kazarinoff and Y-H. Wan: *Theory and Applications of Hopf Bifurcation.* Cambridge University Press, (1981).

[10] Johnson, D. M., Bjørnstad, O. N. and Liebhold, A. M. Landscape mosaic induces travelling waves of insect outbreaks. Oecologia 148, (2006) 51-60.

[11] J. E. Marsden and M. McCracken: *The Hopf Bifurcation and its Applications.* Applied Math. Series, Vol. **19**, New-York, Springer-Verlag, (1976).

[12] Oli, M. K. Population cycles of small rodents are caused by specialist predators: or are they? Trends Ecol. Evol. 18, (2003) 105-107.

[13] J. A. Sherratt and M. J. Smith, Periodic travelling waves in cyclic populations: field studies diffusion models and reaction-diffusion models, J. R. Soc. Interface, 5 (2008) pp: 483-505.

[14] Smith Hal L., Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems. Mathematical Surveys and Monographs. V. 41. American Mathematical Society, 1995.

[15] Turchin, P., Briggs, C. J., Ellner, S. P., Fischlin, A., Kendall, B. E., McCauley, E.,Murdoch,W.W. and Wood, S. N. Population cycles of the larch budmoth in Switzerland. In Population cycles: the case for trophic interactions (ed. A. A. Berryman), pp. (2002) 130-141. Oxford, UK: Oxford University Press.

[16] Ranta, E. and Kaitala, V. Travelling waves in vole population dynamics. Nature (1997) 390-456.

[17] R. Xu, M.A.J. Chaplain, Persistence and global stability in a delayed predator-prey system with MichaelisŰMenten type functional response, Appl. Math. Comput. 130 (2002) 441-455.

[18] R. Yafia, F. El Adnani and H. Talibi Limit cycle and numerical similations for small and large delays in a predator-prey model with modified Leslie-Gower and Holling-type II scheme, Nonlinear Analysis: Real World Applications Vol.9, 2055-2067 (2008).

[19] R. Yafia, F. El Adnani and H. Talibi, Stability of limit cycle in a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay. Applied Mathematical Sciences, Vol. 1, no. 3, pp 119 - 131 (2007)