Nonlinear Dyn
DOI 10.1007/s11071-012-0516-x

ORIGINAL PAPER

Cluster synchronization analysis of complex dynamical networks by input-to-state stability

Junchan Zhao · M.A. Aziz-Alaoui · Cyrille Bertelle

Received: 6 January 2012 / Accepted: 19 June 2012
© Springer Science+Business Media B.V. 2012

Abstract Cluster synchronization is an interesting issue in complex dynamical networks with community structure. In this paper, we study cluster synchronization of complex networks with non-identical systems by input-to-state stability. Some sufficient conditions that ensure cluster synchronization of complex networks are provided. We show that the cluster synchronization is difficult to achieve if there are some links among different clusters. The analysis is then extended to the case where the outer coupling strengths are adaptive. Finally, numerical simulations are given to validate our theoretical analysis.

Keywords Complex network · Cluster synchronization · Input-to-state stability · Adaptive control

J. Zhao (✉)
College of Mathematica and Computer Science, Wuhan Textile University, Wuhan 430073, China
e-mail: junchanzhao@gmail.com

J. Zhao · M.A. Aziz-Alaoui
Applied Mathematics Laboratory, University of Le Havre, 76058 Le Havre Cedex, France

C. Bertelle
LITIS, University of Le Havre, 76058 Le Havre Cedex, France

1 Introduction

Synchronization of complex networks has received wide attention and make great progress in the past decade [1–7]. Usually, the identical synchronization is considered as the complete coincidence of the states of individual systems. Other synchronization phenomena have been proposed, such as lag synchronization, phase synchronization, generalized synchronization [8–10].

Cluster synchronization, as a special synchronization on complex dynamical networks, has been observed in biological science and social contact networks. Because most of these networks have the clustering characteristic, many individuals maintain close contact with others in a same cluster, while only a few individuals link with an outside cluster. Hence, the individuals are synchronized inside the same cluster, but there is no synchronization among the clusters. Moreover, the results from the latest study suggest that the cluster of non-vaccinating in social contact networks will increase disease prevalence [11]. Cluster synchronization has got broad concern recently [12–16]. Considering two-dimensional lattices of diffusively coupled identical chaotic oscillators, Belykh et al. [12] concluded that such cluster synchronization regimes persist when the chaotic oscillators have slightly different parameters. Constructing a new coupling scheme with cooperative and competitive weight-couplings, Ma et al. [13] investigated cluster synchronization patterns with several clusters for

Published online: 12 July 2012

connected chaotic networks. Applying the connection graph stability method, Chen and Lu [14] found that complete synchronization within each cluster is possible only if each node from one cluster receives the same input from nodes in other cluster. Based on the transverse stability analysis, Lu et al. [15] presented sufficient conditions for local cluster synchronization of networks.

In this paper, we address cluster synchronization of a class of complex dynamical networks with community structure. The individuals in the same cluster are identical, while those in different clusters are diverse. First, we denote the coupling strength in the same cluster as ϵ_i, $i = 1, \ldots, N_{\text{clu}}$, where N_{clu} is the number of clusters in the whole network, and the coupling strength among clusters to be c_i. Using the theory of input-to-state stability (ISS) [17, 18], we show that the dynamical network is difficult to achieve synchronization if the outer coupling strength c_i is not convergent to zero. When the c_i vanishes, the dynamical network is cluster synchronization for a sufficiently large ϵ. Moreover, an adaptive law is designed to achieve cluster synchronization. Additionally, our controllers are only applied to a small fraction of nodes.

The rest of this paper is organized as follows. Section 2 introduces a model of complex dynamical network with two clusters and some mathematical preliminaries used in this work. The main results on the cluster synchronization are presented in Sects. 3 and 4, respectively. In Sect. 5, illustrative simulations are provided by taking the chaotic systems in [19, 20] as the node dynamics in the network. Finally, some concluding remarks are given in Sect. 6.

2 Model and preliminaries

In this section, let us assume that the network only has two clusters, that is, $N_{\text{clu}} = 2$. The individuals are identical in the same cluster, and non-identical otherwise. Consider the unweighted and undirected network of N linearly coupled oscillators:

$$\begin{cases} \dot{x}_i = f(t, x_i) + \epsilon_1 \sum_{j=1, j\neq i}^{M} c_{ij}(x_j - x_i), \\ \quad i = 1, \ldots, M - k_1, \\ \dot{x}_i = f(t, x_i) + \epsilon_1 \sum_{j=1, j\neq i}^{M} c_{ij}(x_j - x_i) \\ \qquad + c_i(t) \sum_{j=M+1}^{M+k_2} c_{ij}(y_j - x_i), \\ \quad i = M - k_1 + 1, \ldots, M, \\ \dot{y}_i = g(t, y_i) + \epsilon_2 \sum_{j=M+1, j\neq i}^{N} c_{ij}(y_j - y_i) \\ \qquad + c_i(t) \sum_{j=M-k_1+1}^{M} c_{ij}(x_j - y_i), \\ \quad i = M + 1, \ldots, M + k_2, \\ \dot{y}_i = g(t, y_i) + \epsilon_2 \sum_{j=M+1, j\neq i}^{N} c_{ij}(y_j - y_i), \\ \quad i = M + k_2 + 1, \ldots, N, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^m$ is the state variables of the first M nodes, $y_i \in \mathbb{R}^m$ is the state variables of the others, $f, g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ are continuous functions; $\epsilon_i > 0$ are the inner coupling strength in a cluster, while c_i are the outer coupling strengths between two clusters. k_i are the number of nodes of cluster i which have connection with other clusters. $C = (c_{ij})_{N\times N}$ is an adjacency matrix of the network, in which c_{ij} is one if there is a connection from node i to node j $(i \neq j)$, and is zero otherwise. Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm for vectors and the Frobenius norm for matrices. I_n is the identity matrix of order n.

To begin with, for the convenience of the reader, we recall several necessary mathematical preliminaries.

Assumption 1 [21–23] There exists a positive constant L, such that

$$(x - y)^T \big(\psi(t, x) - \psi(t, y) - \Delta(x - y)\big) \\ \quad \leq -L(x - y)^T (x - y),$$

for any $x, y \in \mathbb{R}^m$ and $t \geq t_0$. Here $\Delta = \text{diag}\{\delta_1, \ldots, \delta_m\}$ is a diagonal matrix.

Remark 1 All linear and piecewise-linear continuous functions satisfy this condition. In addition, if a solution of an ODE system is bounded for any initial condition, the above condition is satisfied. Thus, it includes many well-known chaotic systems, such as the Lorenz system [19], Rössler system [20].

Definition 1 The network (1) is said to achieve cluster synchronization if nodes in the same cluster are synchronized, i.e., for all nodes i, j in the same cluster, $\|x_i - x_j\| \to 0$ as $t \to \infty$. While if nodes i and j belong to different clusters, $\|x_i - y_j\| \nrightarrow 0$ as $t \to \infty$.

Definition 2 A continuous function $\gamma : [0,a) \to [0,\infty)$ is said to be a class $\mathcal{K}$ function if it is strictly increasing and $\gamma(0)=0$. It is said to be a $\mathcal{K}_\infty$ function if $a=\infty$ and $\gamma(t)\to\infty$ as $t\to\infty$. A continuous function $\beta : [0,a)\times[0,\infty)\to[0,\infty)$ is said to be a $\mathcal{KL}$ function if, for each fixed s, $\beta(\cdot,s)$ is a class $\mathcal{K}$ function, and for each fixed r, $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\to 0$ as $s\to\infty$ [25, Chap. 3, pp. 145].

Definition 3 [26] A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ is persistently exciting (PE) if there exist $T_0, \tilde{\delta}_1, \tilde{\delta}_2 > 0$ such that

$$\tilde{\delta}_1 I_n \le \int_t^{t+T_0} \varphi(\tau)\varphi^T(\tau)\,d\tau \le \tilde{\delta}_2 I_n \tag{2}$$

holds for all $t \ge 0$.

Lemma 1 [27] *Given a system of the following form:*

$$\begin{cases} \dot{e}_1 = \beta_1(t) + \zeta(t)e_2; & e_1 \in \mathbb{R}^p \\ \dot{e}_2 = \beta_2(t) \end{cases} \tag{3}$$

such that

(i)
$$\lim_{t\to\infty}\|e_1(t)\| = 0, \qquad \lim_{t\to\infty}\|\beta_1(t)\| = 0, \qquad \lim_{t\to\infty}\|\beta_2(t)\| = 0;$$

(ii) $\zeta(t), \dot{\zeta}(t)$ *are bounded, and* $\zeta^T(t)$ *is PE.*

Then $\lim_{t\to\infty}\|e_2(t)\| = 0$.

The Kronecker product [28] of an n by m matrix A and a p by q matrix B is the np by mq matrix $A\otimes B$ defined as

$$A\otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \dots & A_{nm}B \end{pmatrix}.$$

3 Cluster synchronization by input-to-state stability

In the second equation of system (1), we regard the term $c_i(t)\sum_{j=M+1}^{M+k_2} c_{ij}(y_j - x_i)$ as an input perturbation $u_i(t)$ in the first cluster, and definite a coupling matrix

$$B = \begin{cases} b_{ij} = c_{ij}, & i \ne j,\ i,j = 1,\dots,M, \\ b_{ii} = c_{ii} = -\sum_{j=1,j\ne i}^{M} c_{ij}, & i = 1,\dots,M. \end{cases} \tag{4}$$

Rewrite the first M systems in the following form:

$$\dot{x}_i = f(t,x_i) + \epsilon_1 \sum_{j=1}^{M} b_{ij}x_j + u_i, \quad i = 1,\dots,M. \tag{5}$$

Remark 2 In each cluster, the individuals keep in close contact with each other. It is reasonable to assume that each cluster is connected. Thus, the coupling matrix B of the first cluster is negative semi-definite. It has a simple eigenvalue zero and all the other eigenvalues are negative [29].

Then the cluster synchronization can be transferred to the synchronization of each cluster. We now discuss the problem of synchronization for the system (5). The average state trajectory of uncoupled systems is regarded as the synchronization trajectory [30],

$$s = \frac{1}{M}\sum_{j=1}^{M} x_j \tag{6}$$

and

$$\dot{s} = \frac{1}{M}\sum_{j=1}^{M} f(t,x_j). \tag{7}$$

Define error vectors as

$$e_i = x_i - s, \quad i = 1,\dots,M \tag{8}$$

and notice that the diffusive coupling condition $\sum_{j=1}^{M} b_{ij} = 0$, the error systems are thus described by

$$\dot{e}_i = f(t,x_i) - \frac{1}{M}\sum_{j=1}^{M} f(t,x_j) + \epsilon_1\sum_{j=1}^{M} b_{ij}e_j + u_i, \quad i = 1,\dots,M. \tag{9}$$

Definition 4 The dynamical network (5) is said to achieve ISS if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any $e(t_0)$ and bounded u,

$$\|e(t)\| \le \beta\big(\|e(t_0)\|, t-t_0\big) + \gamma\Big(\sup_{t_0\le\tau\le t}\|u(\tau)\|\Big), \tag{10}$$

for all $t \ge t_0$, where $e^T = (e_1^T,\dots,e_M^T)$ and $u^T = (0,\dots,0,u_{M+1-k_1}^T,\dots,u_M^T)$.

Remark 3 Considering the coupling terms among clusters as perturbations, we can use the result of ISS to analyze the cluster synchronization of the network (5). When all the perturbations go to zero, form (10), each cluster will synchronize as time increases, otherwise, it is difficult to realize cluster synchronization for the network.

Our main result is given in the following theorem.

Theorem 1 *If the function f of Eq.* (5) *satisfies Assumption* 1 *and u_i are bounded, then the first cluster of network given by Eq.* (5) *is ISS for a sufficiently large coupling strength ϵ_1.*

Proof Construct the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{M} e_i^T e_i. \tag{11}$$

The derivative of V along trajectories of the dynamical network (9) is given by

$$\begin{aligned}
\dot{V}|_{(9)} &= \sum_{i=1}^{M} e_i^T \dot{e}_i \\
&= \sum_{i=1}^{M} e_i^T \left[f(t,x_i) - \frac{1}{M}\sum_{j=1}^{M} f(t,x_j) \right] \\
&\quad + \epsilon_1 \sum_{i=1}^{M}\sum_{j=1}^{M} b_{ij} e_i^T e_j + \sum_{i=1}^{M} e_i^T u_i \\
&\le \sum_{i=1}^{M} e_i^T \left[f(t,x_i) - f(t,s) + f(t,s) \right. \\
&\quad \left. - \frac{1}{M}\sum_{j=1}^{M} f(t,x_j) \right] + \epsilon_1 \sum_{i=1}^{M}\sum_{j=1}^{M} b_{ij} e_i^T e_j \\
&\quad + \sum_{i=1}^{M} e_i^T u_i.
\end{aligned}$$

Since $\sum_{i=1}^{M} e_i^T(t) = 0$, one has $\sum_{i=1}^{M} e_i^T [f(t,s) - \frac{1}{M}\sum_{j=1}^{M} f(t,x_j)] = 0$. By Assumption 1, let $e^T = (e_1^T, \ldots, e_M^T)$, we have

$$\begin{aligned}
\dot{V}|_{(9)} &\le -L\sum_{i=1}^{M} e_i^T e_i + \sum_{i=1}^{M} e_i^T \Delta e_i + \epsilon_1 \sum_{i=1}^{M}\sum_{j=1}^{M} b_{ij} e_i^T e_j \\
&\quad + \frac{1}{2}\sum_{i=1}^{M} e_i^T e_i + \frac{1}{2}\sum_{i=1}^{M} u_i^T u_i \\
&\le \sum_{i=1}^{M}\left(-L e_i^T e_i + \frac{1}{2} u_i^T u_i \right) + e^T \Big[I_M \otimes \Delta I_m \\
&\quad + \epsilon_1 (B \otimes I_m) + \frac{1}{2} I_M \otimes I_m \Big] e.
\end{aligned}$$

As the matrix B is symmetric, there exists a unitary matrix $P = (p_1, \ldots, p_M)$ such that $P^T B P = \Lambda$, where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_M\}$ is a diagonal matrix. Let $y = (P^T \otimes I_m)e$; then

$$\begin{aligned}
\dot{V}|_{(9)} &\le \sum_{i=1}^{M}\left(-L e_i^T e_i + \frac{1}{2} u_i^T u_i \right) \\
&\quad + e^T \Big[\frac{1}{2} I_M \otimes I_m + I_M \otimes \Delta I_m \\
&\quad + \epsilon_1 (P \otimes I_m)(\Lambda \otimes I_m)\big(P^T \otimes I_m\big) \Big] e \\
&= \sum_{i=1}^{M}\left(-L e_i^T e_i + \frac{1}{2} u_i^T u_i \right) \\
&\quad + e^T \Big[\frac{1}{2} I_M \otimes I_m + I_M \otimes \Delta I_m \Big] e \\
&\quad + \epsilon_1 y^T (\Lambda \otimes I_m) y.
\end{aligned}$$

Now, the matrix B is semi-negative and only have one eigenvalue zero, thus we obtain

$$\begin{aligned}
&y^T (\Lambda \otimes I_m) y \\
&\quad \le \lambda_2(B) y^T (I_M \otimes I_m) y \\
&\quad \le \lambda_2(B) e^T (P \otimes I_m)(I_M \otimes I_m)\big(P^T \otimes I_m\big) e \\
&\quad = \lambda_2(B) e^T (I_M \otimes I_m) e.
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{V}|_{(9)} &\le \sum_{i=1}^{M}\left(-L e_i^T e_i + \frac{1}{2} u_i^T u_i \right) \\
&\quad + e^T \Big[\frac{1}{2} I_M \otimes I_m + I_M \otimes \Delta I_m \\
&\quad + \epsilon_1 \lambda_2(B)(I_M \otimes I_m) \Big] e.
\end{aligned}$$

Choose ϵ_1 sufficiently large such that $\epsilon_1\lambda_2(B) + \frac{1}{2} + \max_j \delta_j \le 0$. Hence,

$$\dot V|_{(9)} \le \sum_{i=1}^{M}\Big(-Le_i^Te_i+\frac{1}{2}u_i^Tu_i\Big)$$
$$=\sum_{i=1}^{M}\Big[-L(1-\eta)e_i^Te_i-L\eta e_i^Te_i+\frac{1}{2}u_i^Tu_i\Big]$$
$$=-L(1-\eta)\|e\|^2-L\eta\|e\|^2+\frac{1}{2}\|u\|^2,$$

where $0<\eta<1$, $u^T=(0,\dots,0,u_{M+1-i}^T,\dots,u_M^T)$. Therefore, whenever $\|e\|\ge\frac{\|u\|}{\sqrt{2L\eta}}$,

$$\dot V|_{(9)}\le -L(1-\eta)\|e\|^2.$$

By a similar reasoning as in the proofs of Theorems 4.18 and 4.19 in [25], we conclude that there exist class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ, such that, for any t,

$$\big\|e(t)\big\|\le\beta\big(\big\|e(t_0)\big\|,t-t_0\big)+\gamma\Big(\sup_{t_0\le\tau\le t}\big\|u(\tau)\big\|\Big). \quad (12)$$

In the same way, we can also show that the second cluster is ISS. Although the dynamical network (1) contains two clusters only, the results also hold for complex dynamical network with many clusters following the same process of our proof above. □

Remark 4 Note that $c_i(t)\sum_{j=M+1}^{M+k_2}c_{ij}(y_j-x_i)$, the term u_i is not convergent to zero because the trajectories x_i and y_i are different chaotic systems. From the results of ISS, each cluster cannot guarantee to achieve synchronization if the perturbation u_i do not vanish.

4 Cluster synchronization by adaptive method

In this section, an adaptive law on the coupling strength $c_i(t)$ is proposed so that each cluster can achieve synchronization.

Theorem 2 *Suppose that Assumption 1 holds. Then the first cluster of network (5) is synchronized for a sufficiently large coupling strength ϵ_1 and the following adaptive control laws*:

$$\dot c_i(t)=-\sum_{j=M+1}^{M+k_2}c_{ij}e_i^T(y_j-x_i),$$
$$i=M+1-k_1,\dots,M. \quad (13)$$

Proof Construct the Lyapunov function candidate

$$V=\frac{1}{2}\sum_{i=1}^{M}e_i^Te_i+\frac{1}{2}\sum_{i=M+1-k_1}^{M}c_i^2(t). \quad (14)$$

The derivative of V along trajectories of the dynamical network (9) and the adaptive laws (13) is given by

$$\dot V|_{(9,13)}=\sum_{i=1}^{M}e_i^T\dot e_i+\sum_{i=M+1-k_1}^{M}c_i(t)\dot c_i(t)$$
$$=\sum_{i=1}^{M}e_i^T\Big[f(t,x_i)-\frac{1}{M}\sum_{j=1}^{M}f(t,x_j)\Big]$$
$$+\epsilon_1\sum_{i=1}^{M}\sum_{j=1}^{M}b_{ij}e_i^Te_j+\sum_{i=M+1-k_1}^{M}e_i^Tu_i$$
$$-\sum_{i=M+1-k_1}^{M}c_i\sum_{j=M+1}^{M+k_2}c_{ij}e_i^T(y_j-x_i)$$
$$\le\sum_{i=1}^{M}e_i^T\Big[f(t,x_i)-f(t,s)+f(t,s)$$
$$-\frac{1}{M}\sum_{j=1}^{M}f(t,x_j)\Big]+\epsilon_1\sum_{i=1}^{M}\sum_{j=1}^{M}b_{ij}e_i^Te_j$$
$$\le\sum_{i=1}^{M}\big(-Le_i^Te_i\big)$$
$$+e^T\big[I_M\otimes\Delta I_m+\epsilon_1(B\otimes I_m)\big]e$$
$$\le\sum_{i=1}^{M}\big(-Le_i^Te_i\big)$$
$$+e^T\big[I_M\otimes\Delta I_m+\epsilon_1\lambda_2(B)(I_M\otimes I_m)\big]e.$$

Using the same reasoning as in the proof of Theorem 1, we obtain $\epsilon_1\lambda_2(B)+\max_j\delta_j\le 0$ for sufficiently large ϵ_1. Hence,

$$\dot V|_{(9,13)}\le -Le^Te, \quad (15)$$

which means that $\|e\|$, $|c_i|$ are bounded. From (15), we have for any t,

$$\int_0^t e^T(\tau)e(\tau)\,d\tau\le-\frac{1}{L}\int_0^t\dot V(\tau)\,d\tau$$
$$=\frac{1}{L}\big(V(0)-V(t)\big)\le\frac{1}{L}V(0).$$

From (9) and the Assumption 1, $\dot e(t)$ is clearly bounded. It follows from Barbalat's lemma [25] that

$$\lim_{t\to\infty}e^T(t)e(t)=0,$$

then $\lim_{t\to\infty}\|e_i(t)\|=0$ for $i=1,\dots,M$. Therefore, the first cluster of network (1) is synchronized.

In the same way, using the adaptive law,

$$\dot{c}_i(t)=-\sum_{j=M+1-k_1}^{M} c_{ij}e_i^T(x_j-y_i),\quad i=M+1,\dots,M+k_2. \tag{16}$$

We see that the second cluster is synchronized for a sufficiently large ϵ_2. As the synchronization states of both clusters are different, that is, $\|x_i-y_j\|\nrightarrow 0$. Therefore, the dynamical network (1) is cluster synchronization. □

Remark 5 The adaptive laws have been added only to the nodes having links with outside clusters, so this method can be considered as pinning control [22–24].

Now, we discuss the dynamical evolution of the adaptive coupling strength c_i when (1) is cluster synchronization. Let us assume that the trajectory $\zeta_i(t)=:\sum_{j=M+1}^{M+k_2}c_{ij}(y_j-x_i)$ do not converge to zero, and the solutions of the underlying systems x_i, y_j are bounded, which hold, for example, if these dynamical system are chaotic. Hence, we can assume that $\zeta_i(t)$ satisfy the PE condition and that $\dot{\zeta}_i(t)$ are bounded.

From Eq. (9) and Eq. (13), we have

$$\dot{e}_i=\eta_i(t)+c_i\zeta_i(t),$$
$$\dot{c}_i=-\sum_{j=M+1}^{N}c_{ij}e_i^T(y_j-x_i),$$

where $\eta_i(t)=f(t,x_i)-\frac{1}{M}\sum_{j=1}^{M}f(t,x_j)+\epsilon_1\times\sum_{j=1,j\neq i}^{M}b_{ij}e_j$. It is easy to see that $\lim_{t\to\infty}\eta_i(t)=0$ and $\lim_{t\to\infty}\sum_{j=M+1}^{M+k_2}c_{ij}e_i^T(y_j-x_i)=0$ when $\lim_{t\to\infty}e_i(t)=0$. Since $\zeta_i^T(t)$ is PE, and $\zeta_i(t), \dot{\zeta}_i(t)$ are bounded, by Lemma 1 we can conclude that $\lim_{t\to\infty}c_i(t)=0$ for all $i=M+1-k_1,\dots,M$. Thus, the input perturbation $-c_i\sum_{j=M+1}^{M+k_2}c_{ij}(y_j-x_i)$ go to zero, therefore, from the ISS theory, each cluster is completely synchronization.

5 Numerical simulations

In this section, we show some illustrative examples that validate our results in Sects. 3 and 4. Consider a network with two clusters, assuming each cluster is a small-world network which is generated by the well-known NW model [31]. Define $c_{ij}=c_{ji}=1$ if there exists a connection from the jth node to the ith node, and $c_{ij}=c_{ji}=0$ otherwise. Let us assume that both clusters are connected only by one link. In all our examples, each node represents a Lorenz system in the first cluster, and Rössler system in the second cluster. The size of the networks is $N=200$.

The equation of Lorenz system is

$$\begin{cases}\dot{x}=\sigma(y-x),\\ \dot{y}=\rho x-xz-y,\\ \dot{z}=xy-\beta z,\end{cases} \tag{17}$$

where the parameters are fixed as $\sigma=10$, $\beta=\frac{8}{3}$, $\rho=28$. It is a 3-dimensional structure corresponding to the long-term behavior of a chaotic flow, noted for its butterfly shape.

The Rössler system is described by

$$\begin{cases}\dot{x}=-(y+z),\\ \dot{y}=x+ay,\\ \dot{z}=b+z(x-c),\end{cases} \tag{18}$$

when $a=0.2$, $b=0.2$, $c=8.5$, the system also has a chaotic attractor.

From the results in [23], the two systems satisfy Assumption 1.

Example 1 Because there is only one link between both clusters, we assume that the 100th Lorenz system is connected to the first Rössler system. Let the coupling strengths be $\epsilon_1=\epsilon_2=2$, $c_{100}=c_{101}=0.3$, and the initial conditions of Eq. (1) are randomly selected in the interval $[-5,5]$. From the result of Theorem 1, these Lorenz and Rössler systems do not achieve synchronization in each cluster due to the effect of perturbation. Figures 1(a) and (b) show that the synchronization errors are not convergent to zero. We also give the attractors of the 100th Lorenz system and the first Rössler system, Figs. 1(c) and (d), and both systems exhibit similar chaotic attractors themselves.

Example 2 Now, let the outer coupling strengths c_{100}, c_{101} satisfy the following adaptive laws:

$$\dot{c}_{100}(t)=-e_{100}^T(y_{101}-x_{100}),$$
$$\dot{c}_{101}(t)=-e_{101}^T(x_{100}-y_{101}). \tag{19}$$

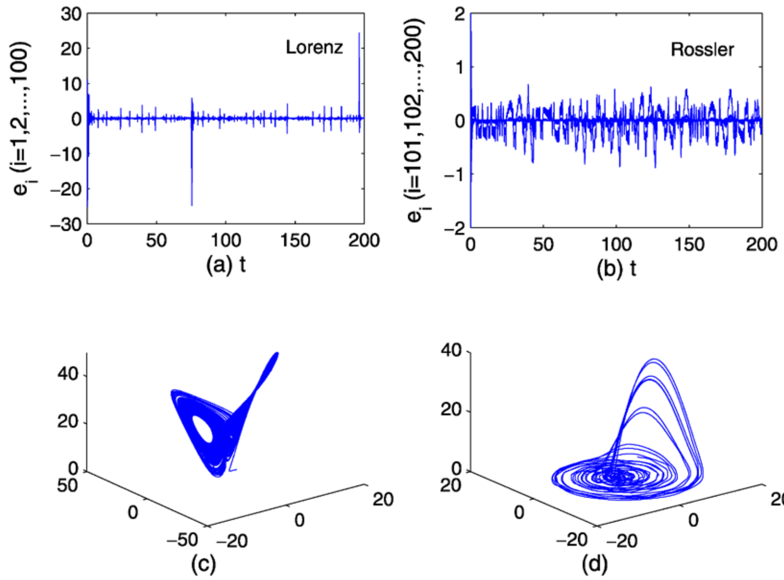

Fig. 1 Cluster synchronization with $N = 200$, $N_{\mathrm{clu}} = 2$, $k_1 = k_2 = 1$ using fixed outer coupling strengths. (**a**) and (**b**) Synchronization errors for both clusters in the dynamical network (1), which do not converge to zero. (**c**) The chaotic attractor of the 100th Lorenz system in the first cluster. (**d**) The chaotic attractor of the first Rössler system in the second cluster. Both attractors are slight perturbed because of the connection between clusters

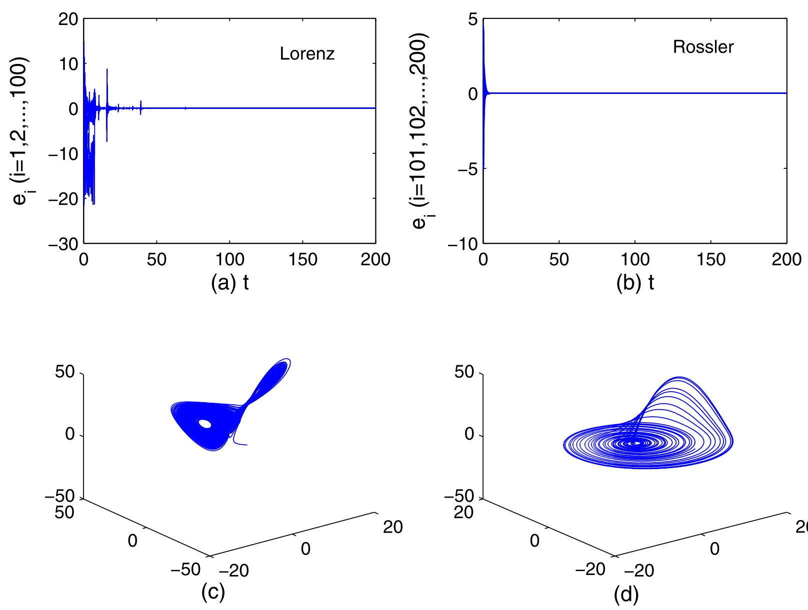

Fig. 2 Cluster synchronization with $N = 200$, $N_{\mathrm{clu}} = 2$, $k_1 = k_2 = 1$ using adaptive laws of outer coupling strengths, the initial values $c_{100}(0) = c_{101}(0) = 1$. Lorenz systems for the first cluster and Rössler for the second. (**a**) and (**b**) Synchronization errors for two clusters in the dynamical network (1), which go to zero. (**c**) The chaotic attractor of the 100th Lorenz system in the first cluster. (**d**) The chaotic attractor of the first Rössler system in the second cluster. Both chaotic systems have normal attractors with adaptive laws (19)

The coupling strengths inside each cluster are $\epsilon_1 = \epsilon_2 = 1$, the initial conditions of Eq. (1) are randomly selected in the interval $[-5, 5]$, and set $c_{100}(0) = c_{101}(0) = 1$. The synchronization errors of each cluster go to zero. Our simulation results show that the dynamical network is cluster synchronization with adaptive laws (19), as shown in Figs. 2(a) and (b). To check the PE condition, define $W_i(t) = \int_t^{t+T_0} \zeta_i(\tau)\zeta_i^T(\tau)\,d\tau$. Figures 3(a) and (b) show that the minimal eigenvalue of matrix $W_i(t)$ is larger than zero for all i and t with $T_0 = 5$, which indicates that ζ_i satisfies the PE condition for all i. The outer coupling strengths $c_i(t)$ vanish as time increases, as can be seen in Figs. 3(c) and (d).

6 Conclusions

In this paper, we have studied the cluster synchronization of complex networks consisting of linearly

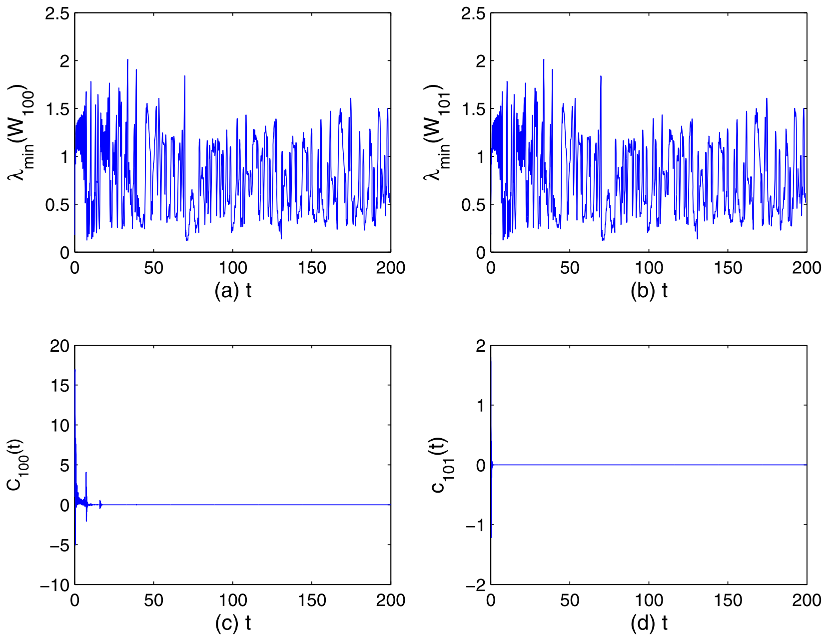

Fig. 3 (**a**) and (**b**) The minimum eigenvalue of the matrix $W_i(t)$ is larger than zero, thus PE condition is satisfied. (**c**) and (**d**) The evolution of the outer coupling strengths $c_{100}(t)$, $c_{101}(t)$, which tend to zero

coupling clusters. Regarding the connections among clusters as perturbation, we have proposed a general criterion to guarantee cluster synchronization by the method of ISS. An effective adaptive strategy to tune the coupling strengths among clusters is designed based on information of nodes' dynamics. As the topological structures play an important role in the synchronization of complex networks. In the next work, we will use the synchronizability matrix to characterize the maximum synchronizability for each cluster of complex network [32, 33], find the sensitive nodes and add the pinning controls to achieve cluster synchronization. It is very useful to explore the effect of topological structure of complex network in the process of cluster synchronization.

Acknowledgements This work was jointly supported by Région Haute-Normandie France and FEDER-RISC, the National Natural Science Foundation of China under Grant Nos. 11172215, 11071280, 50739003, the Youth Project of Hubei Education Department under Grant Nos. Q20111607, Q20111611, and the Foundation of Wuhan Textile University under Grant No. 113073. The authors wish to thank the reviewers and editor for their valuable comments and suggestions.

References

1. Wu, C.W., Chua, L.O.: Synchronization in an array of linearly coupled dynamical systems. IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **42**, 430–447 (1995)
2. Watts, D.J., Strogatz, S.H.: Collective dynamics of 'small-world' networks. Nature **391**, 440–442 (1998)
3. Pecora, L.M., Carroll, T.L.: Master stability function for synchronized coupled systems. Phys. Rev. Lett. **80**, 2109–2112 (1998)
4. Belykh, V.N., Belykh, I.V., Hasler, M.: Connection graph stability method for synchronized coupled chaotic systems. Physica D **195**, 159–187 (2004)
5. Xiang, L., Zhu, J.: On pinning synchronization of general coupled networks. Nonlinear Dyn. **64**, 339–348 (2011)
6. Lü, J., Chen, G.: A time-varying complex dynamical network model and its controlled synchronization criteria. IEEE Trans. Autom. Control **50**, 841–846 (2005)
7. Zhou, J., Chen, T.: Synchronization in general complex delayed dynamical networks. IEEE Trans. Circuits Syst. I **53**, 733–744 (2006)
8. Aziz-Alaoui, M.A.: Synchronization of chaos. In: Encyclopedia of Mathematical Physics, vol. 5, pp. 213–226. Elsevier, Amsterdam (2006)
9. Arenas, A., Díaz-Guilera, A., Kurths, J., Moreno, Y., Zhou, C.: Synchronization in complex networks. Phys. Rep. **469**, 93–153 (2008)
10. Chen, J., Lu, J., Wu, X., Zheng, W.: Generalized synchronization of complex dynamical networks via impulsive control. Chaos **19**, 043119 (2009)
11. Ndeffo Mbah, M., Liu, J., Bauch, C., Tekel, Y., Medlock, J., Meyers, L., Galvani, A.: The impact of imitation on vaccination behavior in social contact networks. PLoS Comput. Biol. **8**, e1002469 (2012)
12. Belykh, I., Belykh, V., Nevidin, K., Hasler, M.: Persistent clusters in lattices of coupled nonidentical chaotic systems. Chaos **13**, 165–178 (2003)
13. Ma, Z., Liu, Z., Zhang, G.: A new method to realize cluster synchronization in connected chaotic networks. Chaos **16**, 023103 (2006)

14. Chen, L., Lu, J.: Cluster synchronization in a complex dynamical network with two nonidentical clusters. J. Syst. Sci. Complex. **21**, 20–33 (2008)
15. Lu, W., Liu, B., Chen, T.: Cluster synchronization in networks of distinct groups of maps. Eur. Phys. J. B **77**, 257–264 (2010)
16. Wang, J., Feng, J., Xu, C., Zhao, Y.: Cluster synchronization of nonlinearly-coupled complex networks with nonidentical nodes and asymmetrical coupling matrix. Nonlinear Dyn. **67**, 1635–1646 (2012)
17. Sontag, E.D.: Smooth stabilization implies coprime factorization. IEEE Trans. Autom. Control **34**, 435–443 (1989)
18. Sontag, E.D., Wang, Y.: On characterizations of the input-to-state stability property. Syst. Control Lett. **24**, 351–359 (1995)
19. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
20. Rössler, O.E.: An equation for continuous chaos. Phys. Lett. A **57**, 397–398 (1976)
21. Lu, W., Chen, T., Chen, G.: Synchronization analysis of linearly coupled systems described by differential equations with a coupling delay. Physica D **221**, 118–134 (2006)
22. Chen, T., Liu, X., Lu, W.: Pinning complex networks by a single controller. IEEE Trans. Circuits Syst. I, Regul. Pap. **54**, 1317–1326 (2007)
23. Yu, W., Chen, G., Lü, J.: On pinning synchronization of complex dynamical networks. Automatica **45**, 429–435 (2009)
24. Zhou, J., Lu, J., Lü, J.: Pinning adaptive synchronization of a general complex dynamical network. Automatica **44**, 996–1003 (2008)
25. Khalil, H.: Nonlinear Systems, 3rd edn. Prentice Hall, Englewood Cliffs (2002)
26. Sastry, S., Bodson, M.: Adaptive Control-Stability, Convergence, and Robustness. Prentice Hall, Englewood Cliffs (1989)
27. Besançon, G.: Remarks on nonlinear adaptive observer design. Syst. Control Lett. **41**, 271–280 (2000)
28. Pease, M.C.: Method of Matrix Algebra. Academic Press, New York (1965)
29. Bollobás, B.: Modern Graph Theory. Springer, New York (1998)
30. Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D.U.: Complex networks: structure and dynamics. Phys. Rep. **424**, 175–308 (2006)
31. Newman, M.E.J., Watts, D.J.: Scaling and percolation in the small-world network model. Phys. Rev. E **60**, 7332 (1999)
32. Lü, J., Yu, X., Chen, G.: Chaos synchronization of general complex dynamical networks. Physica A **334**, 281–302 (2004)
33. Lü, J., Yu, X., Chen, G., Cheng, D.: Characterizing the synchronizability of small-world dynamical networks. IEEE Trans. Circuits Syst. I, Regul. Pap. **51**, 787–796 (2004)