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GLOBAL ATTRACTOR OF COMPLEX NETWORKS OF REACTION-DIFFUSION SYSTEMS OF FITZHUGH-NAGUMO TYPE

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Abstract. We focus on the long time behavior of complex networks of reaction-diffusion systems. We prove the existence of the global attractor and the L^∞-bound for networks of n reaction-diffusion systems that belong to a class that generalizes the FitzHugh-Nagumo reaction-diffusion equations.

1. **Introduction.** Networks of dynamical systems appear naturally in the modeling of numerous applications. In this paper, we focus on reaction-diffusion (RD) systems networks that can be seen as neural networks. More precisely, in the network, each node is a RD system which models the electrical activity of one neuron or a group of neurons. RD systems which we consider here belong to a class that generalizes the FitzHugh-Nagumo (FHN) equations. Recall that Fitzugh-Nagumo equations are a two dimensional simplification of the four dimensional Hodgkin-Huxley (HH) equations, that were introduced in 1952 to model action potential propagation in the squid giant axon, and awarded with the 1963 Nobel Prize in Physiology and Medicine [7, 10, 17]. In the network, the coupling between nodes represent the synaptic activity which can be either electrical or chemical. A classical question in dynamical systems is the existence of the attractor: basically, a set that attracts all the trajectories for large time. This is the question we deal with, by proving the existence of the network global attractor, whatever the topology (i.e. graph connectivity which here, represents the synaptic connectivity) of the network is. The existence of the attractor and the L^∞- bound appear crucially in the related article [4] in which we study the synchronization phenomena for these networks.

Mathematical framework and preliminaries. First of all, we introduce the mathematical framework we use throughout this paper. Mathematically speaking,

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the network is represented by a graph, whose nodes are d-dimensional RD systems and whose edges correspond to the coupling functions between these systems. The general system reads as:

$$U_{it} = \tilde{Q}\Delta U_i + \tilde{F}(U_i) + \tilde{H}_i(U_1, ..., U_n), i \in \{1, ..., n\} \tag{1}$$

with boundary conditions. In this equation, each variable U_i represents a function from $\Omega \times \mathbb{R}^+$ into $\mathbb{R}^d$, Ω is a bounded domain of $\mathbb{R}^N$ and $\tilde{F} : \mathbb{R}^d \to \mathbb{R}^d$ is the nonlinear reaction term. For all $i \in \{1, ..., n\}$, $\tilde{H}_i : \mathbb{R}^{nd} \to \mathbb{R}^d$ is the coupling function between nodes, whereas $\tilde{Q}$ is a diagonal matrix of $\mathbb{R}^{d\times d}$ with non-negative coefficients. We do not go into details concerning the existence of the semi-group of (1). We refer to [14, 15, 23] or [9, 21, 22], for classical results on the existence of semi-group in $L^p(\Omega)$ or in $C^{k,\alpha}(\Omega)$ spaces. Our main concern in this work is the proof of the existence of the global attractor for complex networks of type (1), for which the FHN equations are a particular case. FHN equations are a two dimensional model for oscillations and excitability. They allow to generate action potential propagation, see [1]. One can obtain FHN equations from Hodgkin-Huxley equations, by substituting a variable by its asymptotic value, using a linear correlation between two other variables and exploiting the cubic and linear shape of null-clines. They capture excitability and oscillatory regime found in the HH model, see [6, 7, 11, 17]. Good qualitative analysis of the FHN reaction-diffusion system can be found for example in [12, 19]. In [2], we gave a first analysis of a particular network of FHN RD systems. Here, we consider networks of RD systems with partially diffusive components, that generalize FHN RD equations. This is typical in neuroscience models for which only the membrane potential diffuses. Using techniques coming from [16, 23], we prove the existence of the global attractor for a general complex system representing neural activity. The study of the attractor of RD systems has a quite long history. First works, motivated by biological or fluid mechanics applications, appear in the articles [5, 13], while in [16] the attractor of a partially diffusive system with two scalar equations has been considered. As far as we know, the question of the existence of the attractor of a complex network of RD is new. As we have already pointed out, this question arises naturally from neuroscience since each neuron or group of neurons can be represented by a RD system, whereas coupling terms take account of synaptic interactions between these neurons. Note that we use the result proved here, namely the existence of the attractor and the L^∞-bound, to study, in [4], theoretically and numerically, the synchronization phenomenon, for complex networks of RD systems. We present here results for a network of n partially diffusive systems with d equations. We assume that we can split the system (1) into two subsystems, diffusive and non-diffusive, with s and $d - s$ equations, respectively. Therefore, we set for all $i \in \{1, ..., n\}$, $U_i = (u_i, v_i)$, and write (1) in the following form:

$$\begin{cases} u_{it} &= F(u_i, v_i) + Q\Delta u_i + H_i(u_1, ..., u_n), \text{ on } \Omega \times (0, +\infty), \, i \in \{1, ..., n\} \\ v_{it} &= -\sigma(x)v_i + \Phi(x, u_i) \text{ on } \Omega \times (0, +\infty), \end{cases} \tag{2}$$

with Neumann boundary conditions (NBC) on $\partial\Omega$, and where u_i take values in $\mathbb{R}^s$, $1 \le s < d$, whereas v_i take values in $\mathbb{R}^{d-s}$, we use the classical notation u_t for $\frac{\partial u}{\partial t}$, Q is a diagonal matrix in $\mathbb{R}^{s\times s}$ with coefficients $q^j, j \in \{1, ..., s\}$, and H_i takes values in $\mathbb{R}^s$. This means that diffusion and coupling terms appear only in the s first variables of each subsystem of the network. Finally, $\sigma(x)$ is a diagonal matrix with positive (diagonal) coefficients and with bounded derivatives. The application

Φ takes values in $\mathbb{R}^{(d-s)}$. Under certain conditions, the system (2) generates a semi-group on $\mathcal{H} = (L^2(\Omega))^{nd}$, see [16, 18, 23]. Before going into details of the analysis of system (2), we present some key features appearing in the proof of the global attractor for some systems with one and two variables. These techniques will be generalized to system (2) in section 2. Let us start with the following equation:

$$u_t = \Delta u - u^3 + u^2 + u, \tag{3}$$

considered in a bounded domain Ω with Neumann boundary conditions. Multiplying (3) by u gives,

$$\frac{d}{dt}\int_\Omega u^2 + 2\int_\Omega |\nabla u|^2 = -2\int_\Omega (u^4 - u^3 - u^2) \tag{4}$$

$$\leq -\delta \int_\Omega u^2 + K, \tag{5}$$

for some constants δ and K. By Gronwall lemma, there exists a constant K' such that,

$$\int_\Omega u^2 \leq K',$$

for all initial conditions in $L^2(\Omega)$ and for t large enough. Now, integrating (5) between t and $t+r$ for a given constant r gives,

$$\int_t^{t+r}\int_\Omega |\nabla u|^2 \leq K \text{ for another } K. \tag{6}$$

Multiplying (3) by u^{2k-1}, by analog computations, we find that there exists a constant K'' such that

$$\int_\Omega u^{2k} \leq K'', \tag{7}$$

for all initial conditions in $L^2(\Omega)$ and for t large enough.
Also, multiplying (3) by $-\Delta u$ gives,

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla u|^2 &= -2\int_\Omega (\Delta u)^2 + 2\int_\Omega (u^3 - u^2 - u)\Delta u \\ &\leq -\int_\Omega (\Delta u)^2 + K\int_\Omega (u^6 + u^4 + u^2) \text{ by using Young inequality,}\end{aligned}$$

for some positive constant K. Therefore, thanks to (6) and (7), we deduce, by using the uniform Gronwall lemma (see appendix and the classical paper [8]), that

$$\int_\Omega |\nabla u|^2 < K,$$

for another given constant K and for all initial conditions in $L^2(\Omega)$ for t large enough.
This gives the compacity of trajectories of (3) thanks to the compact injection of H^1 in L^2. Now, we consider the system with two variables,

$$\begin{cases} u_t &= \Delta u - u^3 + u^2 + u + v \\ v_t &= -\delta v + u \end{cases} \tag{8}$$

Multiplying the first equation of (8) by u and the second by v, integrating, using Green formula, Young inequality and Gronwall lemma leads to:

$$\int_\Omega (u^2 + v^2) < K,$$

for a constant K, for all initial conditions in $L^2(\Omega)\times L^2(\Omega)$, and time large enough. Then, we seek to obtain a similar bound in $L^{2k}(\Omega)\times L^{2k}(\Omega)$ for all $k\in\mathbb{N}, k\geq 1$. From the computation above, this result is true for $k=1$. We multiply the first equation of (8) by u^{2k-1} and the second by v^{2k-1}, sum the two equations and integrate, we obtain,

$$\begin{array}{rcl}\frac{1}{2k}\frac{d}{dt}\int_\Omega(u^{2k}+v^{2k}) & = & -(2k-1)\int_\Omega u^{2k-2}|\nabla u|^2-\int_\Omega u^{2k+2}+\int_\Omega(u^{2k+1}+u^{2k})+\int_\Omega vu^{2k-1}\\ & & +\int_\Omega uv^{2k-1}-\delta\int_\Omega v^{2k}.\end{array}$$

Using Young inequality, $ab\leq\frac{c^pa^p}{p}+\frac{b^q}{c^qq}$, leads to,

$$\begin{array}{rcl}\frac{1}{2k}\frac{d}{dt}\int_\Omega(u^{2k}+v^{2k}) & \leq & -\int_\Omega u^{2k+2}+\int_\Omega(u^{2k+1}+u^{2k})+\gamma_1\int_\Omega v^{\frac{2k+2}{3}}+\gamma_2\int_\Omega u^{2k+2}\\ & & +\gamma_3\int_\Omega u^{2k}+\gamma_4\int_\Omega v^{2k}-\delta\int_\Omega v^{2k},\end{array}$$

with $\gamma_2<1$ and $\gamma_4<\delta$. Then, since $\frac{2k+2}{3}<2k$, there exists two constants γ and K such that,

$$\frac{d}{dt}\int_\Omega(u^{2k}+v^{2k})\leq-\gamma\int_\Omega(u^{2k}+v^{2k})+K.$$

It follows that there exists a constant K (depending on k) such that:

$$\int_\Omega(u^{2k}+v^{2k})<K,$$

for all initial conditions in $L^2(\Omega)\times L^2(\Omega)$, and time large enough. By analog computations as done for equation (3), we obtain,

$$\int_\Omega|\nabla u|^2<K.$$

It remains to consider $\int_\Omega|\nabla v|^2$. By multiplying the gradient of the second equation of (8) by ∇v, we obtain:

$$\frac{d}{dt}\int_\Omega|\nabla v|^2=-2\delta|\nabla v|^2+2\nabla u.\nabla v,$$

it follows that,

$$\frac{d}{dt}\int_\Omega|\nabla v|^2\leq-\delta|\nabla v|^2+K,$$

which gives,

$$\int_\Omega|\nabla v|^2<K,$$

for t large enough. Thus, we have the bounds in L^q for all $q\in[2,+\infty[$. We can obtain the bounds in $L^\infty(\Omega)$ thanks to a result in [21].

Note that we can apply the same techniques for the generalized system:

$$\left\{\begin{array}{rcl}u_t & = & \Delta u+f(u,v)\\ v_t & = & -\delta v+g(x,u)\end{array}\right. \tag{9}$$

if the following conditions hold

$$uf(u,v)\leq-\delta_1|u|^p+\delta_2|u||v|+\delta_3 \tag{10}$$

and,

$$|f(u,v)|\leq\delta_1|u|^{p-1}+\delta_2|v|+\delta_3 \tag{11}$$

$$|\frac{\partial g}{\partial x}(x,u)|\leq c|u| \text{ and } |\frac{\partial g}{\partial u}(x,u)|\leq K, \tag{12}$$

where $p > 2$, $\delta_i > 0, i \in \{1, 2, 3\}$.

2. **Existence of the global attractor.** Now, we prove the existence of the global attractor for the dynamical system (2) in $\mathcal{H} = (L^2(\Omega)^d)^n$. The global attractor is a compact invariant set for the flow that attracts all trajectories (see for example [23]). Practically, it is very important since it is the set where all the solutions asymptotically evolve. In particular, all the patterns and solutions relevant for applications belong, asymptotically, to the global attractor, see [2, 3]. Now, we specify some assumptions that we assume throughout the paper. First, we assume that for all $i \in \{1, ..., n\}$, and for all $j \in \{1, ..., s\}$,

$$u_i^j F^j(u_i, v_i) \leq -\delta_1 |u_i^j|^p + \delta_2 |u_i^j| \sum_{k=1}^{s} |u_i^k|^{p_1} + \delta_3 |u_i^j| \sum_{k=1}^{d-s} |v_i^k| + \delta_4, \tag{13}$$

with $p > 2$, $\delta_1, \delta_2, \delta_3 > 0$, $0 \leq p_1 < p - 1$, and,

$$|F^j(u_i, v_i)| \leq \delta_1 |u_i^j|^{p-1} + \delta_2 \sum_{k=1}^{s} |u_i^k|^{p_1} + \delta_3 \sum_{k=1}^{d-s} |v_i^k| + \delta_4. \tag{14}$$

Condition (13) generalizes (10). It indicates a decrease of order p at infinity and allows to obtain bounds in L^q spaces. Condition (14) generalizes (11) and allows us to apply Young inequalities in order to obtain bounds in H^1. A typical example for which (13)-(14) hold, is given by a function F where the component j reads as:

$$\begin{aligned} F^j(u_i^1, ..., u_i^s, v_i^1, ..., v_i^{d-s}) &= -a_{p-1}(u_i^j)^{p-1} \\ &+ \textstyle\sum_{k=0}^{p-2} \sum_{\alpha_{k1}+...+\alpha_{ks}=k} a_{k_1...k_s} \prod_{l=1}^{s} (u_i^l)^{\alpha_{kl}} \\ &+ \sum_{l=1}^{d-s} b_k v_i^l, \end{aligned}$$

with $a_{p-1} > 0$, $a_{k_1...k_s}$, $b_k \in \mathbb{R}$ and p even. This simply means that, relatively to u_i, F^j is polynomial of several variables, with the dominant term given by $(u_i^j)^{p-1}$ with negative coefficient, and p even. The other terms have a degree lower than $p - 1$, whereas, F^j is a linear function of v_i.

Moreover, in order to maintain the effect of the decrease condition (13), we suppose that the coupling functions have a polynomial increase lower than $p - 1$. This reads as:

$$|H_i^j(u_1, ..., u_n)| \leq \delta_4 (1 + \sum_{k=1}^{n} |u_k^j|^{p_1}), \quad 0 < p_1 < p - 1. \tag{15}$$

Finally, we assume that for all $j \in \{1, ..., d - s\}$,

$$\left| \frac{\partial \Phi^j}{\partial x_k}(x, u_i) \right| \leq \delta_5 (1 + \sum_{j=1}^{s} |u_i^j|), \quad k \in \{1, ..., N\}, \tag{16}$$

and,

$$\left| \frac{\partial \Phi^j}{\partial u_i^k}(x, u_i) \right| \leq \delta_5. \tag{17}$$

Conditions (16) and (17) generalize condition (12). They allow to choose functions Φ with spatial heterogeneity, which may lead to rich behavior, bifurcations and pattern formation (see [1]). We deduce from (17) that for all $j \in \{1, ..., d-s\}$,

$$\left|\Phi^j(x, u_i)\right| \leq \delta_6(1 + \sum_{l=1}^{s} \left|u_i^l\right|). \tag{18}$$

The following theorem gives the existence of the global attractor.

Theorem 2.1. *Under assumptions (13)-(17), the semi-group associated with (2) possesses a connected global attractor $\mathcal{A}$ in $\mathcal{H} = (L^2(\Omega))^{nd}$. Furthermore, $\mathcal{A}$ is bounded in $(L^\infty(\Omega))^{nd}$.*

The proof of theorem 2.1 relies on a general result that gives the existence of the global attractor in Banach Spaces, see [23]. If there exists a bounded absorbing set $\mathcal{B}$ in $\mathcal{H}$, which means that $\mathcal{B}$ verifies the following condition,

$$\text{for all bounded set } B \subset \mathcal{H}, \exists t_B; \forall t > t_B, S(t)B \subset \mathcal{B}, \tag{19}$$

and if,

$$\text{for all bounded set } B \subset \mathcal{H}, \exists t_B; \cup_{t \geq t_B} S(t)B \text{ is relatively compact in } \mathcal{H}, \tag{20}$$

then the ω-limit set of $\mathcal{B}$, is an invariant connected compact set that attracts all the trajectories. Therefore, we split the proof of theorem 2.1 into four parts, which for the reader's convenience, are presented as different lemmas. Before going into details, let us briefly present the sketch of the proof. We first show, in lemma 2.2, the existence of a bounded absorbing set in $\mathcal{H}$, that is (19). Then, in lemma 2.3, we obtain $(L^q(\Omega))^{nd}$-bounds for all $q \in \mathbb{N}$. Next, in lemma 2.4, we prove the result of compacity for trajectories, that is (20). More precisely, we establish the existence of a bounded absorbing set in $(H^1(\Omega))^{nd}$. Then, the result follows from the compact injection of $H^1(\Omega)$ into $L^2(\Omega)$. Note that proving lemmas 2.2 and 2.4 gives the global attractor existence. Finally, the theorem 2.1 follows form result that links L^∞ and L^q-norms for linear parabolic equations, see [21].
Let us introduce the following notations,

$$|u|_{p,\Omega} = \Big(\int_\Omega |u|^p dx\Big)^{\frac{1}{p}}, \tag{21}$$

if u is a real or vector valued function.

The lemma below establishes the existence of a bounded absorbing set in $\mathcal{H}$.

Lemma 2.2. *There exists an absorbing bounded set in $\mathcal{H}$, that is, there is a constant K, such that for all initial conditions in $\mathcal{H}$:*

$$(|u|_{2,\Omega}^2 + |v|_{2,\Omega}^2)(t) \leq K \text{ for } t \text{ large enough.}$$

Proof. The proof mainly relies on the presence of $-\delta_1|u_i^j|^p$ in (13) and the fact that $\sigma(x)$ is diagonal with positive coefficients. In the following, parameters δ, c and K are generic constants. We multiply each scalar equation in (2), for all i and j, by u_i^j or v_i^j, and sum. We show, by using Young inequality, $ab \leq \frac{a^p}{\epsilon^p p} + \frac{\epsilon^q b^q}{q}$, and as it has been done for equation (8), that,

$$\frac{d}{dt}(|u|_{2,\Omega}^2 + |v|_{2,\Omega}^2) + \delta_1(|u|_{2,\Omega}^2 + |v|_{2,\Omega}^2) + \delta_2|u|_{p,\Omega}^p + \delta_3|\nabla u|_{2,\Omega}^2 \leq \delta_4. \tag{22}$$

Then by Gronwall inequality, we have for t large enough,

$$(|u|_{2,\Omega}^2 + |v|_{2,\Omega}^2)(t) \leq K. \tag{23}$$

□

Lemma 2.3. *For all $q \in \mathbb{N}$, there exists an absorbing bounded set in $(L^q(\Omega))^{nd}$: for all $q \in \mathbb{N}$, there is a constant K_q such that for all initial conditions in $\mathcal{H}$,*

$$(|u|^q_{q,\Omega} + |v|^q_{q,\Omega})(t) \leq K_q \text{ for } t \text{ large enough.}$$

Proof. In the following, K is a generic constant. We multiply the first equation of (2) by $|u_i^j|^{2k-2}u_i^j$, for all i, j and integrate. We obtain,

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega |u_i^j|^{2k} = \int_\Omega F(u_i, v_i)|u_i^j|^{2k-2}u_i^j + \int_\Omega q^j \Delta u_i^j |u_i^j|^{2k-2}u_i^j + H_i|u_i^j|^{2k-2}u_i^j. \quad (24)$$

Besides,

$$\int_\Omega q^j \Delta u_i^j |u_i^j|^{2k-2}u_i^j = -(2k-1)\int_\Omega |\nabla u_i^j|^2 |u_i^j|^{2k-2}. \quad (25)$$

Thanks to (13) and to (15), it follows that:

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega |u_i^j|^{2k} \leq -\delta_1{}'\int_\Omega |u_i^j|^{2k-2+p} + \delta_2{}'\sum_{k=1}^{d-s}\int_\Omega |u_i^j||v_i^k| + \sum_{k=1}^{n}\int_\Omega |u_i^k|^{p_1}|u_i^j|^{2k-2}u_i^j. \quad (26)$$

Hence, by Young inequality,

$$\frac{1}{2k}\frac{d}{dt}|u|^{2k}_{2k,\Omega} \leq -\delta_1 |u|^{2k}_{2k,\Omega} + K. \quad (27)$$

By Gronwall lemma,

$$|u|^{2k}_{2k,\Omega} \leq K, \quad (28)$$

for t large enough.

Besides, by similar techniques, we have,

$$\frac{d}{dt}|v|^{2k}_{2k,\Omega} = -\sigma|v|^{2k}_{2k,\Omega} + K. \quad (29)$$

Then, the final result follows thanks to Gronwall lemma. □

Now, we prove the compactness of trajectories in $\mathcal{H}$, by establishing (20).

Lemma 2.4. *There exists an absorbing bounded set in $(H^1(\Omega))^{nd}$: there is a constant K, such that for all initial conditions in $\mathcal{H}$ and t large enough,*

$$|\nabla u(t)|_{2,\Omega} \leq K.$$

Proof. By integrating (22) between t and $t + r$, we have for t large enough and $\forall r > 0$:

$$\int_t^{t+r} |\nabla u|^2_{2,\Omega} + \delta\int_t^{t+r} |u|^p_{p,\Omega} \leq K + \delta r. \quad (30)$$

Now, we multiply each component of the first equation of (2) by $-\Delta u_i^j$, integrate and sum over i and j to obtain,

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2_{2,\Omega} = -\sum_{i=1}^{n}\sum_{j=1}^{s}\left(\int_\Omega (F^j(u_i, v_i)\Delta u_i^j + q_j \Delta u_i^j \Delta u_i^j + H_i^j(u_1, ..., u_n)\Delta u_i^j\right) \quad (31)$$

which, thanks to (14) and (15), leads to:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla u|^2_{2,\Omega} + q|\Delta u|^2_{2,\Omega} \\
&\qquad \leq c\sum_{i=1}^{n}\sum_{j=1}^{s}\int_\Omega \Big(1+\sum_{l=1}^{s}|u_i^l|^{p-1}+\sum_{k=1}^{d-s}|v_i^k|+\sum_{l=1}^{n}|u_l^j|^{p_1}\Big)|\Delta u_i^j| \qquad (32)\\
&\qquad \leq c\sum_{i=1}^{n}\sum_{j=1}^{s}\int_\Omega \Big(\frac{c}{2q}\big(1+\sum_{l=1}^{s}|u_i^l|^{p-1}+\sum_{k=1}^{d-s}|v_i^k|+\sum_{l=1}^{n}|u_l^j|^{p_1}\big)^2+\frac{q}{2c}(\Delta u_i^j)^2\Big), \qquad (33)
\end{aligned}$$

where $q=\min_{i\in\{1,...,s\}} q_i$ and c is a generic constant. It follows that:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\nabla u|^2_{2,\Omega} &\leq c\sum_{i=1}^{n}\sum_{j=1}^{s}\int_\Omega \big(1+\sum_{l=1}^{s}|u_i^l|^{2p-2}+\sum_{k=1}^{d-s}|v_i^k|^2+\sum_{l=1}^{n}|u_l^j|^{2p_1}\big) \qquad (34)\\
&\leq c(1+|u|^{2p-2}_{2p-2,\Omega}+|v|^2_{2,\Omega}). \qquad (35)
\end{aligned}$$

Thanks to lemma 2.3, we have for t large enough:

$$|u|^{2p-2}_{2p-2,\Omega} \leq K. \qquad (36)$$

Then we can apply the uniform Gronwall lemma (see appendix and [8]), and show that:

$$|\nabla u|^2_{2,\Omega}(t) \leq K \text{ for t large enough.}$$

It remains to find a bound for $|\nabla v|_{2,\Omega}$. For all $i\in\{1,...,n\}$, and $k\in\{1,...,N\}$, we have:

$$\begin{aligned}
\frac{d}{2dt}|v_{ix_k}|^2_{2,\Omega} \;=\; &\int_\Omega(-\sigma'_{x_k}(x)v_i\cdot v_{ix_k}-\sigma(x)v_{ix_k}\cdot v_{ix_k}+\Phi'_{x_k}(x,u_i)\cdot v_{ix_k}\\
&+\textstyle\sum_{j=1}^{s}u^j_{ix_k}\Phi'_{u_i^j}(x,u_i)\cdot v_{ix_k}),
\end{aligned} \qquad (37)$$

with $v_{ix_k}=\frac{\partial v_i}{\partial x_k}$. Using Young and Cauchy-Schwarz inequalities, we find, thanks to (16) and (17), for time large enough,

$$\frac{d}{dt}|v_{ix_k}|^2_{2,\Omega} \leq -\frac{\sigma}{2}|v_{ix_k}|^2_{2,\Omega}+K. \qquad (38)$$

Finally, the result follows by using the Gronwall inequality and summing over i. □

Proof of theorem 2.1. It remains to prove the L^∞-bound. For all $i\in\{1,...,n\}$, and for all $j\in\{1,...,s\}$, we have:

$$u_i^j(t)=\mathcal{T}(t)u_{i0}^j+\int_0^t\mathcal{T}(t-\tau)\{F^j(u_i^j,v_i^j)(\tau)+H_i^j(u_1(\tau),...,u_n(\tau))+u_i^j(\tau)\} \qquad (39)$$

where $\mathcal{T}$ represents the semi-group associated with $\frac{\partial\varphi}{\partial t}-q^j\Delta\varphi+\varphi=0$ and NBC. We know, see [21], lemma 3 p 25, that $\mathcal{T}$ verifies:

$$|\mathcal{T}(t)\varphi|_{\infty,\Omega} \leq cm(t)^{-\frac{1}{2}}e^{-\lambda t}|\varphi|_{2N,\Omega}, \qquad (40)$$

where $m(t)=\min(1,t)$, λ is the smallest eigenvalue of the operator $I-q^j\Delta$, and c is a positive constant. This allows us to conclude. □

3. **Conclusion.** In this paper, we have considered a network of n coupled reaction-diffusion systems. We have proven the existence of the network global attractor whatever the topology is.

Appendix A. The Gronwall uniform lemma.

Theorem A.1. *Let g,h and $y \in L^1_{loc}$ three positive functions. We assume that for all $t \geq t_0$,*

$$\frac{dy}{dt} \leq gy + h \tag{41}$$

and,

$$\int_t^{t+r} g(s)ds \leq a_1, \int_t^{t+r} h(s)ds \leq a_2, \int_t^{t+r} y(s)ds \leq a_3,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \leq (a_2 + \frac{a_3}{r})e^{a_1}, \forall t \geq t_0 + r.$$

Proof. Let $s_0 \geq t_0$. By (41), we have:

$$\frac{d}{dt}\big(e^{-\int_{s_0}^t g(s)ds} y(t)\big) \leq e^{-\int_{s_0}^t g(s)ds} h. \tag{42}$$

We integrate (42) between t and $s_0 + r$ for $t \in [s_0, s_0 + r]$. We obtain:

$$e^{-\int_{s_0}^{s_0+r} g(s)ds} y(s_0 + r) - e^{-\int_{s_0}^t g(s)ds} y(t) \leq \int_t^{s_0+r} \exp\{-\int_{s_0}^{t'} g(s)ds\} h(t')dt'.$$

It follows that:

$$e^{-a_1} y(s_0 + r) - e^{-\int_{s_0}^t g(s)ds} y(t) \leq \int_t^{s_0+r} h(t')dt'.$$

By multiplying by $\exp\{\int_{s_0}^t g(s)ds\}$, we find:

$$e^{\int_{s_0}^t g(s)ds} e^{-a_1} y(s_0 + r) - y(t) \leq e^{\int_{s_0}^t g(s)ds} \int_t^{s_0+r} h(t')dt'.$$

Then, we integrate between s_0 and $s_0 + r$ with respect to t. This gives

$$e^{-a_1} y(s_0 + r) \leq a_2 + \frac{a_3}{r},$$

thus,

$$y(s_0 + r) \leq (a_2 + \frac{a_3}{r})e^{a_1}.$$

□

Corollary 1. *Let $y, h \in L^1_{loc}$ two positive functions. We assume that for $t \geq t_0$:*

$$\frac{dy}{dt} \leq h \tag{43}$$

and,

$$\int_t^{t+r} h(s)ds \leq a_2, \int_t^{t+r} yds \leq a_3,$$

where r, a_2, a_3 are positive constants. Then

$$y(t+r) \leq \frac{a_3}{r} + a_2, \forall t \geq t_0 + r.$$

Proof. It follows obviously from theorem A.1 with $a_1 = 0$. For sake of clarity, we give here a direct proof: for all $s_0 \in [t_0, +\infty[$, for all $t \in [s_0, s_0 + r]$,

$$y(s_0 + r) - y(t) \leq \int_t^{s_0+r} h ds.$$

Then, we integrate between s_0 et $s_0 + r$ with respect to t. We obtain,

$$y(s_0 + r) \leq \frac{a_3}{r} + a_2.$$

□

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