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SYNCHRONIZATION AND CONTROL OF A NETWORK OF COUPLED REACTION-DIFFUSION SYSTEMS OF GENERALIZED FITZHUGH-NAGUMO TYPE *

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Résumé. Nous considérons un réseau de systèmes de réaction-diffusion de type FitzHugh-Nagumo généralisés. Nous nous intéressons au comportement asymptotique et à la synchronisation du réseau. Ces résultats nous permettent d'étendre d'autres résultats obtenus pour un type particulier de systèmes de FitzHugh-Nagumo.

Abstract. We consider a network of reaction diffusion systems of generalized FitzHugh-Nagumo type, where the cubic function is replaced by a polynomial function with odd degree. We deal with asymptotic behaviour and synchronization of the whole network. These results extend a previous work in which we considered particular systems of FitzHugh Nagumo type.

1. Introduction

Let us denote by w_t the time derivative of the function w. The FitzHugh-Nagumo model,

$$\begin{cases} x_t = c(F(x) + y + z) \\ y_t = \dfrac{1}{c}(x - a + by) \end{cases}$$

where F is a cubic function with positive leading coefficient, z constant, and $a, b, c > 0$, is a simplification of the well known Hodgkin-Huxley model describing the propagation of action potential in neurons, see for example, [1–3,5,7]. In [9], we considered a network of coupled reaction-diffusion systems of the following FitzHugh-Nagumo type (FHN),

$$\begin{cases} \epsilon u_t &= f(u) - v + d_u \Delta u \\ v_t &= u - \delta v + d_v \Delta v \end{cases} \tag{1}$$

where,

$$f(u) = -u^3 + 3u,$$

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and where $\epsilon, \delta > 0$, are small parameters. In this case, the underlying (ODE) part of system (1) induces an asymptotic evolution to a unique limit cycle for the trajectories diferent from $(0,0)$. We showed some results on asymptotic behaviour and synchronization for the network. Here, we will generalize some of these results. Let us consider a network of coupled reaction diffusion systems of the following generalized FHN type,

$$\begin{cases} \epsilon u_t &= f(u) - v + d_u \Delta u + \gamma \\ v_t &= au - bv + d_v \Delta v + \mu \end{cases} \tag{2}$$

where,

$$f(u) = \sum_{k=1}^{p} d_k u^k$$

is a polynomial function of odd degree with negative leading coefficient, $d_p < 0, p \geq 3$. The parameter $\epsilon > 0$ is small. The underlying (ODE) part of system (1) can induce a very more complicated asymptotic behaviour. We assume that a, b, d_u are positive, and d_v is non negative. We look for solutions $u = u(x,t)$, $v = v(x,t)$ on a smooth bounded domain $\Omega \subset \mathbb{R}^n$, with zero-flux Neumann boundary conditions on the boundary of Ω :

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0,$$

where ν denotes the exterior normal to the boundary. The coupling is chosen such that, for all $i = 2, ..., N$, subsystem (u_{i-1}, v_{i-1}) drives subsystem (u_i, v_i). This means that the whole system reads as,

$$\begin{cases} \epsilon u_{1t} &= f(u_1) - v_1 + d_{u_1} \Delta u_1 + \gamma_1 \\ v_{1t} &= a_1 u_1 - b_1 v_1 + d_{v_1} \Delta v_1 + \mu_1 \\ \vdots & \\ \epsilon u_{it} &= f(u_i) - v_i + d_{u_i} \Delta u_i + \alpha_i (u_{i-1} - u_i) + \gamma_i \\ v_{it} &= a_i u_i - b_i v_i + d_{v_i} \Delta v_i + \beta_i (v_{i-1} - v_i) + \mu_i \\ \vdots & \\ \epsilon u_{Nt} &= f(u_N) - v_N + d_{u_N} \Delta u_N + \alpha_N (u_{N-1} - u_N) + \gamma_N \\ v_{Nt} &= a_N u_N - b_N v_N + d_{v_N} \Delta v_N + \beta_N (v_{N-1} - v_N) + \mu_N \end{cases} \tag{3}$$

where $\alpha_i, \beta_i \geq 0$, for $i = 2, ..., N$.

2. Analytical results

2.1. Space homogeneous asymptotic behaviour

Let (u, v) be the solution of system (2), then we have the following result,

Théorème 2.1. *Let,*

$$M = \sup_{x \in \mathbb{R}} f'(x),$$

and λ be the smallest non zero eigenvalue of the Laplacian operator $(-\Delta)$ with zero flux Neumann boundary conditions. If,

$$M - \lambda d_u < 0, \tag{4}$$

then,

$$\lim_{t \to +\infty} \left(||(u - \bar{u}||_{L^2(\Omega)} + ||v - \bar{v}||_{L^2(\Omega)} \right) = 0 \tag{5}$$

where,

$$\bar{u}(t) = \frac{\int_\Omega u(x,t)dx}{|\Omega|}, \qquad \bar{v}(t) = \frac{\int_\Omega v(x,t)dx}{|\Omega|}.$$

Moreover, $\bar{u}$,$\bar{v}$ *are solutions of the following system,*

$$\left\{ \begin{array}{rcl} \epsilon \bar{u}_t &=& f(\bar{u}) - \bar{v} + \gamma + g(t) \\ \bar{v}_t &=& a\bar{u} - b\bar{v} + \mu \end{array} \right. \tag{6}$$

where $g(t)$ *is a function going to* 0 *with exponential rate when* t *goes to* $+\infty$ *.*

Démonstration. Let,

$$\phi(t) = \frac{1}{2}\big(a\epsilon \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2\big),$$

then,

$$\begin{array}{rcl} \dot{\phi} &=& \int_\Omega (\epsilon a \nabla u \nabla u_t + \nabla v \nabla v_t) \\ &=& \int_\Omega (a\nabla u \nabla (f(u) - v + d_u \Delta u) + \nabla v \nabla (au - bv + d_v \Delta v)) \\ &=& \int_\Omega (a(f'(u)|\nabla u|^2 - d_u(\Delta u)^2) - b|\nabla v|^2 - d_v(\Delta v)^2) \end{array}$$

Now, we use the following spectral property of laplacian operator with zero-flux Neumann boundary conditions, see for example [10],

$$\int_\Omega (\Delta u)^2 \geq \lambda \int_\Omega \nabla |u|^2.$$

Then,

$$\begin{array}{rcl} \dot{\phi} &\leq& a\left(\int_\Omega M|\nabla u|^2 - \lambda d_u \int_\Omega |\nabla u|^2\right) - b\int_\Omega |\nabla v|^2 - \lambda d_v \int_\Omega |\nabla v|^2 \\ &\leq& a(M - \lambda d_u)\int_\Omega |\nabla u|^2 - (\lambda d_v + b)\int_\Omega |\nabla v|^2. \end{array}$$

Now, since $\lambda d_u > M$ we have,

$$\dot{\phi} \leq -2\min\left(\frac{\lambda d_u - M}{\epsilon}, \lambda d_v + b\right)\phi,$$

and thus,

$$\phi(t) \leq \phi(0)e^{-c_1 t}$$

where,

$$c_1 = 2\min\left(\frac{\lambda d_u - M}{\epsilon}, \lambda d_v + b\right).$$

Furthermore,

$$\begin{array}{rl} ||u - \bar{u}||^2_{\mathrm{L}^2(\Omega)} + ||v - \bar{v}||^2_{\mathrm{L}^2(\Omega)} &\leq \frac{1}{\lambda}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2\right) \\ &\leq \frac{2}{\lambda}\max\left(\frac{1}{a\epsilon}, 1\right)\phi(t) \end{array}$$

which implies (5). In the remaining of the proof, we show that $\bar{u}$ et $\bar{v}$ are solutions of (6). We have,

$$\begin{cases} \epsilon\bar{u}_t &= \frac{1}{|\Omega|}\int_\Omega f(u) - \bar{v} + \gamma \\ \bar{v}_t &= a\bar{u} - b\bar{v} + \mu \end{cases}$$

thus,

$$\begin{cases} \epsilon\bar{u}_t &= \frac{1}{|\Omega|}\int_\Omega (f(u) - f(\bar{u})) + f(\bar{u}) - \bar{v} + \gamma \\ \bar{v}_t &= a\bar{u} - b\bar{v} + \mu. \end{cases}$$

Let us denote,

$$g(t) = \frac{1}{|\Omega|}\int_\Omega (f(u) - f(\bar{u})).$$

Then, we obtain :

$$\begin{cases} \epsilon\bar{u}_t &= g(t) + f(\bar{u}) - \bar{v} + \gamma \\ \bar{v}_t &= a\bar{u} - b\bar{v} + \mu. \end{cases}$$

But,

$$\begin{aligned} |g(t)| &= |\frac{1}{|\Omega|}\int_\Omega (f(u) - f(\bar{u}))| \\ &\leq \frac{L}{|\Omega|}\int_\Omega |u - \bar{u}| \\ &\leq \frac{L}{|\Omega|^{\frac{1}{2}}}||u - \bar{u}||_{\mathrm{L}^2(\Omega)}, \end{aligned}$$

where,

$$L = \sup_{t\in\mathbb{R}^+} |f'(\bar{u}(t))|,$$

since from a result in [6], we know that $(u, v) \in L^\infty(\Omega) \times L^\infty(\Omega)$. It follows that :

$$\lim_{t\to+\infty} g(t) = 0.$$

Which completes the proof. □

Let (u_i, v_i), $1 \leq i \leq N$ be the solution of system (3)

Théorème 2.2. *Let λ be the smallest non-zero eigenvalue of the Laplacian operator, with zero flux Neumann boundary conditions. Assume that,*

$$M - \lambda d_{u_1} < 0 \text{ and } M - \lambda d_{u_i} - \alpha_i < 0 \quad \forall i \in 2, .., N, \tag{7}$$

(8)

then,

$$\lim_{t\to+\infty} \sum_{i=1}^{N} \left(||u_i - \bar{u}_i||_{\mathrm{L}^2(\Omega)} + ||v_i - \bar{v}_i||_{\mathrm{L}^2(\Omega)}\right) = 0, \tag{9}$$

where,

$$\bar{u}_i(t) = \frac{\int_\Omega u_i(x,t)dx}{|\Omega|}, \qquad \bar{v}_i(t) = \frac{\int_\Omega v_i(x,t)dx}{|\Omega|}, \quad \forall i \in 1, ..., N$$

with ($\bar{u}_i$,$\bar{v}_i$) satisfying,

$$\begin{cases} \epsilon \bar{u}_{it} &= f(\bar{u}_i) - \bar{v}_i + \gamma_i + g_i(t) + \alpha_i(\bar{u}_{i-1} - \bar{u}_i) \\ \bar{v}_{it} &= a_i \bar{u}_i - b_i \bar{v}_i + \mu_i + \beta_i(\bar{v}_{i-1} - \bar{v}_i) \end{cases} \quad (10)$$

and where, $g_i(t) \to 0$ when $t \to +\infty$ with exponential rate decay.

Démonstration. It comes from an induction argument, by using similar techniques as those given in the proof of Theorem 2.1. More precisely, let,

$$\phi_i = \frac{1}{2}\left(\epsilon a_i \int_\Omega |\nabla u_i|^2 + \int_\Omega |\nabla v_i|^2\right),$$

we show that for all $i \in 1, ..., N$ there exists positive constants K_i, c_i such that,

$$\phi_i(t) \leq K_i e^{-c_i t}.$$

From the proof of Theorem 2.1, we know that this result is true for $i = 1$, that is,

$$\phi_1(t) \leq \phi_1(0) e^{-c_1 t}.$$

Let us assume that the result is true until $i - 1$, by algebraic computations we obtain,

$$\begin{aligned} \dot{\phi}_i &\leq a_i(M - \lambda d_{u_i} - \alpha_i + \frac{\alpha_i \kappa_i}{2}) \int_\Omega |\nabla u_i|^2 - (\lambda d_{v_i} + b_i + \frac{\beta_i}{2}) \int_\Omega |\nabla v_i|^2 + a_i \frac{\alpha_i}{2\kappa_i} \int_\Omega |\nabla u_{i-1}|^2 + \frac{\beta_i}{2} \int_\Omega |\nabla v_{i-1}|^2 \\ &\leq a_i(M - \lambda d_{u_i} - \alpha_i + \frac{\alpha_i \kappa_i}{2}) \int_\Omega |\nabla u_i|^2 - (\lambda d_{v_i} + b_i + \frac{\beta_i}{2}) \int_\Omega |\nabla v_i|^2 + s_1 K_{i-1} e^{-c_{i-1} t} \\ &\leq -s_2 \phi + K_{i-1} e^{-c_{i-1} t} \end{aligned}$$

where κ_i is a positive constant satisfying

$$\kappa_i < 2 \frac{\lambda d_{u_i} + \alpha_i - M}{\alpha_i},$$

and $s_1 = \max(\frac{\alpha_i}{\epsilon \kappa_i}, \beta_i)$, $s_2 = 2\min(\frac{\lambda d_{u_i} + \alpha_i(1 - \frac{\kappa_i}{2}) - M}{\epsilon}, \lambda d_{v_i} + b_i + \frac{\beta_i}{2})$, K_{i-1}, c_{i-1} are positive constants. By integration, this yields,

$$\phi_i(t) \leq K_i e^{-c_i t}.$$

The remaining of the proof is similar as this of Theorem 2.1. □

2.2. **Synchronization**

Définition 2.3. Let $S_i = (u_i, v_i)$. We say that S_i and S_j synchronize if,

$$\lim_{t \to +\infty} (||u_i - u_j||_{L^2(\Omega)} + ||v_i - v_j||_{L^2(\Omega)}) = 0.$$

The quantity,

$$(||u_i - u_j||^2_{L^2(\Omega)} + ||v_i - v_j||^2_{L^2(\Omega)})^{\frac{1}{2}}$$

is called the norm of synchronization error between S_i and S_j. Let $S = (S_1, S_2, ..., S_N)$. We say that S synchronize if,

$$\lim_{t\to+\infty} \sum_{i=1}^{N-1} (||u_i - u_{i+1}||_{\mathrm{L}^2(\Omega)} + ||v_i - v_{i+1}||_{\mathrm{L}^2(\Omega)}) = 0$$

The quantity,

$$\Big(\sum_{i=1}^{N-1} (||u_i - u_{i+1}||^2_{\mathrm{L}^2(\Omega)} + ||v_i - v_{i+1}||^2_{\mathrm{L}^2(\Omega)})\Big)^{\frac{1}{2}}$$

is called the norm of synchronization error of S.

Let us consider the system (3) with $d_{u_i} = d_{u_j}$, $d_{v_i} = d_{v_j}$ and $b_i = b_j = b$, $a_i = a_j = a$, $\gamma_i = \gamma_j$, $\mu_i = \mu_j$, for all $i, j \in \{1, ..., N\}$. Let us recall that f is a polynomial function of odd degree with negative leading coefficient,

$$f(u) = \sum_{k=1}^{p} d_k u^k, d_p < 0, p \geq 3.$$

Let,

$$M = \sup_{u\in B, x\in\mathbb{R}} \sum_{k=1}^{p} \frac{f^{(k)}}{k!}(u)x^{k-1},$$

where B is a compact interval in which u_1 remains strictly.

Théorème 2.4. *If,*

$$\alpha_i > M, \quad i = 2, ..., N,$$

then the network $S = ((u_1, v_1), (u_2, v_2), ..., (u_N, v_N))$ synchronize in the sense of definition (2.3).

Démonstration. Let

$$\psi_i(t) = \frac{1}{2}\big(a\epsilon \int_\Omega (u_i - u_{i-1})^2 + \int_\Omega (v_i - v_{i-1})^2\big).$$

Our proof is based on an induction idea. We show that for all $i \in 2, ..., N$,

$$\psi_i(t) \leq K_i e^{-c_i t}.$$

We first consider the subsystem (u_2, v_2). By derivating ψ_2 and using Green formula, we obtain,

$$\begin{aligned}
\dot{\psi_2}(t) &\leq \int_\Omega \big(a(f(u_2) - f(u_1) - \alpha_2(u_2 - u_1))(u_2 - u_1) - (b + \beta_2)(v_2 - v_1)^2\big) \\
&\leq \int_\Omega \big(a(f'(u_1) - \alpha_2 + \sum_{k=2}^{p} \frac{f^{(k)}(u_1)}{k!}(u_2 - u_1)^{k-1})(u_2 - u_1)^2 - (b + \beta_2)(v_2 - v_1)^2\big), \\
&\leq a(M - \alpha_2) \int_\Omega (u_2 - u_1)^2 - (b + \beta_2) \int_\Omega (v_2 - v_1)^2
\end{aligned}$$

this yields,

$$\dot{\psi_2}(t) \leq -c_2\psi.$$

where $c_2 = \min(\frac{\alpha_2 - M}{\epsilon}, b + \beta_2)$ is a positive constant. Thus, we obtain,

$$\psi_2 \leq \psi_2(0)e^{-c_2 t}.$$

Assume the result true until $i-1$, then by algebraic computations we obtain,

$$\begin{aligned}
\dot{\psi}_i(t) &\leq \int_\Omega \big(a(f(u_i) - f(u_{i-1}) - \alpha_i(u_i - u_{i-1}) + \alpha_{i-1}(u_{i-1} - u_{i-2}))(u_i - u_{i-1}) \\
&\quad -(b+\beta_i)(v_i - v_{i-1})^2 + \beta_{i-1}(v_{i-1} - v_{i-2})(v_i - v_{i-1})\big) \\
&\leq \int_\Omega \big(a(M - \alpha_i)(u_i - u_{i-1})^2 + a\alpha_{i-1}(u_{i-1} - u_{i-2})(u_i - u_{i-1}) \\
&\quad -(b+\beta_i)(v_i - v_{i-1})^2 + \beta_{i-1}(v_i - v_{i-1})(v_{i-1} - v_{i-2})\big) \\
&\leq \int_\Omega \big(a(M - \alpha_i)(u_i - u_{i-1})^2 + \frac{a}{2}(\frac{\alpha_{i-1}^2}{\alpha_i - M}(u_{i-1} - u_{i-2})^2 + (\alpha_i - M)(u_i - u_{i-1})^2) \\
&\quad -(b+\beta_i)(v_i - v_{i-1})^2 + \frac{1}{2}(\frac{\beta_{i-1}^2}{\beta_i}(v_{i-1} - v_{i-2})^2 + \beta_i(v_i - v_{i-1})^2)\big) \\
&\leq \int_\Omega \big(a\frac{M - \alpha_i}{2}(u_i - u_{i-1})^2 - (b + \frac{\beta_i}{2})(v_i - v_{i-1})^2 \\
&\quad + \frac{a}{2}\frac{\alpha_{i-1}^2}{\alpha_i - M}(u_{i-1} - u_{i-2})^2 + \frac{\beta_{i-1}^2}{2\beta_i}(v_{i-1} - v_{i-2})^2\big) \\
&\leq -s_1\psi_i + s_2 K_{i-1} e^{-c_{i-1}t}.
\end{aligned}$$

where $s_1 = \min(\frac{\alpha_i - M}{\epsilon}, 2b + \beta_i)$ and $s_2 = \max(\frac{\alpha_{i-1}^2}{\epsilon(\alpha_i - M)}, \frac{\beta_{i-1}^2}{\beta_i})$.
Then, we obtain the result by integration.

□

Corollaire 2.5. *Assume that f is a cubic function, $f(u) = d_3u^3 + d_2u^2 + d_1u$ with $d_3 < 0$. If,*

$$\alpha_i > d_1 - \frac{d_2^2}{2d_3}, \quad i = 2, ..., N,$$

then the network $S = ((u_1, v_1), (u_2, v_2), ..., (u_N, v_N))$ synchronize in the sense of definition (2.3).

Démonstration. In this case, by computation, we obtain that,

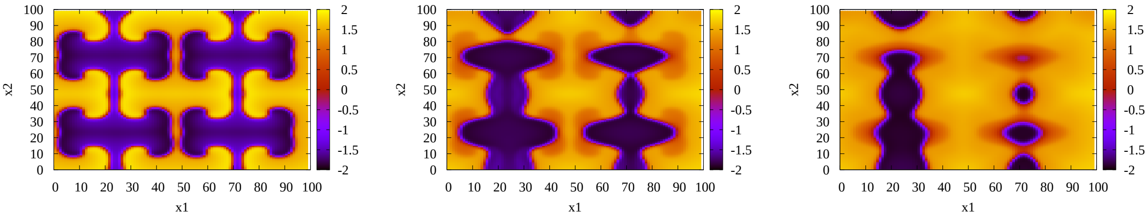
$$M \leq d_1 - \frac{d_2^2}{2d_3}.$$

□

3. Numerical simulations

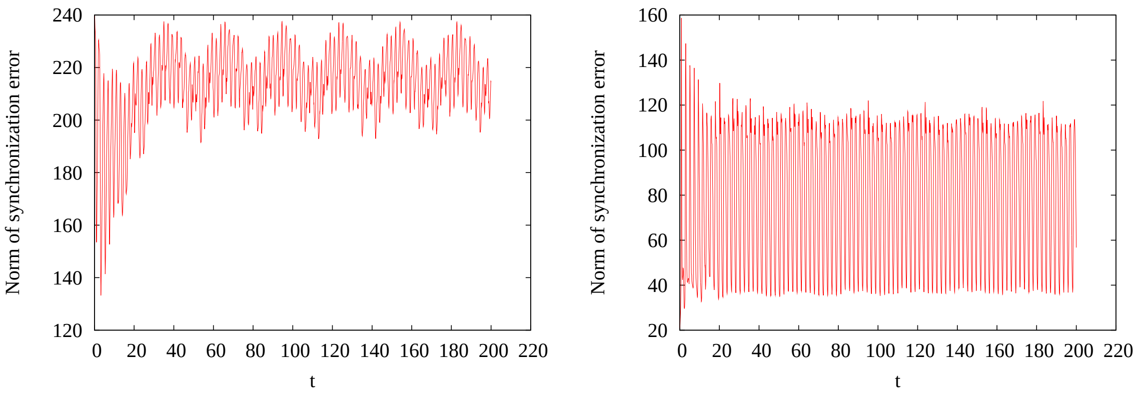
We consider the system (3) for $N = 3$ with for all $i \in \{1, 2, 3\}$, $d_{u_i} = d_{v_i} = 1, a_i = 1, b_i = 0.4$. Moreover for $i \in \{2, 3\}$, we fix $\beta_i = 0$, $\alpha_i > 0$ and $\epsilon = 0.1$. Thus, we consider the following network of three coupled generalized FHN systems,

$$\left\{\begin{array}{lcl}
\epsilon u_{1t} &=& f(u_1) - v_1 + \Delta u_1 \\
v_{1t} &=& au_1 - bv_1 + \Delta v_1 \\
\epsilon u_{2t} &=& f(u_2) - v_2 + \Delta u_2 + \alpha_2(u_1 - u_2) \\
v_{2t} &=& au_2 - bv_2 + \Delta v_2 \\
\epsilon u_{3t} &=& f(u_3) - v_3 + \Delta u_3 + \alpha_3(u_2 - u_3) \\
v_{3t} &=& au_3 - bv_3 + \Delta v_3
\end{array}\right. \tag{11}$$



(a) (b) (c)

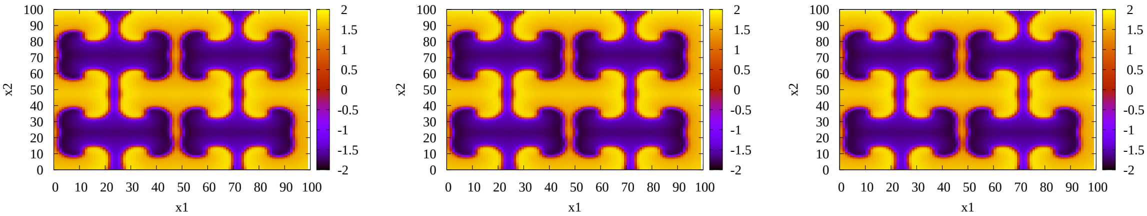
FIGURE 1. Network of three systems of generalized FHN type. Isovalues, of $(a)u_1(x,t)$, $(b)u_2(x,t)$, $(c)u_3(x,t)$ at fixed time $t = 190$ for the coupling strength $\alpha_2 = \alpha_3 = 0.3$.



(a) (b)

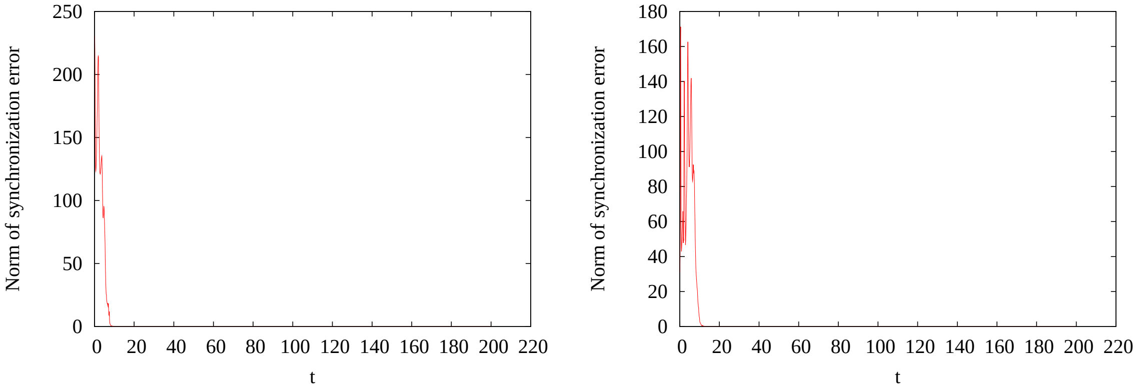
FIGURE 2. Network of three systems of generalized FHN type. The norm of synchronization error given by the definition 2.3 on the interval of time $[0, 200]$ for the coupling strength $\alpha_2 = \alpha_3 = 0.3$: (a) between S_1 and S_2, (b) between S_2 and S_3.

Our numerical simulations, see figure 1, 2, 3, 4, show that system (11) synchronize for a coupling strength $\alpha_2 = \alpha_3$ belonging to the interval $[0.3, 0.4]$. In these figures, the initial conditions are $(u_1(x,0), v_1(x,0))$, particular functions leading to multiple spiral pattern formation, see [8,9], and $(u_2(x,0), v_2(x,0)) = (u_3(x,0), v_3(x,0)) = 1$. Numerical simulations have been performed using an explicit finite difference scheme, with C++ language and with a time step discretization equal to 0.01 and space step discretization equal to 1.



(a) (b) (c)

Figure 3. Network of three systems of generalized FHN type. Isovalues, of $(a)u_1(x,t)$, $(b)u_2(x,t)$, $(c)u_3(x,t)$ at fixed time $t = 190$ for the coupling strength $\alpha_2 = \alpha_3 = 0.4$.



(a) (b)

Figure 4. Network of three systems of generalized FHN type. The norm of synchronization error given by the definition 2.3 on the interval of time $[0, 200]$ for the coupling strength $\alpha_2 = \alpha_3 = 0.4$: (a) between S_1 and S_2, (b) between S_2 and S_3.

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