Generation of interface for solutions of the mass conserved Allen-Cahn equation

Danielle Hilhorst [1], Hiroshi Matano [2], Thanh Nam Nguyen [3], Hendrik Weber [4]

[1] **University of Paris-Sud,** [2] **University of Tokyo,** [3] **NIMS,** [4] **University of Warwick**

Motivation

The singular limit of the nonlocal Allen-Cahn equation

We consider the problem

$$(P^\varepsilon)\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\left(f(u) - \fint_\Omega f(u)\right) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, ∂_ν is the outer normal derivative to $\partial\Omega$ and

$$\fint_\Omega f(u) := \frac{1}{|\Omega|}\int_\Omega f(u(x))\,dx.$$

The singular limit of the nonlocal Allen-Cahn equation

Problem (P^ε) was proposed by Rubinstein and Sternberg as a model for phase separation in a binary mixture.

We assume for the moment that $f(u) = u(1 - u^2)$.

Problem (P^ε) does not possess any comparison principle, which makes its study very difficult.

Propagation of interface

Let Γ_0 be a smooth hypersurface without boundary. There exist a time $T^* > 0$ and a smooth family of initial data $u^\varepsilon(x,0) = u_0^\epsilon(x)$ such that for $t \in [0, T^*)$

$$u^\varepsilon(x,t) \to \begin{cases} -1 & \text{in } \Omega_t^- \\ +1 & \text{in } \Omega_t^+, \end{cases}$$

where

$$\Omega = \Omega_t^- \bigcup \Omega_t^+$$

and where the two subdomains Ω_t^- and Ω_t^+ are separated by a smooth interface Γ_t which propagates according to the law

$$V_n = (N-1)(\kappa - \frac{1}{|\Gamma_t|}\int_{\Gamma_t}\kappa) \quad \text{on } \Gamma_t,\ \ t \in (0, T^*), \Gamma_{t=0} = \Gamma_0.$$

Propagation of interface results

- Chen, Xinfu; Hilhorst, D.; Logak, E. Mass conserving Allen-Cahn equation and volume preserving mean curvature flow. Interfaces Free Bound. 12 (2010), no. 4, 527-549.
- Okada, Koji Dynamical approximation of internal transition layers in a bistable nonlocal reaction-diffusion equation via the averaged mean curvature flow. Hiroshima Math. J. 38 (2008), no. 2, 263-313.

Generation of interface

Today : In the very early stage, the diffusion term is negligible compared with the reaction term, so that the solution of Problem (P^ε) behaves as that of the initial value problem for the corresponding ordinary differential equation

$$(ODE^\varepsilon) \quad \begin{cases} u_t = \dfrac{1}{\varepsilon^2}\left(f(u) - \displaystyle\fint_\Omega f(u)\right) & \text{in } \Omega \times (0, t^\varepsilon), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

With the change of time scale $\tau = \dfrac{t}{\varepsilon^2}$, Problem (ODE^ε) becomes

$$(ODE) \quad \begin{cases} u_\tau = f(u) - \displaystyle\fint_\Omega f(u) & \text{in } \Omega \times (0, \dfrac{t^\varepsilon}{\varepsilon^2}), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Generation of interface

When $t \to t^\varepsilon, \tau \to \tau^\varepsilon := t^\varepsilon/\varepsilon^2 \sim \infty$, the solution u^ε of Problem (P^ε) is such that $u^\varepsilon(t^\varepsilon) \sim v_\infty$ where v_∞ is a stationary solution of Problem (ODE). We will see that in general v_∞ only takes two values a_- and a_+ which are such that

$$f(a_-) = f(a_+) = k$$

and

$$f'(a_-) < 0, \quad f'(a_+) < 0,$$

where k is a constant depending on the initial function. Therefore, the value of u^ε quickly becomes close to either a_- or a_+ with steep interfaces (transition layers) between the regions $\{u^\varepsilon \approx a_-\}$ and $\{u^\varepsilon \approx a_+\}$.

Large time behavior for the nonlocal Allen-Cahn equation

S. Boussaïd, D. Hilhorst, T.-N. Nguyen, Convergence to steady state for the solutions of a nonlocal reaction-diffusion equation, Evolution Equations and Control Theory, Volume **4**, Issue 1, 39–59, (2015).

$$(AC)\begin{cases} v_t = \Delta v + f(v) - \fint_\Omega f(v) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ v(x,0) = v_0(x) & x \in \Omega. \end{cases}$$

We assume here that

$$f(s) = \sum_{i=1}^{n} a_i s^i \quad \text{where } n \geq 3 \text{ is an odd number, } a_n < 0.$$

- Mass conservation property

$$\int_\Omega v(x,t)\,dx = \int_\Omega v_0(x)\,dx.$$

- Lyapunov functional

$$\mathcal{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega F(v)\,dx,$$

where $F(s) = \int_0^s f(\tau)\,d\tau.$

Boussaïd, Hilhorst and Nguyen apply the Lojasiewicz inequality to prove that as $t \to \infty$

$$v(t) \text{ converges to a stationary solution } \varphi \text{ in } H^1(\Omega).$$

In other words, the omega-limit set of Problem (AC) is a singleton.

The corresponding nonlocal ordinary differential equation

D. Hilhorst, H. Matano, T.-N. Nguyen and H. Weber, On the large time behaviour of the solutions of a nonlocal ordinary differential equation, J. Dynam. Differential Equations 28 (2016), 707–731.

We consider the nonlocal ordinary differential equation

$$(ODE)\begin{cases} v_t = f(v) - \fint_\Omega f(v) & \text{in } \Omega \times \mathbb{R}^+, \\ v(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

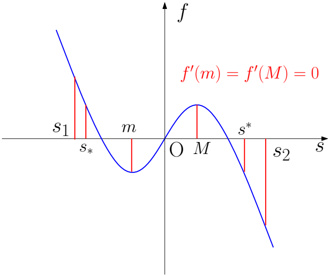
We assume that the function $f \in C^1(\mathbb{R})$ and has exactly three zeros $\alpha_- < \alpha_0 < \alpha_+$ such that

$$f'(\alpha_\pm) < 0, \quad f'(\alpha_0) > 0.$$

We have studied the omega-limit set

$$\omega(u_0) := \{\varphi \in L^1(\Omega) : \exists t_n \to \infty \text{ such that } u(t_n) \to \varphi \text{ in } L^1(\Omega) \text{ as } n \to \infty\}.$$

The nonlocal ordinary differential equation



We choose s_1 (small enough) and s_2 (large enough) such that

$$f(s_2) < f(s) < f(s_1) \quad \text{for all} \quad s \in (s_1, s_2).$$

s_* and s^* satisfy $f(s_*) = f(M), \quad f(s^*) = f(m)$.

Mass conservation and Lyapunov functional

Problem (ODE) has the following properties:

- Mass conservation
- Lyapunov functional

$$E(u) = -\int_{\Omega} F(u)\,dx, \text{ where } F(s) = \int_0^s f(\tau)\,d\tau.$$

- However, the solution of Problem (ODE) is not very smooth so the method used to study Problem (PDE) can not be applied to Problem (ODE). Therefore we use different methods, which are based on studying the profile of $u(t)$ for each time t.

Assumptions on function f

We always suppose that

(F_1) $f \in C^2(\mathbb{R})$, and there exist real numbers $m < M$ such that

$$\begin{cases} f'(s) > 0 \text{ on } (m, M), \\ f'(s) < 0 \text{ on } (-\infty, m) \cup (M, \infty). \end{cases}$$

(F_2) There exist $s_* < s^*$ satisfying

$$\begin{cases} s_* < m < M < s^*, \\ f(s_*) = f(M), \quad f(s^*) = f(m). \end{cases}$$

Assumptions on function f

In some cases, we also consider a special case of f since we can then derive sharper estimates. Such a function f is assumed to satisfy, together with $(\mathbf{F_1}), (\mathbf{F_2})$, the assumption $(\mathbf{F_3})$:

($\mathbf{F_3}$) There exist constants $\underline{m}, \overline{M}$ such that $m < \underline{m} < \overline{M} < M$ that $f'(s) = \mu$ for all $s \in (m, M)$.

Assumptions on the initial data

We always suppose that $s_1 \leq u_0(x) \leq s_2$ for $x \in \Omega$. Moreover, let $\mathcal{H}^{N-1}$ be the $(N-1)$-Hausdroff measure and set

$$A(s) := \int_{\{u_0(\cdot)=s\}} \frac{1}{|\nabla u_0|} \, d\mathcal{H}^{N-1},$$

where for a function $w : \Omega \to \mathbb{R}$, we define $\{w(\cdot) = s\} := \{x \in \Omega : w(x) = s\}$.

We suppose that one of the following sets of hypotheses holds :

$$
(\mathbf{H}_1)\begin{cases}
(\mathbf{H_{11}}) & u_0 \in C^2(\overline{\Omega}) \ \ \textbf{and} \ \ s_* \le u_0 \le s^* \ \ \textbf{on} \ \ \overline{\Omega}, \\
(\mathbf{H_{12}}) & |\{u_0(\cdot) = s\}| = 0 \ \ \textbf{for all} \ \ s \in (m, M), \\
(\mathbf{H_{13}}) & |\nabla u_0| \neq 0 \ \textbf{in} \ \{u_0(\cdot) \in (m, M)\} \ \textbf{and} \ A \in L^\infty_{loc}(m, M).
\end{cases}
$$

$$
(\mathbf{H}_2)\begin{cases}
(\mathbf{H_{21}}) & u_0 \in C^2(\overline{\Omega}) \ \ \textbf{and} \ \ s_* \le \langle u_0 \rangle \le s^*, \\
(\mathbf{H_{22}}) & |\{u_0(\cdot) = s\}| = 0 \ \ \textbf{for all} \ \ s \in \mathbb{R}, \\
(\mathbf{H_{23}}) & |\nabla u_0| \neq 0 \ \textbf{in} \ \Omega \ \textbf{and} \ A \in L^\infty_{loc}(R).
\end{cases}
$$

An ODE vision

We denote by $v = v(x, \tau)$ the solution of the equation without diffusion. Let consider the following initial value problem:

$$v_\tau = f(v) - \langle f(v) \rangle, \quad v(x, 0) = u_0(x), \quad x \in \Omega.$$

Note that v satisfies the mass conservation property:

$$\int_\Omega v(x, \tau)\, dx = \int_\Omega u_0(x)\, dx \;\text{ for all } \tau \geq 0.$$

Let $Y(\tau; s)$ be the unique solution of the initial value problem

$$\dot{Y} = f(Y) - \lambda(\tau), \quad Y(0; s) = s, \text{ with } \dot{Y} := \frac{dY}{d\tau}.$$

Then $Y(\tau; s)$ is strictly increasing in s and $v(x, \tau) = Y(\tau; u_0(x))$ for $x \in \Omega, \tau \geq 0$.

Key property

We will use the notations for each $\tau \geq 0$,

$$\Omega_-(\tau) := \{x \in \Omega, v(x,\tau) \leq m\},$$
$$\Omega_0(\tau) := \{x \in \Omega, m < v(x,\tau) < M\},$$
$$\Omega_+(\tau) := \{x \in \Omega, v(x,\tau) \geq M\}.$$

Let $(\mathbf{H_{11}})$ hold. Then

1. $s_* \leq v(x,\tau) \leq s^*$ for all $x \in \Omega$ and all $\tau \geq 0$.
2. For every $\tau' > \tau \geq 0$,

$$\Omega_-(\tau) \subseteq \Omega_-(\tau'),\ \ \Omega_+(\tau) \subseteq \Omega_+(\tau') \text{ and } \Omega_0(\tau) \supseteq \Omega_0(\tau').$$

   In other words, $\Omega_-(\tau), \Omega_+(\tau)$ are monotonically expanding in τ while $\Omega_0(\tau)$ is monotonically shrinking in τ.

Basic result

We define

$$\Omega_-(\infty) := \bigcup_{\tau \geq 0} \Omega_-(\tau), \quad \Omega_0(\infty) := \bigcap_{\tau \geq 0} \Omega_0(\tau), \quad \Omega_+(\infty) := \bigcup_{\tau \geq 0} \Omega_+(\tau).$$

Let $(\mathbf{H_{11}})$ and $(\mathbf{H_{12}})$ hold. Then there exists a function $\varphi \in L^1(\Omega)$ such that

$$v(\cdot, \tau) \to \varphi \text{ in } L^1(\Omega) \text{ as } \tau \to \infty.$$

Here

$$\varphi = a_- \chi_{\Omega_-(\infty)} + a_+ \chi_{\Omega_+(\infty)},$$

where $\Omega_-(\infty), \Omega_+(\infty)$ are defined as above, χ_A denotes the characteristic function of a set $A \subseteq \Omega$ and a_+, a_- are constants satisfying

$$s_* \leq a_- \leq m, \quad M \leq a_+ \leq s^*, \quad f(a_-) = f(a_+) = \langle f(\varphi) \rangle.$$

Furthermore, we have

$$\int_\Omega \varphi(x)\, dx = \int_\Omega u_0(x)\, dx.$$

The level sets of v

We define $a_0 \in [m, M]$ as the unique solution of the equation $f(s) = \langle f(\varphi) \rangle$ so that

$$a_- \leq m \leq a_0 \leq M \leq a_+.$$

In order to analyze the formation of interface in large time for v, we fix a constant $\eta \in (0, \frac{M-m}{2})$ arbitrarily and consider the sets:

$$\begin{aligned}
\widetilde{\Omega}_-(\tau) &:= \{x \in \Omega : v(x,\tau) \leq a_- + \eta\},\\
\widetilde{\Omega}_0(\tau) &:= \{x \in \Omega : a_- + \eta < v(x,\tau) < a_+ - \eta\},\\
\widetilde{\Omega}_+(\tau) &:= \{x \in \Omega : a_+ - \eta \leq v(x,\tau)\}.
\end{aligned}$$

Some estimates for large time

Our purpose is to study the asymptotic behavior of $\widetilde{\Omega}_\pm(\tau)$ in large time and estimate the decay of $|\widetilde{\Omega}_0(\tau)|$.
We have that $\lambda(\tau) = \langle f(v(\cdot,\tau))\rangle \to \langle f(\varphi)\rangle = f(a_-) = f(a_+)$ as $\tau \to \infty$. We may choose $T_1 = T_1(\eta) > 0$ be such that

$$f(a_+ + \frac{\eta}{2}) \le \lambda(\tau) \le f(a_- - \frac{\eta}{2}) \ \text{ for all } \ \tau \ge T_1.$$

Diffuse interface

Little after little, we narrow down the study of the diffuse interface to the set where $m < v < M$. Let φ be the limit of $v(\cdot, \tau)$ in $L^1(\Omega)$ as $\tau \to \infty$. Then

$$f(m) < \langle f(\varphi) \rangle < f(M).$$

Moreover since $a_- < m < a_0 < M < a_+$, it follows that

$$f'(a_-) < 0, \quad f'(a_0) > 0, \quad f'(a_+) < 0.$$

Narrowing down the diffuse interface

We can choose a constant $\delta = \delta(\eta) > 0$ small enough such that

$$\begin{cases} m < a_0 - \delta < a_0 + \delta < M \\ f(a_- + \eta) < f(a_0 - \frac{\delta}{2}) < f(a_0 + \frac{\delta}{2}) < f(a_+ - \eta). \end{cases}$$

and we set

$$\mu^* := \inf_{s \in [a_0 - \delta, a_0 + \delta]} f'(s) > 0,$$
$$\Omega_0^1(\tau) := \{x \in \Omega : \; v(x,\tau) \in (a_0 - \delta, a_0 + \delta)\}.$$

Level sets of v

Let $u_0 \in C(\overline{\Omega})$ and let $\tau \geq 0$, $s \in \mathbb{R}$. Assume that $\{v(\cdot, \tau) = s\}$ is nonempty. Then

$$\{v(\cdot, \tau) = s\} = \{u_0(\cdot) = Y^{-1}(\tau; \cdot)(s)\}.$$

Proof.

Recall that $v(x, \tau) = Y(\tau, u_0(x))$ and that $Y(\tau, \cdot)$ is strictly increasing. Thus

$$\{x \in \Omega : v(x, \tau) = s\} = \{x \in \Omega : Y(\tau; u_0(x)) = s\}$$
$$= \{x \in \Omega : u_0(x) = Y^{-1}(\tau; \cdot)(s)\}.$$

□

Narrowing down the diffuse interface

Lemma

Let δ *and* μ^* *be as discussed above. Choose* $T_3 > 0$ *such that*

$$f(a_0 - \frac{\delta}{2}) < \lambda(\tau) < f(a_0 + \frac{\delta}{2}) \ \text{ for all } \ \tau \geq T_3,$$

and set

$$C_0 := \sup \left\{ \int_{\{u_0(\cdot)=s\}} \frac{1}{|\nabla u_0(x)|} \, d\mathcal{H}^{N-1} : \right.$$
$$s \in Y^{-1}(T_3; \cdot)([a_0 - \delta, a_0 + \delta]).$$

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Then $C_0 < \infty$ *and*

$$|\Omega_0^1(\tau + T_3)| \leq 2\delta C_0 \exp(-\mu^* \tau) \ \text{ for all } \ \tau \geq 0.$$

Coming back to the nonlocal Allen-Cahn equation

We must evaluate the difference between the solutions of the nonlocal PDE and the nonlocal ODE. We set

$$\hat{u}(x,\tau) := u(x,t) = u(x,\varepsilon^2\tau).$$

The re-scaled function satisfies

$$(\hat{P}^\varepsilon) \begin{cases} \hat{u}_\tau = \varepsilon^2 \Delta \hat{u} + f\left(\hat{u}\right) - \langle f(\hat{u})\rangle & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial \hat{u}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ \hat{u}(0) = u_0 & \text{on } \Omega. \end{cases}$$

To that purpose we must estimate the term $\varepsilon^2 \Delta \hat{u}$.

Recall that

$$s_1 \leq \hat{u}, v \leq s_2.$$

We also prove, by means of maximum principle arguments that

$$|\nabla \hat{u}(x,\tau)|^2 \leq C_4 \exp(2\mu\tau) \text{ for all } x \in \overline{\Omega}, \text{ and all } \tau \geq 0.$$

and that

$$|\Delta \hat{u}(x,\tau)| \leq C_5 \exp(2\mu\tau) \quad \text{for all} \quad x \in \overline{\Omega}, \ \tau \geq 0,$$

which yields

$$|\hat{u}(x,\tau) - v(x,\tau)| \leq C_6 \varepsilon^2 \exp(2\mu\tau) \text{ for all } x \in \overline{\Omega}, \tau \geq 0.$$

Generation of interface

Let $p \geq \frac{1}{2\mu} \log(\frac{C_6}{\eta})$ and set $t_\varepsilon := \frac{1}{\mu}\varepsilon^2 \ln(\frac{1}{\varepsilon}), \tau_\varepsilon := \frac{1}{\mu} \ln(\frac{1}{\varepsilon})$. Choose $\varepsilon_0 > 0$ such that $\tau_{\varepsilon_0} - p \geq T_1 + T_2$. Then for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$|u(x, t_\varepsilon - p\varepsilon^2) - a_-| \leq 2\eta \text{ for all } x \in \widetilde{\Omega}_-(\tau_\varepsilon - p),$$
$$|\widetilde{\Omega}_0(t_\varepsilon - p\varepsilon^2)| \leq C_2 \exp(\mu^* p)\varepsilon^{\mu^*/\mu},$$
$$|u(x, t_\varepsilon - p\varepsilon^2) - a_+| \leq 2\eta \text{ for all } x \in \widetilde{\Omega}_+(\tau_\varepsilon - p).$$

Optimal thickness of interface

Suppose that the hypothesis ($\mathbf{F_3}$) holds, namely that:
there exist constants $\underline{m}, \overline{M}$ such that $m < \underline{m} < \overline{M} < M$ that $f'(s) = \mu$ for all $s \in (m, M)$, let $p \geq \frac{1}{2\mu}\log(\frac{C_6}{\eta})$ and set

$$t_\varepsilon := \frac{1}{\mu}\varepsilon^2 \ln(\frac{1}{\varepsilon}), \quad \tau_\varepsilon := \frac{1}{\mu}\ln(\frac{1}{\varepsilon}).$$

Choose $\varepsilon_0 > 0$ such that $\tau_{\varepsilon_0} - p \geq T_1 + T_2$. Then for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$|u(x, t_\varepsilon - p\varepsilon^2) - a_-| \leq 2\eta \text{ for all } x \in \widetilde{\Omega}_-(\tau_\varepsilon - p),$$
$$|\widetilde{\Omega}_0(t_\varepsilon - p\varepsilon^2)| \leq C_3 \exp(\mu^* p)\varepsilon,$$
$$|u(x, t_\varepsilon - p\varepsilon^2) - a_+| \leq 2\eta \text{ for all } x \in \widetilde{\Omega}_+(\tau_\varepsilon - p).$$

I thank you for your attention!