Generalized travelling waves for a non-autonomous reaction-diffusion system of epidemic type.

Arnaud Ducrot

Université Le Havre Normandie
LMAH FR-CNRS 3335

joint work with
B. Ambrosio (Université Le Havre Normandie)
and
S. Ruan (University of Miami)

Bio Dynamics Days 2020

June 2020, 04

Main goal

Study propagating solution for the RD system of epidemic (or predator-prey) type for $(t, x) \in \mathbb{R}^2$

$$\partial_t u = d_1 \partial_x^2 u + \Lambda - \mu u$$

Main goal

Study propagating solution for the RD system of epidemic (or predator-prey) type for $(t, x) \in \mathbb{R}^2$

$$\partial_t u = d_1 \partial_x^2 u + \Lambda - \mu u$$
$$\partial_t v = d_2 \partial_x^2 v - \gamma v$$

Main goal

Study propagating solution for the RD system of epidemic (or predator-prey) type for $(t, x) \in \mathbb{R}^2$

$$\partial_t u = d_1 \partial_x^2 u + \Lambda - \mu u - \beta uv$$
$$\partial_t v = d_2 \partial_x^2 v - \gamma v \qquad + \beta uv$$

Coupling due to transmission: Mass action incidence

Main goal

Study propagating solution for the RD system of epidemic (or predator-prey) type for $(t,x)\in\mathbb{R}^2$

$$\partial_t u = d_1\partial_x^2 u + \Lambda - \mu u - \beta uv$$
$$\partial_t v = d_2\partial_x^2 v - \gamma v \qquad + \beta uv$$

Here we consider the non-autonomous version

$$\begin{cases}\partial_t u - d(t)\partial_x^2 u = \Lambda(t) - \mu(t)u - \beta(t)uv,\\ \partial_t v - \partial_x^2 v = \beta(t)uv - \gamma(t)v,\end{cases} \quad (t,x)\in\mathbb{R}^2.$$

Main goal

Study propagating solution for the RD system of epidemic (or predator-prey) type for $(t,x) \in \mathbb{R}^2$

$$\partial_t u = d_1 \partial_x^2 u + \Lambda - \mu u - \beta u v$$
$$\partial_t v = d_2 \partial_x^2 v - \gamma v \qquad + \beta u v$$

Here we consider the non-autonomous version

$$\begin{cases} \partial_t u - d(t)\partial_x^2 u = \Lambda(t) - \mu(t)u - \beta(t)uv, \\ \partial_t v - \partial_x^2 v = \beta(t)uv - \gamma(t)v, \end{cases} \quad (t,x) \in \mathbb{R}^2.$$

Aim: Study travelling solution for general time heterogeneities.

Generalized travelling waves

Our problem and main results

Existence

Minimal wave speed

Travelling waves (1)

For a RD equations or systems posed on homogeneous medium (space and time translation invariant)

$$\partial_t U(t,x) = D\partial_{xx}U(t,x)+F(U(t,x)),\ t \geq 0,\ x \in \mathbb{R},\ U(t,x) \in \mathbb{R}^m,$$

a travelling wave is a special entire solution

$$U(t,x) = \tilde{U}(x-ct),\ (t,x) \in \mathbb{R} \times \mathbb{R},$$

where $\tilde{U}$ is the wave profile and $c \in \mathbb{R}$ is the wave speed.
$\tilde{U}(\xi)$ connects two "states" at $\xi = \pm\infty$.
Here "states" can be stationary states, periodic or more complicated solutions.

Travelling waves (2)

The wave profile describes a moving transition (with constant speed c) from one state to another.

Travelling waves (2)

The wave profile describes a moving transition (with constant speed c) from one state to another.

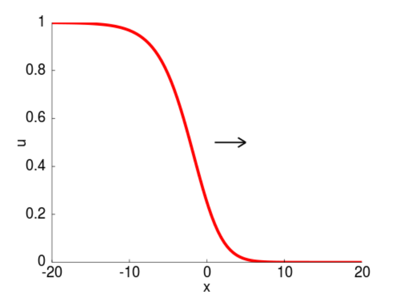
A very huge literature on this rich topic, that arises in various applicative fields in physics and biology:
The combustion theory, neuroscience, population dynamics, and so on (see for instance the monograph of Volpert, Volpert and Volpert).

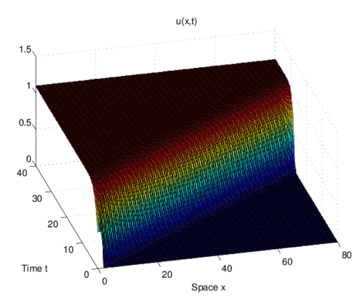
A KPP example

Fisher-KPP equation:

$$\partial_t u = \partial_{xx} u + u(1-u), t > 0,\ x \in \mathbb{R}$$

TW connecting 0 and 1 for all speeds $c \geq 2$.





Heterogeneous medium: transition front

The translation invariance of the medium is no longer true and the propagating profile and the speed have to take into account the heterogeneities.
For a single equation with $x \in \mathbb{R}$

$$\partial_t u = \partial_{xx} u + f(t, x, u(t, x)) \text{ with } f(t, x, 0) = f(t, x, 1) = 0,$$

Heterogeneous medium: transition front

The translation invariance of the medium is no longer true and the propagating profile and the speed have to take into account the heterogeneities.
For a single equation with $x \in \mathbb{R}$

$$\partial_t u = \partial_{xx} u + f(t, x, u(t, x)) \text{ with } f(t, x, 0) = f(t, x, 1) = 0,$$

we define a transition front between $u = 0$ and $u = 1$ as an entire solution $u = u(t, x)$ and an interface $X = X(t)$ such that

$$u(t, x + X(t)) \to \begin{cases} 0 \text{ as } x \to \infty \\ 1 \text{ as } x \to -\infty \end{cases} \quad \text{uniformly for } t \in \mathbb{R},$$

(see Berestycki, Hamel,Nadin etc; Matano for spatially heterogeneous medium)

Time heterogeneous medium: GTW

Among the transition fronts, a special class is those of the so-called *Generalized travelling waves* (GTW).
With the previous example

$$\partial_t u = \partial_{xx} u + f(t, u(t, x)) \text{ with } f(t, 0) = f(t, 1) = 0,$$

A entire solution is said to be a GTW between 0 and 1 if $u(t, x) = U(t, \xi)$ with

$$\xi = x - \int_0^t c(s)ds \text{ with } c = c(t) \in L^\infty(\mathbb{R}) \text{ is the wave speed function,}$$

$$\text{and the profile } U(t, \xi) \to \begin{cases} 0 \text{ as } \xi \to \infty, \\ 1 \text{ as } \xi \to -\infty \end{cases} \text{, uniformly for } t \in \mathbb{R}.$$

Some references

Huge literature for time and/or spatial periodic medium:
Also called pulsating wave introduced by in the book of Shigesada and Kawasaki (*Biological Invasions: Theory and Practice*.

Some references

Huge literature for time and/or spatial periodic medium:
Also called pulsating wave introduced by in the book of Shigesada and Kawasaki (*Biological Invasions: Theory and Practice*.

Some literature for general time dependence:
Nadin and Rossi (JMPA, 2012) $\mapsto$ for KPP nonlinearity
Nadin and Rossi (Anal. PDE, 2015) $\mapsto$ for KPP with general in time and periodic in space
Shen $\mapsto$ stability for KPP, Extensions to non-local – convolution – diffusion,
and others

Average medium

Let $g \in L^\infty(\mathbb{R})$ be given. Define the least and upper mean value respectively as follows

Average medium

Let $g \in L^\infty(\mathbb{R})$ be given. Define the least and upper mean value respectively as follows

$$\text{least mean} \quad \mathcal{M}^-(g) := \lim_{T\to\infty} \inf_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

Average medium

Let $g \in L^\infty(\mathbb{R})$ be given. Define the least and upper mean value respectively as follows

$$\text{least mean} \quad \mathcal{M}^-(g) := \lim_{T\to\infty} \inf_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

$$\text{upper mean} \quad \mathcal{M}^+(g) := \lim_{T\to\infty} \sup_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

Average medium

Let $g \in L^\infty(\mathbb{R})$ be given. Define the least and upper mean value respectively as follows

$$\text{least mean} \quad \mathcal{M}^-(g) := \lim_{T\to\infty} \inf_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

$$\text{upper mean} \quad \mathcal{M}^+(g) := \lim_{T\to\infty} \sup_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

Note that these limits always exist.

Average medium

Let $g \in L^\infty(\mathbb{R})$ be given. Define the least and upper mean value respectively as follows

$$\text{least mean} \quad \mathcal{M}^-(g) := \lim_{T\to\infty} \inf_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

$$\text{upper mean} \quad \mathcal{M}^+(g) := \lim_{T\to\infty} \sup_{s\in\mathbb{R}} \frac{1}{T}\int_s^{s+T} g(l)dl.$$

Note that these limits always exist.
If $\mathcal{M}^-(g) = \mathcal{M}^+(g)$ the function g is said to have a mean value (or uniquely ergodic), namely

$$\lim_{T\to\infty} \frac{1}{T}\int_s^{s+T} g(l)dl \text{ exists uniformly for } s \in \mathbb{R}.$$

A KPP example

Consider GTW for the KPP equation for $g = g(t) \geq 0$

$$\partial_t u = \partial_{xx} u + c(t)\partial_x u + g(t)u(1-u),\ (t,x) \in \mathbb{R}^2,$$
$$u(t,\infty) = 0 \text{ and } u(t,-\infty) = 1.$$

For this problem, when g is uniquely ergodic then $\mathcal{M}c \geq 2\sqrt{\mathcal{M}g}$.

A KPP example

Consider GTW for the KPP equation for $g = g(t) \geq 0$

$$\partial_t u = \partial_{xx} u + c(t)\partial_x u + g(t)u(1-u),\ (t,x) \in \mathbb{R}^2,$$
$$u(t,\infty) = 0 \text{ and } u(t,-\infty) = 1.$$

For this problem, when g is uniquely ergodic then $\mathcal{M}c \geq 2\sqrt{\mathcal{M}g}$. This can be formally observed by the ansatz

$$u(t,x) = e^{a(t)-\lambda x} \text{ for } x \gg 1.$$

A KPP example

Consider GTW for the KPP equation for $g = g(t) \geq 0$

$$\partial_t u = \partial_{xx} u + c(t)\partial_x u + g(t)u(1-u),\ (t,x) \in \mathbb{R}^2,$$
$$u(t,\infty) = 0 \text{ and } u(t,-\infty) = 1.$$

For this problem, when g is uniquely ergodic then $\mathcal{M}c \geq 2\sqrt{\mathcal{M}g}$. This can be formally observed by the ansatz

$$u(t,x) = e^{a(t)-\lambda x} \text{ for } x \gg 1.$$

For general time dependence we get (See Nadin-Rossi)

$$\mathcal{M}^-(c) \geq 2\sqrt{\mathcal{M}^-(g)}$$

The problem

We study GTW for the following

$$\begin{cases}\partial_t u - d(t)\partial_x^2 u = \Lambda(t) - \mu(t)u - \beta(t)uv, \\ \partial_t v - \partial_x^2 v = \beta(t)uv - \gamma(t)v, \end{cases} \quad t \in \mathbb{R},\ x \in \mathbb{R}.$$

that stands either for a predator-prey system u is the prey while v is the predator
or an epidemic system with u the susceptible and v the infectives.

The problem

We study GTW for the following

$$\begin{cases} \partial_t u - d(t)\partial_x^2 u = \Lambda(t) - \mu(t)u - \beta(t)uv, \\ \partial_t v - \partial_x^2 v = \beta(t)uv - \gamma(t)v, \end{cases} \quad t \in \mathbb{R},\ x \in \mathbb{R}.$$

that stands either for a predator-prey system u is the prey while v is the predator
or an epidemic system with u the susceptible and v the infectives.

Aim: GTW connecting the disease free equilibrium to a uniformly positive (endemic) state.

Diffusion reduction

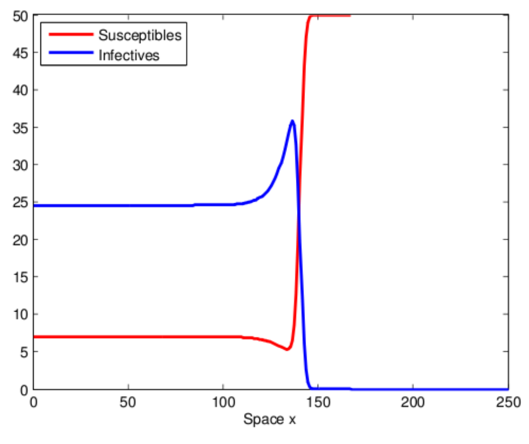
The above system has a normalised time-dependent diffusion for the susceptible.
This follows from a simple rescaling argument from the general problem with two diffusion functions

$$\begin{cases} \partial_t u - d_u(t)\partial_x^2 u = \Lambda(t) - \mu(t)u - \beta(t)uv, \\ \partial_t v - d_v(t)\partial_x^2 v = \beta(t)uv - \gamma(t)v, \end{cases} \quad t \in \mathbb{R},\ x \in \mathbb{R}.$$

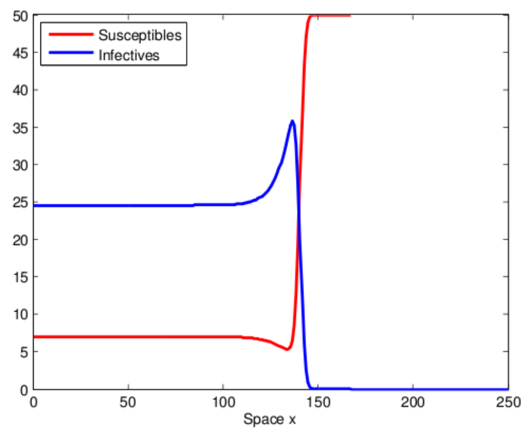
New time variable

$$\tau(t) = \int_0^t d_v(s)\mathrm{d}s \ \hookrightarrow \ \text{yields } d_v(t) \equiv 1.$$

Travelling waves for homogeneous medium



Travelling waves for homogeneous medium



Extension with age since infection in homogeneous medium.

Assumptions and disease free equilibrium

Assumptions:

1. The functions Λ, μ, β and γ are bounded and uniformly positive
2. $d = d(t)$ is uniformly positive and uniformly continuous.

Assumptions and disease free equilibrium

Assumptions:

1. The functions Λ, μ, β and γ are bounded and uniformly positive
2. $d = d(t)$ is uniformly positive and uniformly continuous.

The system has a unique bounded disease free state, namely entire solution for the system with $v = 0$.
It is spatially homogeneous and given by the expression

$$u^*(t) = \int_{-\infty}^{t} e^{-\int_s^t \mu(l)dl} \Lambda(s)\mathrm{d}s, \; t \in \mathbb{R}.$$

Aim

We aim at studying the existence and non-existence of GTW, that is:
(bounded) profile $U(t,\xi) \geq 0$, $V(t,\xi) \geq 0$ and a speed function $c = c(t) \in L^\infty(\mathbb{R})$ satisfying for $(t,\xi) \in \mathbb{R}^2$

$$\begin{cases} \partial_t U = d(t)\partial_\xi^2 U + c(t)\partial_\xi U + \Lambda(t) - \mu(t)U - \beta(t)UV, \\ \partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + \beta(t)UV - \gamma(t)V, \end{cases}$$

Aim

We aim at studying the existence and non-existence of GTW, that is:
(bounded) profile $U(t,\xi) \geq 0$, $V(t,\xi) \geq 0$ and a speed function $c = c(t) \in L^\infty(\mathbb{R})$ satisfying for $(t,\xi) \in \mathbb{R}^2$

$$\begin{cases} \partial_t U = d(t)\partial_\xi^2 U + c(t)\partial_\xi U + \Lambda(t) - \mu(t)U - \beta(t)UV, \\ \partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + \beta(t)UV - \gamma(t)V, \end{cases}$$

together with

$$\lim_{\xi\to\infty} |U(t,\xi) - u^*(t)| + V(t,\xi) = 0 \text{ uniformly for } t \in \mathbb{R},$$

$$\liminf_{\xi\to-\infty} \inf_{t\in\mathbb{R}} V(t,\xi) > 0, \ \liminf_{\xi\to-\infty} \inf_{t\in\mathbb{R}} |U(t,\xi) - u^*(t)| > 0.$$

Transition between the disease free and an endemic "state"

Instability assumptions

We assume that the disease free equilibrium is "unstable", in the sense that

$$\mathcal{T} := \mathcal{M}^{-}\left(\beta(\cdot)u^*(\cdot) - \gamma(\cdot)\right) > 0.$$

This condition is equivalent to

$$\exists a \in W^{1,\infty}(\mathbb{R}), \quad \inf_{t\in\mathbb{R}} \left\{a'(t) + \beta(t)u^*(t) - \gamma(t)\right\} > 0.$$

Instability assumptions

We assume that the disease free equilibrium is "unstable", in the sense that

$$\mathcal{T} := \mathcal{M}^{-}\left(\beta(\cdot)u^{*}(\cdot) - \gamma(\cdot)\right) > 0.$$

This condition is equivalent to

$$\exists a \in W^{1,\infty}(\mathbb{R}), \quad \inf_{t\in\mathbb{R}} \left\{a'(t) + \beta(t)u^{*}(t) - \gamma(t)\right\} > 0.$$

For constant coefficients, it becomes

$$\beta\frac{\Lambda}{\mu} - \gamma > 0 \;\Leftrightarrow\; \mathcal{R}_0 := \frac{\beta\Lambda}{\mu\gamma} > 1.$$

The wave speed

Close to the unstable point $(u^*(t), 0)$ at $\xi = \infty$, V behaves like

$$\partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + \beta(t)u^*(t)V - \gamma(t)V,$$

The wave speed

Close to the unstable point $(u^*(t), 0)$ at $\xi = \infty$, V behaves like

$$\partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + \beta(t)u^*(t)V - \gamma(t)V,$$

Plugging the ansatz $V(t,\xi) = e^{-a(t)-\lambda\xi}$ for some $\lambda > 0$ and $a \in W^{1,\infty}(\mathbb{R})$ yields

$$-a'(t) = \lambda^2 - \lambda c(t) + \beta(t)u^*(t) - \gamma(t),$$

The wave speed

Close to the unstable point $(u^*(t),0)$ at $\xi=\infty$, V behaves like

$$\partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + \beta(t)u^*(t)V - \gamma(t)V,$$

Plugging the ansatz $V(t,\xi) = e^{-a(t)-\lambda\xi}$ for some $\lambda > 0$ and $a \in W^{1,\infty}(\mathbb{R})$ yields

$$-a'(t) = \lambda^2 - \lambda c(t) + \beta(t)u^*(t) - \gamma(t),$$

Hence set $\delta(t) = \beta(t)u^*(t) - \gamma(t)$ and choose

$$c(t) = c_{\lambda,a}(t) = \lambda + \lambda^{-1}\delta(t) + a'(t).$$

Existence result

Note that $\mathcal{M}^-(c_{\lambda,a}) = \lambda + \lambda^{-1}\mathcal{T}$ with $\mathcal{T} = \mathcal{M}^-(\delta) > 0$. Set $\lambda^\star := \sqrt{\mathcal{T}}$ then we have:

Theorem

For each $\lambda \in (0, \lambda^\star)$ and $a \in W^{1,\infty}(\mathbb{R})$ the system admits a GTW for the wave speed function

$$c_{\lambda,a}(t) = \lambda + \lambda^{-1}\delta(t) + a'(t).$$

Remarks

1. Note that for all $\lambda \in (0, \lambda^\star)$ and $a \in W^{1,\infty}(\mathbb{R})$ one has
$$\{\mathcal{M}^-(c_{\lambda,a}),\ \lambda \in (0, \lambda^\star) \text{ and } a \in W^{1,\infty}(\mathbb{R})\} = (2\sqrt{\mathcal{T}}, \infty).$$

Remarks

1. Note that for all $\lambda \in (0, \lambda^\star)$ and $a \in W^{1,\infty}(\mathbb{R})$ one has
$$\{\mathcal{M}^-(c_{\lambda,a}),\ \lambda \in (0, \lambda^\star) \text{ and } a \in W^{1,\infty}(\mathbb{R})\} = (2\sqrt{\mathcal{T}}, \infty).$$

2. If $\delta(t)$ is $T-$periodic, for each $\lambda \in (0, \lambda^\star)$ there exists $a \in W^{1,\infty}(\mathbb{R})$ st
$$c_{\lambda,a}(t) = \text{ constant.}$$
This recovers the known notion of pulsating wave in periodic medium (with constant speed).
However we didn't check that the wave profile is also periodic in time.

Remarks

1. Note that for all $\lambda \in (0, \lambda^\star)$ and $a \in W^{1,\infty}(\mathbb{R})$ one has
$$\{\mathcal{M}^-(c_{\lambda,a}),\ \lambda \in (0, \lambda^\star) \text{ and } a \in W^{1,\infty}(\mathbb{R})\} = (2\sqrt{\mathcal{T}}, \infty).$$

2. If $\delta(t)$ is $T-$periodic, for each $\lambda \in (0, \lambda^\star)$ there exists $a \in W^{1,\infty}(\mathbb{R})$ st
$$c_{\lambda,a}(t) = \text{ constant.}$$
This recovers the known notion of pulsating wave in periodic medium (with constant speed).
However we didn't check that the wave profile is also periodic in time.

3. More generally we didn't study how the heterogeneous time structure (periodic, almost-periodic, uniquely ergodic and so on) is transmitted to the wave profiles.

Minimal wave speed

The quantity $2\sqrt{\mathcal{T}}$ turns out to be the minimal least value for the wave speed.

Theorem

Let (U,V) be a GTW with speed function $c = c(t) \in L^\infty(\mathbb{R})$. Then the following lower estimate holds

$$\mathcal{M}^-(c) \geq 2\sqrt{\mathcal{T}}.$$

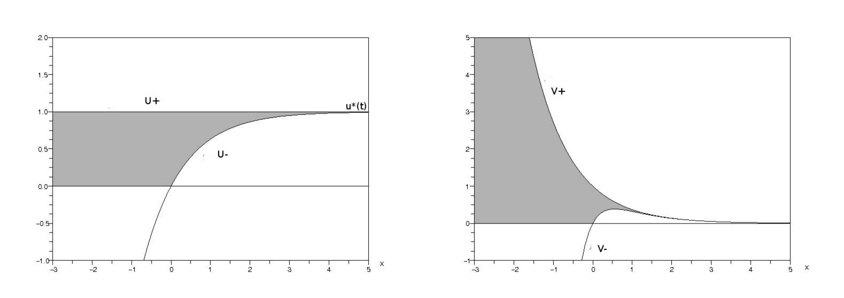
Sub and super-solution pair

No comparison principle for the system
We use a "skew" monotonicity (rather classical for homogeneous system and less classical for heterogeneous)
Fix $c(t) = c_{\lambda,a}(t)$ for some $\lambda \in (0, \lambda^\star)$ and $a \in W^{1,\infty}(\mathbb{R})$.

1. $U(t,\xi) \leq \overline{U}(t,\xi) := u^*(t)$
2. $V(t,\xi) \leq \overline{V}(t,\xi) := e^{a(t)-\lambda\xi}$
3. $U(t,\xi) \geq \underline{U}(t,\xi) := u^*(t) - A(t)e^{-\kappa\xi}$ for some $\kappa > 0$
4. $V(t,\xi) \geq \underline{V}(t,\xi) := e^{a(t)-\lambda\xi}[1 - B(t)e^{-\eta\xi}]$ for some $\eta > 0$

Schematic view of the sub and super-solution pair

At a given time $t \in \mathbb{R}$:



A sequence of initial value problems

For all $n \geq 0$ we consider the initial value problem

$$\begin{cases} \partial_t U^n = d(t)\partial_\xi^2 U^n + c(t)\partial_\xi U^n + \Lambda(t) - \mu(t)U^n - \beta(t)U^n V^n, \\ \partial_t V^n = \partial_\xi^2 V^n + c(t)\partial_\xi V^n + \beta(t)U^n V^n - \gamma(t)V^n, \end{cases}$$

for $\xi \in \mathbb{R}$ and $t \geq -n$ with

$U^n(-n,\xi) = \max(0, \underline{U}(-n,\xi))$ and $V^n(-n,\xi) = \max(0, \underline{V}(-n,\xi))$.

A sequence of initial value problems

For all $n \geq 0$ we consider the initial value problem

$$\begin{cases} \partial_t U^n = d(t)\partial_\xi^2 U^n + c(t)\partial_\xi U^n + \Lambda(t) - \mu(t)U^n - \beta(t)U^n V^n, \\ \partial_t V^n = \partial_\xi^2 V^n + c(t)\partial_\xi V^n + \beta(t)U^n V^n - \gamma(t)V^n, \end{cases}$$

for $\xi \in \mathbb{R}$ and $t \geq -n$ with

$U^n(-n,\xi) = \max(0, \underline{U}(-n,\xi))$ and $V^n(-n,\xi) = \max(0, \underline{V}(-n,\xi))$.

Then U^n and V^n stay between $\max(0, \underline{U}(t,\xi))$, $\overline{U}(t,\xi)$ and $\max(0, \underline{V}(t,\xi))$ and $\overline{V}(t,\xi)$, respectively.

Passing to the limit $n \to \infty$

To obtain a solution we pass to the limit $n \to \infty$.
Main difficulty: the upper estimate for V^n reads as

$$V^n(t,\xi) \leq e^{a(t)-\lambda\xi},\ \forall t \geq -n,\ \xi \in \mathbb{R}.$$

It is unbounded for $\xi \to -\infty$.
We need to prove the boundedness of the solution $V^n(t,\xi)$ with respect to n, $t \geq -n$ and $\xi \in \mathbb{R}$.

Boundedness

Technical arguments based on a contradiction argument.

1. First U^n is bounded by u^*.
2. Next roughly speaking, from the $U-$equation, if V^n becomes large then U^n is close to 0

$$\partial_t U^n = d(t)\partial_\xi^2 U^n + c(t)\partial_\xi U^n + \Lambda(t) - \mu(t)U^n - \beta(t)U^n V^n,$$

   since the decay rate becomes large.
3. Then from the V equation has to decay since

$$\partial_t V^n = \partial_\xi^2 V^n + c(t)\partial_\xi V^n + (\beta(t)U^n - \gamma(t))\, V^n.$$

Conclusion

At that stage we hand-up with the existence of a bounded profile $U(t,\xi) \geq 0$, $V(t,\xi) \geq 0$ for the speed function $c(t) = c_{\lambda,a}(t) \in L^\infty(\mathbb{R})$, satisfying for $(t,\xi) \in \mathbb{R}^2$

$$\begin{cases} \partial_t U = d(t)\partial_\xi^2 U + c(t)\partial_\xi U + \Lambda(t) - \mu(t)U - \beta(t)UV, \\ \partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + \beta(t)UV - \gamma(t)V, \\ \inf_{t\in\mathbb{R}} V(t,\xi) > 0, \ \forall \xi \in \mathbb{R}, \end{cases}$$

together with $U(t,\infty) = u^*(t)$ and $V(t,\infty) = 0$ uniformly for $t \in \mathbb{R}$. This behaviour is obtained from the sub and super solution close to $\xi = \infty$.

Toward the end of the proof

It remains to prove persistence behaviour at $\xi = -\infty$, that is

$$\liminf_{\xi \to -\infty} \inf_{t \in \mathbb{R}} V(t,\xi) > 0,$$

$$\liminf_{\xi \to -\infty} \inf_{t \in \mathbb{R}} |U(t,\xi) - u^*(t)| > 0.$$

Toward the end of the proof

It remains to prove persistence behaviour at $\xi = -\infty$, that is

$$\liminf_{\xi \to -\infty} \inf_{t \in \mathbb{R}} V(t,\xi) > 0,$$

$$\liminf_{\xi \to -\infty} \inf_{t \in \mathbb{R}} |U(t,\xi) - u^*(t)| > 0.$$

This is proved at the same time as the minimal wave speed.

Key result

Both the persistence of the GTW at $\xi = -\infty$ and the minimal wave speed property follow from the next result.

Theorem

Let (U, V) be a bounded solution of the wave profile equation with speed function $c = c(t) \in L^\infty(\mathbb{R})$ st

$$\exists \xi_0 \in \mathbb{R}, \ \inf_{t \in \mathbb{R}} V(t, \xi_0) > 0.$$

Then for all $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ the following holds true

$$\liminf_{t \to \infty} \inf_{\tau \in \mathbb{R}} V\left(t + \tau, \tilde{c}t - \int_\tau^{t+\tau} c(l)dl\right) > 0$$

First consequence: persistence of GTW at $\xi = -\infty$

We choose $\tau = s - t$ and $\tilde{c} = 0$ so that

$$\liminf_{t\to\infty} \inf_{s\in\mathbb{R}} V\left(s, -\int_0^t c(l+s-t)dl\right) > 0$$

while

$$\int_0^t c(l+s-t)dl \geq \inf_{s\in\mathbb{R}} \int_0^t c(l+s)dl > 2\sqrt{\mathcal{T}}t \text{ for } t \gg 1.$$

so that

$$\liminf_{\xi\to-\infty} \inf_{s\in\mathbb{R}} V(s,\xi) > 0,$$

that proves the persistence of the GTW (constructed before) at $\xi = -\infty$.

Second consequence: minimal wave speed

By contradiction, if (U,V) is a GTW with speed $\mathcal{M}^-(c)<2\sqrt{\mathcal{T}}$ then fix

$$\mathcal{M}^-(c)<\tilde{c}<2\sqrt{\mathcal{T}}.$$

Next there exists $t_n\to\infty$ and $(s_n)\subset\mathbb{R}$ st $\gamma_n:=\frac{1}{t_n}\int_0^{t_n}c(l-t_n+s_n)dl-\tilde{c}<0$ so that $\gamma_n t_n\to-\infty$. Next one has

$$\liminf_{n\to\infty}V(s_n,-\gamma_n t_n)>0 \text{ from the theorem,}$$

while $V(s_n,-\gamma_n t_n)\to 0$ from the definition of a GTW (Recall that $-\gamma_n t_n\to\infty$).

Formal ideas for the proof of the key result (1)

Fix $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ then if there exists (τ_n) such that

$$V\left(t+\tau_n, \tilde{c}t - \int_0^t c(l+\tau_n)dl\right) \approx 0 \text{ for } t \gg 1,$$

then $U\left(t+\tau_n, \xi + \tilde{c}t - \int_0^t c(l+\tau_n)dl\right) \approx u^*(t+\tau_n)$ for $t \gg 1$ and ξ bounded.

Formal ideas for the proof of the key result (1)

Fix $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ then if there exists (τ_n) such that

$$V\left(t+\tau_n, \tilde{c}t - \int_0^t c(l+\tau_n)dl\right) \approx 0 \text{ for } t \gg 1,$$

then $U\left(t+\tau_n, \xi + \tilde{c}t - \int_0^t c(l+\tau_n)dl\right) \approx u^*(t+\tau_n)$ for $t \gg 1$ and ξ bounded.

Hence the function $W_n(t,\xi) = V\left(t+\tau_n, \xi + \tilde{c}t - \int_0^t c(l+\tau_n)dl\right)$ satisfies

$$\partial_t W_n \approx \partial_\xi^2 W_n + \tilde{c}\partial_\xi W_n + \delta(t+\tau_n)W_n$$

for $t \gg 1$ and ξ bounded.

Formal ideas for the proof of the key result (1)

Fix $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ then if there exists (τ_n) such that

$$V\left(t+\tau_n, \tilde{c}t - \int_0^t c(l+\tau_n)dl\right) \approx 0 \text{ for } t \gg 1,$$

then $U\left(t+\tau_n, \xi + \tilde{c}t - \int_0^t c(l+\tau_n)dl\right) \approx u^*(t+\tau_n)$ for $t \gg 1$ and ξ bounded.

Hence the function $W_n(t,\xi) = V\left(t+\tau_n, \xi + \tilde{c}t - \int_0^t c(l+\tau_n)dl\right)$ satisfies

$$\partial_t W_n \approx \partial_\xi^2 W_n + \tilde{c}\partial_\xi W_n + \delta(t+\tau_n)W_n$$

for $t \gg 1$ and ξ bounded.

Next construction of an unbounded sub-solution on a large interval $(-R, R)$.

Here we crucially use $\tilde{c} < 2\sqrt{\mathcal{M}^-(\delta)}$, that is the "instability" of $V = 0$ in the moving frame $\tilde{c}$.

Formal ideas for the proof of the key result (2)

The above argument roughly shows that: for all $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ the following holds true

$$\limsup_{t \to \infty} \inf_{\tau \in \mathbb{R}} V\left(t + \tau, \tilde{c}t - \int_\tau^{t+\tau} c(l)dl\right) > 0.$$

Formal ideas for the proof of the key result (2)

The above argument roughly shows that: for all $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ the following holds true

$$\limsup_{t\to\infty} \inf_{\tau\in\mathbb{R}} V\left(t+\tau, \tilde{c}t - \int_{\tau}^{t+\tau} c(l)dl\right) > 0.$$

Then we change from $\limsup$ to $\liminf$ by using dynamical system arguments.

Formal ideas for the proof of the key result (2)

The above argument roughly shows that: for all $\tilde{c} \in [0, 2\sqrt{\mathcal{T}})$ the following holds true

$$\limsup_{t \to \infty} \inf_{\tau \in \mathbb{R}} V\left(t + \tau, \tilde{c}t - \int_{\tau}^{t+\tau} c(l)dl\right) > 0.$$

Then we change from $\limsup$ to $\liminf$ by using dynamical system arguments.
Here we adapt ideas from uniform persistence theory and more precisely some ideas to pass from the so-called weak uniform persistence to the strong version (see Hale and Waltman, Thieme, etc).

Some conclusions

1. Dynamical system arguments allow us to overcome the lack of comparison principle, using the instability of semi-trivial states.

Some conclusions

1. Dynamical system arguments allow us to overcome the lack of comparison principle, using the instability of semi-trivial states.
2. Powerful tools that has also been used and adapted to study the spreading speed for the solutions of systems without comparison principle, such as predator-prey systems (D. JDE 2016; D., Giletti, Matano 2019 CVPDE 2019).

Some conclusions

1. Dynamical system arguments allow us to overcome the lack of comparison principle, using the instability of semi-trivial states.
2. Powerful tools that has also been used and adapted to study the spreading speed for the solutions of systems without comparison principle, such as predator-prey systems (D. JDE 2016; D., Giletti, Matano 2019 CVPDE 2019).
3. Possible extensions for more complicated, non-monotone and monostable diffusive systems in epidemiology and ecology, for instance. (age since infection, chronological age or size structure, logistic growth, non-local diffusion and so on).

Thank you for your attention.