Functional Analysis

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These are lecture notes for the course of Functional Analysis for the students of Master of Mathematics of Le Havre Normandie University and the Hudson School of Mathematics. They are based on the following textbooks:

- Functional Analysis by Peter D. Lax, John Wiley and Sons (2002)
- Functional Analysis by T. Buhler and D.A. Salomon, AMS (2018)
- Functional Analysis by W. Rudin, McGraw-Hill, 2nd ed. (1991)
- Functional Analysis, Sobolev Spaces and Partial Differential Equations by H. Brezis, Springer, (2010)
- Functional Analysis by K. Yosida, Springer-Verlag, 2nd ed. (1968)
- Functional Analysis by G. Bachman and L. Narici, Dover, (2000)
- Functional Analysis, lecture notes by J. Schenker at Michigan University, accessed on the web on July 3rd 2022

I have also used the personal notes of S. Maingot's course at Le Havre Normandie University.

Contents

Chapter 1

Hahn-Banach Theorem and Applications

1.1 The Hahn-Banach Theorem

Theorem 1. *Let X be a linear space over $\mathbb{R}$ and p a real valued function on X with the properties:*

$$(1)\ p(ax) = ap(x) \text{ for all } x \in X \text{ and } a > 0 \text{ (positive homogeneity)}$$

$$(2)\ p(x+y) \leq p(x) + p(y) \text{ for all } x, y \in X \text{ (subadditivity)}$$

If l is a real valued linear functional defined on a linear subspace Y of X and dominated by p, that is

$$l(y) \leq p(y) \text{ for all } y \in Y,$$

then l can be extended to all of X as a linear functional such that $l(x) \leq p(x)$ for all $x \in X$.

To prove the theorem, we will use the Zorn's Lemma.

Theorem 2 (Zorn's Lemma)**.** *Let S be a partially ordered set such that every totally ordered subset has an upper bound. Then S has a maximal element.*

Well, now, to understand Zorn's Lemma, we need to define an upper bound as well as a maximal element.

Definition 1. *A partially ordered set S is a set on which an order relation $a \leq b$ is defined for some (but not necessarily all) pairs $a, b \in S$ with the following properties:*

$$(1)\ \text{transitivity: if } a \leq b \text{ and } b \leq c \text{ then } a \leq c$$

$$(2)\ \text{reflexivity: } a \leq a\ \forall a \in S$$

A subset T of S is totally ordered if

$$x, y \in T \Rightarrow x \leq y \text{ or } y \leq x.$$

An element $u \in S$ is an upper bound for $T \subset S$ if

$$x \in T \Rightarrow x \leq u.$$

A maximal element $m \in S$ satisfies

$$m \leq b \Rightarrow m = b.$$

A concrete and simple illustration of Zorn's Lemma is provided by an interval of real numbers. Let $S = [a, b]$ a bounded interval of $\mathbb{R}$ with $a < b$. Then every totally ordered subset of S has an upper bound in S. And S admits a maximal element which is b. Now, consider the interval $S = [a, b)$. This interval is totally ordered but doesn't admit an upper bound in S.

Proof of Hahn-Banach Theorem

We will first apply the Zorn's Lemma. Let's consider the following partially ordered set (poset), S whose elements are the pairs $(h_i, D(h_i))$ with h_i a linear functional defined on a subspace $D(h_i) \supset Y$, and such that

$$h_i = l \text{ on } Y \text{ and } h_i \leq p \text{ on } D(h_i).$$

The order on S is defined as

$$h_i \leq h_j \iff D(h_i) \subset D(h_j) \text{ and } h_i = h_j \text{ on } D(h_i)$$

Let T be a totally ordered subset of S. Define $(h, D(h)) \in S$ as

$$D(h) = \bigcup \left(D(h_i)\right)_{\{} i \in I\}$$

and,

$$h(y) = h_i(y) \text{ for } y \in D(h_i).$$

Then, $(h, D(h))$ is an upper bound of T. From Zorn's Lemma, it follows that S possesses a maximal element, $(g, D(g))$. We will show that $D(g) = X$. Let us assume that this is not the case. Let $x_0 \in X \setminus D(g)$. Consider the subspace

$$H = \{y + ax_0; y \in D(g), a \in \mathbb{R}\}$$

We look for a linear function w defined on H such that $w \leq p$. For this, it is sufficient to define w s.t.

$$w = g \text{ on } D(g),$$

and

$$w(y + ax_0) \leq p(y + ax_0); y \in D(g), a \in \mathbb{R}$$

By linearity, this is equivalent to

$$g(y) + aw(x_0) \leq p(y + ax_0); y \in D(g), a \in \mathbb{R}$$

and

$$w(x_0) \leq (1/a)(p(y + ax_0) - g(y)), y \in D(g), a > 0 \tag{1.1}$$

along with

$$w(x_0) \geq (1/a)(p(y + ax_0) - g(y)), y \in D(g), a < 0 \tag{1.2}$$

Factorizing by a, and thanks to the positive homogeneity, Equation (1.1) rewrites

$$w(x_0) \leq p(y + x_0) - g(y), y \in D(g) \tag{1.3}$$

Analogously, factorizing by $-a$, Equation (1.2) rewrites

$$w(x_0) \geq -p(y - x_0) + g(y), y \in D(g) \tag{1.4}$$

To find a suitable value for $w(x_0)$ it is sufficient that

$$-p(z - x_0) + g(z) \leq p(y + x_0) - g(y) \ \forall z, y \in D(g) \tag{1.5}$$

or

$$g(y + z) \leq p(y + x_0) + p(z - x_0) + \ \forall y, z \in D(g) \tag{1.6}$$

But we know that

$$g(y + z) \leq p(y + z) = p(z - x_0 + x_0 + y) \leq p(y + x_0) + p(z - x_0).$$

It follows that Equation (1.6) holds. This contradicts the fact that $(l^+, D(g))$ is maximal element which in turns implies that $D(g) = X$. $\square$

Exercise 1:

Let $X = \mathbb{R}^2$ and $Y = \{(x,y) \in X; -2x + y = 0\}$. Let l a real valued linear function defined on Y by $l(1,2) = 1$. Find a p as in the Hahn-Banach theorem such that $l \leq p$ on Y. What can you deduce? Construct explicitly an extension of l on X, with $l \leq p$ on X.

Solution 1:

We remark that

$$l(x, 2x) = x \leq ||(x, 2x)||.$$

Therefore we can set

$$p(x,y) = ||(x,y)||$$

From theorem 1, we can extend l on $\mathbb{R}^2$, with $l(x) \leq ||x||$ on X. To have an explicit definition of l on $\mathbb{R}^2$, we write

$$l(x,y) = l(x(1,2) + (y - 2x)(0,1)$$

which leads to a sufficient condition on the possible values of $l(0,1)$.

Exercise 2:

Let $X = L^2(0,1)$ and $Y = \{u = \sum_i u_i\varphi_i \in X; -2u_0 + u_1 = 0, u_i = 0 \text{ for } i > 1\}$, where $(\varphi_i)_{i\in\mathbb{N}}$ denotes an eigenfunction basis of X. Let l a real valued linear function defined on Y by $l(\varphi_0 + 2\varphi_1) = 1$. Find a p as in the Hahn-Banach theorem such that $l \leq p$ on Y. What can you deduce? Construct explicitly an extension of l on X, with $l \leq p$ on X.

Solution 2:

We remark that

$$l(u_0\varphi_0 + 2u_0\varphi_1) = u_0 \leq ||u_0\varphi_0 + 2u_0\varphi_1||.$$

Therefore we can set

$$p(u) = ||u||_{L^2}$$

From theorem 1, we can extend l on X, with $l(u) \leq ||u||$ on X. To have an explicit definition of l on X, we write ....

What happens if the function p is actually a norm? In this case the Hahn–Banach theorem is an existence result for bounded linear functionals on normed vector spaces.

Corollary 1. *Let X be a normed space over $\mathbb{R}$. If l is a real valued linear function defined on a linear subspace Y of X such that*

$$l(y) \leq c||y|| \text{ for all } y \in Y,$$

with $c \geq 0$. Then l can be extended to all of X as a bounded linear real valued function such that

$$|l(x)| \leq c||x|| \text{ for all } x \in X.$$

Exercise 3:

Prove Corollary 1.

Solution 3:

The corollary results from a direct application of Theorem 1 with $p(x) = c||x||$. We have

$$p(x+y) = c||x+y|| \leq c(||x|| + ||y||) = p(x) + p(y)$$

and for $a > 0$

$$p(ax) = c||ax|| = ap(x).$$

It follows that p satisfies the assumptions of Theorem 1 which in turn allows us to apply the theorem and prove the result. Note that if $l(x) < 0$, then $|l(x)| = -l(x) = l(-x) \leq c||x||$ wich ensures that the linear function l is bounded.

We now provide a complex version of Theorem 1:

Proposition 3. *Let X be a normed space over $\mathbb{C}$ and p a real non negative function on X with the properties:*

$$(1)\ p(ax) = |a|p(x) \text{ for all } x \in X \text{ and } a \in \mathbb{C}$$

$$(2)\ p(x+y) \leq p(x) + p(y) \text{ for all } x, y \in X \text{ (subadditivity)}$$

If l is a linear functional taking values in $\mathbb{C}$ defined on a linear subspace Y of X and dominated by p, that is

$$|l(y)| \leq p(y) \text{ for all } y \in Y,$$

then l can be extended to all of X as a linear functional such that $|l(x)| \leq p(x)$ for all $x \in X$.

Proof
We denote by u the real part of l and by v its imaginary part so that

$$l(y) = u(y) + iv(y).$$

Then u is a linear functional from Y to $\mathbb{R}$. Furthermore, note that

$$|u(y)| \leq |l(y)| \leq p(y).$$

We can therefore apply Theorem 1 to u. It follows that u can be extended over X with $u(x) \leq p(x)$ for all $x \in X$. Next note that for $y \in Y$

$$l(y) = u(y) - iu(iy). \tag{1.7}$$

This follows from the fact that on one hand

$$l(iy) = il(y) = iu(y) - v(y)$$

while on the other hand

$$l(iy) = u(iy) + iv(iy).$$

Identifying the real parts in those equations leads to:

$$v(y) = -u(iy)$$

which gives eq. (1.7). We now extend l to X thanks to Equation (1.7). Note that l is linear. One can indeed check that

$$l(x_1 + x_2) = l(x_1) + l(x_2).$$

And,

$$l(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = i(u(x) - iu(ix)) = il(x).$$

Lastly assume that

$$l(x) = re^{i\theta}$$

so that

$$e^{-i\theta}l(x) = r.$$

Since $r \in \mathbb{R}$, it follows that

$$r = e^{-i\theta}l(x) = l(e^{-i\theta}x) = u(e^{-i\theta}x).$$

From which we deduce that

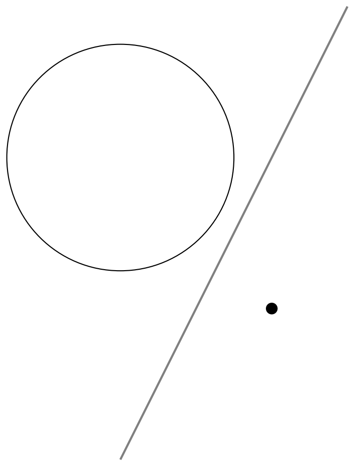
$$l(x) = e^{i\theta}u(e^{-i\theta}x).$$

And finally,

$$|l(x)| \leq |u(e^{-i\theta}x)| \leq p(e^{-i\theta}x) = p(x).$$

□

1.2 Geometric Hahn-Banach Theorems



Definition 2. *A set $S \subset X$ is convex if for all $x, y \in S$ and $t \in [0,1]$ we have $tx + (1-t)y \in S$.*

Definition 3. *A point $x \in S \subset X$ is an interior point of type I of S if for all $y \in X$ there exists $\mu > 0$ s.t.*

$$|t| < \mu \Rightarrow x + ty \in S.$$

Definition 4. *If X is a normed space, a point $x \in S \subset X$ is an interior point of type II of S if there $\exists \epsilon > 0$ s.t.*

$$B(x, \epsilon) \subset S.$$

where

$$B(x, \epsilon) = \{y \in X; ||y - x|| < \epsilon.$$

Remark 4. *Note that if x is an interior point of type II then it is an interior point of type I, for in this case one can choose $\mu < \frac{\epsilon}{||y||}$.*

In this section, since we do not assume a norm on X, all interior point will be of type I.

Theorem 5. *Let X be a linear space over $\mathbb{R}$. Let K be a convex subset of X, and suppose 0 is an interior point (of type I) of K. If $y \notin K$ then there exists a linear functional $l : X \to \mathbb{R}$ s.t.*

$$l(x) \leq l(y) \text{ for all } x \in K$$

with strict inequality for all interior points x of K.

In order to prove this theorem we will need to define the gauge function.

Definition 5. *Let K a subset of X which contains 0 as an interior point, the gauge of K denoted by p_k, is the real valued function defined on X by:*

$$p_k(x) = \inf\{b > 0; \frac{x}{b} \in K\}$$

To prove Theorem 5 we will apply the Hahn-Banach Theorem 1. We first prove that when K is convex, the gauge p_k is positively invariant and sub-additive.

Lemma 1. *We assume that $K \subset X$ is convex. Then the function p_k defined in Definition 5 satisfies*

$$(i) p_k(ax) = ap_k(x) \, \forall x \in X \, \forall a > 0,$$

$$(ii) p_k(x + y) = p_k(x) + p_k(y) \, \forall x, y \in X.$$

Proof
Note that

$$\frac{x}{b} \in K \Leftrightarrow a\frac{x}{ab} \in K$$

from which it follows that

$$p_k(ax) = ap_k(x)\, \forall x > 0.$$

Let $x, y \in X$ and b, c such that

$$\frac{x}{b}, \frac{y}{c} \in K.$$

Since K is convex, it follows that

$$\frac{x+y}{b+c} = \frac{b}{b+c}\frac{x}{b} + \frac{c}{b+c}\frac{y}{c} \in K,$$

which implies that

$$p_K(x+y) \leq b + c.$$

Let $(b_n), (c_n)$ two sequences such that $\frac{x}{b_n}, \frac{y}{c_n} \in K$ and converging toward $p_K(x)$ and $p_K(y)$. Taking the limit in the inequality

$$p_K(x+y) \leq b_n + c_n$$

gives

$$p_K(x+y) \leq p_K(x) + p_K(y).$$

□
We can now proceed with the proof of Theorem 5.
Proof
We want to apply Theorem 1. Since p_k is positively invariant and subadditive, we look for a relevant linear functional l and a subspace $Y \subset X$ such that

$$l(x) \leq p_K(x) \forall x \in Y.$$

Now assume $y \notin K$. We set

$$l(y) = 1.$$

Since l is linear

$$l(ay) = al(y)$$

therefore we can define l as such on the subspace

$$\{ay, a \in \mathbb{R}\}.$$

Note that since $y \notin K$ and $0 \in K$,

$$\frac{y}{b} \in K \Rightarrow b > 1$$

and therefore

$$p_K(y) = \inf\{b > 0; \frac{y}{b} \in K\} \geq 1.$$

This in turn implies that

$$p_K(y) \geq l(y)$$

and therefore

$$p_K(ay) = ap_K(y) \geq al(y) = l(ay)\, \forall a > 0.$$

We have also

$$l(ay) \leq 0 \leq p_k(ay) \forall a \leq 0.$$

We complete the proof by application of Theorem 1: l can be extended to X and

$$l(x) \leq p_K(x)\, \forall x \in X.$$

Now note that

$$x \in K p_K(x) \leq 1,$$

since $l(y) = 1$ we deduce that
$$\forall x \in K\, l(x) \leq l(y).$$
Note finally that if z is an interior point of K
$$p_k(z) < 1.$$
□

Theorem 6. *Let X be a linear space over $\mathbb{R}$. Let K be a convex subset of X with an interior point (of type I). If $y \notin K$ then there exists a linear functional $l : X \to \mathbb{R}$ s.t.*
$$l(x) \leq l(y) \text{ for all } x \in K$$
with strict inequality for all interior points x of K.

Proof
The result follows from a translation. Let z_0 be the interior point of K. We apply theorem 5 with $K' = \{z - z_0, z \in K\}$ and $y' = (y_{z0}) \notin K'$. □

Exercise 4:
Prove that
$$p_K(x) < 1 \Leftrightarrow x \text{ is interior to } K$$

Solution 4:

We already know that
$$x \text{ is interior to } K \Rightarrow p_K(x) < 1.$$
Conversely if $p_K(x) < 1$, there exists some $b < 1$ such that
$$\frac{x}{b} \in K.$$
But then, for all $y \in K$, one can find $\epsilon > 0$ such that
$$|t| < \epsilon \Rightarrow x + ty\; inK.$$
This follows from the fact that 0 is interior to K and the convexity of K (draw a picture!).

Theorem 7. *Let X be a linear space over $\mathbb{R}$. Let H, M be two disjoint convex subsets of X. Assume that at least one of them has an interior point. Then H and M can be separated by a hyperplane $l(x) = c$: there is a real valued linear function l and $c \in \mathbb{R}$ s.t.*
$$l(u) \leq c \leq l(v) \forall u \in H, v \in M.$$

Proof
Assume without loss of generality that H has an interior point. Let
$$K = \{x \in X; x = u - v, u \in H, v \in M.\}$$
Note that since $H \cap M = \emptyset$
$$0 \notin K.$$
Also K has an interior point. We now apply Theorem 5 to K and 0. Therefore there exists a linear functional l such that
$$l(u - v) \leq 0 \forall u \in H, v \in M,$$
which gives
$$l(u) \leq l(v) \forall u \in H, v \in M.$$
If we choose such that
$$\sup_{u \in H} l(u) \leq c \leq \inf_{v \in M} l(v)$$
we obtain the result. □

1.3 Application of the Hahn-Banach Theorem

This section is largely inspired by [Sch, Lax02].

We first consider the linear space $X = B(S)$ of all real valued bounded functions on some set S. $B(S)$ is endowed with the following partial order: for $x, y \in B(S)$

$$x \leq y \text{ if } \forall s \in S\, x(s) \leq y(s).$$

If $0 \leq x$ we say that x is non-negative. On $B(S)$ a linear functional l is said to be positive if it satisfies $l(y) \geq 0$ for all $y \geq 0$. The following theorem holds

Theorem 8. *Let Y be a linear subspace of $B(S)$. We assume that there exists $y_0 \in Y$ such that $y_0 \geq 1$ and a positive linear functional l on Y. Then l can be extended to all of $B(S)$ as a positive linear functional.*

Theorem 8 results from a more general result which we will state below. We first need to define a cone.

Definition 6. *A subset $P \subset X$ of a linear space over $\mathbb{R}$ is a cone if*

$$\forall x, y \in P,\, \forall t, s \geq 0,\, tx + sy \in P.$$

A linear functional on X is P-positive if $P(x) \geq 0$ for all $x \in P$.

Theorem 9. *Let $P \subset X$ be a cone with an interior point y_0. If Y is a subspace containing y_0 on which is defined a $P \cap Y$-positive linear functional l, then l has an extension to X which is P-positive.*

Proof
Again, we want to apply Theorem 1. We define p as follows:

$$p(x) = \inf\{l(y); y - x \in P, y \in Y\}.$$

Note that since y_0 is an interior point of x, by definition there exists $t > 0$ such that

$$y_0 - tx \in P.$$

Since P is a cone

$$\frac{1}{t}(y_0 - tx) = \frac{1}{t}y_0 - x \in P.$$

Therefore the definition makes sense. Next, note that since P is a cone

$$\forall a > 0,\, y - ax \in P \Leftrightarrow a(\frac{y}{a} - x) \in P,$$

and since Y is a subspace

$$y \in P \Leftrightarrow \frac{y}{a} \in Y.$$

Therefore,

$$\begin{aligned} p(ax) = &\ \inf\{l(y); y - ax \in P, y \in Y\} \\ = &\ \inf\{l(y); \tfrac{y}{a} - x \in P, y \in Y\} \\ = &\ \inf\{l(az); z - x \in P, z \in Y\} \\ = &\ \inf\{al(z); z - x \in P, z \in Y\} \\ = &\ ap(x). \end{aligned}$$

We can now look at the sub-additivity of P. Let $x_1, x_2 \in X$, $y_1, y_2 \in Y$, such that $y_i - x_i \in P, i \in \{1, 2\}$. Since P is a cone

$$y_1 - x_1 + y_2 - x_2 = y_1 + y_2 - (x_1 + x_2) \in P.$$

Therefore

$$p(x_1 + x_2) = \inf\{l(z); z - (x_1 + x_2) \in P, z \in Y\} \leq l(y_1 + y_2) = l(y_1) + l(y_2).$$

Taking a limit of appropriate sequences $(l(y_1^n))$, $(l(y_2^n))$ converging toward $p(x_1)$ and $p(x_2)$ provides

$$p(x_1 + x_2) \leq p(x_1) + p(x_2).$$

Next, note that if $x, y \in Y$ and $y - x \in P$

$$l(x) = l(x - y) + l(y) \leq l(y)$$

since $l(x - y) = -l(y - x) \leq 0$ which comes from the P-positivity of l. It follows from the definition of p as an inf that that $l(x) \leq p(x)$ for $x \in Y$. We can therefore extend p on X with $l \leq p$ on X by applying *Theorem* 1. Finally, let $x \in P$. Then

$$p(-x) = \inf\{l(y); y + x \in P, y \in Y\}.$$

But in this case for $y = 0$, $y + x = x \in P$. Therefore

$$p(-x) \leq l(0) = 0.$$

This in turn gives $l(-x) \leq p(-x) \leq 0$ which shows that l is P-positive. ☐

Proof (Theorem 8).

Under the assumptions of Theorem 8, y_0 is an interior point of the cone of the positive functions. Therefore, we can apply Theorem 9 with P defined as this cone. This gives the result. ☐The following theorems use topological properties. We refer to [Bre11] for proofs.

Theorem 10. *Let X be a normed space over $\mathbb{R}$. Let $l : X \to \mathbb{R}$ be a linear functional different from zero and $\alpha \in \mathbb{R}$. Then the hyperplane*

$$H = \{x \in X; f(x) = \alpha\}$$

is closed if and only if l is continuous.

Theorem 11. *Let X be a normed space over $\mathbb{R}$. Let H, M be two disjoint convex subsets of X. Assume that H is open and M is co,pact. Then H and M can be separated strictly by a hyperplane $l(x) = c :$ there is a real valued linear function l and $c \in \mathbb{R}$ s.t.*

$$l(u) < c < l(v) \forall u \in H, v \in M.$$

Chapter 2

Banach Spaces

In the previous chapter, we used a function p which was subadditive and positive homogeneity. In this chapter, we consider normed spaces that induce naturally a metric and a topology.

2.1 Normed and Banach Spaces

Definition 7. *Let X be a linear space over $F = \mathbb{R}$ or $\mathbb{C}$. A norm on X is a function $||\cdot|| : X \to [0, +\infty)$ s.t.*

$$(1)\ ||x|| = 0 \iff x = 0$$

$$(2)\ ||x + y|| \leq ||x|| + ||y|| \textit{ (subadditivity)}$$

$$(3)\ ||ax|| = |a|||x|| \textit{ (homogeneity)}$$

A normed space is a linear space X with a norm $||\cdot||$.

Exercise 5:
Prove that the $d(x, y) = ||x - y||$ defines a distance on X.

Solution 5:
First,

$$d(x, y) = 0 \Rightarrow ||x - y|| = 0 \Rightarrow x = y.$$

Next,

$$d(x, y) = ||x - y|| = ||x - z + z - y|| \leq ||x - z|| + ||z - y|| = d(x, z) + d(z, y).$$

Finally,

$$d(x, y) = ||x - y|| = || - (y - x)|| = ||y - x|| = d(y, x).$$

Thus any normed space is a metric space if we define the distance as above. As a consequence, the following topological properties hold:

- a sequence x_n converges to x if $d(x_n, x) = ||x_n - x|| \to 0$.
- a set $U \subset X$is open if for every $x \in U$ there is a ball $B(x, \epsilon)$ centered at x with radius $\epsilon > 0$ s.t. $B(x, \epsilon) \subset U$.
- a set F is closed if $X \setminus F$ is open.
- a set K is compact if every open cover of K has a finite sub-cover.

Two norms $||\cdot||_1$ and $||\cdot||_2$ on X are equivalent if there is $A, B > 0$ s.t.

$$||x||_1 \leq A||x||_2 \text{ and } ||x||_2 \leq B||x||_1 \ \forall x \in X$$

Exercise 6:
Prove that two equivalent norms define the same topology.

Solution 6:
Assume that two norms equivalent norms $||\cdot||_1$ and $||\cdot||_2$ are defined on X. Let O be an open set of $(X, ||\cdot||_1)$, and $x \in O$. There exists ϵ_1 such that $B_1(x, \epsilon_1) \subset O$ where

$$B_1(x, \epsilon_1) = \{y \in X; ||y - x||_1 < \epsilon_1\}$$

We look for ϵ_2 such that $B_2(x, \epsilon_2) \subset O$ where

$$B_2(x, \epsilon_2) = \{y \in X; ||y - x||_2 < \epsilon_2\},$$

for this, it is sufficient to find ϵ_2 such that

$$||y - x||_2 < \epsilon_2 \Rightarrow ||y - x||_1 < \epsilon_1$$

Since

$$||x||_1 \leq A||x||_2$$

it is sufficient to choose

$$\epsilon_2 = \frac{\epsilon_1}{A},$$

indeed this gives

$$||y - x||_1 < A||y - x||_2 < A\epsilon_2 = \epsilon_1$$

Therefore, any open set of $(X, ||\cdot||_1)$ is an open set of $(X, ||\cdot||_2)$. The converse statement also holds.

Finally recall that a metric space X is complete if every Cauchy sequence (x_n) converges in X.

Definition 8. *A Banach space is a complete normed space.*

Exercise 7:
Assume that $(X, ||\cdot||_1)$ is a Banach space. Assume that $||\cdot||_1$ and $||\cdot||_2$ are equivalent. Prove that then $(X, ||\cdot||_2)$ is a Banach space.

Solution 7:
Let (x_n) be a Cauchy sequence in $(X, ||\cdot||_2)$. It follows from

$$||x_n - x_p||_1 \leq A||x_n - x_p||_2$$

that (x_n) is also a Cauchy sequence in $(X, ||\cdot||_1)$. Since $X, ||\cdot||_1)$ is a Banach space, there exists $x \in X$ such that $||x_n - x||_1$ converges toward 0 as n goes to $+\infty$. Next, since

$$||x_n - x||_2 \leq B||x_n - x||_1,$$

it follows that $||x_n - x||_2$ converges also toward 0.

Banach Spaces and topological properties: a few examples

1) Consider the space $X = C([0, 1])$ of continuous real valued functions on the closed unit interval $[0, 1]$. Then the formulas

$$||f||_\infty = \sup_{0 \leq t \leq 1} |f(t)|, \ ||f||_2 = \Big(\int_0^1 |f(t)|^2 dt\Big)^{\frac{1}{2}}$$

for $f \in C([0, 1])$ define norms on X. The space $C([0, 1])$ is a Banach space with $||\cdot||_\infty$ but is not complete with $||\cdot||_2$. It follows that these two norms are not equivalent.

2) The space $Y = C^1([0,1])$ of continuously differentiable real valued functions on the closed unit interval is a dense linear subspace of C([0, 1]) with the supremum norm and so is not a closed subset of $(C([0,1]), ||\cdot||_\infty)$.

3) Consider the closed unit ball $B = \{f \in C([0,1])|; ||f||_\infty \leq 1\}$ in $C([0,1])$ with respect to the supremum norm. This set is closed and bounded. For every $t \in [0,1]$, $(f(t)_{f\in B} = [-1,1]$ is compact but B is not equicontinuous. More explicitly, consider the sequence (f_n) of B defined by $f_n(t) = \sin(n\pi t)$ for $n \in \mathbb{N}$ and $0 \leq t \leq 1$. For any $\mu > 0$, let n such that $t_n = \frac{1}{2n} < \mu$. Then $|f_n(0) - f_n(t_n)| = 1$. Theorem 20 below shows that the compactness of the unit ball characterizes the finite-dimensional normed vector spaces.

4) For each $p \in [1, \infty)$ let

$$l^p = \{(a_1, a_2, ...); \sum_{j=1}^{+\infty} |a_j|^p < \infty\}.$$

Define a norm on l^p by

$$||a||_p = \big(\sum_{j=1}^{+\infty} |a_j|^p\big)^{\frac{1}{p}}.$$

Then l^p is a Banach space.

5) Let

$$l^\infty = \{(a_1, a_2, ...); \sup |a_j| < \infty.$$

Define a norm on l^∞ by

$$||a||_\infty = \sup |a_j|.$$

Then l^∞ is a Banach space.

6) Let

$$C_0 = \{(a_1, a_2, ...); \lim_{n\to\infty} a_n = 0\}.$$

Choose the norm $||\cdot||_\infty$ on C_0. Then C_0 is a Banach space.

7) Let

$$F_p = \{(a_1, a_2, ...); \exists N \in \mathbb{N} s.t. n \geq N \Rightarrow a_n = 0\}.$$

Choose the norm $||\cdot||_p$ on F_p. Then F_p is a normed space but it is not complete. Its completion is isomorphic to l^p.

8) Let $\Omega \subset \mathbb{R}^d$ be an open bounded set and let $p \in [1, \infty)$. Let $X = C_c(\Omega)$ be the space of continuous functions with compact support in Ω, endowed with the norm

$$||f||_p = \big(\int_\Omega |f(x)|^p dx\big)^{\frac{1}{p}}.$$

Then X is a normed space, which is not complete. Its completion is denoted $L^p(\Omega)$ and may be identified with the set of equivalence classes of measurable functions such that

$$\int_\Omega |f(x)|^p dx < +\infty$$

with two functions f, g called equivalent if $f(x) = g(x)$ for almost every x.

9) Let X denote the set of C^1 functions on Ω such that

$$\int_\Omega |f(x)|^p dx < \infty \text{ and } \int_\Omega |\partial_j f(x)|^p dx < \infty, \;\; j = 1, ..., n$$

Let's define

$$||f||_{1,p} = \left[\int_\Omega |f(x)|^p dx + \sum_{j=1}^{n} \int_\Omega |\partial_j f(x)|^p dx\right]^{\frac{1}{p}}$$

Then X is a normed space which is not complete. Its completion is the so called Sobolev space $W^{1,p}(\Omega)$ which can be identified with the subspace of $L^p(\Omega)$ consisting of (equivalence classes) of functions all of whose first derivatives are in $L^p(\Omega)$ in the sense of distributions. The study of these spaces is part of another course.

Exercise 8:

Let

$$p \in (1, +\infty) \text{ and } q \text{ defined as } \frac{1}{p} + \frac{1}{q} = 1.$$

Prove Young inequality:

$$\forall a, b \geq 0, \ ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Solution 8:

Apply the ln function to the right-hand side of the equation. This gives the result thanks to the concavity of ln.

Exercise 9:

(Holder inequality) Let $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$\int |fg| \leq ||f||_{L^p} ||g||_{L^q}.$$

Solution 9:

Let

$$\tilde{f} = \frac{|f|}{||f||_{L^p}}, \ \tilde{g} = \frac{|g|}{||g||_{L^q}}.$$

Then,

$$||\tilde{f}||_{L^p}^p = ||\tilde{g}||_{L^q}^q = 1.$$

We apply the Young inequality to $\tilde{f}\tilde{g}$ and integrate, we obtain,

$$\int \tilde{f}\tilde{g} \leq \int \Big(\frac{\tilde{f}^p}{p} + \frac{\tilde{g}^q}{q} \Big) = 1.$$

This gives the result.

Exercise 10:

(Minkowsky Inequality) Let $f \in L^p$ and $g \in L^p$. Show that

$$||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p}.$$

Solution 10:

We write

$$\begin{array}{rcl} |||f| + |g|||_{L^p}^p & = & \int (|f| + |g|)^p \\ & = & \int (|f| + |g|)(|f| + |g|)^{p-1} \\ & \leq & (||f||_{L^p} + ||g||_{L^p}) \big(\int (|f| + |g|)^{qp-q} \big)^{\frac{1}{q}} \text{ thanks to Holder inequality,} \\ & \leq & (||f||_{L^p} + ||g||_{L^p}) \big(\int (|f| + |g|)^p \big)^{1-\frac{1}{p}} \text{ using the fact that } \frac{1}{p} + \frac{1}{q} = 1. \end{array}$$

Multiplying both sides by $|||f| + |g|||_{L^p}^{1-p}$ gives the result. The next section provides a little more details about L^p spaces.

2.2 L^p spaces

This section is inspired by [Bre11, Rud87]. Let $(\Omega, \mathcal{M}, \mu)$ denote a measure space, i.e., Ω is a set and
(i) $\mathcal{M}$ is a σ-algebra in Ω, i.e., $\mathcal{M}$ is a collection of subsets of Ω such that:
(a)$\emptyset \in \mathcal{M}$
(b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$,
(c) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$ whenever $\forall n A_n \in \mathcal{M}$,
(ii) μ is a positive measure, i.e., $\mu : \mathcal{M} \to [0, +\infty]$ satisfies
(a) $\mu(\emptyset) = 0$,
(b)$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu(A_n)$ whenever (A_n) is a disjoint countable family of members of $\mathcal{M}$.
The members of $\mathcal{M}$ are called the measurable sets. We shall also assume that
(iii) Ω is σ-finite, i.e., there exists a countable family (A_n) in $\mathcal{M}$ such that $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ and $\forall n\, \mu(A_n) < +\infty$.

The sets $E \in \mathcal{M}$ with the property that $\mu(E) = 0$ are called the null sets. We say that a property holds a.e. (or for almost all $x \in \Omega$) if it holds everywhere on Ω except on a null set. On a measure space $(\Omega, \mathcal{M}, \mu)$, we can define the (Lebesgue) integral of a real valued measurable function f

$$\int_\Omega f d\mu.$$

We refer to [Rud87] for the detailed construction of the integral. A fundamental example of measure space is given by $\Omega = \mathbb{R}^n$ associated with the Lebesgue's measure relying on the natural measure of cubes in $\mathbb{R}^n$. The following theorems hold.

Theorem 12 (Lebesgue-Monotone Convergence)**.** *Let (f_n) be a sequence of measurable functions defined on Ω, and suppose that*
(a) $0 \le f_1(x) \le f_2(x) \le ...$ a.e.,
(b) $f_n(x) \to f(x)$a.e.
Then f is measurable, and

$$\int f_n \to \int f$$

Theorem 13 (Beppo-Levi-Monotone Convergence)**.** *Let (f_n) be a sequence of measurable functions defined on Ω, and suppose that*

$$(a)\, 0 \le f_1(x) \le f_2(x) \le ...a.e.,$$

$$(b) \sup_{n\in\mathbb{N}} \int f_n(x) < +\infty.$$

Then $(f_n(x))$ converges a.e. to a finite limit, which we denote by $f(x)$; the function f belongs to L^1 and $||f_n - f||_1 \to 0$.

Theorem 14 (Dominated Convergence Theorem)**.** *Let (f_n) be a sequence of measure functions defined on Ω, and $g \in L^1(\Omega)$. We assume that (a) $(f_n(x))$ converges a.e. to a finite limit, which we denote by $f(x)$;*
(b) $f_n(x) \le g(x)$ a.e.
Then the function f belongs to L^1 and $||f_n - f||_1 \to 0$.

Theorem 15 (Fatou's Lemma)**.** *If $\forall n \in \mathbb{N}$ $(f_n) : \Omega \to [0, +\infty]$ is measurable, then*

$$\int \liminf f_n \le \liminf \int f_n.$$

Exercise 11:

Prove that L^p is a Banach space.

Solution 11:

We consider Ω with the Lebesgue measure and the borelian sets. We follow the proof in [Rud87]. We start with the case $1 \le p < +\infty$. Let (f_n) a Cauchy sequence in L^p.

$$\forall \epsilon > 0\, \exists N \in \mathbb{N};\, n, p > N \Rightarrow ||f_n - f_p||L^p < \epsilon.$$

From (f_n) we extract a subsequence as follows: there exists n_0 such that

$$||f_n - f_{n_0}||_p < \frac{1}{2} \forall n > n_0.$$

Then pick $n_1 > n_0$ such that

$$||f_n - f_{n_1}||_p < \frac{1}{2^2} \forall n > n_1,$$

and by induction $n_i > n_{i-1}$ such that

$$||f_n - f_{n_i}||_p < \frac{1}{2^{i+1}} \forall n > n_i.$$

By construction one has that

$$||f_{n_{i+1}} - f_{n_i}||_p < \frac{1}{2^{i+1}} \forall i \in \mathbb{N}.$$

Note that

$$\sum_{i \in N} ||f_{n_i+1} - f_{n_i}||_p < \sum_{i \in \mathbb{N}} \frac{1}{2^{i+1}} = 1.$$

Let

$$g_k(x) = \sum_{i=0}^{k} |f_{n_i+1}(x) - f_{n_i}(x)|, \;\; g(x) = \sum_{i=0}^{+\infty} |f_{n_i+1}(x) - f_{n_i}(x)|.$$

We want to apply Theorem 13 to (g_k^p).

$$g_k^p \leq g_{k+1}^p$$

and

$$\int g_k^p \leq \sum_{i=0}^{k} ||f_{n_i+1} - f_{n_i}||_p < 1.$$

Therefore $g_k^p(x)$ converges a.e. on Ω and so do $g_k(x)$. We denote by $g(x)$ the limit of $g_k(x)$. It follows that $\sum_i (f_{n_i+1}(x) - f_{n_i}(x))$ converges absolutely a.e. on Ω. So f_n converges *a.e.*. Let f denotes its limit. Let $\epsilon > 0$. Since f_n is Cauchy, for all $\epsilon > 0$, there exists N such that $n, k \geq N$ implies

$$||f_n - f_k||_p < \epsilon,$$

or

$$\int_\Omega |f_n - f_k|^p < \epsilon^p.$$

From Fatou's lemma we deduce

$$\int_\Omega |f - f_N|^p \leq \liminf \int_\Omega |f_{n_i} - f_N|^p \leq \epsilon^p,$$

from which we conclude that

$$||f_n - f||_p \to 0.$$

Next, we consider the case $p = \infty$. Remind that $L^\infty(\Omega)$ is defined as the set of measurable functions for which there exists a positive constant C such that $f < C$ *a.e.* on Ω. And

$$||f||_\infty = \inf_{C>0} \{f < C a.e.\}.$$

Exercise 12:

Prove that the set of continuous functions with compact support is dense in L^p.

Solution 12:

Coming soon...

2.3 Finite versus Infinite Dimensional Normed Spaces

The following results are specific to finite dimensional normed spaces. The proofs can be found in classical Topology courses.

Proposition 16. *Every finite-dimensional normed space is a Banach space.*

Proposition 17. *Let X be a normed space. Then every finite dimensional linear subspace of X is a closed subset of X.*

Proposition 18. *Let X be a finite-dimensional normed space. Let $K \subset X$. Then K is compact if and only if K is closed and bounded.*

Theorem 19. *Le X be a finite dimensional vector space. Then any two norms on X are equivalent.*

Exercise 13:
Prove the theorem stated above.

Solution 13:
Let $||\cdot||$ be a norm on X. Let $e_1, e_2, ..., e_n$ be a basis of X. For $x = \sum_{i=1}^n \lambda_i e_i$, set

$$||x||_\infty = \max_{i \in 1,...,n} |\lambda_i|.$$

This defines a norm on X. We have

$$||x|| \le ||x||_\infty \sum_{i=1}^n |\lambda_i|.$$

On the other hand, note that $S = \{x \in X; ||x||_\infty = 1$ is a compact set in $(X, ||\cdot||_\infty)$. Since

$$|||x|| - ||y||| \le ||x - y|| \le (\sum_{i=1}^n |\lambda_i|)||x - y||\infty$$

the function

$$f : \begin{cases} (X, ||\cdot||_\infty) & \to & (\mathbb{R}, |\cdot|) \\ x & \to & ||x|| \end{cases}$$

is continuous. Therefore $f(S)$ is a compact set of $(\mathbb{R}, |\cdot|)$. It follows that there exists $x_0 \in S$ such that

$$f(x_0) = \inf_{x \in S} f(x)$$

Let $x \neq 0$.

$$\frac{x}{||x||_\infty} \in S.$$

Therefore

$$||x_0|| \le ||\frac{x}{||x||_\infty}||$$

which in turn implies

$$||x||_\infty \le \frac{1}{||x_0||}||x||.$$

The following lemma is the key tool to prove Theorem 20.

Lemma 2. *Let X be a normed space and let $Y \subset X$ be a closed linear subspace that is not equal to X. Then there exists a vector $x \in X$ such that*

$$||x|| = 1, \ \inf_{y \in Y} ||x - y|| \ge \frac{1}{2}.$$

Proof
Let $x \in X \setminus Y$ and let

$$d = \inf_{y \in Y}\{||x - y||\}.$$

Since Y is closed, $d > 0$ (why?). Let $y_0 \in Y$ such that

$$||y - x|| < 2d.$$

Next, we define z as

$$z = \frac{x - y_0}{||x - y_0||}.$$

For any $y \in Y$, it holds

$$||z - y|| = ||\frac{x - y_0 - y||x - y_0||}{||x - y_0||}|| \geq \frac{d}{2d} = \frac{1}{2}.$$

□

Theorem 20. *Let X be a normed space and denote the closed unit ball in X by*

$$B = \{x \in X; ||x|| \leq 1\}$$

Then B is compact if and only if X is finite dimensional.

Proof
X finite dimensional $\Rightarrow B$ compact has been proven in the undergraduate course of topology. We prove here that X infinite dimensional $\Rightarrow B$ is not compact. For this, we are going to construct a sequence (x_n) in B from which any convergent subsequence cannot be extracted. Let $S = \{x \in X; ||x|| = 1\}$ and let $x_0 \in S$. From Lemma 2, there exists $x_1 \in S$ such that $||y - x_1|| \geq \frac{1}{2}$ for all $y \in span\{x_0\}$. Then we proceed by induction, and assuming that $x_0, x_1, ..., x_n$ are already defined, we choose $x_{n+1} \in S$ such that $||y - x_{n+1}|| \geq \frac{1}{2}$ for all $y \in span\{x_0, x_1, ..., x_n\}$. No convergent subsequence can be extracted from (x_n).

2.4 Linear functionals on a Banach Space

Definition 9. *A linear functional $l : X \to \mathbb{K}$ on a normed space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is bounded if there is $c < \infty$ s.t.*

$$|l(x)| \leq c||x|| \forall x \in X.$$

The inf over all such c is the norm l,

$$||l|| = \sup_{x \in X, x \neq 0} \frac{|l(x)|}{||x||} \tag{2.1}$$

Theorem 21. *A linear functional l on a normed space X is bounded if and only if it is continuous.*

Proof
We assume first that l is bounded. Let $x \in X$. Let (x_n) be a sequence converging to x. Then

$$|l(x) - l(x_n)| \leq ||l|| ||x - x_n||$$

which shows that l is continuous. Conversely, assume that l is not bounded, then there exists a sequence (x_n) such that:

$$\frac{|l(x_n)|}{||x_n||} \geq n$$

Setting

$$y_n = \frac{x_n}{\sqrt{n}||x_n||}$$

we have

$$|l(y_n)| \geq \sqrt{n} \to +\infty.$$

But

$$\lim_{n \to +\infty} y_n = 0$$

which is a contradiction. □

Definition 10. *The set X' of all bounded linear functionals on X is called the dual of X.*

Theorem 22. *Endowed with the norm defined by Equation (2.1), X' is a Banach space.*

Proof
Let (l_n) be a Cauchy sequence in X'. We want to show that there exists a linear functional l on X such that (l_n) converges to l. We know that for all ϵ

$$||l_n - l_m|| < \epsilon$$

for any n, m large enough. It follows that

$$||l_n(x) - l_m(x)|| < \epsilon||x|| \; \forall x \in X.$$

Thus, for all x in X, $l_n(x)$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete $(l_n(x))$ converges toward a value that we call $l(x)$. This defines l as a functional over $\mathbb{K}$. By computations limits, one can deduce that l is linear. We need to prove that l is bounded. Since (l_n) is Cauchy and thanks to the inequality

$$||l_n|| - ||l_m||| \leq ||l_n - l_m||$$

we know that $(||l_n||)$ is a Cauchy sequence in $\mathbb{R}$. Since it is Cauchy, it is bounded by a constant c. Therefore

$$|l_n(x)| \leq ||l_n||||x|| \leq c||x||$$

Taking the limit, we obtain

$$|l(x)| \leq c||x||$$

which shows that l is bounded. Also, from

$$|l_m(x) - l_n(x)| < \epsilon||x||$$

we obtain, taking the limit in m that

$$|l(x) - l_n(x)| \leq \epsilon||x||$$

which implies that

$$||l - l_m|| \to 0.$$

□

Can we always say that for a given Banach space there exists a bounded linear functional? This is the topic of the next exercise which illustrates how the Hahn-Banach theorem theorem 1 provides the answer.

Exercise 14:

Let $y_1, ..., y_N$ be N linearly independent vectors in a normed space X and $\alpha_1, ..., \alpha_N$ arbitrary scalars. Prove that there is a bounded linear functional $l \in X'$ such that

$$l(y_j) = \alpha_j, j = 1, ..., N$$

Solution 14:

Let

$$Y = span\{y_1, y_2, ..., y_N\}.$$

We define l on Y by

$$l(y_j) = \alpha_j.$$

Indeed, thanks to the linearity of l this defines l on Y by

$$l(y) = l(\sum_j \beta_j y_j) = \sum_j \beta_j \alpha_j,$$

for all $y = \sum_j \beta_j y_j \in Y$. One can verify that l is linear. Furthermore l is bounded. This follows from

$$|l(\sum_j \beta_j y_j)| \leq \sum_j |\beta_j||\alpha_j| \leq \max_j |\alpha_j| \sum_j |\beta_j| \leq C||y||.$$

The last inequality holds because $\sum_j |\beta_j|$ is a norm on Y and all the norms on Y are equivalent. To conclude, we apply Corollary 1 if $\mathbb{K} = \mathbb{R}$ or *Proposition* 3 if $\mathbb{K} = \mathbb{C}$.

Proposition 23. *Let l be a bounded linear functional defined on linear subspace Y of X. Then l can be extended to X, furthermore*

$$||l||_{Y'} = ||l||_{X'}$$

Proof
We apply Theorem 1 if $\mathbb{K} = \mathbb{R}$ or Proposition 3 if $\mathbb{K} = \mathbb{C}$ with $p(x) = ||l||_{Y'}||x||$. So we can extend l to X, and

$$|l(x)| \leq ||l||_{Y'}||x|| \forall x \in X$$

We deduce that

$$\sup_{x \in X} \frac{|l(x)|}{||x||} \leq ||l||_{Y'}$$

that is,

$$||l||_{X'} \leq ||l||_{Y'}.$$

On the other hand

$$||l||_{Y'} = \sup_{x \in Y} \frac{|l(x)|}{||x||} \leq \sup_{x \in X} \frac{|l(x)|}{||x||} \leq ||l||_{X'}.$$

□.

Proposition 24. *For every $x_0 \in X$, there exists $\varphi_{x_0} \in X'$ such that*

$$||\varphi_{x_0}|| = ||x_0|| \text{ and } |\varphi_{x_0}(x_0)| = ||x_0||^2$$

Proof
We apply Proposition 23 with $Y = \{\alpha x_0; \alpha \in \mathbb{K}\}$ and $\varphi_{x_0}(\alpha x_0) = \alpha ||x_0||^2$. So, φ_{x_0} can be extended to X with

$$||\varphi_{x_0}||_{X'} = ||\varphi_{x_0}||_{Y'}$$

Note that

$$|\varphi_{x_0}(\alpha x_0)| = |\alpha|||x_0||^2$$

from which we deduce that

$$||\varphi_{x_0}||_{X'} = ||\varphi_{x_0}||_{Y'} = ||x_0||.$$

Finally choosing $\alpha = 1$ shows that

$$\varphi_{x_0}(x_0)| = ||x_0||^2.$$

□

Proposition 25. *For every $x \in X$, we have that*

$$||x|| = \sup_{\varphi \in X', ||\varphi|| \leq 1} |\varphi(x)|.$$

Proof
For every $\varphi \in X'$ with $||\varphi|| \leq 1$, we have

$$|\varphi(x)| \leq ||x||.$$

To prove the result, it remains to find a φ such that $||\varphi|| \leq 1$ and $|\varphi(x)| = ||x||$. From Proposition 24, there exists φ_0 such that

$$||\varphi_0|| = ||x|| \text{ and } |\varphi_0(x)| = ||x||^2$$

Choosing

$$\varphi_1 = \frac{1}{||x||}\varphi_0$$

provides the result.

Chapter 3

The Uniform Boundedness Principle and the Closed Graph Theorem

3.1 The Baire Category Theorem

Theorem 26 (Baire). *Let X be a complete metric space and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets in X. Assume that*

$$\forall n \in \mathbb{N},\ (F_n)^\circ = \emptyset$$

then

$$\Big(\bigcup_{n\in\mathbb{N}} F_n\Big)^\circ = \emptyset$$

Proof
Let

$$U_n = F_n^c$$

then U_n is an open set. Furthermore, since

$$\overline{U_n} = \overline{F_n^c} = \left(F_n^\circ\right)^c = X$$

it is dense in X. Now,

$$\bigg(\Big(\bigcup_{n\in\mathbb{N}} F_n\Big)^\circ\bigg)^c = \overline{\Big(\bigcup_{n\in\mathbb{N}} F_n\Big)^c} = \overline{\bigcap_{n\in\mathbb{N}} U_n}$$

therefore, it is sufficient to prove that $A = \bigcap_{n\in\mathbb{N}} U_n$ is dense in X. Let $x \in X$, and let $\epsilon > 0$. We want to prove $A \cap B(x,\epsilon) \neq \emptyset$. To this end we are going to construct a Cauchy sequence (x_n) whose limit belongs to $A \cap B(x,\epsilon)$. Since U_0 is dense in X, $U_0 \cap B(x,\epsilon) \neq \emptyset$. Let x_0 and $r_0 > 0$ such that $B_c(x_0, r_0) \subset U_0 \cap B(x,\epsilon)$ where B_c denotes the closed ball. This is possible since $U_0 \cap B(x,\epsilon)$ is an open set. Next let x_1 and r_1 such that $B_c(x_1, r_1) \subset B(x_0, r_0) \cap U_1$ and $r_1 < \frac{r_0}{2}$. Next, having defined x_n, r_n, we define x_{n+1}, r_{n+1} such that $B_c(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1}$ and $r_{n+1} < \frac{r_n}{2}$. Clearly, (x_n) is a Cauchy sequence. We note l its limit. For any fixed n, $l \in B_c(x_n, r_n) \subset U_n \cap B(x,\epsilon)$.$\square$

Corollary 2. *Let X be a nonempty complete metric space. Let (F_n) be a sequence of closed subsets such that*

$$\bigcup_{n\in\mathbb{N}} F_n = X$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$F_{n_0}^\circ \neq \emptyset$$

Proof
Assume that for every $n \in \mathbb{N}$
$$(F_n)^\circ = \emptyset$$
then by the Baire theorem,
$$\Big(\bigcup_{n\in\mathbb{N}} F_n\Big)^\circ = \emptyset$$
But this is not possible because
$$\bigcup_{n\in\mathbb{N}} F_n = X = X^\circ$$
since X is open. Therefore since it is not empty the corollary is proved. $\square$

3.2 The Uniform Boundedness Principle

The following theorem is often referred as the Uniform Boundedness Principle or the Banach-Steinhaus theorem.

Theorem 27 (Uniform Boundedness Principle)**.** *Let E and F be two Banach spaces and let $(T_i)_{i\in I}$ be a family of continuous linear operators from E into F. Assume that*
$$\forall x \in E \ \sup_{i\in I} ||T_i x|| < \infty. \tag{3.1}$$
Then
$$\sup_{i\in I} ||T_i|| < \infty.$$

Proof
For $n \in \mathbb{N}$, we define F_n as
$$F_n = \{x \in X; \forall i \in I,\ ||T_i x|| \le n.\}$$
Note that F_n is closed in X Let $x \in X$. From assumption (3.1) there exists a constant C_x such that
$$\sup_{i\in I} ||T_i x|| \le C_x.$$
It follows that for $n \ge C_x$, $x \in F_n$. Therefore $x \in \cup_{n\in\mathbb{N}} F_n$, which implies that
$$X = \bigcup_{n\in\mathbb{N}} F_n.$$
From Corollary 2, it follows that there exits n_0 such that $F_{n_0}^\circ \neq \emptyset$. Let $x \in X$ and $r > 0$ such that
$$B_c(x, r) \subset F_{n_0},$$
where $B_c(x, r)$ is the closed ball of center x and radius r. By definition of F_{n_0}, one has that
$$\sup_{i\in I} ||T_i(x + ry)|| \le n_0$$
for all y in X such that $||y|| = 1$. Next, remark that
$$||T_i(ry)|| - || - T_i(x)|| \le ||T_i(ry + x)|| \le n_0$$
which gives
$$||T_i(ry)|| \le n_0 + ||T_i(x)||$$
or
$$||T_i(y)|| \le \frac{1}{r}(n_0 + ||T_i(x)||)$$
which proves that
$$\sup_{i\in I} ||T_i|| \le \frac{1}{r}(n_0 + C_x).$$
$\square$

3.3 The open mapping theorem and the closed graph theorem

Theorem 28 (Open Mapping Theorem)**.** *Let E and F be two Banach spaces and let T be a continuous linear operator from E onto F (onto = surjective). Then there exists a constant $c > 0$ such that the image of the unit ball of E contains the ball of center 0 and radius c of F:*

$$B_F(0, c) \subset T(B_E(0, 1))$$

Proof

We split the proof into two parts.
1) First, we prove that there exists a constant $c_1 > 0$ such that :

$$B(0, c_1) \subset \overline{T(B(0, 1))}. \tag{3.2}$$

Note that since T is surjective,

$$F = \bigcup_{n\in\mathbb{N}} nT(B(0, 1)).$$

To see this, consider any $y \in F$. There exists $x \in E$ such that

$$y = T(x) = 2||x||T(\frac{x}{||2x||})$$

which shows that $y \in nT(B(0, 1)$ as soon as n is large enough. Next, we seek to apply the Baire theorem theorem 26. Since we have

$$F = \bigcup_{n\in\mathbb{N}} n\overline{T(B(0, 1))}$$

there exists n_0 such that

$$\left(n_0\overline{T(B(0, 1))}\right)^{\circ} \neq \emptyset.$$

Now, let $y \in F$ and $c > 0$ such that

$$B(y, c) \subset n_0\overline{T(B(0, 1))}$$

Assume that $||z|| < c$, we can write

$$y + z = \lim_{k\to+\infty} n_0T(x_k)$$

with

$$x_k \in B(0, 1)$$

therefore,

$$\frac{y + z}{n_0} = \lim_{k\to+\infty} T(x_k)$$

which means that

$$B(\frac{y}{n_0}, \frac{c}{n_0}) \subset \overline{T(B(0, 1))}$$

WLOG, assume that

$$B(y, c) \subset \overline{T(B(0, 1))}.$$

In particular $y \in \overline{T(B(0, 1))}$. It follows that $-y \in \overline{T(B(0, 1))}$ too. Therefore:

$$B(0, c) = -y + B(y, c) \subset \overline{T(B(0, 1))} + \overline{T(B(0, 1))}.$$

Next, we remark that

$$\overline{T(B(0, 1))} + \overline{T(B(0, 1))} = 2\overline{T(B(0, 1))}. \tag{3.3}$$

Let us detail this last statement. Let $y, z \in \overline{T(B(0, 1))}$ and $(x_{1k}), (x_{2k})$ two sequences in $B(0, 1)$ such that $(T(x_{1k}))$ and $(T(x_{2k}))$ converge respectively toward y and z. It follows that

$$y + z = \lim_{k\to+\infty} 2T(\frac{x_{1k} + x_{2k}}{2}) \in 2\overline{T(B(0, 1))}$$

which implies

$$\overline{T(B(0,1))} + \overline{T(B(0,1))} \subset 2\overline{T(B(0,1))}.$$

Conversely, let $y = \lim_{k\to+\infty} 2T(x_k)$ with $x_k \in B(0,1)$. We have that

$$y = \lim_{k\to+\infty} (T(x_k + x_k)$$

which implies

$$2\overline{T(B(0,1))} \subset \overline{T(B(0,1))} + \overline{T(B(0,1))}$$

which proves Equation (3.3). This gives eq. (3.2) with $c_1 = \frac{c}{2}$.
2) Now, we know that for any given linear application T (we did not use the fact that T is continuous yet), there exists a constant c_1 such that

$$B(0,c_1) \subset \overline{T(B(0,1))}$$

Let, $y \in B(0, \frac{c_1}{4})$. From step 1), we deduce that there exists $x \in B(0,1)$ such that

$$||4y - Tx|| < \frac{c_1}{2}$$

We set

$$x_0 = \frac{x}{4}.$$

Note that

$$||x_0|| < \frac{1}{4} \text{ and } ||y - Tx_0|| < \frac{c_1}{8}.$$

Next, analogously, there exists $x \in B(0,1)$ (in general, different from the previous one of course!) such that

$$||8(y - Tx_0) - Tx|| < \frac{c_1}{2}.$$

We set

$$x_1 = \frac{x}{8}.$$

It follows that

$$||x_1|| < \frac{1}{8} \text{ and } ||y - Tx_0 - Tx_1|| < \frac{c_1}{16}.$$

By induction, we construct a sequence (x_n) such that

$$||x_n|| < \frac{1}{2^{n+2}} \text{ and } ||y - \sum_{k=0}^{n} Tx_k|| < \frac{c_1}{2^{n+3}}. \tag{3.4}$$

The second inequality induces that

$$y = \sum_{k=0}^{+\infty} Tx_k$$

Also, by construction, $(\sum_{k=0}^{n} x_k)$ is a Cauchy sequence, so it converges to a limit x with

$$||x|| \le \sum_{k=0}^{+\infty} ||x_k|| \le \frac{1}{4}\sum_{k=0}^{+\infty} \frac{1}{2^k} = \frac{1}{2}.$$

Since T is continuous, we have

$$y = \sum_{k=0}^{+\infty} Tx_k = T\sum_{k=0}^{+\infty} x_k = Tx$$

with $||x|| < 1$. This proves the theorem. □
The following exercise explains why Theorem 28 is called the open mapping theorem.

Exercise 15:

Let E and F be two Banach spaces and let T be a bounded linear operator from E onto F (onto = surjective). Prove that the image of an open set of E by T is an open set of F.

Solution 15:

Let U be an open set of E. Let $y \in T(U)$. Let $x \in F$ such that $y = T(x)$. Let c such that

$$B(0,c) \subset T(B(0,1)).$$

Since U is open, there exists $\delta > 0$ such that

$$B(x,\delta) \subset U$$

But

$$T(B(x,\delta)) = y + \delta T(B(0,1))$$

We deduce that

$$y + \delta B(0,c) \subset T(B(x,\delta)) \subset T(U),$$

which proves the result.

Exercise 16:

Assume that T is a bijective (injective (one to one) +surjective (onto)) bounded linear operator from a Banach space E to a Banach space F. Then T^{-1} is bounded.

Solution 16:

Let c as in Theorem 28, so that

$$B(0,c) \subset T(B(0,1))$$

Since

$$||\frac{cy}{2||y||}|| < c$$

There exists $x \in B(0,1)$ such that

$$x = T^{-1}(\frac{cy}{2||y||}).$$

Therefore

$$||T^{-1}(\frac{cy}{2||y||})|| < 1$$

from which we deduce that

$$||T^{-1}(y)|| < \frac{2}{c}||y||.$$

Exercise 17:

Let E be a linear space. Assume that $||\cdot||_1$ and $||\cdot||_2$ are two norms on E such that E is Banach for each of the two norms. Assume further that there exists a constant $c > 0$ such that

$$\forall x \in E \;\; ||x||_2 \leq c||x||_1.$$

Then the two norms are equivalent: there exists a constant $d > 0$ such that

$$\forall x \in E \;\; ||x||_1 \leq d||x||_2.$$

Solution 17:

We apply the result of the previous exercise with $F = E$, $T = Id$ from $(E, ||\cdot||_1)$ to $(E, ||\cdot||_2)$.

Theorem 29 (The Closed Graph Theorem)**.** *Let E and F be two Banach spaces. Let T be a linear operator from E into F. Assume that the graph of T, $G(T) = \{(x, Tx); x \in E\}$, is closed in $E \times F$. Then T is bounded.*

Proof

We consider the following norm on E:

$$\forall x \in E,\ ||x||_2 = ||x||_E + ||Tx||_F$$

To prove the theorem, it is sufficient to prove that $(E, ||\cdot||_2)$ is a Banach space. If this is true, then applying the result of the previous exercise we deduce that there exists a constant $d > 0$ such that

$$||x||_E + ||Tx||_F \leq d||x||_E$$

which gives

$$||Tx||_F \leq (d-1)||x||_E.$$

Let us then prove that $(E, ||\cdot||_2)$ is a Banach space. Assume (x_n) is a Cauchy sequence in $(E, ||\cdot||_2)$. Then (x_n, Tx_n) is a Cauchy sequence in $E \times F$. Since E and F are Banach, (x_n, Tx_n) converges to some $(x, y) \in E \times F$. Since G is closed we deduce that $(x, y) \in G(T)$, *i.e* $y = Tx$. It follows that (x_n) converges to x in $(E, ||\cdot||_2)$. □

Exercise 18:

Definition 11. *A family $\{S(t); t \geq 0\}$ of bounded linear operators from a Banach space into itself is called a semigroup if*

$$(i)\ S(0) = I$$

$$(ii)\ S(t+s) = S(t)S(s) \textit{ for each } t, s \geq 0.$$

Definition 12. *Let X be a Banach space. A semigroup $\{S(t); t \geq 0\}$ is called a semigroup of class C_0, or C_0-semigroup if for each $x \in X$ we have*

$$\lim_{t \to 0^+} S(t)x = x$$

Prove that if $\{S(t); t > 0\}$ is a C_0-semigroup, then there exists a constant $M > 1$ and $\omega \in \mathbb{R}$ such that

$$||S(t)|| \leq Me^{\omega t}\ \forall t \geq 0.$$

Solution 18:

We first prove that there exists $M > 1$, $\mu > 0$ such that for

$$\forall t \in [0, \mu]\ ||S(t)|| \leq M$$

where $||.||$ denotes the operator norm. Assume that this is not the case. Then for all $M \geq 1$ and for all $\mu > 0$ there exists $t_{M,\mu} \in (0, \mu]$ such that

$$||S(t_{M,\mu})|| > M.$$

So we can construct a sequence t_n such that for all $n \in \mathbb{N}$

$$0 < t_n < \frac{1}{n} \text{ and } ||S(t_n)|| > n. \tag{3.5}$$

Note that since $(S(t))$ is a C_0-semigroup,

$$\forall x \in X\ \lim_{n \to +\infty} S(t_n)x = x$$

It follows that

$$\forall x \in X \sup_{n\in\mathbb{N}} ||S(t_n)x|| < +\infty.$$

From the uniform boundedness theorem, we deduce that

$$\sup_{n\in\mathbb{N}} ||S(t_n)|| < +\infty,$$

which contradicts eq. (3.5). Next, for $t > 0$, we write $t = n\mu + \delta$, with $\delta \in (0, \mu)$. We have

$$S(t) = S(\delta)S(\mu)^n$$

and therefore

$$\begin{aligned} ||S(t)|| &\leq M^{n+1} \\ &\leq e^{(n+1)\ln M} \\ &\leq e^{(\frac{t-\delta}{\mu}+1)\ln M} \\ &\leq M e^{\frac{t}{\mu}\ln M} \end{aligned}$$

with gives the result with $\omega = \frac{\ln M}{\mu}$.

Chapter 4

Weak and Weak* Topologies

4.1 Topological Spaces, Comparison of topologies and the initial topology

4.1.1 Topological Spaces

Previously, we dealt with metric spaces. Metric spaces are naturally endowed with the metric topology. The aim of this short paragraph is to recall how to define a topological space more generally without a distance.

Definition 13. *A topology on a set X is a set $\mathcal{T}$ of subsets of X, called open sets, that satisfy:*

1. $\emptyset, X \in \mathcal{T}$
2. *If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$*
3. *If for all $i \in I$ $U_i \in \mathcal{T}$ then $\cup_{i\in I} U_i \in \mathcal{T}$.*

Endowed with a topology $\mathcal{T}$ the set $(X, \mathcal{T})$ is called a topological space.

For the purpose of what is coming next, we will now recall the definition of basis of a topology.

Definition 14. *Let $(X, \mathcal{T})$ be a topological space. Then a subset of open sets $\mathcal{B} \subset \mathcal{T}$ is called a basis of open sets of $\mathcal{T}$ if every nonempty open set of X can be written as an union of sets of $\mathcal{B}$:*

$$\forall U \in \mathcal{T}, \exists (U_i)_{i\in I} \subset \mathcal{B},\, U = \cup_{i\in I} U_i.$$

Exercise 19:
Let $(X, \mathcal{T})$ be a topological space.

1. Prove that the two following statements are equivalent
   (a) $\mathcal{B}$ is a basis of $\mathcal{T}$.
   (b)
   $$\forall x \in X,\, \forall U \in \mathcal{T}, x \in U \Rightarrow \exists V \in \mathcal{B}\, s.t.\, x \in V \subset U.$$
2. We assume that $\mathcal{B}$ is a basis of $\mathcal{T}$. Prove that $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $V \in \mathcal{B}$ such that $x \in V \subset U$.

Solution 19:

1. Assume that $\mathcal{B}$ is a basis of $\mathcal{T}$. Let $x \in X$, $U \in \mathcal{T}$ such that $x \in U$. Since $\mathcal{B}$ is a basis of $\mathcal{T}$, we write $U = \cup_{i\in I} V_i$ with $\forall i \in I, V_i \in \mathcal{B}$. It follows that there exists $i \in I$ such that $x \in V_i$ which proves 1-b. Conversely, let $U \in \mathcal{T}$. Then we can write $U = \cup_{x\in U}\{x\} \subset \cup_{x\in U}\{V_x\} \subset U$ where for all x, $V_x \in \mathcal{B}$ and $x \in V_x \subset U$, which proves that $\mathcal{B}$ is a basis of $\mathcal{T}$.

2. Assume that $\mathcal{B}$ is a basis of $\mathcal{B}$ and let $U \in \mathcal{T}$. Then the conclusion follows from 1-b. Conversely, let $U \in \mathcal{T}$ for which for all $x \in U$ there exists $V \in \mathcal{B}$ such that $x \in V \subset U$. Then we can write $U = \cup_{x\in U}\{x\} \subset \cup_{x\in U}\{V_x\} \subset U$ where for all x, $V_x \in \mathcal{B}$ and $x \in V_x \subset U$. Therefore U is an open set as a union of open sets.

Remark 30. *For a metric space, the open balls are a basis of the metric topology.*

4.1.2 Comparison of topologies

Definition 15. *Let E be a set, and $\mathcal{T}_1$, $\mathcal{T}_2$ two topologies on E. We say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ (or weaker) if $\mathcal{T}_1 \subset \mathcal{T}_2$. Alternatively, $\mathcal{T}_2$ is said to be finer (or stronger) than $\mathcal{T}_1$.*

Next, we will see how to define a topology upon a given basis of open sets. Assume that $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ a basis of $\mathcal{T}$, then

$$\forall x \in X\, \exists U \in \mathcal{B}\, s.t.\, x \in U \tag{4.1}$$

and

$$\forall U_1, U_2 \in \mathcal{B}\, \forall x \in U_1 \cap U_2\, \exists U \in \mathcal{B}\, s.t.\, x \in U \subset U_1 \cap U_2. \tag{4.2}$$

Conversely if $\mathcal{B} \subset \mathcal{P}(X)$ satisfies eq. (4.1) and eq. (4.2), there exists a unique topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a basis.

Exercise 20:
Prove the previous statement.

Solution 20:
Assume that $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ a basis of $\mathcal{T}$. Since X is open, eq. (4.1) holds. eq. (4.2) comes from the fact that $U_1 \cap U_2$ is an open set. Conversely if $\mathcal{B} \subset \mathcal{P}(X)$ satisfies eq. (4.1) and eq. (4.2), we define $\mathcal{T}$ as the set of sets which write as an union of elements of $\mathcal{B}$ plus the empty set. Then one can check that $\mathcal{T}$ is a topology. In particular, if $U_1 \in \mathcal{T}$ and $U_2 \in \mathcal{T}$ then

$$\begin{aligned} U_1 \cap U_2 &= \big(\cup_{i\in I} U_{1i}\big) \cap \big(\cup_{j\in J} U_{2j}\big) \\ &= \cup_{i\in I, j\in J} U_{1i} \cap U_{2j} \\ &= \cup_{i\in I, j\in J} \cup_{x\in U_{1i}\cap U_{2j}} V_x \end{aligned}$$

where $x \in V_x \subset U_{1i} \cap U_{2j}$ and $V_x \in \mathcal{B}$. The other verifications are left to the reader.

4.1.3 The initial topology

Let E be a set and $(F_i, \mathcal{T}_i)_{i\in I}$, a family of topological spaces, and for all $i \in I$ let φ_i an application from E to F_i. Let $\mathcal{B}$ the family of sets defined as finite intersection of the sets $\varphi_i^{-1}(O_j^i)$ with $i \in I$ and O_j^i open set of F_i. Then $\mathcal{B}$ satisfies the assumptions eq. (4.1) and eq. (4.2). Indeed, for any $i \in I$, $\varphi_i^{-1}(F_i) = E \in \mathcal{B}$, which shows that eq. (4.1) holds. Next, let $U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$. By definition, we can write U_1, U_2 as finite intersections: $U_1 = \cap_{i\in I_1, j\in J_1^i} \varphi_i^{-1}(O_j^i)$, $U_2 = \cap_{i\in I_2, j\in J_2^i} \varphi_i^{-1}(O_j^i)$. It follows that $U_1 \cap U_2$ writes also as a finite intersection of the required form, and therefore $U_1 \cap U_2 \in \mathcal{B}$. It follows that eq. (4.2) is also satisfied. The initial topology is now defined as the unique topology for which $\mathcal{B}$ is a basis.

Definition 16. *The initial topology on E is the unique topology for which $\mathcal{B}$ as defined as above is a basis.*

By construction, we have that,

Proposition 31. *The initial topology is the coarsest topology that contains the sets*

$$\big(\varphi_i^{-1}(\omega)\big)_{i\in I, \omega\in\mathcal{T}_i}$$

4.2 The Weak Topology

Definition 17. *The weak topology $\sigma(E, E')$ on E is the initial topology associated with the linear functionals $f \in E'$.*

Exercise 21:

Show that the topology associated with the usual norm $||\cdot||$ on E is stronger than the weak topology $\sigma(E, E')$.

Solution 21:

Since all linear functional φ is continuous from $(E, ||\cdot||)$ into $\mathbb{K}$, $\varphi_i^{-1}(\omega)$ is an open set of $(E, ||\cdot||)$ for all open set $\omega \in \mathbb{K}$. Then, the result follows from Proposition 31.

Definition 18. *We say that a topology $\mathcal{T}$ defined on a space X is Hausdorff if for any $x, y \in X$ with $x \neq y$ there exists two open sets $U, V \in \mathcal{T}$ with $x \in U$, $y \in V$ such that $U \cap V = \emptyset$.*

Exercise 22:

We assume that $\mathbb{K} = \mathbb{R}$. Show that the weak topology $\sigma(E, E')$ is Hausdorff.

Solution 22:

Let $x, y \in E$ with $x \neq y$. We look for two open sets U, V of $\sigma(E, E')$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Let us define φ on $\mathrm{span}(y - x)$ by

$$\varphi(y - x) = 1.$$

Then φ can be extended to a bounded linear function on E, see Proposition 23. Note then that

$$\begin{aligned}\varphi(y) &= \varphi(x) + \varphi(y - x)\\ &= \varphi(x) + 1.\end{aligned}$$

Next, let

$$\begin{aligned}U &= \{z \in E; \varphi(z) < \varphi(x) + 1/2\} = \varphi^{-1}((-\infty, \varphi(x) + 1/2))\\ V &= \{z \in E; \varphi(z) > \varphi(x) + 1/2\} = \varphi^{-1}((\varphi(x) + 1/2, +\infty.)\end{aligned}$$

Then U, V are two open sets of $\sigma(E, E')$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Exercise 23:

We assume that $\mathbb{K} = \mathbb{C}$. Show that the weak topology $\sigma(E, E')$ is Hausdorff.

Solution 23:

Let $x, y \in E$ with $x \neq y$. We look for two open sets U, V of $\sigma(E, E')$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Let us define φ on $\mathrm{span}(y - x)$ by

$$\varphi(y - x) = 1.$$

Then φ can be extended to a bounded linear function on E, see Proposition 23. Note then that,

$$|\varphi(x) - \varphi(y)| = 1$$

$$\begin{aligned}U &= \{z \in E; \varphi(z) \in B(\varphi(x), 1/2)\} = \varphi^{-1}(B(\varphi(x), 1/2))\\ V &= \{z \in E; \varphi(z) \in B(\varphi(y), 1/2)\} = \varphi^{-1}(B(\varphi(y), 1/2))\end{aligned}$$

Then U, V are two open sets of $\sigma(E, E')$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 19. *Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. We say that $V \subset X$ is a neighborhood of x if there exists $U \in \mathcal{T}$ such that $x \in U \subset V$.*

Definition 20. *Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. We say that $\mathcal{B}$ is a basis of neighborhoods of x if for any neighborhood V of x we can find a neighborhood U of x $U \in \mathcal{B}$ such that $U \subset V$.*

Exercise 24:

Let E be a Banach space. Show that

$$\{z \in E; |\varphi_i(z) - \varphi_i(x)| < \epsilon,\, i \in \{0, ..., n\}, \varphi_0, \varphi_1, ..., \varphi_n \in E', n \in \mathbb{N}, \epsilon > 0\}$$

defines a basis of neighborhoods of x for the weak topology.

Solution 24:

Let $x \in E$, and V a neighborhood of x. By definition of a neighborhood, V contains an open set which contains x. By definition of the weak topology there exists a finite number of applications $(\varphi_i)_{i \in I}$ and a finite number of open sets in $\mathbb{K}$, $(O_j)_{j \in J_i}$ such that

$$x \in \cap_{i \in I, j \in J_i} \varphi_i^{-1}(O_j).$$

Therefore for all $i \in I, j \in J$,

$$\varphi_i(x) \in O_j$$

Since O_j is open in $\mathbb{K}$, there exists ϵ_{ij} such that

$$B(\varphi_i(x), \epsilon_{ij}) \in O_j$$

Let $\epsilon = \min\{\epsilon_{ij}\}$, then

$$x \in \bigcap_{i \in I, j \in J_i} \varphi_i^{-1}(B(\varphi_i(x), \epsilon)).$$

When a sequence (x_n) converges weakly *i.e* in the weak topology $\sigma(E, E')$ toward x we will write

$$x_n \rightharpoonup x.$$

To say that a sequence (x_n) converges strongly means the convergence in the usual norm:

$$||x_n - x|| \to 0.$$

Proposition 32. *A sequence (x_n) converges weakly toward $x \in E$ in the $\sigma(E, E')$ topology if and only if for every $\varphi \in E'$, $(\varphi(x_n))$ converges toward $\varphi(x)$.*

Proof

Assume that (x_n) converges weakly toward x in the $\sigma(E, E')$ topology. Let U be an open set containing $\varphi(x)$ in $\mathbb{K}$. Then $\varphi^{-1}(U)$ is an open set of $\sigma(E, E')$ which contains x. Therefore, there exists $N \in \mathbb{N}$ such that $n > N$ implies $x_n \in \varphi^{-1}(U)$ which in turns implies that $\varphi(x_n) \in U$.

Conversely, assume that for every $\varphi \in E'$, $(\varphi(x_n))$ converges toward $\varphi(x)$. Let $U \in \sigma(E, E')$ an open set containing x. Then by definition there exists a finite number of $\varphi_i \in E'$ and open sets in $\mathbb{K}$, V_i, $i \in \{1, ..., p\}$, such that

$$x \in \bigcap_{i=1}^{p} \varphi_i^{-1}(V_i) \subset U.$$

For all $i \in \{1, ..., p\}$, let N_i be such that

$$n > N_i \Rightarrow \varphi_i(x_n) \in V_i.$$

Let $N = \max_i N_i$. Then

$$n > N \Rightarrow \forall i \in \{1, ..., p\}\, \varphi_i(x_n) \in V_i.$$

which in turn implies that

$$x_n \in \bigcap_{i=1}^{p} \varphi_i^{-1}(V_i) \subset U.$$

□

Proposition 33. *Assume that (x_n) converges strongly toward x in E, then it converges weakly in the $\sigma(E,E')$ topology.*

Proof

This follows from the continuity of φ for the strong topology. More precisely, let $\varphi \in E'$, then

$$|\varphi(x_n) - \varphi(x)| \leq ||\varphi|| ||x_n - x||$$

which proves the result. Another way to express it is as follows. Consider an open set U of $(E, \sigma(E,E'))$ containing x. Then it is also an open set in $(X, ||\cdot||)$ since $\sigma(E,E')$ is coarser than the norm topology. It follows that

$$\exists N\, s.t.\, n > N \Rightarrow x_n \in U.$$

which proves the result.□

Proposition 34. *Assume that a sequence (x_n) converges weakly toward $x \in E$ in the $\sigma(E,E')$. Then (x_n) is bounded for the usual norm in E and*

$$||x|| \leq \liminf_{n\to+\infty} ||x_n||.$$

Proof

Since

$$x_n \rightharpoonup x \text{ in } \sigma(E,E')$$

it follows that $\forall \varphi \in E'$ $\varphi(x_n) \to \varphi(x)$ which in turn implies

$$\forall \varphi \in E' \sup_{n\in\mathbb{N}} |\varphi(x_n)| < c_\varphi$$

for some positive constant c_φ. Defining the bounded linear functional g_{x_n} from E' to $\mathbb{K}$ as $g_{x_n}(\varphi) = \varphi(x_n)$, we deduce from the uniform boundedness principle that

$$\sup_{n\in\mathbb{N}} ||g_{x_n}||_{\mathcal{L}(E',\mathbb{K})} = \sup_{n\in\mathbb{N}} \sup_{||\varphi||\leq 1} ||\varphi(x_n)|| \leq C$$

for some constant C. Now since, from Proposition 25

$$\sup_{||\varphi||\leq 1} ||\varphi(x_n)|| = ||x_n||$$

it follows that

$$\sup_{n\in\mathbb{N}} ||x_n|| \leq C,$$

which proves that (x_n) is bounded in the usual norm in E. Next,

$$|\varphi(x_n)| \leq ||\varphi|| ||x_n||$$

which implies that for every φ such that $||\varphi|| \leq 1$

$$|\varphi(x_n)| \leq ||x_n||$$

Taking the lim inf in both sides, we obtain that for every φ such that $||\varphi|| \leq 1$

$$|\varphi(x)| \leq \liminf ||x_n||$$

taking the sup in the left hand side of this inequality and using the fact that

$$\sup_{||\varphi||\leq 1} |\varphi(x)| = ||x||$$

we obtain that

$$||x|| \leq \liminf_{n\to+\infty} ||x_n||$$

□

Exercise 25:

Prove that if (x_n) converges toward x in $\sigma(E,E')$ and (φ_n) converges toward φ in E', then $\varphi_n(x_n)$ converges toward $\varphi(x)$ in $\mathbb{K}$.

Solution 25:

$$\begin{aligned}|\varphi(x)-\varphi_n(x_n)| &= |\varphi(x)-\varphi(x_n)+\varphi(x_n)-\varphi_n(x_n)| \\ &\leq |\varphi(x)-\varphi(x_n)|+|\varphi(x_n)-\varphi_n(x_n)| \\ &\leq |\varphi(x)-\varphi(x_n)|+||\varphi-\varphi_n||||x_n||.\end{aligned}$$

Proposition 35. *When E is of finite dimension the weak topology $\sigma(E,E')$ and the usual topology on E are the same.*

Proof

Let $\mathcal{T}$ denote the usual topology on E. We already know that

$$U \in \sigma(E,E') \Rightarrow U \in \mathcal{T}.$$

Conversely assume that $U \in \mathcal{T}$. We want to prove that $U \in \sigma(E,E')$. Let $x \in U$, we want to prove that there exists an open set $V \in \sigma(E,E')$ such that $x \in V \subset U$. Let $r > 0$ such that $B(x,r) \subset U$. Let $e_1, e_2, ..., e_n$ be a basis of E, with $\forall i \in \{1,..,n\}$, $||e_i|| = 1$. We define φ_i from E to $\mathbb{K}$ as $\varphi_i(x) = x_i$ where $x = \sum_{i=1}^n x_i e_i$. Without loss of generality, we assume that $||x|| = \sum_{i=1}^n |x_i|$. Note that $\varphi_i \in E'$. Consider the open set $V \in \sigma(E,E')$ defined as

$$V = \bigcap_{i=1}^n \varphi_i^{-1}(B(x_i,\epsilon)).$$

Note that $x \in V$. Furthermore, for $y \in V$,

$$||y-x|| = \sum_{i=1}^n |y_i - x_i| < n\epsilon.$$

Choosing $\epsilon < \frac{r}{n}$ provides the result. □

The proposition proposition 35 is only valid in finite dimensions.

Exercise 26:

Prove that if E is infinite dimensional, the unit sphere

$$S = \{x \in E; ||x|| = 1\}$$

is not a closed set for the weak topology $\sigma(E,E')$.

Solution 26:

We shall prove in fact that the closure of S in $\sigma(E,E')$ denoted by $\bar{S}$ is equal to the closed unit ball $B_c = \{x \in E; ||x|| \leq 1\}$. We first start to prove that the unit open ball $B(0,1) \subset \bar{S}$ which proves that S is not closed since $S \cap B(0,1) = \emptyset$ which implies that $S \neq \bar{S}$. Let $x_0 \in B(0,1)$. Let $U \in \sigma(E,E')$ with $x_0 \in U$. We want to prove that $U \cap S \neq \emptyset$. Since $U \in \sigma(E,E')$ and $x_0 \in U$, there exists $\varphi_1, ..., \varphi_n \in E'$, and $\epsilon > 0$ such that

$$\bigcap_{i=1}^n \varphi_i^{-1}(B(\varphi_i(x_0),\epsilon)) \subset U.$$

Let $y \in E$ such that $\forall i \in \{1,...,n\}$, $\varphi_i(y) = 0$. Such a y exists because E is infinite dimensional. If this was not the case one could define a bijective application $\varphi = (\varphi_1, ..., \varphi_n)$ from E into $\mathbb{K}^n$ which would imply that E is of dimension $n < \infty$. Now, consider $g(t) = ||x_0 + ty||$. We have that $g(0) < 1$ and $\lim_{t \to +\infty} g(t) = +\infty$. Since g is continuous, there exists t_0 such that $g(t_0) = 1$. This implies that $x_0 + t_0 y \in S$. Note also that $\forall i \in \{1,...,n\}$, $\varphi_i(x_0 + t_0 y) = \varphi_i(x_0)$, therefore $x_0 + t_0 y \in U$. Finally, $U \cap S \neq \emptyset$ and therefore S is not a closed set for $\sigma(E, E')$. Finally, note that

$$B_c = \cap_{\varphi \in E', ||\varphi|| \leq 1} \varphi^{-1}(B_c(0,1))$$

is closed in $\sigma(E, E')$ as an intersection of closed sets. This means that $S \subset B_c \subset \bar{S}$ which implies $B_c = \bar{S}$.

Exercise 27:
Prove that if E is infinite dimensional, the unit ball

$$B = \{x \in E; ||x|| < 1\}$$

is not an open set for the weak topology $\sigma(E, E')$.

Solution 27:
Proceed by contraction and remark that

$$S = B_c(0,1) \cap (B(0,1))^c.$$

4.3 The Weak* Topology

We are now interested in the topologies defined on E'. First, we can define the usual (strong) topology associated to the dual norm on E'. We can also define the weak topology $\sigma(E', E'')$, by analogy with the construction of $\sigma(E, E')$ as the initial topology associated with the elements of E''. But we are now going to define a third topology on E' called the weak* topology and denoted by $\sigma(E', E)$. For every $x \in E$, consider the linear functional $\varphi_x : E' \to \mathbb{K}$ defined by

$$\varphi \to \varphi_x(\varphi) = \varphi(x)$$

As x runs through E, we obtain a collection $(\varphi_x)_{x \in E}$ of maps from E' into $\mathbb{K}$.

Definition 21. *The weak* topology* $\sigma(E', E)$ *on* E' *is the initial topology associated with the linear functionals* $(\varphi_x)_{x \in E}$.

Exercise 28:
Show that the weak* topology $\sigma(E', E)$ is Hausdorff.

Solution 28:

Let $\varphi_1, \varphi_2 \in E'$ with $\varphi_1 \neq \varphi_2$. Then there exists $x \in E$ such that $\varphi_1(x) \neq \varphi_2(x)$. Since K is Hausdorff, there exits two open balls, $B_1(\varphi_1(x), \epsilon_1)$ and $B_2(\varphi_2(x), \epsilon_2)$ such that $B_1 \cap B_2 = \emptyset$. Let $U_1 = \varphi_x^{-1}(B_1) = \{\varphi \in E'; \varphi(x) \in B_1\}$, and $U_2 = \varphi_x^{-1}(B_2) = \{\varphi \in E'; \varphi(x) \in B_2\}$. Then $U_1, U_2 \in \sigma(E', E)$, $\varphi_1 \in U_1$, $\varphi_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$

Exercise 29:
Show that

$$\{\varphi \in E'; |\varphi(x_i) - \varphi_0(x_i)| < \epsilon, \, i \in \{0,...,n\}, \, x_0, x_1, ..., x_n \in E, n \in \mathbb{N}, \epsilon > 0\}$$

defines a basis of neighborhoods of φ_0.

Solution 29:

Proceed as in the analogous result for $\sigma(E, E')$.

When a sequence (φ_n) converges weakly $i.e$ in the weak* topology $\sigma(E', E)$ toward φ we will write

$$\varphi_n \overset{*}{\rightharpoonup} \varphi.$$

The few next propositions are analogous as those in the previous section. Their proof are left as an exercise

Proposition 36. *A sequence (φ_n) converges weakly toward $\varphi \in E'$ in the $\sigma(E', E)$ topology if and only if for every $x \in E$, $(\varphi_n(x))$ converges toward $\varphi(x)$.*

Proposition 37. *Assume that (φ_n) converges strongly toward φ in E', then it converges in $\sigma(E', E'')$ and in the $\sigma(E', E)$ topology.*

Proposition 38. *Assume that a sequence (φ_n) converges weakly toward φ in the $\sigma(E', E)$ topology. Then (φ_n) is bounded for the norm topology in E' and*

$$||\varphi|| \leq \liminf n \to +\infty ||\varphi_n||$$

Exercise 30:

Prove that if (φ_n) converges toward φ in $\sigma(E', E)$ and (x_n) converges toward x strongly in E, then $\varphi_n(x_n)$ converges toward $\varphi(x)$ in $\mathbb{K}$.

Solution 30:

Proceed as in the analogous result for $\sigma(E, E')$.

Exercise 31:

Let $g : E' \to \mathbb{K}$ be a linear functional that is continuous for the weak* topology. Then there exists some $x_0 \in E$ such that

$$g(\varphi) = \varphi(x_0) \forall \varphi \in E'$$

Solution 31:

Left to the reader.

Exercise 32:

Assume that H is a hyperplane in E' that is closed in $\sigma(E', E)$. Then H has the form

$$H = \{\varphi \in E'; \varphi(x_0) = \alpha\}$$

for some $x_0 \in E$, and some $\alpha \in \mathbb{K}$.

Solution 32:

Left to the reader.

The product topology

Let $(E_i, \tau_i)_{i\in I}$, be a family of topological spaces. We consider the cartesian product space

$$E = \prod_{i\in I} E_i$$

which means that an element $x \in E$ writes as

$$x = (x_i)_{i\in I}$$

where for each $i \in I$, $x_i \in E_i$. Next, for each $i \in I$, we note

$$p_i : E \to E_i$$

the canonical projection from E into E_i, which at $x = (x_j)_{j\in I}$ associates x_i.

Definition 22. *The initial topology associated with the family of projections $(p_i)_{i\in I}$ is called the product topology on E.*

Remark that for each open set U_i of E_i,

$$p_i^{-1}(U_i) = \prod_{j\in I} U_j$$

with $U_j = U_i$ if $j = i$ and $U_j = X_j$ if $j \neq i$. Therefore a finite intersection of open sets $p_i^{-1}(U_i)$, with U_i an open set of E_i, is an open set of E which write

$$\prod_{j\in I} U_j$$

with U_i open set of E_i for all $i \in I$ and $U_i = E_i$ for all but at most a finite number of indexes. Recall that those open sets are a basis for the product topology in E.

In the case where $F_i = F$ for all $i \in I$ we have

$$E = \prod_{i\in I} F = F^I$$

and the product topology is also called the topology of point wise convergence since it corresponds to the point wise convergence for functions.

Theorem 39. *The closed unit ball*

$$B_{E'} = \{\varphi \in E'; ||\varphi|| \leq 1\}$$

is compact in the weak topology $\sigma(E', E)$.*

Proof

The proof goes as follows. We first prove that $B_{E'}$ is compact in $\mathbb{K}^E$ endowed with the product topology. Then we remark that the topology $\sigma^*(E', E)$ is the induced topology by the product topology on E'.

$B_{E'}$ is compact in $\mathbb{K}^E$ endowed with the product topology

First note that

$$B_{E'} = \{\varphi \in E'; |\varphi(x)| \leq 1, \text{ for } ||x||| \leq 1\}$$
$$= \{\varphi \in \mathbb{K}^E; \forall x, y \in E, \forall \lambda \in \mathbb{C} |\varphi(x)| \leq ||x||, \varphi(\lambda x) = \lambda x,\ \varphi(x+y) = \varphi(x) + \varphi(y)\}$$

Next, notice that

$$\{\varphi \in \mathbb{K}^E; \forall x \in E, \forall \lambda \in \mathbb{K} |\varphi(x)| \leq ||x||\}$$

is product of compact sets which is compact by Tichonov's Theorem. To conclude, we prove that

$$\{\varphi \in \mathbb{K}^E; \forall x, y \in E, \forall \lambda \in \mathbb{K} \varphi(\lambda x) = \lambda\varphi(x),\ \varphi(x+y) = \varphi(x) + \varphi(y)\}$$

is a closed set. Indeed, for fixed $x \in E$ and $\lambda \in X$,

$$\Big(\{\varphi \in \mathbb{K}^E; \varphi(\lambda x) = \lambda\varphi(x)\}\Big)^c$$

$$= \{\varphi \in \mathbb{K}^E; \varphi(\lambda x) \neq \lambda\varphi(x)\}$$

which is an open set of the product topology as an union of elementary open sets $(U_y)_{y\in E}$ with $U_y = \mathbb{K}$ if $y \notin \{x, \lambda x\}$, and $U_x \times U_{\lambda x} = \mathbb{K}^2 \setminus \{(z, \lambda z)_{z\in\mathbb{K}}\}$ ($\mathbb{K}^2 \setminus \{(z, \lambda z)_{z\in\mathbb{K}}\}$ writes as union of products of balls $B_x \times B_{\lambda x}$). Taking the intersection over x and λ gives the result. An analog argument holds for $\varphi(x + y) = \varphi(x) + \varphi(y)$.

$B_{E'}$ is compact in E' endowed with the topology $\sigma(E', E)$

We remark that the topology $\sigma(E', E)$ is the induced topology by the product topology in E'. Let U be an open set of (E', σ^*) and $\varphi \in U$. Then, there exists $x_1, ..., x_n \in E$ and $V_1, ..., V_k$ open sets of $\mathbb{K}$ such that

$$\varphi \in \bigcap_{i\in\{1,..,n\}, l\in\{1,...,k\}} g_{x_i}^{-1}(V_l)$$

which writes as a finite intersection of sets of the form

$$E' \cap \prod_{y\in E} W_y$$

with $W_y = \mathbb{K}$ for all $y \in E$ except if $y = x_i$ for some i in which case $W_{x_i} = V_k$ for some k. This proves that $\sigma^*(E', E)$ is the topology induced by the product topology on E'.□

4.4 Reflexive spaces

Definition 23. *Let E be a Banach space and J the canonical injection from E into E'' which at each $x \in E$ associates $\varphi_x \in E''$ by $\varphi_x(\varphi) = \varphi(x)$. We say that E is reflexive if $J(E) = E''$.*

Remark 40. *We recall that J is well define because:*

$$|\varphi_x(\varphi)| = |\varphi(x)| \leq ||x||||\varphi||.$$

So E is reflexive means that J is onto.

Remark 41. *Examples of reflexive spaces are L^p and l^p for $1 < p < \infty$. Note however that $L^1, L^\infty, l^1, L^\infty$ are not reflexive. We shall see in the next chapter that Hilbert spaces are reflexive.*

Theorem 42. *A Banach space E is reflexive if and only if the closed ball $B_c = \{x \in E; ||x|| \leq 1\}$ is compact for the weak topology $\sigma(E, E')$.*

Proof (Necessity)
We assume that E is reflexive, and because $||\varphi_x|| = ||x||$ we have that $J(B_c) = \{\varphi_x; ||x|| \leq 1\}$. Furthermore, from Theorem 39, we know that $J(B_c)$ is compact for the topology $\sigma(E'', E')$. To prove the result, it is therefore sufficient to prove that J^{-1} is continuous from $(E'', \sigma(E'', E'))$ into $(E, \sigma(E, E'))$. This is equivalent to prove that for all $\varphi \in E'$, $\varphi \circ J^{-1}$ is continuous $(E'', \sigma(E'', E'))$ into $\mathbb{K}$. Now, note that for every $\varphi_x \in E''$,

$$\varphi \circ J^{-1}(\varphi_x) = \varphi(x).$$

But by definition, $(\sigma(E'', E')$ makes φ continuous from $(E'', \sigma(E'', E'))$ into $\mathbb{K}$. □

The proof of the sufficiency of Theorem 42 relies on two lemmas.

Lemma 3 (Helly)**.** *Let E be a Banach space, $\varphi_1, ..., \varphi_n \in E'$ and $\gamma_1, ..., \gamma_n \in \mathbb{K}$. The following properties are equivalent.*

1. *$\forall\varepsilon > 0$, $\exists x_\epsilon$ such that $||x_\epsilon|| \leq 1$ and*

$$\forall i \in \{1, ..., n\}\, |\varphi_i(x_\epsilon) - \gamma_i| < \epsilon.$$

2. $\forall \beta_1, ..., \beta_n \in \mathbb{K}$,

$$|\sum_{i=1}^{n} \beta_i \gamma_i| \leq ||\sum_{i=1}^{n} \beta_i \varphi_i||$$

Proof
$1 \Rightarrow 2$
Let $\epsilon > 0$ and x_ϵ such as in 1. Then,

$$|\sum_{i=1}^{n} \beta_i \gamma_i| = |\sum_{i=1}^{n} \beta_i (\gamma_i - \varphi_i(x_\epsilon)) + \sum_{i=1}^{n} \beta_i \varphi_i(x_\epsilon|$$

$$\leq \epsilon \sum_{i=1}^{n} |\beta_i| + ||\sum_{i=1}^{n} \beta_i \varphi_i||.$$

Taking the limit as ϵ goes to 0 provides the result.
$2 \Rightarrow 1$
Let $\gamma = (\gamma_1, ..., \gamma_n)$ and for $x \in E$, $\phi(x) = (\varphi_1(x), ..., \varphi_n(x))$. Property 1 says that for all $\epsilon > 0$ there exists x_ϵ such that $||x_\epsilon|| \leq 1$ and

$$||\phi(x_\epsilon) - \gamma||_\infty < \epsilon,$$

where $|| \cdot ||_\infty$ refers to the classical max norm in $\mathbb{K}^n$. In other words, this means that $\gamma \in \overline{\Phi(B_c(0,1))}$ with respect to the closure in $(\mathbb{K}^n, || \cdot ||_\infty$. We proceed now by contradiction. Assume that

$$\gamma \notin \overline{\phi(B_c(0,1))}.$$

Then we can find $\beta_1, ...\beta_n \in \mathbb{K}$ such that $\forall x \in B_c(0,1)$

$$|\sum_{i=1}^{n} \beta_i \varphi_i(x)| < \alpha < |\sum_{i=1}^{n} \beta_i \gamma_i|$$

(why? Exhibit explicitly $\beta_1, ..., \beta_n$). Taking the sup over x in the left hand side leads to a contradiction. □

Lemma 4 (Goldstine)**.** *Let E be a Banach space. Then $J(E)$ is dense in $(E'', \sigma(E'', E'))$.*

Proof
We are going to prove that $J(B_c)$ is dense in $B_{c,E''} = \{g \in E''; ||g|| \leq 1\}$ for the topology $\sigma(E'', E')$. Let $g \in B_{c,E''}$ and $U \in \sigma(E'', E')$ with $g \in U$. Then there exists $\varphi_1, ..., \varphi_n \in E'$ and ϵ such that

$$\{\eta \in E''; \forall i \in \{1, ..., n\}\, |\varphi_i(g) - \varphi_i(\eta)| < \epsilon\} \subset U.$$

We look for $x \in E$ such that

$$\forall i \in \{1, ..., n\}\, |\varphi_i(g) - \varphi_i(x)| < \epsilon$$

Thanks to lemma 3, it is sufficient to prove that $\forall \beta_1, ..., \beta_n \in \mathbb{K}$,

$$|\sum_{i=1}^{n} \beta_i \gamma_i| \leq ||\sum_{i=1}^{n} \beta_i \varphi_i||$$

with $\gamma_i = \varphi_i(g)$. But this is true because

$$|\sum_{i=1}^{n} \beta_i \varphi_i(g) \leq ||\sum_{i=1}^{n} \beta_i \varphi_i||||g||||\sum_{i=1}^{n} \beta_i \varphi_i||.$$

Proof(end of theorem 42)
The canonical injection J from $(E, \sigma(E, E'))$ into $(E'', \sigma(E'', E'))$ is continuous, since for $x \in E$ and $\varphi \in E'$, $J(x)(\varphi) = \varphi(x)$ and φ is continuous from $(E, \sigma(E, E'))$ to $\mathbb{K}$. Assuming that B_c is compact in $(E, \sigma(E, E'))$, it follows that $J(B_c)$ is compact in $(E'', \sigma(E'', E'))$, and therefore closed. This ends the proof since $\overline{J(B_c)} = B_{c,E''}$.

4.5 Separable spaces

Definition 24. *Let E be a Banach space. We say that E is separable if there exists a countable subspace of E which is dense in E.*

Theorem 43. *Let E be a separable Banach space. Then the closed unit ball in E' is metrizable in the weak* topology (E', E). Conversely, if the closed unit ball in E' is metrizable in the weak* topology (E', E), then E is separable.*

Proof
We assume that E is separable. We assume that $(a_n)_{n\in\mathbb{N}}$ is dense in the unit ball of E. For all $_1, \varphi_2 \in E'$, we set

$$|\varphi_1 - \varphi_2| = \sum_{n=0}^{+\infty} \frac{1}{2^n} |\varphi_1(a_n) - \varphi_2(a_n)|$$

This defines a norm on E'. Assume that U is an open in $\sigma(E', E)$. We shall prove that $U \cap B_{c,E'}$ is an open set for the topology induced by $|\cdot|$. Let $\varphi_0 \in U \cap B_{c,E'}$. Since U is an open set in $\sigma(E', E)$, there exists $x_1, ..., x_k$ and $\epsilon > 0$ such that

$$\{\varphi \in E'; |\varphi_(x_i) - \varphi_0(x_i)| < \epsilon, \, i \in \{1, ..., k\}\} \subset U.$$

We want to prove that there exists $\mu > 0$, such that $|\varphi - \varphi_0 < \mu$ and $\varphi \in B_{c,E'}$ implies $\varphi \in U \cap B_{c,E'}$. Now,

$$\begin{aligned} |\varphi(x_i) - \varphi_0(x_i)| &\leq |\varphi(x_i) - \varphi_(a_{n_i})| + |\varphi_(a_{n_i}) - \varphi_0(a_{n_i})| + |\varphi_0(a_{n_i}) - \varphi_0(x_i)| \\ &\leq 2|x_i - a_{n_i}| + 2^{n_i}|\varphi - \varphi_0| \\ &< \epsilon, \end{aligned}$$

where the $n_i's$ were chosen such that $||x_i - a_{n_i}|| < \frac{\epsilon}{4}$ and μ such that $|\varphi - \varphi_0| < \min_{i\in\{1,...,k\}} \frac{\epsilon}{2^{n_i+1}}$. This proves that every open set induced by $\sigma(E', E)$ in the ball $B_{c,E'}$ is an open set in the topology induced by the norm $|\cdot|$ on $B_{c,E'}$.

Next, we want to prove that any open set in the topology induced by the norm $|\cdot|$ on $B_{c,E'}$ is an open set induced by $\sigma(E', E)$ on $B_{c,E'}$. Let U be an open set of E' for $|\cdot|$. Let $\varphi_0 \in B_{c,E'} \cap U$. Let μ such that

$$\{\varphi \in E'; |\varphi - \varphi_0| < \mu\} \subset U.$$

We want to prove that there exists $\epsilon > 0$ and some $x_1, ..., x_k \in E$ such that

$$\left(B_{c,E'} \cap \{\varphi \in E'; |\varphi(x_i) - \varphi_0(x_i)| < \epsilon, \, i \in \{1, ..., k\}\}\right) \subset \left(B_{c,E'} \cap \{\varphi \in E'; |\varphi - \varphi_0| < \mu\}\right)$$

Recall that

$$|\varphi - \varphi_0| = \sum_{n=0}^{+\infty} \frac{1}{2^n} |\varphi(a_n) - \varphi_0(a_n)|.$$

Since

$$|\varphi(a_n) - \varphi_0(a_n)| \leq ||\varphi - \varphi_0||_{E'} ||a_n|| \leq 2,$$

we deduce that

$$\frac{1}{2^n} |\varphi(a_n) - \varphi_0(a_n)| \leq \frac{1}{2^{n-1}}$$

Let n_0 such that

$$\sum_{n=n_0+1}^{+\infty} \frac{1}{2^{n-1}} < \frac{\mu}{2}.$$

Then it is sufficient to ensure that

$$|\varphi(a_n) - \varphi_0(a_n)| < (2 - \frac{1}{2^{n_0+1}}) \frac{\mu}{2} \; \forall n \in \{0, ..., n_0\}$$

to obtain the result.

Chapter 5

Hilbert Spaces

In this chapter we work with $\mathbb{K} = \mathbb{R}$. The last section discusses however the case $\mathbb{K} = \mathbb{C}$.

5.1 Scalar Product and Hilbert Spaces. Projection on a Closed Convex Set

Definition 25. *A (real) scalar product on a linear space H over $\mathbb{R}$ is a real valued function $(\cdot, \cdot) : H \times H \to \mathbb{R}$ which satisifies*

- *Bilinearity: $u \to (u, v)$ and $v \to (u, v)$ are linear.*
- *Symmetry : $(u, v) = (v, u)$.*
- *Positivity : $(u, u) > 0$ whenever $u \neq 0$.*

Exercise 33:
Prove the Cauchy-Schwarz inequality:

$$|(u, v)| \leq (u, u)^{\frac{1}{2}} (v, v)^{\frac{1}{2}}$$

Solution 33:
We consider,

$$P(\lambda) = (u + \lambda v, u + \lambda v).$$

Note that $P(\lambda) \geq 0$. Furthermore

$$P(\lambda) = (u, u) + 2\lambda(u, v) + \lambda^2(v, v).$$

Since $P(\lambda) \geq 0$ its discriminant $\Delta = b^2 - 4ac$ is non-positive. This gives

$$P(\lambda) = 4((u, v))^2 - 4(u, u)(v, v) \leq 0.$$

Which in turn implies

$$|(u, v)| \leq (u, u)^{\frac{1}{2}} (v, v)^{\frac{1}{2}}.$$

For all $u \in H$, we set

$$|u| = (u, u)^{\frac{1}{2}}$$

Exercise 34:

Prove that $|\cdot|$ defines a norm on H.

Solution 34:
We prove that
$$|u+v| \leq |u| + |v|.$$
The two other verifications are left to the reader.
$$\begin{aligned}|u+v|^2 =&(u+v, u+v)\\ =&|u|^2 + 2(u,v) + |v|^2\\ \leq&|u|^2 + 2|u||v| + |v|^2 \text{ by Cauchy-Schwarz}\\ \leq&(|u|+|v|)^2.\end{aligned}$$

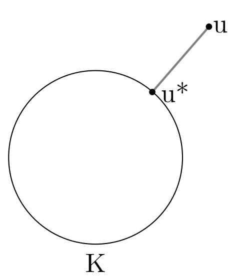
Definition 26. *We say that a linear space H endowed with a scalar product is a Hilbert space if H is complete for the norm $|\cdot|$ defined above.*

Classical examples of Hilbert spaces are $\mathbb{R}^n$, L^2, H^1 to cite only but a few. Hilbert spaces are reflexive but we are not giving the proof here. See [Bre11].

Exercise 35:
Prove that
$$|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2) \tag{5.1}$$
This equality is known as the parallelogram's law.

Solution 35:
Just write
$$|a+b|^2 = (a+b, a+b) = |a|^2 + |b|^2 + 2(a,b)$$
and
$$|a-b|^2 = (a+b, a+b) = |a|^2 + |b|^2 - 2(a,b).$$
Summing the two previous equalities provides the result.

Theorem 44. *Let K be a closed convex subset of H. Then for all $u \in H$ there exists a unique $u^* \in K$ such that*
$$|u-u^*| = \inf_{v \in K} |u-v|.$$
Furthermore u^* is characterized by $u^* \in K$ and
$$(u-u^*, v-u^*) \leq 0 \, \forall v \in K. \tag{5.2}$$



Proof
Let v_n a sequence such that
$$d = \inf_{v \in K} |u-v| = \lim_{n \to +\infty} |u-v_n|$$
We will prove that (v_n) is a Cauchy sequence. We apply the parallelogram's law (5.1) to
$$a = u - v_n,\ b = u - v_m.$$
We obtain
$$|2u - (v_n+v_m)|^2 + |v_m - v_n|^2 = 2|u-v_n|^2 + 2|u-v_m|^2$$

and subsequently,

$$|u - \frac{v_n + v_m}{2}|^2 + |\frac{v_m - v_n}{2}|^2 = \frac{1}{2}|u - v_n|^2 + \frac{1}{2}|u - v_m|^2$$

which gives

$$|\frac{v_m - v_n}{2}|^2 = \frac{1}{2}|u - v_n|^2 + \frac{1}{2}|u - v_m|^2 - |u - \frac{v_n + v_m}{2}|^2$$

and since, $\frac{v_n+v_m}{2} \in K$, we can write

$$|\frac{v_m - v_n}{2}|^2 \leq \frac{1}{2}|u - v_n|^2 + \frac{1}{2}|u - v_m|^2 - d^2.$$

Since the right hand side converges toward 0, we have that (v_n) is Cauchy. Since E is complete, (v_n) converges. We set

$$u^* = \lim_{n \to +\infty} v_n.$$

It follows that

$$|u - u^*| = \inf_{v \in K} |u - v|.$$

Now, for all $t \in [0, 1]$, $w \in K$ we have

$$\begin{aligned} |u - u^*| &\leq |u - ((1 - t)u^* + tw)| \\ &\leq |u - u^* + t(u^* - w)| \end{aligned}$$

which implies

$$|u - u^*|^2 \leq |u - u^*|^2 + 2t(u - u^*, u^* - w) + t^2|u^* - w|^2.$$

Therefore,

$$0 \leq 2t(u - u^*, u^* - w) + t^2|u^* - w|^2.$$

Dividing by t and taking the limit as t goes to 0 we obtain

$$(u - u^*, w - u^*) \leq 0.$$

Conversely, assume that

$$(u - u^*, w - u^*) \leq 0$$

for some $u^* \in K$. Then, for all $w \in K$

$$\begin{aligned} |w - u|^2 &= (w - u, w - u) \\ &= (w - u^* + u^* - u, w - u^* + u^* - u) \\ &= |u - u^*|^2 + |w - u^*|^2 + 2(u^* - u, w - u^*) \\ &\geq |u - u^*|^2. \end{aligned}$$

Finally, we need to prove the uniqueness. Let $u_1^*, u_2^* \in K$ satisfying eq. (5.5), then

$$\forall w \in K,\ (u - u_1^*, w - u_1^*) \leq 0$$

and

$$\forall w \in K,\ (u - u_2^*, w - u_2^*) \leq 0.$$

Choosing $w = u_2^*$ in the first equation above and $w = u_1^*$ in the latter, summing the two, we obtain

$$|u_1^* - u_2^*|^2 \leq 0,$$

which implies $u_1^* = u_2^*$. □

The above element u^* is called the projection of u in K and denoted by

$$u^* = P_K(u)$$

The following inequality holds

Proposition 45.

$$|P_K(u) - P_K(v)| \leq |u - v|$$

Proof

$$\forall w \in K, \ (u - P_K(u), w - P_K(u))$$

and

$$\forall w \in K, \ (v - P_K(v), w - P_K(v)).$$

As before, choosing $w = P_K(v)$ in the first equation above and $w = P_K(u)$ in the latter, summing the two, we obtain

$$(u - P_K(u) - v + P_K(v), P_K(v) - P_K(u)) \leq 0$$

which implies

$$|P_K(v) - P_K(u)|^2 \leq (v - u, P_K(v) - P_K(u)).$$

Applying the Cauchy-Schwarz inequality to the right-hand side gives the result. □

In the case where H is a linear subspace of H, we have also:

Proposition 46. *If K is a linear closed subspace of H then*

$$(u - P_K(u), v) = 0 \, \forall v \in K.$$

Proof
Let $w \in K$, then

$$(u - u^*, w) = (u - u^*, w + u^* - u^*) \leq 0$$

and

$$(u - u^*, -w) = (u - u^*, -w + u^* - u^*) \leq 0$$

which implies

$$(u - u^*, w) = 0.$$

□

5.2 Riesz-Frechet, Stampacchia and Lax-Milgram Theorems

5.2.1 The Riesz-Frechet theorem

In a Hilbert space H, for any $\varphi \in H'$ there exists an element $u \in H$ such that the product scalar with u equals φ. This is the $Riesz - Frechet$ representation theorem stated below.

Theorem 47. *[Riesz-Frechet Representation Theorem] Let $\varphi \in H'$, there exists a unique $u \in H$ such that*

$$(u, v) = \varphi(v) \, \forall v \in H.$$

Proof
Let

$$M = \varphi^{-1}(0).$$

Note that M is a linear closed subspace of H. If $M = H$ then $u = 0$. We assume from now on that $M \neq H$. Let $w_0 \in H \setminus M$ and let $w_1 = P_M w_0$ the projection of w_0 on M. Now consider

$$w = \frac{w_0 - w_1}{|w_0 - w_1|}.$$

We remark that

$$|w| = 1$$

and

$$(w, z) = 0 \, \forall z \in M.$$

Now for any $v \in H$, we define h as

$$h = v - \frac{\varphi(v)}{\varphi(w)} w.$$

Note that $\varphi(h) = 0$ and therefore $h \in M$. It follows that

$$(w, h) = 0.$$

and therefore for all $v \in H$

$$(w, v) = \frac{\varphi(v)}{\varphi(w)}.$$

Setting $u = \varphi(w)w$, we obtain that for all $v \in H$,

$$(u, v) = \varphi(v)$$

For the uniqueness, assume that u_1, u_2 satisfy the condition then

$$\forall v \in H,\ (u_1 - u_2, v) = 0.$$

which implies

$$|u_1 - u_2| = 0.$$

5.2.2 The Stampacchia theorem

For the next theorem, it is useful to remind the follwoing fixed point theorem.

Theorem 48. *Let (X, d) be a complete metric space. We assume that S is an application from X into X such that for all $x, y \in X$*

$$d(S(x), S(y)) \leq k d(x, y)$$

with $k < 1$, then there exists a unique $x^* \in X$ such that

$$S(x) = x.$$

Proof
Let $x_0 \in X$. We consider the sequence defined iteratively by

$$x_{n+1} = S(x_n).$$

We have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}) \\ &\leq k^n d(x_0, x_1) \sum_{i=0}^{p-1} k^i \\ &\leq k^n d(x_0, x_1) \frac{1}{1-k} \xrightarrow[n \to +\infty]{} 0. \end{aligned}$$

Therefore, (x_n) is a Cauchy sequence, and since X is complete it converges toward some $x \in X$. Taking the limit in the expression $d(x_n, x_{n+1})$ gives

$$d(x, S(x)) = 0.$$

Finally, if x, y are two fixed points, we obtain

$$d(x, y) \leq k d(x, y)$$

which implies $x = y$.

Definition 27. *We say that a bilinear form a on H is continuous if there exists a constant C such that*

$$|a(u,v)| \leq C|u||v| \;\forall u,v \in H.$$

We say that a is coercive if there exists $\alpha > 0$ such that

$$a(u,u) \geq \alpha |u|^2.$$

Theorem 49 (Stampacchia)**.** *Let a be a continuous and coercive bilinear form on H and K be a closed convex. Then for any $\varphi \in H'$ there exists a unique $u \in K$ such that for all $v \in K$*

$$a(u, v-u) \geq \varphi(v-u) \tag{5.3}$$

Proof
From Theorem 47 there exists a unique $z \in H$ such that $(z,v) = \varphi(v)$ for all $v \in H$. Next, since for fixed u, $a(u,v)$ defines a continuous linear form on H, Theorem 47 provides a linear map from H into H such that $(Au,v) = a(u,v) \forall v \in H$. Note that from the continuity of $a(\cdot,\cdot)$ we deduce that

$$|Au|^2 = a(u, Au) \leq C|u||Au|$$

which implies

$$|Au| \leq C|u|.$$

It follows that the problem (5.3) is equivalent to finding $u \in K$ such that

$$(Au, v-u) \geq (z, v-u) \,\forall v \in K.$$

Note that for any $\rho > 0$, this is equivalent to

$$(-\rho Au + \rho z + u - u, v - u) \leq 0 \,\forall v \in K.$$

So it is equivalent to find u such that

$$P_K(-\rho Au + \rho z + u) = u.$$

We are therefore looking for a fixed point of the map $S : v \to P_K(-\rho Av + \rho v + v)$. Next, note that

$$|S(v_2) - S(v_1)| \leq |-\rho A(v_2 - v_1) + (v_2 - v_1)|$$

and therefore

$$\begin{aligned}
|S(v_2) - S(v_1)|^2 &\leq |v_2 - v_1|^2 - 2\rho(A(v_2 - v_1), (v_2 - v_1)) + \rho^2 |A(v_2 - v_1)|^2 \\
&\leq |v_2 - v_1|^2 - 2\rho\alpha |v_2 - v_1|^2 + \rho^2 a(v_2 - v_1, A(v_2 - v_1)) \\
&\leq |v_2 - v_1|^2 - 2\rho\alpha |v_2 - v_1|^2 + \rho^2 C|v_2 - v_1||A(v_2 - v_1)| \\
&\leq |v_2 - v_1|^2 - 2\rho\alpha |v_2 - v_1|^2 + \rho^2 C^2 |v_2 - v_1|^2.
\end{aligned}$$

Choosing $\rho < \frac{2\alpha}{C^2}$ implies that

$$|S(v_2) - S(v_1)|^2 \leq k^2 |v_2 - v_1|^2$$

with $k^2 = 1 - 2\rho\alpha + \rho^2 C^2 < 1$. Therefore, we deduce from Theorem 48 the existence of unique u such that $S(u) = u$. □

Proposition 50. *Under the assumptions of Theorem 51, if we assume furthermore that a is symmetric then u is characterized by $u \in K$ and*

$$\frac{1}{2}a(u,u) - \varphi(u) = \inf_{v \in K} \big(\frac{1}{2}a(v,v) - \varphi(v)\big).$$

Proof
If a is symmetric then it defines a scalar product. Applying Theorem 47 with this product scalar provides the existence of z such that

$$a(z, v) = \varphi(v)\, \forall v \in H.$$

In this case inequation (5.3) is equivalent to

$$a(u, v - u) \geq a(z, v - u)$$

which is equivalent to $u = P_K(z)$ (for the scalar product a). We know that this is also equivalent to find u such that

$$a(u - z, u - z) = \inf_{v \in K} a(z - v, z - v).$$

Since

$$a(z - v, z - v) = a(z, z) - 2a(z, v) + a(v, v)$$

this is equivalent to find u which minimizes

$$\frac{1}{2}a(v, v) - \varphi(v).$$

5.2.3 The Lax-Milgram theorem

Theorem 51 (Lax-Milgram)**.** *Let a be a continuous and coercive bilinear form on H. Then for any $\varphi \in H'$ there exists a unique $u \in H$ such that for all $v \in H$*

$$a(u, v) = \varphi(v)\, \forall v \in H. \tag{5.4}$$

Furthermore if a is symmetric then u is characterized by

$$\frac{1}{2}a(u, u) - \varphi(u) = \inf_{v \in K} \big(\frac{1}{2}a(v, v) - \varphi(v)\big).$$

Proof
Let $\varphi \in H'$. From Theorem 51 with $K = H$ there exists a unique $u \in H$ such that

$$\forall v \in H, a(u, v - u) \geq \varphi(v - u).$$

Therefore,

$$\forall v \in H, a(u, v) = a(u, v + u - u) \geq \varphi(v + u - u) = \varphi(v).$$

And also,

$$a(u, -v) \geq \varphi(-v)$$

which provides

$$a(u, v) \leq \varphi(v)$$

and therefore

$$a(u, v) = \varphi(v).$$

The characterization is as in Proposition 50.

5.3 Hilbert Sums

Definition 28. *Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of closed linear subspaces of H. We say that H is the Hilbert sum of the E_n's if*

1. *the spaces E_n are mutually orthogonal, i.e.,*

$$(u, v) = 0\ \forall u \in E_n, \forall v \in E_m, n \neq m$$

2. *the linear space spanned by finite linear combinations of elements of E_n's is dense in H.*

Theorem 52. *Assume that H is the Hilbert sum of the E_n's. Then,*

$$\forall u \in H, \sum_{k=0}^{+\infty} P_{E_k} u = u$$

and

$$\sum_{k=0}^{+\infty} |P_{E_k} u|^2 = |u|^2 \text{ (Bessel-Parseval's Identity).}$$

We will use the following lemma.

Lemma 5. *Assume that (v_n) is a sequence in H such that*

$$(v_n, v_m) = 0 \text{ if } n \neq m,$$

and

$$\sum_{k=0}^{+\infty} |v_k|^2 < +\infty.$$

Let the series

$$\sum_{k=0}^{+\infty} v_k$$

converges in H and

$$|\sum_{k=0}^{+\infty} v_k|^2 = \sum_{k=0}^{+\infty} |v_k|^2.$$

Proof(Proof of Lemma 5)
Let

$$S_n = \sum_{k=0}^{n} v_k.$$

We remark that (S_n) is a Cauchy sequence. Indeed, for $m > n$,

$$|S_m - S_n|^2 = \sum_{k=n+1}^{m} |v_k|^2.$$

Since $\sum_{k=0}^{+\infty} |v_k|^2 < +\infty$, we deduce that (S_n) is a Cauchy sequence. Taking the limit in

$$|S_n|^2 = \sum_{k=0}^{n} |v_k|^2$$

proves the lemma. □

Proof(Proof of Theorem 52)
We are going to apply Lemma 5 Let

$$u_n = P_{E_n} u \text{ and } S_n = \sum_{k=0}^{n} v_k.$$

From assumptions, since $u_n \in E_n$ and $u_m \in E_m$, we have

$$(u_n, u_m) = 0 \text{ if } n \neq m.$$

Next, we remark that

$$(u - u_n, v) = 0 \forall v \in E_n.$$

Choosing $v = u_n$ gives

$$(u, u_n) = |u_n|^2$$

Summing up from 0 to n gives

$$(u, S_n) = |S_n|^2.$$

By the Cauchy-Schwarz inequality,

$$|S_n|^2 = (u, S_n) \leq |u||S_n|,$$

and therefore

$$|S_n| \leq |u|,$$

which in turn implies

$$\sum_{k=0}^{n} |u_k|^2 \leq |u|^2.$$

From lemma Lemma 5, we deduce that $\sum_{k=0}^{+\infty} u_k$ converges. Let us call it S. It remains to prove that $u = S = \sum_{k=0}^{+\infty} u_k$. We remark that

$$(u - S_n, v) = 0 \forall v \in E_m, m \leq n.$$

This is because

$$u - S_n = (u - u_m) - \sum_{k=0, k\neq m}^{n} u_k.$$

Taking the limit gives

$$(u - S, v) = 0$$

for all v in the linear space spanned by finite linear combinations of elements of $E_n, n \in \mathbb{N}$ Since this space is dense in H, this implies that

$$(u - S, v) = 0 \forall v \in H.$$

It follows that $S = u$. ☐

Definition 29. *A sequence $(e_n)_{n\in\mathbb{N}}$ in H is said to be an orthonormal basis if it satisfies the following properties:*

1. $$|e_n| = 1, \text{ and } (e_m, e_n) = 0, \forall m \neq n,$$

2. *the linear space spanned by the $(e_n)'s$ is dense in H.*

Proposition 53. *Let (e_n) be an orthonormal basis. Then for every $u \in H$, we have*

$$u = \sum_{k=0}^{+\infty} (u, e_k) e_k$$

and

$$|u|^2 = \sum_{k=0}^{+\infty} |(u, e_k)|^2$$

Proof
We apply Theorem 52 with $E_n = span\{e_n\}$. We have oonly to prove that

$$P_{E_n} u = (u, e_n) e_n.$$

But

$$\begin{aligned}(u - (u, e_n)e_n, e_n) =& (u, e_n) - (u, e_n) \\ =& 0.\end{aligned}$$

☐

5.4 Hilbert spaces on $\mathbb{C}$

In this chapter we briefly discuss the case $\mathbb{K} = \mathbb{C}$.

Definition 30. *A scalar product on a linear space H over $\mathbb{C}$ is a complex valued function $(\cdot, \cdot) : H \times H \to \mathbb{R}$ which satisifies*

- *Sesquilinearity: $u \to (u, v)$ is linear and $v \to (u, v)$ is skewlinear: $(x, ay) = \bar{a}(x, y)$.*
- *Skew Symmetry : $(u, v) = \overline{(v, u)}$.*
- *Positivity : $(u, u) > 0$ whenever $u \neq 0$.*

Exercise 36:
Prove the Cauchy-Schwarz inequality:

$$|(u, v)| \leq (u, u)^{\frac{1}{2}} (v, v)^{\frac{1}{2}}$$

Solution 36:
Left to the reader.

For all $u \in H$, we set

$$|u| = (u, u)^{\frac{1}{2}}$$

Exercise 37:

Prove that $| \cdot |$ defines a norm on H.

Solution 37:
Left to the reader.

Theorem 54. *Let K be a closed convex subset of H. Then for all $u \in H$ there exists a unique $u^* \in K$ such that*

$$|u - u^*| = \inf_{v \in K} |u - v|.$$

Furthermore u^* is characterized by $u^* \in K$ and

$$\Re(u - u^*, v - u^*) \leq 0 \, \forall v \in K. \tag{5.5}$$

Bibliography

[Bre11] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* 2011.

[Lax02] Peter. D. Lax. *Functional Analysis.* 2002.

[Rud87] W. Rudin. *Real and Complex Analysis.* 1987.

[Sch] J. Schenker. Functional analysis, lecture notes (michigan university).