Ordinary Differential Equations

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Contents

Chapter 1

Introduction

In these lectures, we will focus on equations of the following type

$$U' = f(U) \tag{1.1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a given function. We look for a solution $U(t)$ of (1.1) defined and differentiable on an interval $I \subset \mathbb{R}$. This equation is called an ODE because it deals with a function and its derivative. The word ordinary comes from the fact that only one variable is considered (here, t). This contrasts with Partial Differential Equations where different variables and their corresponding partial derivatives are considered. ODEs describe a wide range of phenomena from physics, chemistry, biology... Before delving into more theoretical aspects of ODEs, we want to start theses lectures by providing some examples. The goal here is to provide a concrete approach of the problems before building a rigorous theoretical framework. A good example t start with, is the trajectories of planets. Each planet evolves in the three dimensional space according to physical laws. Those trajectories can be determined by solving the gravitational Newton's laws. Those are ODEs. The main objective of ODEs is to describe the trajectories of solutions of (1.1). Let's start with examples.

1.1 Newton's equations

In classical mechanics, the movement of particle is described by a function $x(t)$ from $\mathbb{R}$ into $\mathbb{R}^3$. The particle moves under the action of a force F which depends on the position x. the second Newton's states that

$$mx'' = F(x) \tag{1.2}$$

where m is a constant (the mass).

Exercise 1.

1. We set $y = x'$. Rewrite (1.2) as an order 1 equation (with just first derivatives).
2. We assume that
$$F(x) = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} \tag{1.3}$$
where m and g are constants. We assume further that initial conditions satisfy
$$x(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } y(0) = \begin{pmatrix} 0 \\ k_2 \\ k_3 \end{pmatrix} \tag{1.4}$$
with $k_2 > 0,\ k_3 > 0$. Compute the solution of (1.2)-(1.4).
3. Write $x_3(t)$ as a function of $x_2(t)$ and represent the trajectory in the upper quarter plane $x_2 > 0, x_3 > 0$

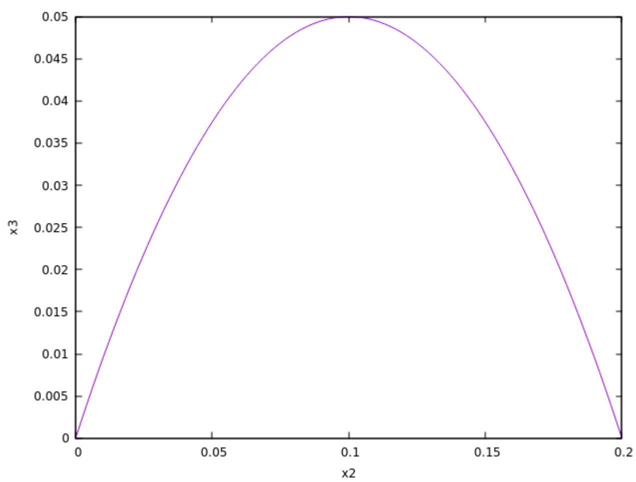


Figure 1.1: Évolution de x_3 en fonction de x_2.

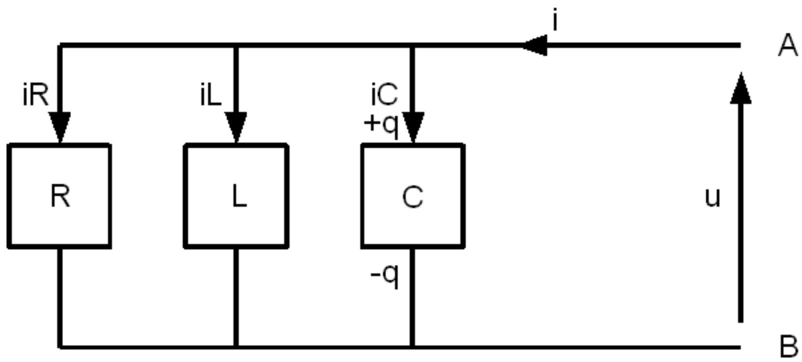
1.2 Parallel RLC Circuit

We consider a parallel RLC circuit with a resistor R, a capacitor C, and an inductor L , see 1.2. According to Kirchoff's law, we have:

$$i = i_R + i_L + i_C.$$

Next,

$$i_R = \frac{u}{R}\,, i_L' = \frac{1}{L}u\,, i_C = Cu',$$



taking the derivatives, it follows that

$$i' = \frac{1}{R}u' + \frac{1}{L}u + Cu'' \tag{1.5}$$

Exercise 2.

- Set $u' = v$ and rewrite equation (1.5) as an order 1 ODE.
- We assume $i' = k$, constant. Find the stationary solutions ($i.e.$ $u' = v' = 0$) of the resulting equation.

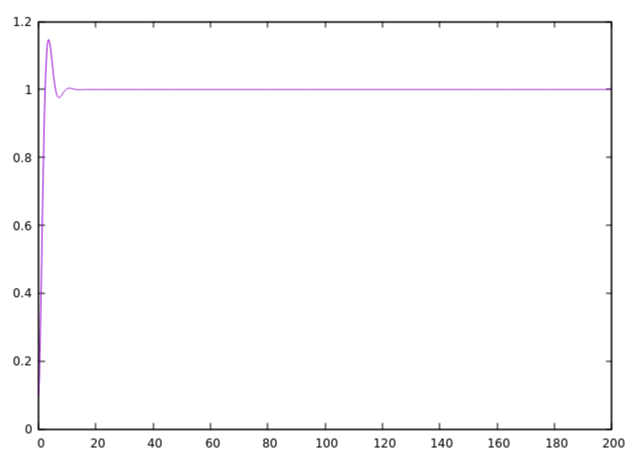


Figure 1.2: Évolution de u en fonction du temps, pour $i' = 1$ $u(0) = v(0) = 0.1$, et $R = L = C = 1$.

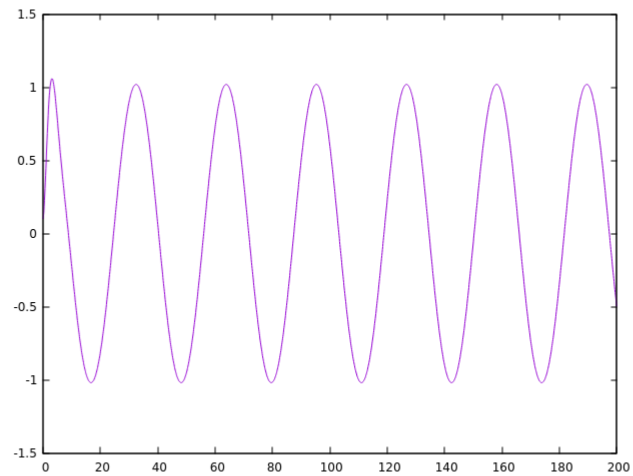


Figure 1.3: Évolution de u en fonction du temps, pour $i'(t) = \cos(0.2t)$ $u(0) = v(0) = 0.1$

1.3 Radioactivity

The time evolution of the quantity N of a radioactive nucleus is given by:

$$N' = -\lambda N \tag{1.6}$$

where λ is the rate of disintegration per unit of time.
Exercise 3.
Compute the solution of (1.6), for a given initial quantity N_0 of nucleus.

1.4 A few concepts

This section contains a lot of information. It is intended to provide a general idea of concepts which will be studied and redefined in more detail later. Some of definition are taken or adpated from [4]. We assume that f is of class C^1. The notation

$$Df(u)$$

refers to the jacobian matrix associated to f at u.

Definition 1. *A point $u^* \in \mathbb{R}^m$ is said to be a stationary point (or equilibrium or fixed point) of equation* (1.1) *if*

$$f(u^*) = 0.$$

A stationary point $u^* \in \mathbb{R}^m$ is called hyperbolic if none of the eigenvalues of $Df(u^*)$ has a real part with value 0.

Definition 2. *A stationary point $u^* \in \mathbb{R}^m$ is called a sink if all the eigenvalues of*

$$Df(u^*)$$

have a negative real part. It is called a source if all the eigenvalues have a positive real part. It is called a saddle node if it is hyperbolic and has at least one eigenvalue with a negative real part and one eigenvalue with a positive real part.

Definition 3. *We assume that f is C^2 and that $m = 2$. An hyperbolic stationary point $u^* \in \mathbb{R}^m$ for which all the eigenvalues of*

$$Df(U^*)$$

are real is called a node. If it is hyperbolic and the eigenvalues are complex, it is called a focus.

Definition 4. *We assume that for all $z \in \mathbb{R}^m$, the equation* (1.1)*, with initial condition $u(0) = z$ has a unique regular solution $u_z(s), s \in \mathbb{R}$. We define for all $t \in \mathbb{R}$ the application:*

$$\phi_t : \begin{cases} \mathbb{R}^m & \rightarrow & \mathbb{R}^m \\ z & \mapsto & \phi_t(z) = u_z(t) \end{cases} \tag{1.7}$$

Then the set of maps ϕ_t are called the flow associated to (1.1)*; and $(\phi_t(z))_{t \geq 0}$ is called the trajectory ensued from z.*

Definition 5 (Stability)**.** *A stationary point $u^* \in \mathbb{R}^m$ is said to be stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\forall z \in B(u^*, \delta) \forall t \geq 0 \, \phi_t(z) \in B(u^*, \epsilon).$$

A stationary point which is not stable is unstable. It is asymptotically stable if it is stable and if furthermore for all $z \in B(u^*, \delta)$,

$$\lim_{t \to +\infty} \Phi_t(z) = u^*.$$

Theorem 1. *A sink is asymptotically stable. A source and a saddle node are unstables.*

Definition 6 (Differentiable Manifold)**.** *An n-dimensional differentiable manifold, M, is a connected metric space with an open covering $\{U_\alpha\}$, $i.e.$, $M = \bigcup_\alpha U_\alpha$, such that*

1. *for all α, U_α is homeomorphic to the unit ball B in R^m, $i.e.$ for all α there exists an homeomorphism h_α from U_α onto B, and*
2. *if $U_\alpha \cap U_\beta \neq \emptyset$ then*
$$h = h_\alpha \circ h_\beta^{-1}$$
is differentiable and for all $x \in h_\beta(U_\alpha \cap U_\beta)$, $Det\, h(x) \neq 0$

The pair (U_α, h_α) is called a chart for the manifold M and the set of all charts is called an atlas for M. The differentiable manifold M is called orientable if there is an atlas with $Det(D(h_\alpha \circ h_\beta^{-1})(x)) > 0$ for all α, β and $x \in h_\alpha(U_\alpha \bigcap U_\beta)$.

For simplicity of the exposition, at this stage we avoid the case of complex values

Definition 7. *In the case where f is linear, we denote E^s, E^i, E^c, the eigenspaces associated to the eigenvalues with respectively negative, positive and zero values.*

Theorem 2 (Stable Manifold Theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open set containing $(0,0)$, $f \in C^1(E)$ and ϕ_t the flow associated with* (1.1)*. We assume $f(0) = 0$ and $Df(0)$ has k negative eigenvalues and $m - k$ positive eigenvalues. Then, there exists a diffrentiable manifold S with dimension k tangeant to the subspace E^s of the linear system associated to* (1.1)*, such that for $t \geq 0$, $\phi_t(S) \subset S$ and for all $z \in S$*

$$\lim_{t \to +\infty} \phi_t(z) = 0.$$

And there exists a differentiable manifold I with dimension $n - k$ tangeant to the subspace E^i of the linear system associated to (1.1)*, such that for $t \leq 0$, $\phi_t(I) \subset I$ and for all $z \in S$*

$$\lim_{t \to -\infty} \phi_t(z) = 0.$$

Definition 8. *We set*

$$W^s = \cup_{t \leq 0} \phi_t S$$

$$W^i = \cup_{t \geq 0} \phi_t I$$

Theorem 3 (Center Manifold)**.** *Let $U \subset \mathbb{R}^n$ be a open set containing $(0,0)$, $f \in C^1(E)$ and ϕ_t the flow associated with* (1.1)*. We assume $f(0) = 0$ and $Df(0)$ has k negative eigenvalues, j positive eigenvalues and $l = m - k - j$ eigenvalues with zero real parts. Then, there exists a differentiable manifold W^c with dimension l tangeant to the subspace E^c of the linear system associated to* (1.1)*, a differentiable manifold W^s with dimension k tangeant to E^s, and a differentiable manifold W^i with dimension j tangeant to E^i . Furthermore W^s, W^i et W^c are invariant under the flow.*

Definition 9. *We assume $m = 1$ and consider in* (1.1) *a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (t,u) \mapsto f(t,u)$. A function u^+ of class C^1 is called an upper solution (see [12]) of* (1.1) *on $[0, +\infty)$ if*

$$\forall t, \geq 0 \, (u^+)' > f(t, u^+) \tag{1.8}$$

Proposition 1. *We assume $m = 1$. Assume that u^+ is an upper solution of* (1.1) *and that u is a solution of* (1.1)*. For simplicity, we assume at this stage that both are defined on $[0, +\infty)$. Then $u^+(0) \geq u(0)$ implies that*

$$\forall t > 0 \, u^+(t) > u(t)$$

A lower solution is defined in the same way and an analogous result holds. See [12] p 24.

1.5 The Hodgkin Huxley model (1952)

$$\left\{\begin{array}{rcl} C\dfrac{dV}{dt} &=& I+\overline{g}_{Na}m^3h(E_{Na}-V)+\overline{g}_K n^4(E_K-V)+\overline{g}_L(E_L-V),\\ \dfrac{dn}{dt} &=& \alpha_n(V)(1-n)-\beta_n(V)n,\\ \dfrac{dm}{dt} &=& \alpha_m(V)(1-m)-\beta_m(V)m\\ \dfrac{dh}{dt} &=& \alpha_h(V)(1-h)-\beta_h(V)h, \end{array}\right. \tag{1.9}$$

$$\begin{array}{rl} \alpha_n(V)=0.01\frac{-V+10}{\exp(1-0.1V)-1}, & \beta_n(V)=0.125\exp(-V/80),\\ \alpha_m(V)=0.1\frac{-V+25}{\exp(2.5-0.1V)-1}, & \beta_m(V)=4\exp(-V/18),\\ \alpha_h(V)=0.07\exp(-V/20), & \beta_h(V)=\frac{1}{1+\exp(-0.1V+3))}. \end{array} \tag{1.10}$$

$$E_K=-12\ mV,\ \ E_{Na}=120\ mV,\ \ E_L=10.6\ mV$$

$$\overline{g}_K=36,\overline{g}_{Na}=120,\overline{g}_L=0.3$$

Exercise 4.

1. Are α et β well defined?.
2. Compute their sign.
3. Show that if $(m,n,h)(0)\in(0,1)^3$, then $(m,n,h)(t)\in(0,1)^3$ for all $t\in(0,+\infty)$.
4. We assume that $I\geq 0$ is a constant. Prove that there exists V_m et V_M such that $V(0)\in(V_m,V_M)$, then $V(t)\in(V_m,V_M)$ for all $t\in(0,+\infty)$.
5. Prove that the computation of stationary solutions can be found by solving an equation $f(V)=0$ (write $f:\mathbb{R}\to\mathbb{R}$).
6. Prove that equation (1.9) has a stationary solution.

1.6 The FitzHugh-Nagumo Model

A FitzHugh-Nagumo model type (1961)

$$\left\{\begin{array}{rcl} \epsilon\dfrac{du}{dt} &=& f(u)-v,\\ \dfrac{dv}{dt} &=& au-bv-c \end{array}\right. \tag{1.11}$$

$$f(u)=-u^3+3u,\ a>0,\ b\geq 0, 0<\epsilon<<1.$$

Exercise 5. We consider (1.11) with $a=1, b=0, c\leq 0$.

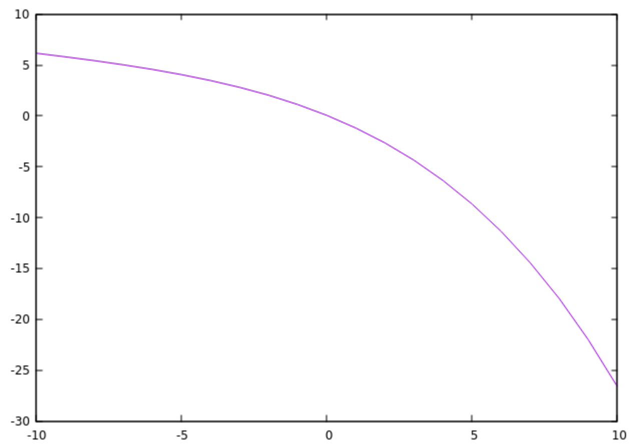


Figure 1.4: The function $f(V),\ I=0$

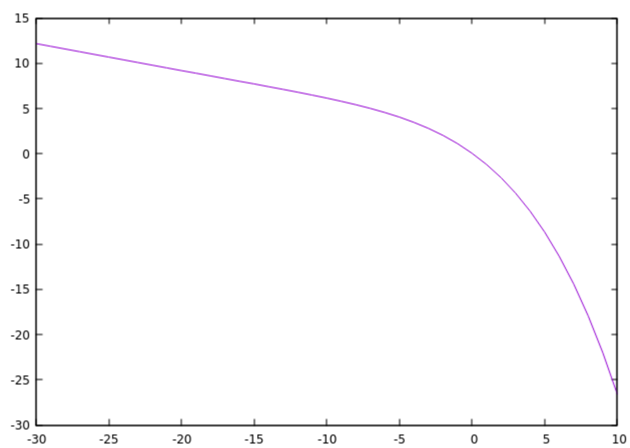


Figure 1.5: The function $f(V),\ I=0$

1. What are the stationary solutions?
2. Discuss the nature of the stationary solutions.
3. Compute the nullclines.
4. Sketch typical trajectories of the system.

The following proposition is useful to compute the eigenvalues of a 2×2 matrix.

Proposition 2. *The eigenvalues of a* 2×2 *matrix* A *are*

$$\lambda_{1,2} = 0.5(tr(A) + \sqrt{tr^2(A) - 4\det A})$$

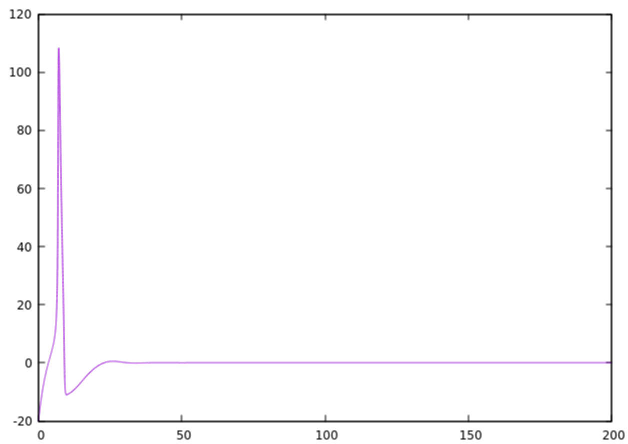


Figure 1.6: V as a function of time. $I = 0$

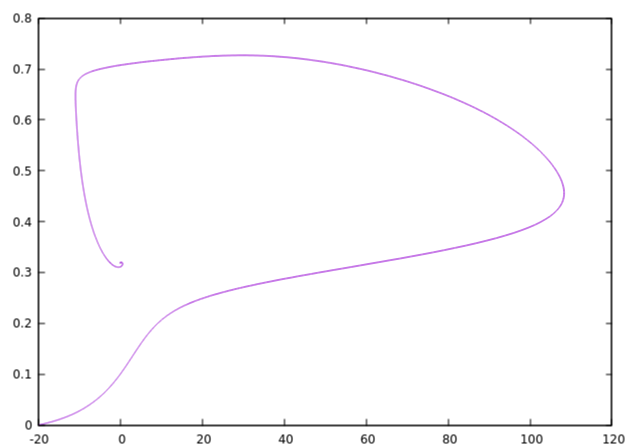


Figure 1.7: $(V, n).I = 0$

1.7 The logistic equation

For the biological models to come, we recommend the famous book of Murray [3].

The Malthus model (1798)
The Malthus model describes the growth of a population. It says that the birth and death rate are proportional to the population. It writes.

$$\begin{cases} \frac{dx}{dt} & = & (b-d)x, \; b, d > 0 \\ x(0) & = & x_0 \end{cases} \tag{1.12}$$

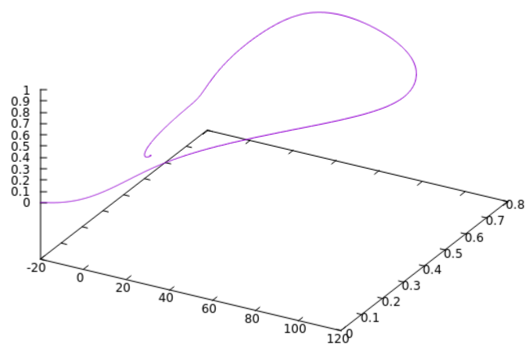


Figure 1.8: (V, n, m). $I = 0$.

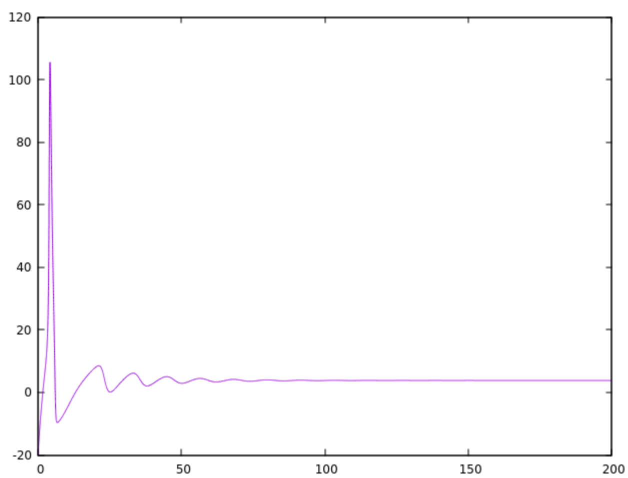


Figure 1.9: V as a function of time. $I = 6$

where b is the birth rate and d the death rate. **Exercise 6.** Compute the solution of (1.12). Discuss the aymptotic evolution of the solution.

The Verhulst model (1838,1845)

To prevent an infinite exponential growth, Verhulst [13, 14] proposed that a self-limiting process should operate when a population becomes too large. The Verhulst equation– or logistic growth– writes

$$\begin{cases} \dfrac{dx}{dt} &= ax(1-\frac{x}{K}),\ a>0, K>0,\ N \in \mathbb{N} \\ x(0) &= x_0 \in [0, N] \end{cases} \tag{1.13}$$

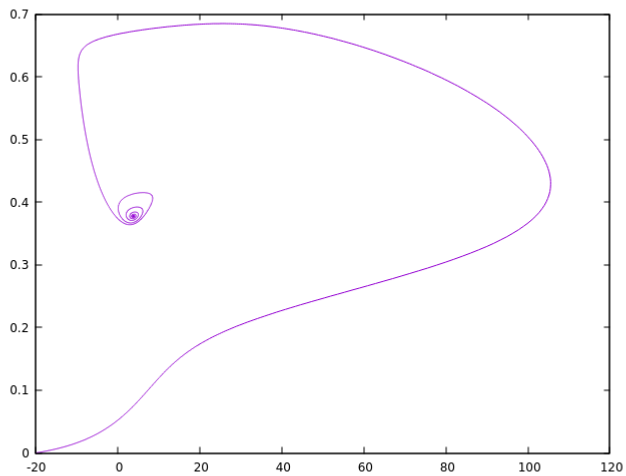


Figure 1.10: $(V, n).I = 6$

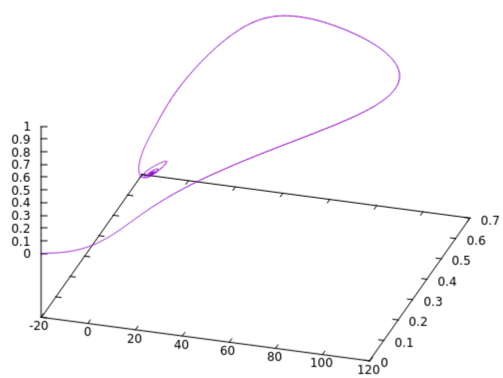


Figure 1.11: (V, n, m). $I = 6$.

In this equation the birth rate is $a(1 - \frac{x}{K})$. The constant K is called the carrying capacity. **Exercise 7.**

1. Look for stationary solutions of (1.13).

2. Discuss their study nature and stability.

3. Show that if $x_0 \in (0, K)$, then $x(t) \in (0, K)$ for all $t \geq 0$.

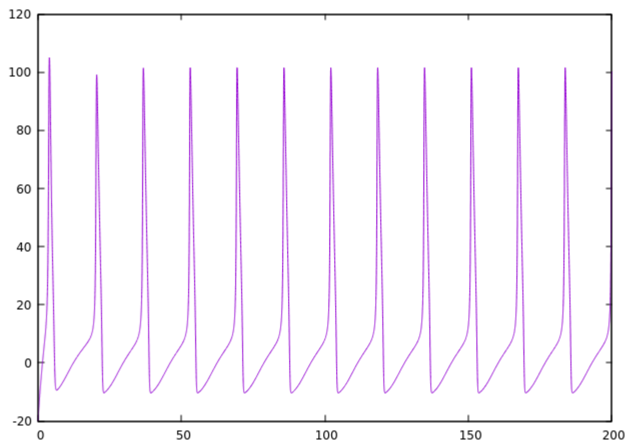


Figure 1.12: V as a function of time. $I = 7$

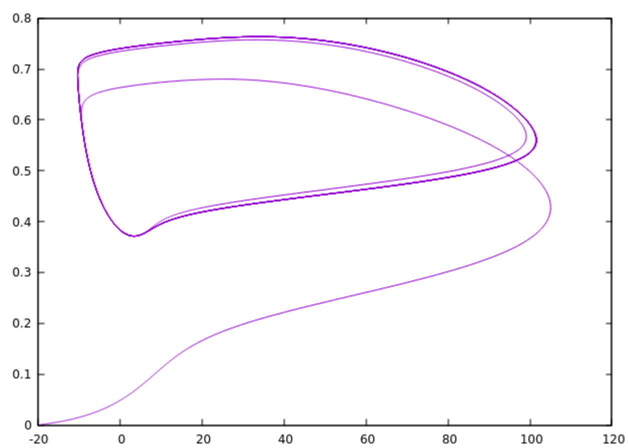


Figure 1.13: (V, n). $I = 7$

4. We assume $x_0 \in (0, K)$. Show that

$$\lim_{t \to +\infty} x(t) = 1.$$

5. We assume $x_0 \in (0, K)$. Compute the solution explicitely.

1.8 The Lotka-Volterra model (1925-1926)

Between 1910 and 1925, Lokta [8, 9, 10] published several works on mathematical models for periodic oscillations in chemical concentrations and extended it to species

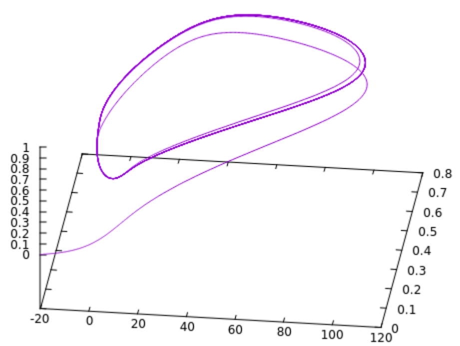


Figure 1.14: (V, n, m). $I = 7$.

populations In 1926, Volterra [**?**] proposed a simple prey-predator to explain the oscillatory levels of fish caught in the Adriatic sea. The equations write

$$\left\{ \begin{array}{rcl} \dfrac{du}{dt} & = & au - buv = f_1(u,v),\ a,b,c,d > 0. \\ \dfrac{dv}{dt} & = & cuv - dv = f_2(u,v) \end{array} \right. \tag{1.14}$$

where u denotes the prey and v the predator. The assumptions in the model are that: the prey in the absence of predation grows exponentially in a Malthusian way (the au term). The predation reduces the prey's growth rate by a term proportional to the prey and predator populations $(-buv)$. (iii) In th ere is no prey to eat, the predator's population decreases exponentially (tem $-dv$). If there is preys to eat, the resulting predator's growth rate is proportional to the product prey-predator (cuv). Equation (1.14) was also derived by Lotka in 1920,1925, [9, 10] from a theoretical chemical reaction which he could exhibit periodic behavior in the chemical concentrations.

Exercise 8.

1. Find the stationary solutions of (1.14).
2. Study their nature and stability.
3. Compute the solutions such that $u_0 = 0$ (no prey).
4. Compute the solutions such that $v_0 = 0$ (no predator).
5. Prove that any solution starting with IC $u_0 > 0$ and $v_0 > 0$ remain in the positive quadrant $u(t) > 0,\ v(t) > 0$ for $t > 0$.
6. Compute and draw the nullclines

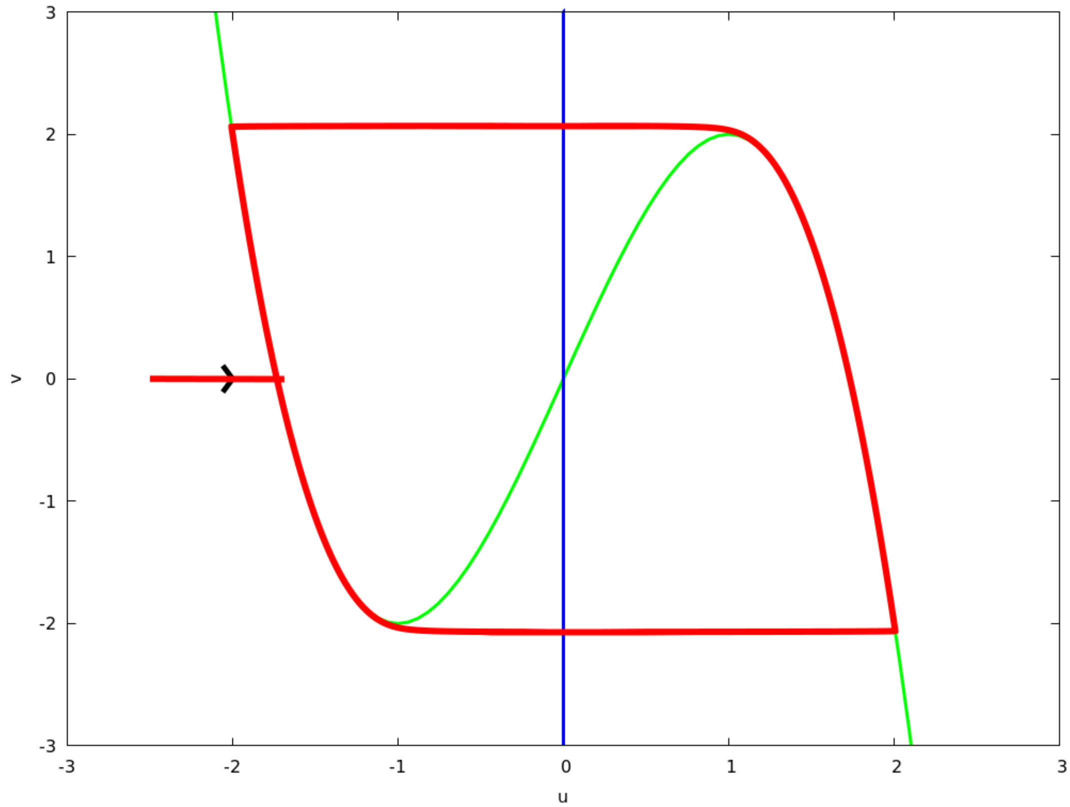


Figure 1.15: a=1,b=0,c=0

7. Find a real function $h(u, v)$ such that

$$u_t h_u + v_t h_v = 0?$$

8. Sketch the trajectories of the solutions.

1.9 The SIR model

The SIR model is a simple model that describe epidemics. Here, S stands for Susceptible, I for infected and R for recovered. It goes back to an article from Kermack– and McKendrick in 1927. The model writes

$$\begin{cases} S_t &= -kIS \\ I_t &= kIS - rI \\ R_t &= rI \end{cases} \tag{1.15}$$

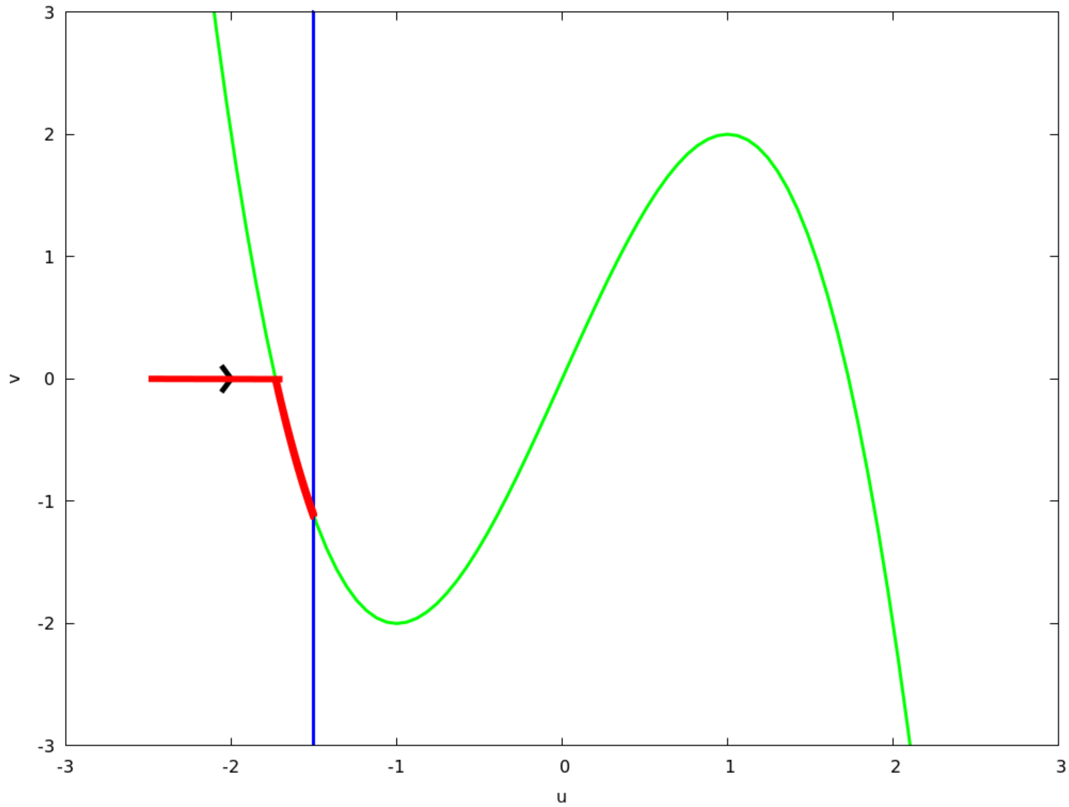


Figure 1.16: a=1,b=0,c=-1.5

We assume that S_0, I_0 et R_0 are positive. Then the following results hold.

Theorem 4. *For $t > 0$, $S(t), I(t)$ and $R(t)$ remain in the interval*

$$[0, S(0) + I(0) + R(0)$$

$$S(t) + I(t) + R(t) \text{ is constant}$$

There exist two constant positive values S^* and R^* such that

$$\lim_{t\to+\infty} S(t) = S^*, \lim_{t\to+\infty} I(t) = 0, \lim_{t\to+\infty} R(t) = R^*$$

furthermore

$$S^* - \frac{r}{k}\ln(S^*) = I_0 + S_0 - \frac{r}{k}\ln S_0$$

$$R^* = S_0 + I_0 + R_0 - S^*.$$

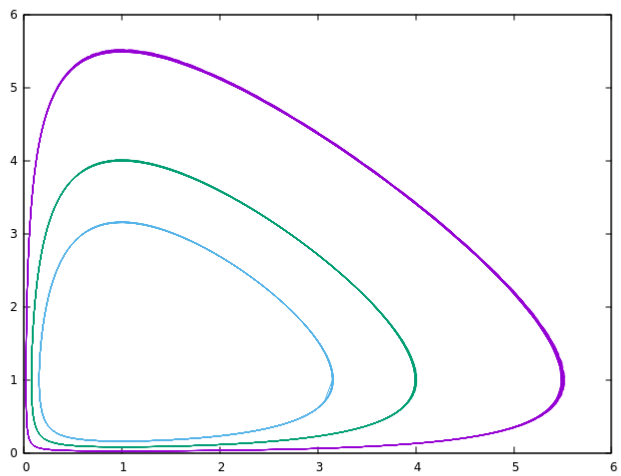


Figure 1.17: .

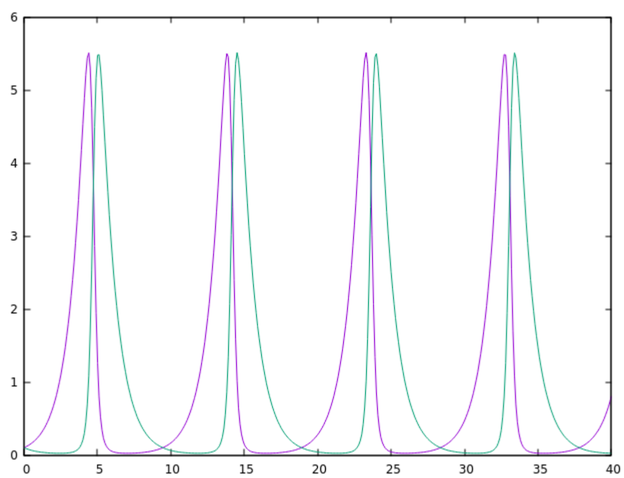


Figure 1.18: u and v as functions of time.

Also, $S(t)$ is decreasing, $R(t)$ is increasing. If

$$S_0 < \frac{r}{k},$$

$I(t)$ is decreasing, and if

$$S_0 > \frac{r}{k},$$

there exists t_0 such that $I(t)$ is decreasing on $(0, t_0)$ and decreasing on $(t_0, +\infty)$.

1.10 The Oregonator model for the Belousov-Zhabotinsky chemical reaction

The following model is called the Oregonator model. Its aim is to model the Belousov-Zhabotinsky chemical reaction for which chemical concentrations can oscillate and lead to pattern formation when placed in a Petri dish. The model writes:

$$\begin{cases} \epsilon_1 \dfrac{dx}{dt} &= qy - xy + x(1-x), \\ \epsilon_2 \dfrac{dy}{dt} &= -qy + fz - xy, \\ \dfrac{dz}{dt} &= (x - z) \end{cases} \tag{1.16}$$

$$\epsilon_1 = 9.9 \times 10^{-3}, \epsilon_2 = 1.98 \times 10^{-5}, q = 7.62 \times 10^{-5}, f = 1. \tag{1.17}$$

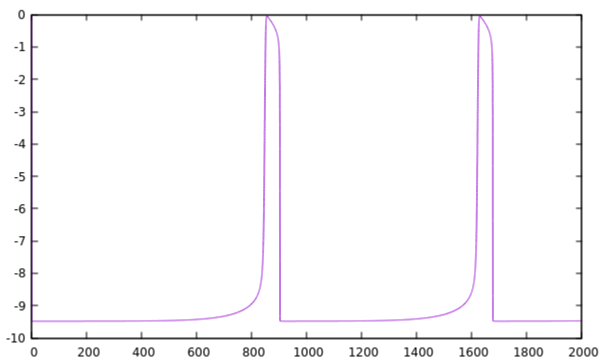


Figure 1.19: $\ln x$ as a function of time

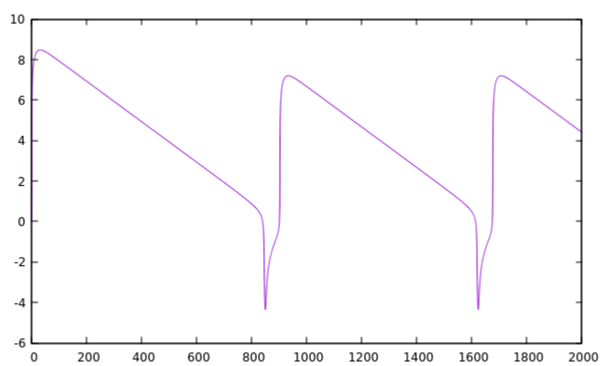


Figure 1.20: $\ln y$ as a function of time

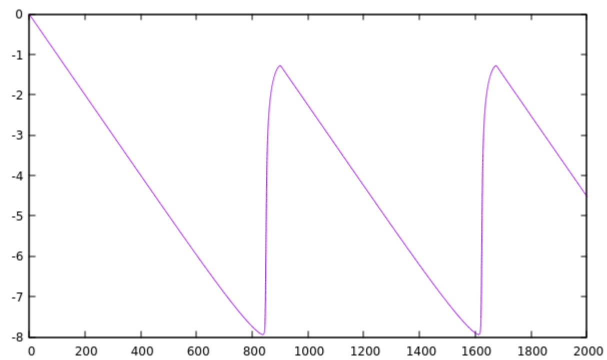


Figure 1.21: $\ln x$ as a function of time

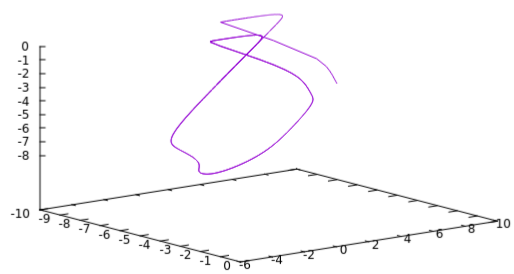


Figure 1.22: $(\ln x, \ln y, \ln z)$

1.11 Lorenz equations

Lorenz equations go back to 1963 when Edward Lorenz working then in forecast weather introduced it as a simplification of Navier-Stokes equations. They write:

$$\begin{cases} \frac{dx}{dt} & = & \sigma(y-x), \\ \frac{dy}{dt} & = & \rho x - y - xz, \\ \frac{dz}{dt} & = & xy - \beta z \end{cases} \tag{1.18}$$

$$\alpha = 10,\ \beta = \frac{8}{3},\ \rho > 0 \tag{1.19}$$

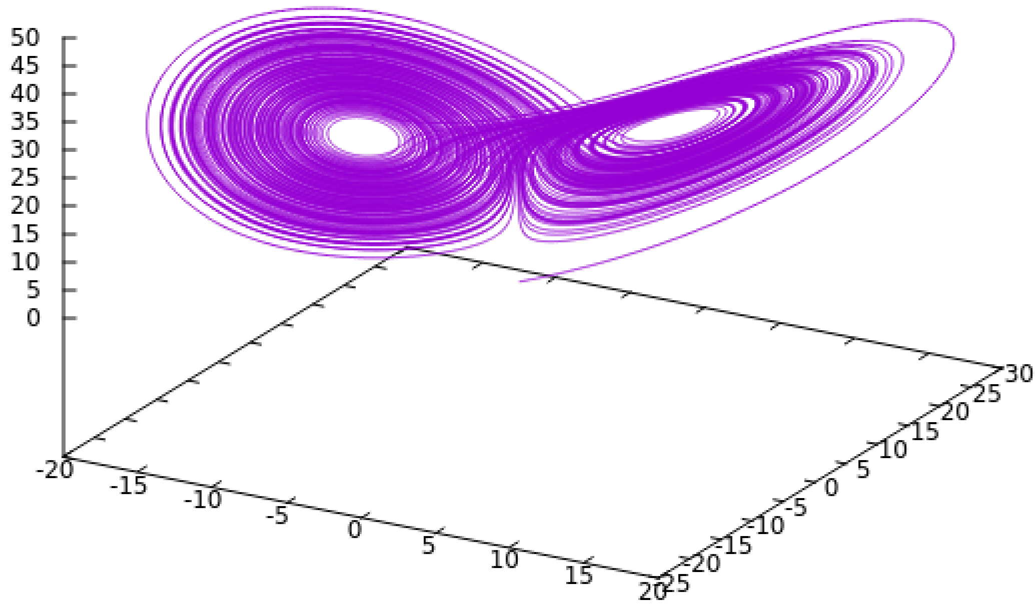


Figure 1.23: $\alpha = 10$, $\beta = \frac{8}{3}$, $\rho = 28$

1.12 The Runge-Kutta-4 method

We briefly introduce here the Runge-Kutta 4 to approximate solutions of ODEs. We consider a time interval $[0, T]$ where the solution has to be approximated. We divite this time interval into N segments of length $h = T/N$. We set $t_n = nh, n \in \{0, ..., N\}$. We assume that the initial condition u_0 is given. We then approximate $u(t_n)$ by a

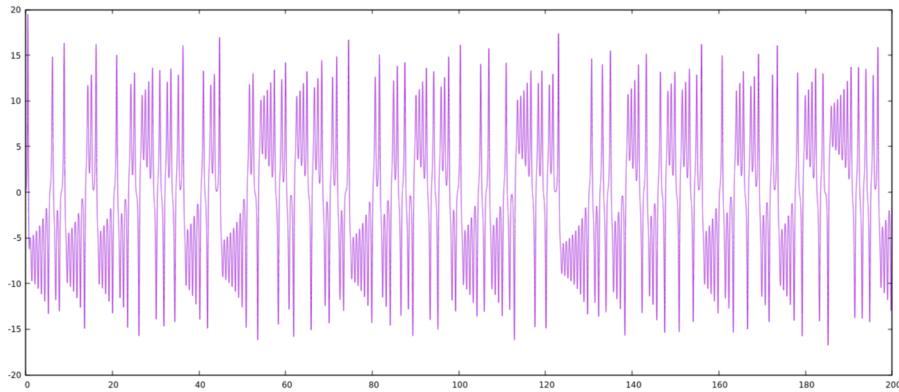


Figure 1.24: $\alpha = 10$, $\beta = \frac{8}{3}$, $\rho = 28$

vector u_n thanks to the iteration:

$$K_1 = F(u_n)$$
$$K_2 = F(u_n + \frac{h}{2}K_1)$$
$$K_3 = F(u_n + \frac{h}{2}K_2)$$
$$K_4 = F(u_n + hK_3)$$

$$u_{n+1} = u_n + \frac{h}{6}(K_1 + 2k_2 + 2K_3 + 2K_4)$$

List of Projects

1. HH equations- How did they obtain their equations? Theoretical and numerical analysis
2. Stability of a limit cycle. The proof of Andronov
3. Proof of the stable invariant manifold in Perko
4. FHR- general
5. FHR- slow-fast
6. SIR Models
7. Lorenz model
8. The 3 body problem
9. Lotka-Volterra Models and beyond

Chapter 2

Fundamental results

2.1 Definitions. Maximal and global solutions

2.1.1 Initial value problem (IVP)

Let $I \subset \mathbb{R}$ be an open interval containing 0 and $U \subset \mathbb{R}^m$ an open set. We set $O = I \times U$. Let f be a continuous function from O into $\mathbb{R}^m$. We consider the initial value problem (IVP):

$$\begin{cases} u' &= f(t,u) \\ u(0) &= u_0,\, u_0 \in U \end{cases} \tag{2.1}$$

Definition 10. *A solution of* (2.1) *on an open interval $J \subset I$ is a differentiable function u from J into $\mathbb{R}^m$ which satisfies* (2.1) *on J:*

$$u(0) = u_0 \text{ and } \forall t \in J,\, u'(t) = f(t, u(t))$$

Remark 1. *To simplify the presentation, we consider here without loss of generality the initial time t_0 to be 0; note however that any time t_0 could be considered instead.*

2.1.2 Maximal Solutions

Exercise 9. Compute the solutions of the following equations. For each equation specify the larger interval on which the solution is well defined. Draw a picture of the solutions.

1. $u' = u$ with $u(0) = u_0$
2. $u' = u^2$ with $u(0) = u_0$.

Correction

1. The solution is $u(t) = u_0 e^t$. It is defined for every $t \in \mathbb{R}$.

2. First note that $u(t) = 0$ is a solution defined on $\mathbb{R}$, with $u(0) = u_0 = 0$. Next if $u \neq 0$,
$$\begin{aligned} u' &= u^2 \\ \Leftrightarrow u^{-2}u' &= 1 \\ \Leftrightarrow -(u^{-1})' &= 1 \\ \Leftrightarrow u^{-1}(t) - u^{-1}(0) &= -t, \quad \text{by integrating on} (0,t), \\ \Leftrightarrow u(t) &= \frac{1}{u_0^{-1} - t} \end{aligned}$$
If $u_0 > 0$, the solution $u(t) = \frac{1}{u_0^{-1}-t}$ is defined on $(-\infty, \frac{1}{u_0})$ (with $\lim_{t\to\frac{1}{u_0}} = +\infty$). Lastly, if $u_0 < 0$, the solution $u(t) = \frac{1}{u_0^{-1}-t}$ is defined on $(\frac{1}{u_0}, +\infty)$ (with $\lim_{t\to\frac{1}{u_0}} = -\infty$).

We can see that solutions may not be defined for all times. This leads to the definition of maximal solutions.

Definition 11. *Let $u_1 : J_1 \to \mathbb{R}^m$ et $u_2 : J_2 \to \mathbb{R}^m$ two solutions of* (2.1). *We say that u_2 is an extension of u_1 if $J_1 \subset J_2$ and for all t in J_1, $u_1(t) = u_2(t)$.*

Definition 12. *We say that a solution of* (2.1) *is a maximal solution if it cannot be extended.*

Theorem 5. *Every solution of* (2.1) *extends into a maximal solution.*

2.1.3 Global solutions

Definition 13. *A solution is said to be global if it satisfies* (2.1) *for all $t \in I$.*

Remark 2. *In the question 2 of the above exercise, $u(t) = 0$ is a global solution; the other solutions are maximal but not global.*

2.1.4 Regularity of solutions

Theorem 6. *If f is of class C^k, every solution of* (2.1) *is of classe C^{k+1}.*

Proof. We consider first the case $k = 0$. According to the definition, if u is a solution of (2.1) on J, then u is differentiable on J. It follows that, u is a continuous function. Since f is a continuous function from O to $\mathbb{R}^m$, it follows from the composition of continuous functions that $t \to f(t, u(t))$ is a continuous function from J to $\mathbb{R}^m$. It follows that u' is continuous and that u is of class C^1.

Now consider the case $k \in \mathbb{N}^*$. We proceed with an induction argument. We assume that the theorem is true up to $k-1$. Now, assume that f is of class C^k. Then it is of class C^{k-1} and u is of class C^k thanks to our induction assumption. Now, by composition $f(t, u(t))$ is of class C^k on J. It follows that u' est de classe C^k and therefore u is of class C^{k+1} since $u^{(k+1)} = (u')^{(k)}$ is a continuous function. □

2.2 Existence Theorem (Cauchy-Peano-Arzela)

2.2.1 IVP and Integral Equation

Lemma 1. *A continuous function $u : J \to \mathbb{R}^m$ is a solution of* (2.1) *if and only if*

$$\forall t \in J, u(t) = u_0 + \int_0^t f(s, u(s))ds. \tag{2.2}$$

Proof. If u is a solution of (2.1), integrating between 0 and t the equation

$$u' = f(s, u(s))$$

gives (2.2). Conversely, assuming that (2.2) is satisfied, one can differentiate the right hand side, which gives that u is differentiable and provides

$$u'(t) = f(t, u(t))$$

□

2.2.2 Approximate solutions and exact solution

In this paragraph, we consider a sequence of piecewise continuously differentiable functions $(u^p)_{p\in\mathbb{N}}$ all defined on an interval $[0, T]$, such that $u^p(0) = u_0$.This means that we assume that for all $p \in \mathbb{N}$, u^p is continuous and that there exists $t_0 = 0 < t_1 < ... < t_N = T$ such that u^p is of class C_1 on $(t_n, t_{n+1}), n \in \{0, ..., N-1\}$. We assume furthermore that for all $p \in \mathbb{N}$

1. $\forall t \in [0, T]$, $(t, u^p(t)) \in O$ and there exists $M > 0$ such that for all p, for all t, $||u^p(t)|| \leq M$.
2. $\forall n \in \{0, ..., N-1\}, \forall t \in (t_n, t_{n+1})$,

   $$||(u^p)'(t) - f(t, u^p(t))|| \leq \epsilon_p,$$

   with

   $$\lim_{p\to+\infty} \epsilon_p = 0.$$

Under those assumptions, the following theorem holds:

Theorem 7. *We assume that the sequence $(u^p)_{p\in\mathbb{N}}$ converges uniformly toward a function u on $[0, T]$. Then u is a solution of* (2.1).

Proof. The proof splits into two main steps:

1. First, we prove that

   $$\forall t \in (0, T] \Big|\Big| u^p(t) - u_0 - \int_0^t f(s, u^p(s))ds \Big|\Big| \leq T\epsilon_p.$$

2. Then, we take the limit as p goes to $+\infty$.

Let $t \in (0, T)$. We have:

$$\begin{aligned}
&||u^p(t) - u_0 - \int_0^t f(s, u^p(s))ds|| \\
&= ||\int_0^t (u^p)'(s)ds - \int_0^t f(s, u^p(s))ds|| \\
&\leq T\epsilon_p
\end{aligned}$$

Furthermore, since f is continuous on the compact set $[0.T] \times \bar{B}(0, M)$, f is uniformly continuous. Consider $\epsilon > 0$, there exists μ such that $|t - t'| < \mu$ and $||u_1 - u_2|| < \mu$ implies

$$||f(t, u_1) - f(t', u_2)|| \leq \epsilon$$

Let P such that $p > P$ implies

$$||u^p(t) - u(t)|| \leq \mu \text{ for all } t \in [0, T]$$

then $p > P$ implies

$$||f(t, u^p(t)) - f(t, u(t))|| \leq \epsilon \text{ for all } t \in [0, T].$$

It follows that the sequence of functions $(f(t, u^p(t)))$ converges uniformly toward $f(t, u(t))$ on $[0, T]$ as p goes to $+\infty$. Therefore

$$\lim_{p \to +\infty} \int_0^t f(s, u^p(s))ds = \int_0^t f(s, u^p(s))ds.$$

One could have also applied the Dominated Convergence Theorem. Now, since the application $|| \cdot ||$ is continuous, it follows that

$$u(t) = u_0 + \int_0^t f(s, u(s))ds$$

□

2.2.3 Ascoli's Theorem

This section deals with Ascoli's theorem. We shall state and prove it in a form particularly suitable for the proof of the existence theorem for solutions of (2.1). The framework relies on two general compact metric sets (E, d), and (F, d'). To see this in a more concrete way, it is useful to think of E and F as $E = [0, T]$ and F a closed ball of $\mathbb{R}^m$. Then, we shall prove the existence of a convergence subsequence $(u^{\varphi(p)})_{p\in\mathbb{N}}$ of $k-$lipschitz applications. Note that in the proof of existence of solutions of (2.1), this sequence will be provided by the Euler method.

Theorem 8. *Let $(u^p)_{p\in\mathbb{N}}$, be a sequence of $k-$lipschitz applications from a compact metric space (E, d) into a compact metric space (F, d'). There exists a subsequence of $(u^p)_{p\in\mathbb{N}}$ which converges uniformly. Furthermore, the limit is also a $k-$lipschitz application.*

Proof. We have to construct the subsequence. To this end, we are going to construct by induction a decreasing sequence of subsets of $\mathbb{N}$:

$$S_0 = \mathbb{N} \supset S_1 \supset ... \supset S_n ...$$

such that every of this subsets has an infinite cardinal. The subsequence will be then indexed thanks to these subsets. We assume that S_{n-1} is constructed. Then we extract S_n from S_{n-1} as follows. Since E and F are compact sets, we can cover them with a finite number of balls of radius $\frac{1}{n}$.

$$E \subset U_{i=1}^{N} B(t_i, \frac{1}{n}),$$

$$F \subset U_{j=1}^{N'} B(z_j, \frac{1}{n}).$$

Next, we remark that for all $p \in \mathbb{N}$ and $i \in \{1, ..., N\}$ there exists $j \in \{1, ..., N'\}$ such that

$$u^p(t_i) \in B(z_j, \frac{1}{n})$$

We therefore define an application,

$$\begin{array}{rl} \phi : S_{n-1} \to & \{z_1, ..., z_{N'}\}^N \\ p \mapsto & (\phi_1(p), ..., \phi_N(p)) \end{array} \tag{2.3}$$

such that for every $p \in S^{n-1}$, for every $i \in \{1, ..., N\}$

$$u^p(t_i) \in B(\phi_i(p), \frac{1}{n}).$$

Now, S^{n-1} is infinite whereas $\{z_1, ..., z_{N'}\}^N$ is finite. Therefore, there exists $l = (l_1, ..., l_N) \in \{z_1, ..., z_{N'}\}^N$ such that $\phi^{-1}(l_1, ..., l_N)$ is infinite. We define:

$$S_n = \phi^{-1}(l_1, ..., l_N).$$

Next, note that for every $\forall p, q \in S_n$, $\forall t \in E$,

$$\begin{array}{rcl} d'(u^p(t), u^q(t)) & \leq & d'(u^p(t), u^p(t_i)) + d'(u^p(t_i), u^q(t_i)) + d'(u^q(t_i), u^q(t)) \\ & & \text{where } i \text{ is such that } d(t, t_i) < \frac{1}{n} \\ & \leq & k\frac{2}{n} + d'(u^p(t_i), l_i) + d'(l_i, u^q(t_i)) \\ & \leq & \frac{2}{n}(k+1) \end{array}$$

Now, we extract the subsequence $(u^{\varphi(p)})_{p \in \mathbb{N}}$ as follows. We set $\varphi(0) = 0$ and for all $p > 0$, we define $\varphi(p)$ as:

$$\varphi(p) > \varphi(p-1) \text{ and } \varphi(p) \in S_p$$

Then,

$$\forall q > p, \varphi(p), \varphi(q) \in S_p,$$

It follows that

$$\forall t \in E, d'(y^{\varphi(p)}(t), y^{\varphi(q)}(t)) \leq \frac{2}{p}(k+1).$$

Therefore $(u^{\varphi(p)}(t))$ is a Cauchy sequence in F. Since F is compact, it is complete. Therefore it converges to some $u(t)$ in F. Since the above inequalities do not depend on t, the convergence is uniform. Finally for t, s in E

$$\begin{aligned} d'(u(t), u(s)) &\leq d'(u(t), u^{\varphi(p)}(t)) + d'(u^{\varphi(p)}(t), u^{\varphi(p)}(s)) + d'(u^{\varphi(p)}(s), u(s)) \\ &\leq d'(u(t), y^{\varphi(p)}(t)) + kd(t,s) + d'(u^{\varphi(p)}(s), y(s)) \end{aligned}$$

which gives the result as $p \to +\infty$. □

2.2.4 Euler's Method

Euler's method is a basic method to construct an approximate solution of (2.1). Here, it will be used to prove the existence of the exact solution of (2.1). We divide the interval $[0, T]$ into N sub-intervals of $h = \frac{T}{N}$. We set:

$$t_0 = 0, t_1 = h, t_2 = 2h, ..., t_N = Nh = T.$$

Then, we define a linear function as follows: we set $u(0) = u_0$, and then by induction for $t \in (t_n, t_{n+1}]$,

$$u(t) = u(t_n) + (t - t_n) f(t_n, u(t_n)).$$

2.2.5 Uniform Bounds

Let T_0 and r_0 such that

$$[0, T_0] \times \bar{B}(u_0, r_0) \subset O.$$

Exact solutions

Assume u is a solution of (2.1). Then:

$$||u(t) - u_0|| \leq \int_0^t ||f(s, u(s))|| ds \leq Mt$$

as long as $u(t)$ remains in the compact set $[0, T_0] \times \bar{B}(u_0, r_0)$, and where

$$M = \sup_{(t,z) \in [0,T_0] \times \bar{B}(u_0, r_0)} ||f(t, z)||$$

. Then for

$$t < \max\{\frac{r_0}{M}, T_0\}$$

$(t, u(t))$ remains in $[0, T_0] \times \bar{B}(y_0, r_0) \subset O$.

Approximate solution provided by Euler's method

If u is given by the Euler's method, the same calculation holds: assume $t \in (t_n, t_{n+1}]$, $n \in \{0, ..., N-1\}$

$$||u(t) - u_0|| \leq (h \sum_{k=0}^{n-1} ||f(t_k, u(t_k))|| + ||f(t_n, u(t_n))||(t - t_n)$$
$$\leq M(t_n + t - t_n) \leq Mt.$$

as long as $u(t)$ remains in $[0, T_0] \times \bar{B}(u_0, r_0)$ with

$$M = \sup_{(t,z) \in [0,T_0] \times \bar{B}(u_0, r_0)} ||f(t, z)||.$$

It follows that for

$$t < \max\{\frac{r_0}{M}, T_0\},$$

$(t, u(t))$ remains in $[0, T_0] \times \bar{B}(u_0, r_0) \subset O$.

2.2.6 Cauchy-Peano-Arzela's Theorem

We assume

$$T < \max\{\frac{r_0}{M}, T_0\}.$$

Lemma 2. *Let $\epsilon > 0$, there exists $h > 0$ such that the function provided by Euler's method satisfies the points 1 and 2 of section 2.2.2.*

Proof. Let u the function provided by Euler's method. The Uniform bound follows from the previous section. The point 2 follows from the uniform continuity of f on the compact set $[0, t_0] \times B(u_0, r_0)$. We provide hereafter a few more details.

We want to find h such that for all $n \in \{0, ..., N-1\}$, for all $t \in (t_n, t_{n+1})$:

$$||u'(t) - f(t, u(t))|| \leq \epsilon \tag{2.4}$$

Note that for $t \in (t_n, t_{n+1})$, (2.4) is equivalent to

$$||f(t_n, u(t_n)) - f(t, u(t))|| \leq \epsilon \tag{2.5}$$

Now since f is continuous on the compact $[0, T_0] \times \bar{B}(u_0, r_0)$, it is uniformly continuous. Therefore there exists $\mu > 0$ such that

$$|t - t_n| + ||u(t_n) - u(t)|| < \mu$$

implies

$$||f(t_n, u(t_n)) - f(t, u(t))|| \leq \epsilon.$$

But

$$||u(t_n) - u(t)|| = ||f(t_n, u(t_n)\big(u(t) - u(t_n)\big)|| \leq M|t - t_n|,$$

therefore, if

$$h < \frac{\mu}{M+1}$$

ineguality (2.4) is satisfied. □

Theorem 9. *Under the assumption that f is continuous on the open set O, there exists $T > 0$ and a function u such that u is a solution of* (2.1) *on* $[0, T]$.

Proof. We define T as in the previous section. Then we construct a sequence of functions with Euler's method on $[0, T]$, associated with a sequence $\epsilon_p)$ with limit 0. Note that all those functions are M-lipschitz. This follows from the arguments below. Consider a function u defined by Euler's method. Assume furthermore that $t > s$ with $s \in (t_m, t_{m+1})$ and $t \in (t_n, t_{n+1})$. Then,

$$||u(t) - u(s)|| \leq f(t_m, u(t_m))(t_{m+1} - s) + ... + f(t_n, u(t_n))(t - t_n)||$$

which leads to

$$||u(t) - u(s)|| \leq M(t - s).$$

Then we extract a converging sequence thanks to Ascoli's theorem. This provides the result thanks to the theorem 7. □

Exercise 10.
What are the solutions of the following IVP?

$$y' = 3|y|^{\frac{2}{3}},\ y(0) = 0 \tag{2.6}$$

2.3 The Cauchy-Lipschitz Theorem

2.3.1 The Cauchy-Lipschitz Theorem

Definition 14. *Let $(E, d), (F, d')$ two metric spaces. A function $f : E \to F$ is said to be locally lipschitz if*

$$\forall x \in E\, \exists \mu, k > 0 \text{ such that}$$

$$\forall (y, z) \in E \times E,\ y, z \in B(x, \mu) \Rightarrow d'(f(y), f(z)) \leq k d(y, z)$$

Theorem 10 (Cauchy-Lipschitz)**.** *We assume that f is locally lipschitz on an open set $O \subset \mathbb{R} \times R^m$. We assume that $(0, u_0) \in O$. Then, there exists $T > 0$ such that* (2.1) *admits a unique solution on* $[0, T]$.

Proof. **Existence**
Since f is locally lipschitz, f is continuous. From theorem 9, there exists a solution of (2.1) on some interval $[0, T]$.
Uniqueness
Let r_0 such that solutions of (2.1) remain in $\bar{B}(u_0, r_0)$ for $t \in [0, T]$. Since $[0, T] \times \bar{B}(u_0, r_0)$ is compact, there exists $k > 0$ such that f is k-lipschitz on $[0, T] \times \bar{B}(u_0, r_0)$. Let us assume that there are two solutions u_1, u_2 defined on $[0, T]$. Let

$$v(t) = \int_0^t ||u_1(s) - u_2(s)|| ds.$$

We have

$$||u_1(t) - u_2(t)|| \leq k \int_0^t ||u_1(s) - u_2(s)||ds.$$

which means that

$$v'(t) \leq kv(t). \tag{2.7}$$

From (2.7), by multiplying both sides by e^{-kt} and integrating, we deduce that

$$v(t) \leq 0. \tag{2.8}$$

which proves the theorem. □

Exercise 11. Prove that a function locally lipschitz in a compact set is lipschitz.
Solution
Assume that this is not the case. Then

$$\forall k > 0 \, \exists y_k, z_k; \, ||f(y_k) - f(z_k)|| \geq k||y_k - z_k||.$$

Since the space is compact, there exists y, z and subsequences $(y_k), (z_k)$, such that

$$y = \lim_{k \to +\infty} y_{\varphi(k)}, \;\; z = \lim_{k \to +\infty} z_{\varphi(k)}.$$

There is now two possibilities, either $y = z$ or $y \neq z$. If $y \neq z$, then

$$||f(y_k) - f(z_k)|| \xrightarrow{k \to +\infty} ||f(y) - f(z)||.$$

Meanwhile,

$$||f(y_k) - f(z_k)|| \geq k||y_k - z_k|| \xrightarrow{k \to +\infty} +\infty$$

which is a contradiction. So, $y = z$. In this case, let μ_y and K_y such that

$$\forall v, w \in B(y, \mu_y) ||f(v), f(w)|| \leq K_y ||v - w||.$$

Then, for k large enough,

$$y_{\varphi(k)}, z_{\varphi(k)} \in B(y, \mu_y),$$

and,

$$||f(y_{\varphi(k)}) - f(z_{\varphi(k)})|| \geq k||y_{\varphi(k)} - z_{\varphi(k)}||$$

with $k > K_y$ which is a contradiction.

It is worth noting that when f is locally lipschitz, the sequence provided by the Euler's method converges towards the solution of (2.1). We are going to prove this result. To this end we need to prove the discrete Gronwall lemma.

2.3.2 The discret Gronwall Lemma

Lemma 3. *Let u^1, u^2 two regular functions defined on $[0,T]$. We assume that f is $k-$lipshitz. We assume furthermore that*

$$||(u^i)'(t) - f(t, u^i(t))|| < \epsilon_i, \, \forall t \in [0,T], i \in \{1,2\}$$

Then for all $t \in [0,T]$

$$||u^1(t) - u^2(t)|| \leq (\epsilon_1 + \epsilon_2)\frac{e^{kt} - 1}{k}$$

Proof. We set

$$v(t) = \int_0^t ||u^1(s) - u^2(s)||ds.$$

Then, after computations, we find that

$$||u^1(t) - u^2(t)|| \leq (\epsilon_1 + \epsilon_2)t + k\int_0^t ||u^1(s) - u^2(s)||ds.$$

which means

$$v'(t) \leq (\epsilon_1 + \epsilon_2)t + kv(t). \tag{2.9}$$

From (2.9), multiplying by e^{-kt} and integrating, we deduce that

$$v(t) \leq (\epsilon_1 + \epsilon_2)\frac{e^{kt} - (1 + kt)}{k^2}. \tag{2.10}$$

Using again (2.9), we deduce that

$$||u_1(t) - u_2(t)|| \leq (\epsilon_1 + \epsilon_2)\frac{e^{kt} - 1}{k}.$$

□

2.3.3 Convergence of the sequence provided by the Euler method

Proposition 3. *We assume that f is locally lipschitiz on the open set O. Let $T > 0$ given by the Cauchy-Lipschitz Theorem. Then the sequence provided by the Euler method converges uniformly towards the solution of* (2.1).

Proof. Let r_0 as defined in the proof of the Cauchy-Peano-Arzela. Since $[0,T] \times \bar{B}(y_0, r_0)$ is compact, there exists $k > 0$ such that f is k-lipschitz on $[0,T] \times \bar{B}(u_0, r_0)$. Let (y^p) the sequence provided by the Euler method. The Gronwall ensures that (u^p) is a uniform Cauchy sequence. Since $\bar{B}(y_0, r_0)$ is complets, we deduce that (u^p) converges uniformly towards u. Thanks to the theorem 7, it follows that u is a solution of (2.1). □

2.3.4 Another proof thanks to a fixed point theorem

Another classical way to prove the Cauchy-Lipschitz theorem is to make use of a fixed point theorem. We refer to [12, 4] for such proofs.

2.4 Uniqueness of maximal solutions

Theorem 11. *Under the assumptions of the Cauchy-lipshitz theorem, the solution of* (2.1) *can be extended to a maximal interval. This extension is unique.*

Proof. Let u_1, u_2 two maximal solutions of (2.1). Assume that these two solutions are different. We set:

$$t_0 = \sup\{s \in \mathbb{R}; u_1(t) = u_2(t), t \in [0, s]\}$$

By continuity, $u_1(t_0) = u_2(t_0)$. Then we apply the Cauchy-lipschitz theorem to extend both solutions. This is a contradiction. We can also deduce that the maximal interval of definition is open. □

2.5 Theorem of existence and uniqueness of global solutions

Theorem 12. *We assume that* $O = \mathbb{R} \times R^m$ *and that* f *is* k*-lipschitz on* O*. Then* (2.1) *admits a unique global solution.*

Proof. Let

$$v(t) = \int_0^t ||u(t) - u_0||$$

We can prove in this case that

$$v(t) \leq Ce^{kt}$$

fro some constant C. If u was defined on a bounded interval $[0, b_0)$ with $b_0 < +\infty$, one could define u by continuity in b_0. This is a contradiction. See [4, 12] for more details.

□

Chapter 3

Qualitative analysis of linear systems in dimension 2

This chapter deals with a detailed analysis of linear systems of dimension 2, $i.e.$ systems which write

$$X' = AX \tag{3.1}$$

with

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

and A is a 2×2 real matrix. As we will see the qualitative behavior of (3.1) relates with the eigenvalues of the matrix A. As such this chapter is divided according to the nature of the eigenvalues of A.

3.1 Non zero distinct real eigenvalues

3.1.1 Saddle-nodes

Exercise 12. We consider the system

$$\left\{ \begin{array}{lcl} x' & = & -x \\ y' & = & 2y \end{array} \right. \tag{3.2}$$

1. Find the solutions of (3.2).

2. Sketch the trajectories in the $x - y$ plane.

Solution

1) The solutions are $x(t) = x_0 e^{-t}$, $y(t) = y_0 e^{2t}$.

2) The solutions with $x_0 = 0$ or $y_0 = 0$ lie respectively on the y axis and the x axis. The other solutions write

$$x = \frac{x_0^2 y_0}{x^2}.$$

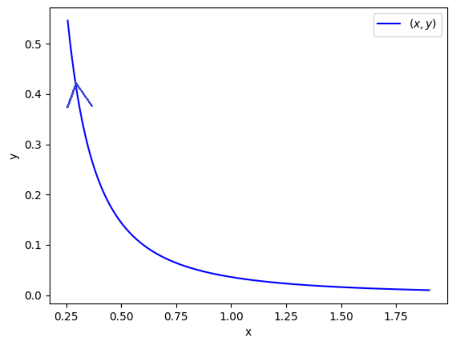


Figure 3.1: Saddle-Node. A Trajectory of (3.2) in the quarter of plane $x > 0, y > 0$.

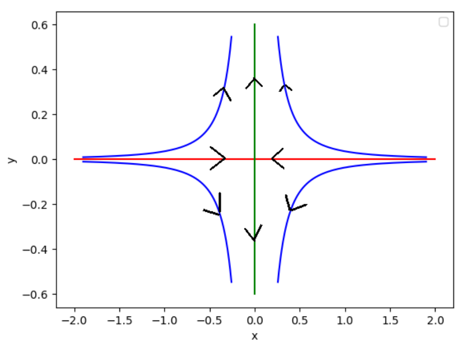


Figure 3.2: Saddle-Node. Trajectories of system (3.2). We observe that trajectories that lie in $y = 0$ converge toward $(0, 0)$. The other go to ∞

Note that

$$Det(A - \lambda I) = \lambda^2 - (TrA)\lambda + DetA$$

therefore the eigenvalues of A are

$$\lambda_1 = \frac{1}{2}(TrA - \sqrt{(TrA)^2 - 4DetA}),\ \lambda_2 = \frac{1}{2}(TrA + \sqrt{(TrA)^2 - 4DetA}).$$

For 2d systems, you can use this formula to compute the eigenvalues in a fast way.

Exercise 13. Same questions with

$$\begin{cases} x' &= x + 3y \\ y' &= x - y \end{cases} \tag{3.3}$$

Solution

1) Classicaly we look for eigenvalues and their associated eigenvectors. If λ_1 is an eigenvalue and V_1 its associated eigenvector, then on one hand,

$$Ae^{\lambda_1 t}V_1 = \lambda_1 e^{\lambda_1 t}V_1$$

and on the other hand

$$(e^{\lambda_1 t}V_1)' = \lambda_1 e^{\lambda_1 t}V_1.$$

Therefore, $\lambda_1 e^{\lambda_1 t}V_1$ is a solution. We deduce here that the solutions write

$$X(t) = \alpha e^{\lambda_1 t}V_1 + \beta e^{\lambda_2 t}V_2$$

where $\lambda_1 = -2$, $\lambda_2 = 2$ are the two eigenvalues of A, and

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ V_2 = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$$

2) The diagram of the solutions is represented in figure 3.1.1

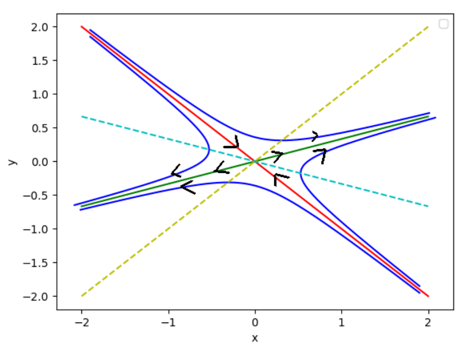


Figure 3.3: Saddle-Node. Trajectories of system (3.4). We observe that trajectories that lie in $\{\alpha V_1; \alpha \in \mathbb{R}\}$ converge toward $(0,0)$. The other go to ∞

3.1.2 Stable node (sink)

Exercise 14. Same questions with

$$\begin{cases} x' &= -2x \\ y' &= -y \end{cases} \tag{3.4}$$

Solution

1) The solutions are $x(t) = x_0 e^{-2t}$, $y(t) = y_0 e^{-t}$.

2) The solutions with $x_0 = 0$ or $y_0 = 0$ lie respectively on the y axis and the x axis. The other solutions write

$$y = \frac{x_0}{y_0^2} y^2.$$

The trajectories lying in the x axis go to 0 as t go to $+\infty$. The trajectories lying in the y axis go to 0 as t go to $-\infty$. At $t = +\infty$, the trajectories are asymptotic to the y axis.

Exercise 15. Same questions with

$$\begin{cases} x' &= -y \\ y' &= 2x - 3y \end{cases} \tag{3.5}$$

Solution

1) The solutions are

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2$$

where $\lambda_1 = -2$, $\lambda_2 = -1$ are the two eigenvalues of A, and

$$V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \; V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

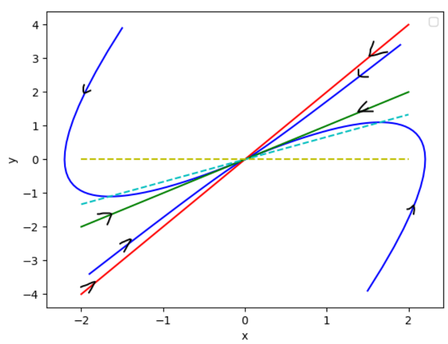


Figure 3.4: Stable Node. Trajectories of system (3.5). All solutions converge toward $(0,0)$. We observe that trajectories that lie in $\{\alpha V_1; \alpha \in \mathbb{R}\}$ stay there. Other are tangent to V_2 at $t = +\infty$

Exercise 16. Numerical simulations

1. Run a few simulations for system (3.5). Sketch some relevant solutions.
2. Prove that (3.5) rewrites
$$Y' = DY \tag{3.6}$$
with D a diagonal matrix for which the diagonal elements are the eigenvalues of the matrix associated to (3.5) and with
$$X = PY$$
with P a matrix to explicit.

3. Simulate system (3.6) and plot a few relevant solutions.
4. Plot PY for these solutions.

3.1.3 Unstable node (source)

Exercise 17. Same exercice with

$$\begin{cases} x' &= y \\ y' &= -2x+3y \end{cases} \tag{3.7}$$

3.2 Complex eigenvalues

Exercise 18. Same exercice with

$$\begin{cases} x' &= y \\ y' &= -x \end{cases} \tag{3.8}$$

Exercise 19. Same exercice with

$$\begin{cases} x' &= ax-by \\ y' &= bx+ay \end{cases} \tag{3.9}$$

Exercise 20. We consider the system

$$X' = AX \tag{3.10}$$

We assume that the matrix A has two complex eigenvalues λ et $\bar{\lambda}$. Prove that there exists a change of variables $X = PY$ such that (3.10) is equivalent to:

$$Y' = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(3.11)

with P is a matrix to explicit.
Exercise 21. Provide an analysis of

$$\begin{cases} x' &= 3x-2y \\ y' &= 4x-y \end{cases} \tag{3.12}$$

Solution

We find that the eigenvalues are complex and equal to

$$\lambda = 1-2i \ \bar{\lambda} = 1+2i$$

An eigenvector associated to λ is

$$V = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Setting

$$U = PX$$

we obtain

$$U' = HU$$

with

$$H = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

whose solutions write

$$z(t) = z_0 e^{2it}$$

with

$$z(t) = u + iv.$$

Exercise 22. Numerical simulations

Simulate (3.12) and plot a few relevant solutions.

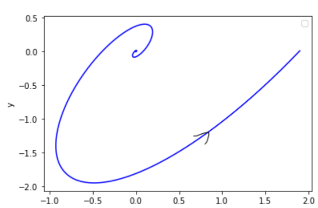


Figure 3.5: Unstable Focus. Trajectories of system (3.12).

3.3 Other cases

3.3.1 Two distinct real eigenvalues. One zero-eigenvalue

Exercise 23. Same exercise with

$$\left\{ \begin{array}{lcl} x' & = & x - y \\ y' & = & 2x - 2y \end{array} \right. \tag{3.13}$$

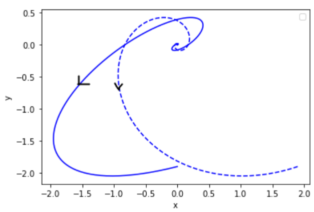


Figure 3.6: Unstable Focus. Trajectories of system (3.12) and of the associated system (3.2) (dashed line)

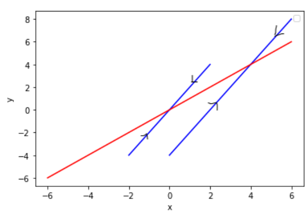


Figure 3.7: Non hyperbolic stationary point. Trajectories of system (3.13)

3.3.2 Two equal eigenvalues

A diagonal

Exercise 24. Same exercise with

$$\left\{ \begin{array}{lcl} x' & = & -x \\ y' & = & -y \end{array} \right. \tag{3.14}$$

Solution

In this case the solution write

$$X(t) = e^{-t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

and all trajectories go toward the origin along a line. See figure

A non diagonalizable

Exercise 25. Same exercise with

$$\left\{ \begin{array}{lcl} x' & = & x + y \\ y' & = & -4x - 3y \end{array} \right. \tag{3.15}$$

Solution

We find a unique eigenvalue $\lambda = -1$ with multiplicity 2. An eigenvector associated to λ is

$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Next, we look for a vector V_2 such that

$$AV_2 = \lambda V_2 + V_1.$$

We find that

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

works. It follows that

$$AP = PJ$$

with

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

and

$$J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(J is the canonical Jordan form of A.) As before, we set $X = PU$. Therefore, Eq. 3.15 becomes

$$\left\{ \begin{array}{lcl} u' & = & -u + v \\ v' & = & -v \end{array} \right. \tag{3.16}$$

After computations we find that

$$u(t) = u_0 e^{-t} + v_0 e^{-t} t,\ v_0 = v_0 e^{-t}.$$

Or

$$U(t) = u_0 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_0 e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Since $X(t) = PU(t)$, on find also that

$$\frac{y(t)}{x(t)} = \frac{u_0 p_{21} + v_0(p_{21} t + p_{22})}{u_0 p_{11} + v_0(p_{11} t + p_{12})}$$

which proves that

$$\lim_{t \to +\infty} \frac{y(t)}{x(t)} = \frac{p_{21}}{p_{11}}$$

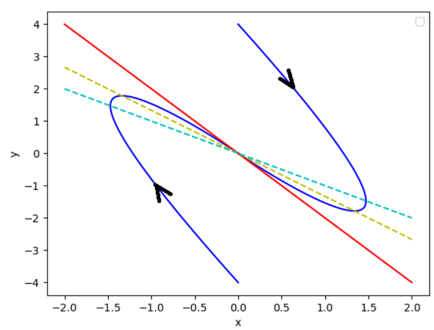


Figure 3.8: Stable Node. Trajectories of system (3.16).

Chapter 4

Qualitative analysis of nonlinear systems in dimension 2

Definition 15. *A point $u^* \in \mathbb{R}^m$ is called a stationary point (or equilibrium point) of equation*

$$u' = f(u)$$

if

$$f(u^*) = 0.$$

We assume that f is of class C^1. The notation

$$Df(u)$$

stands for the jacobian matrix of f at u.

Definition 16. *A stationary point is said to be hyperbolic if no eigenvalue of $Df(U^*)$ has a zero real part.*

Definition 17. *A stationary point $u^* \in \mathbb{R}^m$ is called a sink if all the eigenvalues of*

$$Df(U^*)$$

have a negative real part. It is called a source if all the eigenvalues have a positive real part. It is called a saddle-node if at least one eigenvalue has a negative real part and one eigenvalue has a positive real part.

In the following, for sake of simplicity for the students, we provide Pour simplifier, we provide a definiton of nodes and focus related to the jacobian. For a more standard geometric definition we suggest the reader to refer to [4]. Both definitions are equivalent for the cases considered in this lecture.

Definition 18. *We assume that f is of classe C^2 and that $m = 2$. A stationary point $u^* \in \mathbb{R}^2$ is called a node if*

$$Df(U^*)$$

has two real negative eigenvalues or two positive real eigenvalues. It is called a focus if the eigenvalues are complex with negative or positive real parts.

Definition 19. *We assume that for every $z \in \mathbb{R}^m$, equation* (1.1) *with initial condition $u(0) = z$ has a unique solution $u(s)$ for $s \in \mathbb{R}$. We define for $t \in \mathbb{R}$ an application ϕ_t from $\mathbb{R}^m$ into $\mathbb{R}^m$ which for each $z \in \mathbb{R}^m$ (the initial condition) associates $u(t)$ (the solution ensued from z at time t). The application ϕ is called the flow associated to* (1.1)*.*

Definition 20 (Stability)**.** *A stationary point u^* is said to be stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\forall z \in B(u^*, \delta) \forall t \geq 0\, \Phi_t(z) \in B(u^*, \epsilon).$$

A stationary point u^* is said to be unstable if it is not stable. It is said to be asymptotically stable if it is stable and if, furthermore, for each $z \in B(u^*, \delta)$,

$$\lim_{t \to +\infty} \Phi_t(z) = u^*.$$

Theorem 13. *A sink is asymptotically stable. A source and a saddle-node are unstable.*

Exercise 26. Prove the theorem in the case where the two eigenvalues are real, distinct and negative.
Hint: look for a change of variables as in the linear case, then compute $(x^2(t) + y^2(t))'$...
Solution
Assume that (u^*, v^*) is a sink with two negative real eigenvalues $\lambda_1 < \lambda_2 < 0$. After a change of variables around (u^*, v^*) the system,

$$\begin{aligned} u' &= f(u, v) \\ v' &= g(u, v) \end{aligned}$$

rewrites

$$\begin{aligned} u' &= f_u(u^*, v^*)u + f_v(u^*, v^*)u + h_1(u, v) \\ v' &= g_u(u^*, v^*)u + g_v(u^*, v^*)u + h_2(u, v) \end{aligned}$$

where $h(u, v) = (h_1(u, v), h_2(u, v))$ satisfies $\lim_{||(u,v)|| \to 0} \frac{||h(u,v)}{||(u,v)||} = 0$. Let P the matrix made of the two eigenvectors associated with λ_1 and λ_2, and let $X = P^{-1}U$. Then the system becomes

$$\begin{aligned} x' &= \lambda_1 x + h_1(x, y) \\ y' &= \lambda_2 y + h_2(x, y) \end{aligned}$$

where the function h changed but still satisfies $\lim_{||X|| \to 0} \frac{||h(X)}{||X||} = 0$. Now

$$(x^2 + y^2)' = 2xx' + 2yy' = 2\lambda_1 x^2 + 2\lambda_2 y^2 + 2h_1(x, y)x + 2h_2(x, y)y$$

$$\leq 2\lambda_2(x^2 + y^2) + ||h(x, y)||||(x, y)||$$

$$\leq (x^2 + y^2)(2\lambda_2 + \frac{||h(x, y)||}{||(x, y)||})$$

Then there exists $\delta > 0$ such that for $||X|| < \delta$, implies $\frac{||h(x,y)||}{||(x,y)||} + \lambda_2 < 0$. Assuming $||X(0)|| < \delta$ implies

$$(x^2 + y^2)' \leq \lambda_2(x^2 + y^2)\lambda_2$$

which implies

$$(x^2 + y^2)(t) \leq e^{\lambda_2 t}(x^2 + y^2)(0).$$

This implies the asymptotic stability.

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