Semigroups and Evolutionary problems

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Contents

These are lecture notes for the course of Semigroups for the students of Master of Mathematics of Le Havre Normandie University and the Hudson School of Mathematics. The course strongly relies on the book of Vrabie listed below. We list also other textbooks which consultation may be helpful.

- C_0 Semigroups and Applications, by Ioan I. Vrabie, Elsevier (2003)
- Functional Analysis by W. Rudin, McGraw-Hill, 2nd ed. (1991)
- Real and Complex Analysis by W. Rudin, McGraw-Hill, 3rd ed. (1987)
- Principles of Mathematical Analysis by W. Rudin, McGraw-Hill, 3rd ed. (1964)
- Functional Analysis, Sobolev Spaces and Partial Differential Equations by H. Brezis, Springer, (2010)
- Functional Analysis by K. Yosida, Springer-Verlag, 2nd ed. (1968)
- Functional Analysis and semigroups by E. Hille and R.S. Phillips, AMS (1957)
- Abstract Parabolic Evolution Equations and their Application, by A. Yagi, Springer (2010)
- Semigroups of Linear Operators and Applications to Partial Differential Equations, by A. Pazy, Springer (1983)
- An introduction to semilinear evolution equations, by T. Cazenave and A. Haraux, Oxford Press (1998)

Chapter 1

semigroups of Linear Operators

Consider the differential equation

$$x' = ax,$$

with $a \in \mathbb{R}$. The solution of this equation is

$$x(t) = x(0)e^{at}$$

and satisfies $x(t+s) = e^{as}e^{at}x(0)$. If we define for $t \in \mathbb{R}$, $S(t)$ by

$$S(t) : \begin{cases} \mathbb{R} & \to \mathbb{R} \\ x & \mapsto e^{at}x \end{cases}$$

then, we have,

$$S(t+s) = S(t)S(s)$$

and

$$S(0) = I.$$

The main purpose of this chapter is to analyze what happens when a is replaced by an operator acting on functional spaces. We refer to [Gre06] for more details on the historical links between the original works of Peano on ordinary differential equations and the further developments in functional analysis.

1.1 Uniformly Continuous semigroups

Let X be a Banach space and let $\mathcal{L}(X)$ be the set of all linear bounded operators from X to X. Endowed with the operator norm $||\cdot||_{\mathcal{L}(X)}$, defined for $U \in \mathcal{L}(X)$ by

$$||U||_{\mathcal{L}(X)} = \sup_{||x|| \leq 1} ||U(x)||.$$

It is known that $\mathcal{L}(X)$ is a Banach space.

Definition 1. *A family $\{S(t); t \geq 0\}$ in $\mathcal{L}(X)$ is a semigroup of linear operators on X, or simply semigroup if*

$$(i)\ S(0) = I$$

$$(ii)\ S(t+s) = S(t)S(s) \text{ for each } t, s \geq 0.$$

If, in addition, it satisfies the following continuity condition at $t = 0$

$$\lim_{t \to 0^+} S(t) = I$$

in the norm topology of $\mathcal{L}(X)$, the semigroup is called uniformly continuous.

A first important example of an uniformly continuous semigroup is given by $S(t) = e^{tA}$, where e^{tA} is the exponential of the matrix tA.

Exercise 1:
Let A be a $n \times n$ real matrix. Define, for each $t > 0$, $S(t) : \mathbb{R}^n \to \mathbb{R}^n$ y $S(t)x = e^{tA}x$, where

$$e^{tA} = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}$$

1. Prove that $\{S(t); t \geq 0\}$ is a uniformly continuous semigroup of linear operators.

2. Prove that $t \to S(t)$ if of class C^1 from $[0, +\infty)$ to $\mathcal{L}(\mathbb{R}^n)$ and satisfies

$$\frac{d}{dt} S(t) = AS(t) \tag{1.1}$$

Solution 1:
1. The fact that $e^{(t+s)A} = e^{tA}e^{sA}$ follows from the absolute convergence of the exponential series. The computations are analog to the equality for the exponential of complex numbers, see for example the prologue in [Rud87]. For the reader's convenience we provide here a few details. First for all $t \geq 0$

$$\sum_{k=0}^{+\infty} \frac{1}{k!} ||At||^k = e^{t||A||}.$$

Therefore the series

$$\sum_{k=0}^{+\infty} \frac{1}{k!} (At)^k$$

converges absolutely so it converges in $\mathcal{L}(\mathbb{R}^n)$ (to a matrix). Next we want to compare

$$e^{(t+s)A} \text{ and } e^{tA}e^{sA}.$$

To do that, we compare

$$\sum_{k=0}^{N} \frac{1}{k!}(At)^k \sum_{k=0}^{N} \frac{1}{k!}(As)^k,$$

and

$$\sum_{k=0}^{N} \frac{1}{k!}((A(t+s))^k.$$

We remark that

$$||\sum_{k=0}^{N} \frac{1}{k!}(At)^k \sum_{k=0}^{N} \frac{1}{k!}(As)^k - \sum_{k=0}^{N} \frac{1}{k!}((A(t+s))^k||$$

$$= ||\sum_{k=0}^{N} \frac{1}{k!}(At)^k \sum_{k=0}^{N} \frac{1}{k!}(As)^k - \sum_{k=0}^{N} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} ||At||^i ||As||^{k-i}||$$

$$= ||\sum_{k=0}^{N} \frac{1}{k!}(At)^k \sum_{k=0}^{N} \frac{1}{k!}(As)^k - \sum_{k=0}^{N} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} ||A||^k ||t^i s^{k-i}||$$

$$\leq || \sum_{k=N}^{2N} \frac{1}{k!} \big(||A||(t+s)\big)^k|| \xrightarrow[N \to +\infty]{} 0.$$

One have also $S(0) = Id$. To prove that the semigroup is uniformly continuous we need to prove that $||e^{tA} - Id||$ converge toward 0 as $t \to 0^+$. Note that

$$\begin{aligned} ||S(t) - Id|| = & \ ||\textstyle\sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} - Id|| \\ = & \ ||\textstyle\sum_{k=1}^{+\infty} \frac{t^k A^k}{k!}|| \\ \leq & \ \textstyle\sum_{k=1}^{+\infty} \frac{|t|^k ||A||^k}{k!} \\ \leq & \ \textstyle\sum_{k=0}^{+\infty} \frac{|t|^k ||A||^k}{k!} - 1 \\ \leq & \ e^{||A||t} - 1 \xrightarrow[t \to 0^+]{} 0 \end{aligned}$$

2. Next, we consider the quantity

$$S(t+h) - S(t).$$

We have

$$\begin{aligned} S(t+h) - S(t) = & \ \textstyle\sum_{k=0}^{+\infty} \frac{(t+h)^k A^k}{k!} - \sum_{k=0}^{+\infty} \frac{(t)^k A^k}{k!} \\ = & \ \textstyle\sum_{k=0}^{+\infty} \sum_{i=0}^{k} \frac{\binom{k}{i}}{k!} t^{k-i} h^i A^k - \sum_{k=0}^{+\infty} \frac{(t)^k A^k}{k!} \\ = & \ \textstyle\sum_{k=0}^{+\infty} \frac{(t)^k A^k}{k!} + \sum_{k=1}^{+\infty} \sum_{i=1}^{k} \frac{\binom{k}{i}}{k!} t^{k-i} h^i A^k - \sum_{k=0}^{+\infty} \frac{(t)^k A^k}{k!} \\ = & \ \textstyle\sum_{k=1}^{+\infty} \sum_{i=1}^{k} \frac{\binom{k}{i}}{k!} t^{k-i} h^i A^k \end{aligned} \tag{1.2}$$

It follows that

$$\frac{S(t+h) - S(t)}{h} = \ \textstyle\sum_{k=1}^{+\infty} \sum_{i=1}^{k} \frac{\binom{k}{i}}{k!} t^{k-i} h^{i-1} A^k \tag{1.3}$$

When $h \to 0$ this quantity converges toward

$$\sum_{k=1}^{+\infty} \frac{k}{k!} t^{k-1} A^k$$

which is equal to

$$A \sum_{k=1}^{+\infty} \frac{1}{(k-1)!} t^{k-1} A^{k-1}$$

or

$$Ae^{At}.$$

It follows that $S(t)$ admits a derivative and that

$$\frac{d}{dt} e^{At} = Ae^{At}.$$

To prove that $S'(t)$ is continuous, we write

$$\begin{aligned} ||Ae^{A(t+h)} - Ae^{At}|| = & \ ||Ae^{At}(e^{Ah} - Id)|| \\ \leq & \ ||A||e^{||A||t}||e^{Ah} - Id|| \\ \leq & \ ||A||e^{||A||t}||e^{Ah} - Id|| \\ & \xrightarrow[h \to 0]{} 0 \end{aligned} \tag{1.4}$$

Exercise 2:

Let X be the space of all bounded and uniformly continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$, endowed with the sup-norm $|| \cdot ||_\infty$, and let $\{S(t); t \geq 0\} \subset \mathcal{L}(X)$ be defined by

$$(S(t)f)(s) = f(t+s)$$

for each $f \subset X$ and each $t, s \subset \mathbb{R}^+$.

1. Prove that S is a semigroup.
2. Prove that S is not uniformly continuous.

Solution 2:
1. Write it down!
2. Consider the sequence of functions

$$f_n(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{n} \\ n(t-\dfrac{1}{n}) & \text{if } \frac{1}{n} \leq t \leq \frac{2}{n} \\ 1 & \text{if } t \geq \frac{2}{n} \end{cases}$$

and remark that

$$||S(\frac{1}{n})f_n - f||_\infty = 1.$$

Definition 2. *The infinitesimal generator, or generator of the semigroup of linear operators $\{S(t); t \geq 0\}$ is the operator $A : D(A) \subseteq X \to X$, defined by*

$$D(A) = \{x \in X; \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists}\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}$$

Equivalently, we say that A generates $\{S(t); t \geq 0\}$.

1.2 Generators of Uniformly Continuous Semigroups

Theorem 1. *A linear operator $A : D(A) \subset X \to X$ is the generator of a uniformly continuous semigroup if and only if $D(A) = X$ and $A \in \mathcal{L}(X)$.*

Proof
Assume that $\{S(t); t \geq 0\}$ is a uniformly continuous semigroup. By definition,

$$\lim_{t\to 0} S(t) = I$$

in $\mathcal{L}(X)$. Next remark that

$$\frac{1}{\rho}\int_0^\rho S(t)dt - I$$
$$= \frac{1}{\rho}\int_0^\rho (I + S(t) - I)dt - I$$
$$= \frac{1}{\rho}\int_0^\rho (S(t) - I)dt$$

Since

$$\frac{1}{\rho}||\int_0^\rho (S(t) - I)dt|| \leq \frac{1}{\rho}\int_0^\rho ||S(t) - I||dt \to 0 \text{ as } \rho \to 0^+$$

we obtain that

$$||\frac{1}{\rho}\int_0^\rho S(t)dt - I|| < 1$$

for ρ small enough. It follows that $\frac{1}{\rho}\int_0^\rho S(t)dt$ and therefore $\int_0^\rho S(t)dt$ is invertible (this is because if $||B|| < 1$ then $(I+B)\sum_{n=0}^{+\infty}(-1)^n B^n = I$). Next, we write

$$\frac{1}{h}\big(S(h)-I\big)\int_0^\rho S(t)dt = \frac{1}{h}\int_0^\rho S(t+h)dt - \frac{1}{h}\int_0^\rho S(t)dt$$

We set $s = t+h$ in the right-hand side of the equation. This gives

$$\frac{1}{h}\big(S(h)-I\big)\int_0^\rho S(t)dt = \frac{1}{h}\int_h^{\rho+h} S(s)ds - \frac{1}{h}\int_0^\rho S(t)dt$$

$$= \frac{1}{h}\int_h^\rho S(s)ds + \frac{1}{h}\int_\rho^{\rho+h} S(s)ds - \frac{1}{h}\int_0^h S(t)dt - \frac{1}{h}\int_h^\rho S(t)dt$$

$$= \frac{1}{h}\int_\rho^{\rho+h} S(s)ds - \frac{1}{h}\int_0^h S(t)dt$$

and then

$$\frac{1}{h}\big(S(h)-I\big) = \Big(\frac{1}{h}\int_\rho^{\rho+h} S(s)ds - \frac{1}{h}\int_0^h S(t)dt\Big)\Big(\int_0^\rho S(t)dt\Big)^{-1}$$

Now, the right-hand side converges in $\mathcal{L}(X)$ as $h \to 0$. So do the left hand side then. It follows that

$$A = \big(S(\rho)-I\big)\Big(\int_0^\rho S(t)dt\Big)^{-1} \tag{1.5}$$

Now assume that $A \in \mathcal{L}(X)$. We define:

$$S(t) = \sum_{k=0}^{+\infty}\frac{t^k A^k}{k!}$$

$S(t)$ is a semigroup of linear operators (see exercise 1). We need to proof that it is uniformly continuous.

$$S(t) - I = \sum_{k=1}^{+\infty}\frac{t^k A^k}{k!}$$

$$S(t) - I = At\sum_{k=1}^{+\infty}\frac{t^{k-1} A^{k-1}}{k!}$$

It follows that

$$||S(t)-I|| \le t||A||e^{||A||t}$$

which shows that S is uniformly continuous. It remains to prove that A is the infinitesimal generator of S.

$$\frac{1}{h}\big(S(h)-I\big) - A = \sum_{k=1}^{+\infty}\frac{h^{k-1}A^k}{k!} - A$$

$$= \sum_{k=2}^{+\infty}\frac{h^{k-1}A^k}{k!}$$

$$= hA^2\sum_{k=2}^{+\infty}\frac{h^{k-2}A^{k-2}}{k!}$$

It follows that

$$||\frac{1}{h}\big(S(h)-I\big) - A|| \le h||A||^2 e^{||A||h}.$$

This completes the proof. $\square$

Consider now the Cauchy problem

$$\begin{cases} u' &= Au + f \\ u(0) &= \xi \end{cases} \tag{1.6}$$

where $A \subset \mathcal{L}(X)$ and $f \in C([0,T];X)$.

Theorem 2. *For any $(\xi, f) \in X \times C([0,T];X)$, Equation (1.6) has a unique solution $u \in C([0,T];X)$ given by the so called variation of constants, or Duhamel, formula*

$$u(t,\xi,f) = S(t)\xi + \int_0^t S(t-s)f(s)ds$$

for each $t \in [0,T]$ where $\{S(t); t \geq 0\}$ is the semigroup generated by A.

1.3 C_0-semigroups

Definition 3. *A semigroup of linear operators $\{S(t); t \geq 0\}$ is called a semigroup of class C_0, or C_0-semigroup if for each $x \in X$ we have*

$$\lim_{t \to 0^+} S(t)x = x$$

Theorem 3. *If $\{S(t); t > 0\}$ is a C_0-semigroup, then there exists a constant $M > 1$ and $\omega \in \mathbb{R}$ such that*

$$||S(t)|| \leq Me^{\omega t} \; \forall t \geq 0.$$

Proof

We first prove that there exists $M > 1$, $\mu > 0$ such that for

$$||S(t)||_{\mathcal{L}(X)} \leq M \forall t \in [0, \mu]$$

Assume that this is not the case. Then for all $M \geq 1$ and for all $\mu > 0$ there exists $t_{M,\mu} \in (0, \mu]$ such that

$$||S(t_{M,\mu})||_{\mathcal{L}(X)} > M.$$

So we can construct a sequence t_n such that for all $n \in \mathbb{N}$

$$0 < t_n < \frac{1}{n} \text{ and } ||S(t_n)||_{\mathcal{L}(X)} > n. \tag{1.7}$$

Note that since $(S(t))$ is a C_0-semigroup,

$$\forall x \in X \quad \lim_{n \to +\infty} S(t_n)x = x$$

It follows that

$$\forall x \in X \sup_{n \in \mathbb{N}} ||S(t_n)x|| < +\infty.$$

From the uniform boundedness theorem, we deduce that

$$\sup_{n \in \mathbb{N}} ||S(t_n)||_{\mathcal{L}(X)} < +\infty,$$

which contradicts eq. (1.7). Next, for $t > 0$, we write $t = n\mu + \delta$, with $\delta \in (0, \mu)$. We have

$$S(t) = S(\delta)S(\mu)^n$$

and therefore

$$\begin{aligned} ||S(t)|| &\leq M^{n+1} \\ &\leq e^{(n+1)\ln M} \\ &\leq e^{(\frac{t-\delta}{\mu}+1)\ln M} \\ &\leq Me^{\frac{t}{\mu}\ln M} \end{aligned}$$

with gives the result with $\omega = \frac{\ln M}{\mu}$. □

Definition 4. *A C_0 semigroup is called a C_0 semigroup of type (M, ω) if*

$$||S(t)||_{\mathcal{L}(X)} \leq Me^{\omega t} \forall t \geq 0$$

Definition 5. *A C_0 semigroup is called a C_0 semigroup of contractions if*

$$||S(t)||_{\mathcal{L}(X)} \leq 1 \,\forall t \geq 0$$

Exercise 3:

1. Discuss the notion of Riemann integral do define

$$\int_0^t S(s)ds$$

where $(S(t))$ is a uniformly continuous semigroup.
What can you say if S is a C_0 semigroup?
2. Look for the definition of the Bochner Integral, for example in [Yos68]. See also [Eva10].

Solution 3:

1. Let $t > 0$. We consider the sum

$$K_n = \frac{t}{n}\sum_{k=0}^{n-1} S(k\frac{t}{n}).$$

We can prove that (K_n) is a Cauchy sequence in $\mathcal{L}(X)$. Indeed, assume $n > p$

$$K_n - K_p = \frac{t}{n}\sum_{k=0}^{n-1} S(k\frac{t}{n}) - \frac{t}{p}\sum_{k=0}^{p-1} S(k\frac{t}{p}).$$

Reordering $(\frac{kt}{p})$ and $(\frac{nt}{p})$ as a subdivision (t_i), we can rewrite this last equality as

$$K_n - K_p = \sum_{i=0}^{l-1}(t_{i+1} - t_i)S(\phi^n(t_i)) - \sum_{i=0}^{l-1}(t_{i+1} - t_i)S(\phi^p(t_i)).$$

where $|\phi^p(t_i) - \phi^n(t_i)| < \frac{t}{p}$. Now, since $(S(t))$ is uniformly continuous, we have that

$$\forall \epsilon > 0 \exists \delta \, s.t. \, \tau < \delta \Rightarrow \forall s \in [0, t - \tau) \, ||S(s+\tau) - S(s)|| \leq ||S(s)|| ||S(\tau) - Id|| < e^{||A||s}\epsilon$$

It follows that for all $\epsilon > 0$, for p, n large enough

$$\leq \sum_{i=0}^{l-1}(t_{i+1} - t_i)||S(\phi^n(t_i)) - S(\phi^p(t_i))||$$

$$\leq \sum_{i=0}^{l-1}(t_{i+1} - t_i)||S(\min(\phi^n(t_i), \phi^p(t_i))|| ||S(|\phi^n(t_i) - \phi^p(t_i)|) - I||$$

$$\leq te^{|A|t}\epsilon.$$

Therefore (K_n) is Cauchy. So it converges. We set

$$\int_0^t S(s)ds = \lim_{n\to+\infty} K_n.$$

If $(S(t))$ is a C_0-semigroup one can define in the same way

$$\int_0^t S(s)xds \forall x \in X.$$

Exercise 4:

Let $\{S(t); t \geq 0\}$ be a C_0 semigroup. Show that the map

$$\begin{aligned} [0, +\infty) \times X &\to X \\ (t, x) &\to S(t)x \end{aligned}$$

is continuous.

Solution 4:

Let $h > 0$ and $y \in X$

$$\begin{aligned} ||S(t+h)y - S(t)x|| &= ||S(t+h)y - S(t+h)x + S(t+h)x - S(t)x|| \\ &\leq ||S(t+h)y - S(t+h)x|| + ||S(t+h)x - S(t)x|| \\ &\leq ||S(t+h)|| ||y - x|| + ||S(t)|| ||S(h)x - x|| \\ &\leq Me^{\omega(t+h)} ||y - x|| + ||S(t)|| ||S(h)x - x|| \\ &\xrightarrow[(h,x)\to(0^+,x)]{} 0 \end{aligned}$$

Let $h < 0$ and $y \in X$

$$\begin{aligned} ||S(t+h)y - S(t)x|| &= ||S(t+h)y - S(t+h)x + S(t+h)x - S(t)x|| \\ &\leq ||S(t+h)y - S(t+h)x|| + ||S(t+h)x - S(t)x|| \\ &\leq ||S(t+h)|| ||y - x|| + ||S(t+h)|| ||x - S(-h)x|| \\ &\leq Me^{\omega(t+h)} \big(||y - x|| + ||x - S(-h)x|| \big) \\ &\xrightarrow[(h,x)\to(0^-,x)]{} 0 \end{aligned}$$

Exercise 5:

Check Equation (1.5) if $S(t) = e^{at}$.

Solution 5:

Left to the reader.

Exercise 6:
Prove Theorem 2.

Solution 6:

Left to the reader.

Theorem 4. *Let $A : D(A) \subseteq X \to X$, be the generator of a C_0 semigroup of linear operators $\{S(t); t \geq 0\}$. Then*

1. $$\forall t > 0 \forall x \in X \lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} S(s)xds = S(t)x \tag{1.8}$$

2. $$\forall t > 0 \forall x \in X, \int_0^t S(s)xds \in D(A) \text{ and } A \int_0^t S(s)xds = S(t)x - x$$

3. *for each $x \in D(A)$ and each $t \geq 0$, $S(t)x \in D(A)$. In addition, the mapping $t \to S(t)x$ is of class C^1 on $[0, +\infty)$, and satisfies*
$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax$$

4. *for each $x \in D(A)$ and each $0 \leq s \leq t < +\infty$, we have*

$$\int_s^t AS(\tau)x d\tau = \int_s^t S(\tau)Ax d\tau = S(t)x - S(s)x$$

Proof

1.
$$\begin{aligned} & ||\frac{1}{h}\int_t^{t+h} S(s)x ds - S(t)x|| \\ = & ||\frac{1}{h}\big(\int_t^{t+h} S(s)x ds - \int_t^{t+h} S(t)x ds\big)|| \\ \leq & \frac{1}{h}||S(t)||_{\mathcal{L}(X)}\int_t^{t+h} ||S(s-t)x - x|| ds \end{aligned}$$

which converges to 0 as h goes to 0^+.

2.
$$\begin{aligned} & \frac{1}{h}\big(S(h)\int_0^t S(s)x ds - \int_0^t S(s)x ds\big) \\ = & \frac{1}{h}\big(\int_0^t S(s+h)x ds - \int_0^t S(s)x ds\big) \\ = & \frac{1}{h}\big(\int_h^{t+h} S(s)x ds - \int_0^t S(s)x ds\big) \\ = & \frac{1}{h}\big(\int_t^{t+h} S(s)x ds - \int_0^h S(s)x ds\big) \end{aligned}$$

Now from Equation (1.8), we deduce that,

$$\lim_{h\to 0^+}\frac{1}{h}\big(S(h)\int_0^t S(s)x ds - \int_0^t S(s)x ds\big) = S(t)x - x$$

which gives the result.

3.
$$\begin{aligned} & ||\frac{1}{h}\big(S(t+h)x - S(t)x\big) - S(t)Ax|| \\ \leq & ||S(t)||_{\mathcal{L}(X)}||\frac{1}{h}\big(S(h)x - x\big) - Ax|| \end{aligned}$$

which proves that, since $x \in D(A)$,

$$S(t)x \in D(A), \text{ and } \frac{d}{dt^+}S(t)x = AS(t)x = S(t)Ax$$

Now, for $h < 0$ and $t + h > 0$, we write

$$\begin{aligned} & ||\frac{1}{h}\big(S(t+h)x - S(t)x\big) - S(t)Ax|| \\ \leq & ||S(t+h)||||\frac{1}{-h}(S(-h)x - x) - Ax + Ax - S(-h)Ax|| \\ \leq & ||S(t+h)||\big(||\frac{1}{-h}(S(-h)x - x) - Ax|| + ||Ax - S(-h)Ax||\big) \end{aligned}$$

which show the left differentiability. Since $S(t)Ax$ is continuous, we obtain that $S(t)x$ is of class C^1 with respect to t.

4. for each $x \in D(A)$ and each $0 \leq s \leq t < +\infty$, we have

$$\int_s^t AS(\tau)x d\tau = \int_s^t S(\tau)Ax d\tau = \int_s^t \frac{d}{d\tau}S(\tau)x d\tau = S(t)x - S(s)x$$

□

1.4 The Infinitesimal Generator of C_0 semigroups

Definition 6. *An operator $A : D(A) \subset X \to X$ is called closed, if its graph is closed in $X \times X$.*

Theorem 5. *Assume that $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $\{S(t); t \geq 0\}$. Then $D(A)$ is dense in X, and A is a closed operator.*

Proof
Let $x \in X$. For $\varepsilon > 0$, it follows from Theorem 4 2) that

$$\frac{1}{\epsilon} \int_0^{\epsilon} S(t)x \in D(A)$$

and from Theorem 4 1) that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^{\epsilon} S(t)x dt = x$$

which proves that $D(A)$ is dense in X. Next, we prove that A is closed. Let $(x_n, Ax_n)_{n\in\mathbb{N}}$ a sequence with $x_n \in A$ for all $n \in \mathbb{N}$. We assume that (x_n, Ax_n) converges to (x, y) in $X \times X$. We want to prove that $y = Ax$. We know from Theorem 4 that

$$S(h)x_n - x_n = \int_0^h S(s)Ax_n ds,$$

taking the limit in n in this last equation, we obtain

$$S(h)x - x = \int_0^h S(s)yds$$

Dividing both sides by h and taking the limit as h goes toward zero shoes that $x \in D(A)$ and gives

$$Ax = y.$$

□

Theorem 6. *Assume that $A : D(A) \subset X \to X$ is the infinitesimal generator of two C_0-semigroups $\{S(t); t \geq 0\}$ and $\{T(t); t \geq 0\}$. Then $S(t) = T(t)$ for $t \geq 0$.*

Proof
Let $t > 0$. For $x \in D(A)$, let us consider the function

$$f(s) = T(t-s)S(s)x$$

for $s \in [0, t]$. Then

$$f'(s) = -AT(t-s)S(s)x + T(t-s)AS(s)x = 0.$$

Therefore f is constant in $[0, T]$ and $f(t) = f(0)$ gives $S(t) = T(t)$. Since $D(A)$ is dense in X the result is true for $x \in X$. □

Exercise 7:
Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup. The goal of this exercise is to prove that $\bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in X.

1. Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a function with a compact support $[a, b] \subset (0, +\infty)$. We define $x(\varphi)$ as

$$x(\varphi) = \int_0^{+\infty} \varphi(t)S(t)xdt.$$

   Prove that

$$x(\varphi) \in \bigcap_{n\in\mathbb{N}} D(A^n).$$

2. Prove that there exists a sequence (φ_n) of functions as defined above that converge toward x.

Solution 7:

Left to the reader.

Exercise 8:

We say that a semigroup of operators is a group of operators is the properties of the semigroup can be extended to $t \in \mathbb{R}$. Prove that a uniformly continuous semigroup can be extended to a uniformly continuous group.

Solution 8:

...

Chapter 2

The Hille-Yosida Theorem

2.1 The Hille-Yosida Theorem. Statement

Definition 7. *Let $A : D(A) \subset X \to X$ a linear operator. The resolvent set $\rho(A)$ is the set of all the complex numbers λ, called regular values, for which $\lambda I - A$ is one-to-one and onto, and for which $R(\lambda; A) = (\lambda I - A)^{-1}$ is continuous from X to X.*

Theorem 7 (Hille-Yosida)**.** *A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if*

$$(i) A \text{ is densely defined, closed,}$$
$$(ii)(0, +\infty) \subset \rho(A) \text{ and for each } \lambda > 0$$
$$||R(\lambda; A)||_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

2.2 Proof of the Hille-Yosida Theorem. Necessity

ProofAssume that A is the infinitesimal generator of a C_0 semigroup, then $D(A)$ is dense and A is closed. This results from Theorem 5. Therefore (i) holds. In order to prove (ii), we define

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} S(t) x dt. \tag{2.1}$$

We are going to prove that $R(\lambda) = R(\lambda; A)$ and that $||R(\lambda)|| \leq \frac{1}{\lambda}$. First note that

$$\begin{aligned} ||R(\lambda)x|| &\leq \int_0^{+\infty} e^{-\lambda t} ||S(t)|| ||x|| dt \\ &\leq \frac{1}{\lambda} ||x||. \end{aligned}$$

So the integral in Equation (2.1) is well defined. Next, we compute

$$\lim_{h \to 0+} \frac{1}{h} \Bigg(S(h) R(\lambda) x - R(\lambda) x \Bigg).$$

We have that

$$\begin{aligned} & S(h)R(\lambda)x - R(\lambda)x \\ &= \int_0^{+\infty} e^{-\lambda t} S(t+h) x dt - \int_0^{+\infty} e^{-\lambda t} S(t) x dt \\ &= \int_h^{+\infty} e^{-\lambda (t-h)} S(t) x dt - \int_0^{+\infty} e^{-\lambda t} S(t) x dt \end{aligned}$$

$$= \int_0^{+\infty} e^{-\lambda(t-h)} S(t)xdt - \int_0^h e^{-\lambda(t-h)} S(t)xdt - \int_0^{+\infty} e^{-\lambda t} S(t)xdt$$
$$= (e^{\lambda h} - 1) \int_0^{+\infty} e^{-\lambda(t)} S(t)xdt - e^{-\lambda h} \int_0^h e^{-\lambda t} S(t)xdt$$

from which we deduce that

$$\lim_{h\to 0+} \frac{1}{h} \left(S(h)R(\lambda)x - R(\lambda)x \right) = \lambda R(\lambda)x - x.$$

This implies that $R(\lambda)x \in D(A)$ and that

$$AR(\lambda)x = \lambda R(\lambda)x - x$$

which means that

$$(\lambda Id - A)R(\lambda) = Id.$$

It remains to prove that

$$R(\lambda)(\lambda Id - A) = Id.$$

To do that, assume that $x \in D(A)$, we compute

$$R(\lambda)Ax = \int_0^{+\infty} e^{-\lambda t} S(t)Axdt$$
$$= \int_0^{+\infty} e^{-\lambda t} AS(t)xdt$$
$$= \int_0^{+\infty} e^{-\lambda t} \frac{d}{dt}(S(t)x)dt$$
$$= \int_0^{+\infty} e^{-\lambda t} \frac{d}{dt}(S(t)x)dt$$
$$= -\int_0^{+\infty} \frac{d}{dt}(e^{-\lambda t})S(t)xdt + [e^{-\lambda t}S(t)x]_0^{+\infty}$$
$$= \lambda R(\lambda)x - x,$$

which gives

$$\lambda R(\lambda)x - R(\lambda)Ax = x$$

that is

$$R(\lambda)(\lambda Id - A) = Id.$$

□

2.3 Proof of the Hille-Yosida Theorem. Sufficiency

Definition 8. *Let $A : D(A) \subset X \to X$ be a linear operator satisfying (i) and (ii) in Theorem 7. Then the operator A_λ defined by $A_\lambda = \lambda AR(\lambda; A)$ is called the Yosida approximate of A*

In order to prove the sufficiency, we will provide two lemmas.

Lemma 8. *Let $A : D(A) \subset X \to X$ be a linear operator satisfying (i) and (ii) in Theorem 7. Then*

$$\forall x \in X,\ \lim_{\lambda\to+\infty} \lambda R(\lambda; A)x = x \tag{2.2}$$

$$\forall x \in X,\ A_\lambda x = \lambda^2 R(\lambda; A)x - \lambda x \tag{2.3}$$

$$\forall x \in D(A),\ \lim_{\lambda\to+\infty} A_\lambda x = Ax \tag{2.4}$$

Proof
Let $x \in D(A)$. Since
$$R(\lambda; A)(\lambda Id - A) = Id$$
we have also
$$\lambda R(\lambda; A) - Id = R(\lambda; A)A$$
Therefore, for all $x \in D(A)$,
$$||\lambda R(\lambda; A)x - x|| = ||R(\lambda; A)Ax|| \leq \frac{1}{\lambda}||Ax||$$
so that Equation (2.2) holds for $x \in D(A)$. For $x \in X$, we use the fact that $D(A)$ is dense. Let $\epsilon > 0$, and $y \in D(A)$ such that $||x - y|| < \frac{\epsilon}{4}$. Since
$$||\lambda R(\lambda; A)x - x|| = ||\lambda R(\lambda; A)x - \lambda R(\lambda; A)y + \lambda R(\lambda; A)y - y + y - x||$$
$$\leq ||\lambda R(\lambda; A)||||x - y|| + ||\lambda R(\lambda; A)y - y|| + ||y - x||$$
$$\leq 2||y - x|| + ||\lambda R(\lambda; A)y - y||$$
$$\leq \epsilon$$
if λ is large enough which proves eq. (2.2).
Next, we remark that
$$\begin{aligned} \lambda^2 R(\lambda; A) - \lambda Id = & \ \lambda\big(\lambda R(\lambda; A) - (\lambda Id - A)R(\lambda; A)\big) \\ = & \ \lambda\big(\lambda Id - (\lambda Id - A)\big)R(\lambda; A) \\ = & \ \lambda AR(\lambda; A) \\ = & \ A_\lambda \end{aligned}$$
Finally, for $x \in D(A)$,
$$\begin{aligned} A_\lambda x = & \ \lambda AR(\lambda; A)x \\ = & \ \lambda(\lambda R(\lambda; A) - Id)x \\ = & \ \lambda(\lambda R(\lambda; A) - R(\lambda; A)(\lambda Id - A))x \\ = & \ \lambda R(\lambda; A)Axm \end{aligned}$$
which converges toward Ax as λ converges to $+\infty$ thanks to eq. (2.2). □

Lemma 9. *Let $A : D(A) \subset X \to X$ be a linear operator satisfying (i) and (ii) in Theorem 7. Then for each $\lambda > 0$. A_λ is the infinitesimal generator of a uniform continuous semigroup $\{e^{tA_\lambda}; t \geq 0\}$ which satisfies*
$$\forall t \geq 0, \ ||e^{tA_\lambda}|| \leq 1.$$
Furthermore,
$$\forall x \in X, \forall \lambda, \mu > 0, \ ||e^{tA_\lambda}x - e^{tA_\mu}x|| \leq t||A_\lambda x - A_\mu x||. \tag{2.5}$$

Proof
From eq. (2.3), we deduce that
$$||A_\lambda x|| \leq \lambda(\lambda||R(\lambda; A)|| + 1)||x||)$$
$$\leq 2\lambda||x||$$
therefore A_λ is the infinitesimal generator of a uniform continuous semigroup that we call $\{e^{tA_\lambda}; t \geq 0\}$. Furthermore,
$$\begin{aligned} ||e^{tA_\lambda}|| = & \ ||e^{t\lambda^2 R(\lambda;A) - t\lambda I}|| \\ \leq & \ ||e^{t\lambda^2 R(\lambda;A)}||||e^{-t\lambda I}|| \\ \leq & \ e^{t\lambda^2||R(\lambda;A)||}e^{-\lambda t} \\ \leq & \ e^{\lambda t}e^{-\lambda t} \\ \leq & \ 1. \end{aligned}$$

In order to prove eq. (2.5), we remark that

$$e^{tA_\lambda}x - e^{tA_\mu}x = \int_0^1 \frac{d}{ds}\left(e^{stA_\lambda}e^{(1-s)tA_\mu}x\right)ds$$

$$= \int_0^1 tA_\lambda e^{stA_\lambda}e^{(1-s)tA_\mu}x - tA_\mu e^{stA_\lambda}e^{(1-s)tA_\mu}x$$

It follows that

$$||e^{tA_\lambda}x - e^{tA_\mu}x|| \leq t||A_\lambda x - A_\mu x||.$$

□

Proof(Hille-Yosida, sufficiency)
First, we need to define a good candidate $S(t)$ to be the semigroup generated by A. From, eq. (2.5) and eq. (2.4) we deduce that for any (λ_n) converging to $+\infty$, for fixed $t > 0$ and $x \in D(A)$, $e^{tA_{\lambda_n}}x$ is a Cauchy sequence in X. Therefore it converges. We set:

$$S(t)x = \lim_{\lambda \longrightarrow +\infty} e^{tA_\lambda}.$$

We note here that the convergence is uniform with respect to t on any closed interval $[0, T]$. Note that since $S(t)$ is defined as a limit, it follows from the properties of e^{tA_λ} that for all $x \in D(A)$:

$$S(t+s)x = \lim_{\lambda \longrightarrow +\infty} e^{(t+s)A_\lambda}x = \lim_{\lambda \longrightarrow +\infty} e^{tA_\lambda}e^{sA_\lambda}x = S(t)S(s)x.$$

Analogously, $S(0)x = x$ and $||S(t)|| \leq 1$. Since $S(t)$ is uniformly continuous on $D(A)$ and $D(A)$ is dense, we can extend $S(t)$ by continuity on X. We now want to prove that $S(t)$ is a C_0 semigroup. Let $\epsilon > 0$, and let $x_\epsilon \in D(A)$ such that $||x - x_\epsilon|| < \frac{\epsilon}{4}$. Then,

$$S(t)x - x = S(t)x - S(t)x_\epsilon + S(t)x_\epsilon - e^{tA_\lambda}x_\epsilon + e^{tA_\lambda}x_\epsilon - x_\epsilon + x_\epsilon - x$$

from which we deduce that

$$||S(t)x - x|| \leq \frac{\epsilon}{2} + ||S(t)x_\epsilon - e^{tA_\lambda}x_\epsilon|| + ||e^{tA_\lambda}x_\epsilon - x_\epsilon||$$

$$||S(t)x - x|| \leq \frac{\epsilon}{2} + ||S(t)x_\epsilon - e^{tA_\lambda}x_\epsilon|| + ||e^{tA_\lambda} - Id||||x_\epsilon||$$

Choosing λ such that $||S(t)x_\epsilon - e^{tA_\lambda}x_\epsilon|| < \frac{\epsilon}{4}$ on $[0, T]$, and $t_0 < T$ such that $||e^{tA_\lambda} - Id|| < \frac{\epsilon}{4||x_\epsilon||}$ on $(0, t_0)$ proves that

$$||S(t)x - x|| < \epsilon \, \forall t \in [0, t_0).$$

This proves that $(S(t))$ is a C_0 semigroup of contractions. Next, we need to prove that A generates $(S(t))$. Let $x \in D(A)$. We consider

$$Bx = \lim_{h \to 0} \frac{S(h)x - x}{h},$$

whenever this limit exists. Note that

$$S(h)x - x = \lim_{\lambda \to +\infty} e^{hA_\lambda}x - x$$

$$= \lim_{\lambda \to +\infty} \int_0^h \frac{d}{dt} e^{tA_\lambda}x dt$$

$$= \lim_{\lambda \to +\infty} \int_0^h e^{tA_\lambda}A_\lambda x dt$$

We will prove below that

$$\lim_{\lambda \to +\infty} e^{tA_\lambda}A_\lambda x = S(t)Ax. \tag{2.6}$$

Accordingly, we obtain

$$S(h)x - x = \int_0^h S(t)Axdt$$

Dividing both sides by h and taking the limit at 0 gives

$$\lim_{h\to 0^+} \frac{1}{h}\big(S(h)x - x\big) = Ax.$$

In other words, $D(A) \subset D(B)$, and for all $x \in D(A)$, $Bx = Ax$. Let us prove eq. (2.6).

$$\begin{aligned} &\forall x \in D(A),\ ||e^{tA_\lambda}A_\lambda x - S(t)Ax|| \\ &= ||e^{tA_\lambda}A_\lambda x - e^{tA_\lambda}Ax + e^{tA_\lambda}Ax - S(t)Ax|| \\ &\leq ||e^{tA_\lambda}||||A_\lambda x - Ax|| + ||e^{tA_\lambda}Ax - S(t)Ax|| \\ &\leq ||A_\lambda x - Ax|| + ||e^{tA_\lambda}Ax - S(t)Ax|| \end{aligned}$$

which converges to 0 when λ goes to 0.

Finally, we need to prove that $D(A) = D(B)$. We first remark that, since B generates a C_0 semigroup of contractions, from the necessity condition, we deduce that $1 \in \rho(B)$. It follows that $(I - B)D(B) = X$ and subsequently $D(B) = (I - B)^{-1}X$. On the other hand, from the assumptions, we know that $1 \in \rho(A)$ and $(I - A)D(A) = X$. Note also that $(I - A)D(A) = (I - B)D(A)$. Therefore $D(A) = (I - B)^{-1}X$ which provides the result. □

Exercise 9:

Prove that

$$R(\lambda; A)R(\mu; A) =$$

Solution 9:

$$\begin{aligned} &(\lambda Id - A)(\mu Id - A) = (\mu Id - A)(\lambda Id - A) \\ &\Rightarrow Id = (\mu Id - A)(\lambda Id - A)R(\mu; A)R(\lambda; A) \\ &\Rightarrow R(\lambda; A)R(\mu; A) = R(\mu; A)R(\lambda; A) \end{aligned}$$

Chapter 3

Classical Operators from Physics generating C_0 semigroups

3.1 The Heat equation -1d

In this section we consider the equation:

$$\begin{array}{rcl} u_t & = & u_{xx}\ \forall (x,t) \in (0,1) \times \mathbb{R}^{+*} \\ u(x,0) & = & u_0(x) \forall x \in (0,1) \\ u(0,t) & = & u(1,t) = 0 \forall t \geq 0 \end{array} \tag{3.1}$$

This is the heat equation in one dimensional space which describes the evolution of the temperature in a rod. Its study goes back to J. Fourier in the nineteenth century, see for example [GG05].. In order to clarify the well-posedeness of this equation, we consider the following operator A.

$$\begin{array}{rl} A : D(A) \subset L^2(0,1) \to & L^2(0,1) \\ u \to & u_{xx} \end{array} \tag{3.2}$$

with

$$D(A) = H_0^1(0,1) \cap H^2(0,1)$$

Exercise 10:

1. Prove that the operator A generates a C_0 semigroup of contractions.
2. What do you conclude with respect to Equation (3.1)?

Solution 10:

1) The idea here is to use the theorem of Hille-Yosida (theorem 7) to prove the result. To prove that the sufficient conditions i) and ii) hold, we shall study the equation

$$-Au + \lambda u = f$$

with $\lambda > 0$ and $f \in L^2(0,1)$. We look for solutions $u \in D(A)$ of

$$-u'' + \lambda u = f. \tag{3.3}$$

One could apply the Lax-Milgram theorem. However, since we work with dimension 1, it is worth to compute the solution explicitly. We first assume that f is regular to perform computations and then return to $f \in L^2$ by a limit process. Equivalently, we look at

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ f \end{pmatrix},$$

along with

$$u(0) = u(1) = 0. \tag{3.4}$$

Since the eigenvalues of $\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$ are $-\sqrt{\lambda}$ and $\sqrt{\lambda}$, this gives,

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = P \begin{pmatrix} e^{-\sqrt{\lambda}x} & 0 \\ 0 & e^{\sqrt{\lambda}x} \end{pmatrix} P^{-1} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - \int_0^x P \begin{pmatrix} e^{-\sqrt{\lambda}(x-y)} & 0 \\ 0 & e^{\sqrt{\lambda}(x-y)} \end{pmatrix} P^{-1} \begin{pmatrix} 0 \\ f \end{pmatrix} dy.$$

where

$$P = \begin{pmatrix} 1 & 1 \\ -\sqrt{\lambda} & \sqrt{\lambda} \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{\lambda}} \\ \frac{1}{2} & \frac{1}{2\sqrt{\lambda}} \end{pmatrix}$$

After a few computations, and taking into account that $u(0) = u(1) = 0$, we obtain

$$u(x) = -\frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}x} v_0 + \frac{1}{2\sqrt{\lambda}} e^{\sqrt{\lambda}x} v_0 + \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}x} \int_0^x e^{\sqrt{\lambda}y} f(y)dy - \frac{1}{2\sqrt{\lambda}} e^{\sqrt{\lambda}x} \int_0^x e^{-\sqrt{\lambda}y} f(y)dy$$

where v_0 is given by

$$\frac{e^{-\sqrt{\lambda}} \int_0^1 e^{\sqrt{\lambda}y} f(y)dy - e^{\sqrt{\lambda}} \int_0^1 e^{-\sqrt{\lambda}y} f(y)dy}{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}.$$

Therefore, for each continuous f, there exists a unique $u \in C^1(0,1)$ solution of (3.3)-(3.4). Note that for $f \in L^2(0,1)$, the expression found for the solution makes sense. Now, we approach f in L^2 by a sequence of continuous function (f_n). Solving (3.3)-(3.4), we obtain a sequence of functions (u_n) which converges toward u. Also u satisfies (3.3) in the sense of distributions and thanks to

$$(\lambda Id - A)u = f,$$

we deduce that $u \in H^2$ (since u and f are in L^2) and in H^1 (see the computation below). Furthermore, from the boundary conditions, we deduce that $u \in H_0^1$. Now,

$$\lambda u - u'' = f$$

implies

$$\lambda \int_0^1 u^2 + \int_0^1 (u')^2 = \int_0^1 fu$$

and therefore

$$\int_0^1 u^2 \leq \frac{1}{\lambda} ||f||_{L^2} ||u||_{L^2}$$

which in turn implies that

$$||u||_{L^2} \leq \frac{1}{\lambda} ||f||_{L^2}.$$

This gives the uniqueness and also the estimate

$$||(\lambda Id - A)^{-1}|| \leq \frac{1}{\lambda}.$$

Finally, to prove the sufficiency conditions, we remark that $H^2 \cap H_0^1$ is dense in L^2 (see [Bre11, Eva10]). Also $(\lambda Id - A)^{-1}$ is well defined and continuous in L^2, it is therefore closed. This implies that $(\lambda Id - A)$ is closed in $D(A)$, indeed let $(x_n) \subset D(A)$ a sequence converging toward x in L^2, such that $(\lambda x_n - Ax_n)$ converges toward y. Then by continuity,

$$x_n = (\lambda Id - A)^{-1}(\lambda x_n - Ax_n)$$

converges toward $(\lambda Id - A)^{-1} y$ which gives

$$(\lambda Id - A)x = y$$

and proves that $(\lambda Id - A)$ is closed. This implies that A is closed.

3.2 The Wave equation -1d

In this section we consider the equation:

$$\begin{array}{rcl} u_{tt} & = & u_{xx}\ \forall (x,t) \in (0,1)\times \mathbb{R}^{+*} \\ u(x,0) & = & \xi_1(x) \forall x \in (0,1) \\ u_t(x,0) & = & \xi_2(x) \forall x \in (0,1) \\ u(0,t) & = & u(1,t) = 0 \forall t \geq 0 \end{array} \tag{3.5}$$

This is the wave equation in one dimensional space. It describes the variations of a string fixed at its ends. It involves a second derivative with respect to the time. In order to study Equation (3.5) as previously, we set $v = u_t$, so that Equation (3.5) becomes

$$\begin{array}{rcl} u_t & = & v\ \forall (x,t) \in (0,1)\times \mathbb{R}^{+*} \\ v_t & = & u_{xx}\ \forall (x,t) \in (0,1)\times \mathbb{R}^{+*} \\ u(x,0) & = & \xi_1(x)\ \forall x \in (0,1) \\ v(x,0) & = & \xi_2(x)\ \forall x \in (0,1) \\ u(0,t) & = & u(1,t) = 0 \forall t \geq 0 \end{array} \tag{3.6}$$

Setting $z = (u,v)$, Equation (3.6) rewrites $z_t = Az$, $z(0) = z_0$ where we define A as

$$\begin{array}{rl} A : D(A) \subset H_0^1(0,1)\times L^2(0,1) \rightarrow & H_0^1(0,1)\times L^2(0,1) \\ (u,v) \rightarrow & (v, u_{xx}) \end{array} \tag{3.7}$$

with

$$D(A) = H_0^1(0,1) \cap H^2(0,1) \times L^2(0,1).$$

Furthermore, we endow the space $X = H_0^1(0,1)\times L^2(0,1)$ with the scalar product

$$\big((u_1,v_1),(u_2,v_2)\big) = \int_0^1 u_1' u_2' + \int_0^1 v_1 v_2.$$

Endowed with this scalar product, X is a a real Hilbert space.

Exercise 11:

1. Prove that the operator A generates a C_0 semigroup of contractions. What about $-A$?
2. What do you conclude with respect to Equation (3.6)?

Solution 11:

1. As before for all given $w = (f,g) \in X$ we study the equation

$$(\lambda Id - A)z = w$$

for $z = (u,v) \in D(A)$. This rewrites

$$\begin{pmatrix} \lambda u - v \\ \lambda v - u_{xx} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{3.8}$$

Multiplying the first equation by λ and summing up the two equations gives

$$\lambda^2 u - u_{xx} = \lambda f + g$$

We can now use the same computations as in the previous section to obtain:

$$u(x) = -\frac{1}{2\lambda}e^{-\lambda x}h_0 + \frac{1}{2\lambda}e^{\lambda x}h_0 + \frac{1}{2\lambda}e^{-\lambda x}\int_0^x e^{\lambda y}(\lambda f(y)+g(y))dy - \frac{1}{2\lambda}e^{\lambda x}\int_0^x e^{-\lambda y}(\lambda f(y)+g(y))dy$$

with

$$h_0 = \frac{e^{-\lambda}\int_0^1 e^{\lambda y}(\lambda f(y)+g(y))dy - e^{\lambda}\int_0^1 e^{-\lambda y}(\lambda f(y)+g(y))dy}{e^{\lambda} - e^{-\lambda}}.$$

And then v is given by

$$v = \lambda u - f.$$

Therefore, $(\lambda Id - A)^{-1}$ is well defined in X. Next, multiplying the derivative of the first equation in Equation (3.8) by u_x and the second by v, integrating and summing up gives

$$\lambda \int u_x^2 - \int v_x u_x + \lambda v^2 - \int u_{xx} v = \int f_x u_x + \int gv$$

Integrating by parts leads to

$$\lambda \int u_x^2 + \lambda v^2 = \int f_x u_x + \int gv.$$

This rewrites

$$\lambda ||z||_X^2 = (z, w).$$

Applying the Cauchy-Schwartz inequality gives

$$\lambda ||z||_X^2 \leq ||z||_X ||w||_X$$

and therefore

$$||z||_X \leq \frac{1}{\lambda} ||w||_X$$

or equivalently

$$||(\lambda Id - A)^{-1} w||_X \leq \frac{1}{\lambda} ||w||_X.$$

This proves the result.

An analogous result holds in $\mathbb{R}^n$, see [Bre11, Vra03].

3.3 The Heat Equation in 3d with Dirichlet Boundary Conditions

In this section we consider the equation:

$$\begin{aligned} u_t &= \Delta u \ \forall (x,t) \in \Omega \times \mathbb{R}^{+*} \\ u(x,0) &= u_0(x) \forall x \in \Omega \\ u(x,t) &= 0 \forall t \geq 0 \forall x \in \partial\Omega \end{aligned} \tag{3.9}$$

where $\Omega \subset \mathbb{R}^3$ is an open set. We will consider different space of functions in order to write Equation (3.9) as $u_t = Au$ where the operator A generates a C_0 semigroup of contractions, as it was the case in $1d$. From now on, we assume that the reader is familiar with the basics of Sobolev spaces. We refer for example to [Bre11, GT01, Jos13, Eva10, Vra03].

3.3.1 The L^2 setting

Let $X = L^2(\Omega)$, and $A = \Delta$ with $D(A) = \{u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\}$. Then A generates a C^0 semigroup of contractions on X. The proof is suggested as an exercise for the reader.

Exercise 12:
Prove that the operator A generates a C_0 semigroup of contractions.

Solution 12:
For $f \in L^2(\Omega)$ and $\lambda > 0$, we consider the equation

$$-\Delta u + \lambda u = f \tag{3.10}$$

We set

$$a(u,v) = \int_\Omega \nabla u . \nabla v dx + \lambda \int_\Omega uv dx$$

and

$$F(v) = \int_{\Omega} fvdx.$$

We rewrite Equation (3.10) in a weak sense as

$$\int_{\Omega} \nabla u.\nabla vdx + \lambda \int_{\Omega} uvdx = \int_{\Omega} fvdx \, \forall v \in H_0^1. \tag{3.11}$$

Assume that H_0^1 is endowed with scalar product induced by H^1. Then $F \in H^{-1}$ and a is bilinear continuous and coercive on H_0^1. We deduce from the Lax-Milgram theorem (see for example [Bre11, GT01]) that for every $f \in L^2$ the equations admits a unique weak solution in H_0^1. Next, choosing $u = v$ in Equation (3.11), and applying the Cauchy-Schwarz inequality in the right hand side f=gives

$$||(\lambda I - A)^{-1}||_{L^2} \leq \frac{1}{\lambda} \, \forall \lambda > 0.$$

3.3.2 The L^p setting

Let $1 < p < +\infty$. Let $X = L^p(\Omega)$, and $A = \Delta$ with $D(A) = W_0^{1,p}(\Omega) \cap W^{2,p}$. Then A generates a C^0 semigroup of contractions on X. Details can be found in [Vra03, Paz83]. The density of $D(A)$ in X is a classical result. For the existence and uniqueness of the solution of

$$-Au + \lambda u = f$$

with $f \in L^p$, we refer to [GT01]. For the estimate of $R(\lambda, A)$, we propose to solve it as an exercise.
Exercise 13:

Prove that for $\lambda > 0$

$$||\lambda I - A||_X \leq \frac{1}{\lambda}$$

Solution 13:
This exercise is left to the reader.

3.3.3 The $C_0(\bar{\Omega})$ setting

Let $X = C_0(\bar{\Omega})$, and $A = \Delta$ with $D(A) = \{H_0^1(\Omega) \cap C_0(\bar{\Omega}), \Delta u \in C_0(\bar{\Omega})\}$. Then A generates a C^0 semigroup of contractions on X. See [Vra03] and [GT01] for more details.

3.4 The Heat Equation in 3d with Neumann Boundary Conditions

3.5 The Maxwell Equation

In this section, we discuss the existence of solutions of the Mxwell Equation. Maxwell Equations describes the time evolution of electric field E and a magnetic field H. We recall that for $\varphi \in L^2(\mathbb{R}^3)$ and $F(F_1, F_2, F_3) \in (L^2(\mathbb{R}^3))^3$:

$$\nabla \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \varphi}{\partial x_3} \end{pmatrix}$$

$$\nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

$$\nabla \times F == \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ -\frac{\partial F_3}{\partial x_1} + \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

Then, the Maxwell Equation writes

$$\begin{aligned} E_t = & \quad -c\nabla \times H \\ H_t = & \quad c\nabla \times E \\ \nabla \cdot E = & \quad \nabla \cdot H = 0 E(0) = E_0, \\ H(0) = H_0 & \end{aligned}$$

Define

$$D(A) = \{H, E \in (L^2(\mathbb{R}^3))^3, \nabla \times H . \nabla \times E \in (L^2(\mathbb{R}^3))^3\}$$

$$A(E, H) = (-c\nabla \times H, c\nabla \times E)$$

The following theorem holds (see [Vra03]

Theorem 10. *A generates a C_0 semigroup in $(L^2(\mathbb{R}^3))^3 \times (L^2(\mathbb{R}^3))^3$.*

3.6 The Schrodinger Equation

The Schrodinger equation below describes the time evolution of a wave function, the quantum-mechanical characterization of an isolated physical system.

$$\frac{\partial}{\partial t}\varphi(x,t) = i\big(\frac{h}{2m}\varphi_{xx}(x,t) - \frac{1}{h}V(x)\varphi(x,t)\big)$$

where φ is a complex valued function, m is the mass of the particle, $V(x)$ is the potential that represents the environment in which the particle exists and h is the reduced Planck constant. Define $X = L^2(\Omega, \mathbb{C})$, and

$$D(A) = \{u \in H_0^1(\Omega, \mathbb{C}); \Delta u \in X\}$$

$$Au = i\Delta u$$

The following theorem holds (see [Vra03]

Theorem 11. *A generates a C_0 semigroup in X .*

3.7 Some insights about the L^p setting-Fundamental solutions of the Laplace and the Poisson Equations

The L^p setting is much more technical than the L^2 setting. This paragraph is intended to provide the reader with the basic ideas on which rely the more advanced techniques of the L^p setting. A classical reference is [GT01]. Two fundamental papers are [ADN59, ADN64]. A key ingredient is to provide fundamental solutions of the Laplace and Poisson Equations. We follow here [Eva10] and [Jos13]. We recall that the Laplace equation is

$$-\Delta u = 0$$

and the Poisson equation writes

$$-\Delta u = f.$$

Solutions of the Laplace equation are called harmonic functions. Those equations are very important in Physics, see [Eva10].

Exercise 14:
Find radial solutions of the Laplace equation for $n \geq 2$.

Solution 14:

We start with the case $n = 2$. We set

$$u(x) = \varphi(||x||),$$

with $x = (x_1, x_2)$.

$$\frac{\partial u}{\partial x_1} = \varphi'(||x||) \times \frac{1}{2}(x_1^2 + x_2^2)^{-\frac{1}{2}} \times 2x_1$$
$$= \varphi'(||x||)(x_1^2 + x_2^2)^{-\frac{1}{2}} x_1.$$

And,

$$\frac{\partial^2 u}{\partial x_1^2} = \varphi'(||x||)(x_1^2 + x_2^2)^{-1} x_1^2 - \varphi'(||x||)(x_1^2 + x_2^2)^{-\frac{3}{2}} x_1^2 + \varphi'(||x||)(x_1^2 + x_2^2)^{-\frac{1}{2}}.$$

It follows that

$$\Delta u = \varphi'(||x||) + \frac{1}{||x||}\varphi'(||x||).$$

Denoting $||x|| = r$, we look, for a function φ such that

$$\varphi''(r) + \frac{1}{r}\varphi'(r) = 0.$$

This equivalent to

$$\frac{\varphi''(r)}{\varphi'(r)} = -\frac{1}{r}.$$

This gives

$$\ln(|\varphi'|) = -\ln(r) + c,$$
$$\varphi' = \frac{c}{r}, c \in \mathbb{R},$$

and therefore

$$\varphi(r) = c\ln(r) + c_2.$$

For $n \geq 3$ analog computations provide

$$\varphi(r) = cr^{2-n} + c_2.$$

In particular, we found that

$$\phi(x) = \begin{cases} \ln(||x||) & \text{if } n = 2 \\ ||x||^{2-n} & \text{if } n > 2 \end{cases}$$

is harmonic if $x \neq 0$.

Exercise 15:
Prove that for $n \geq 2$, and for K compact

$$\int_K |\phi(x)| dx < +\infty$$

Solution 15:

It is sufficient to prove the result for $K = B(0, 1)$. For $n = 2$ use polar coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. We find that

$$\int_{B(0,1)} |\ln(||x||)| dx_1 dx_2$$
$$= -2\pi \int_0^1 r \ln r dr$$
$$= \frac{\pi}{2}.$$

For $n \geq 3$, we write

$$\begin{aligned}\int_{B(0,R)} \varphi(||x||)dx &= \int_0^R \int_{\partial B(0,r)} \varphi(r) dr d\sigma \\ &= \int_0^R \int_{\partial B(0,1)} r^{n-1}\varphi(r) dr d\sigma \\ &= n\alpha(n) \int_0^R r^{n-1}\varphi(r)dr \\ &= n\alpha(n) \int_0^R r^{n-1} r^{2-n} dr \\ &= n\alpha(n)\frac{R^2}{2}\end{aligned}$$

where $\alpha(n)$ denotes the volume of the unit sphere in $\mathbb{R}^n$, $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

Exercise 16:
Prove that if $u, v \in C^2(\bar{\Omega})$,

$$\int_\Omega \Delta u v dx - \int_\Omega u \Delta v dx = \int_{\partial\Omega} v \nabla u.\vec{n} d\sigma(x) - \int_{\partial\Omega} u \nabla v.\vec{n} d\sigma \tag{3.12}$$

Solution 16:

Hint: Use Green formula We set

$$\phi(x) = \begin{cases} \frac{1}{2\pi}\ln(||x||) & \text{if } n = 2 \\ \frac{1}{n(2-n)\alpha(n)}||x||^{2-n} & \text{if } n > 2 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

Theorem 12. *If $u \in C^2(\bar{\Omega})$, then under the notations above, for $x \in \Omega$*

$$u(x) = \int_{\partial\Omega} \Big(u(y)\frac{\partial \phi}{\partial \nu}(x-y) - \phi(x-y)\frac{\partial u}{\partial \nu}(y)\Big) dy + \int_\Omega \phi(x-y)\Delta u(y) dy \tag{3.13}$$

Exercise 17:
Prove Theorem 12.

Solution 17:
Let $\epsilon > 0$ such that $B(x,\epsilon) \subset \Omega$. The idea is to apply Equation (3.12) with $u(y) = \phi(x-y)$ and $v(y) = u(y)$ on $\Omega \setminus B(x,\epsilon)$, then take the limit at $\epsilon = 0$.

$$\begin{aligned}\int_{\Omega\setminus B(x,\epsilon)} \Phi(x-y)\Delta u(y)dy - \int_{\Omega\setminus B(x,\epsilon)} \Delta\Phi(x-y) u dy &= \int_{\partial\Omega} \Phi\nabla u.\vec{n}d\sigma - \int_{\partial\Omega} u\nabla\Phi.\vec{n}d\sigma \\ &- \int_{\partial B(x,\epsilon)} \Phi\nabla u.\vec{n}d\sigma + \int_{\partial B(x,\epsilon)} u\nabla\Phi.\vec{n}d\sigma\end{aligned}$$

Then we remark that,

$$\begin{aligned}|\int_{\partial B(x,\epsilon)} \Phi\nabla u.\vec{n}d\sigma| &\leq \int_{\partial B(x,\epsilon)} |\Phi|\nabla u|_\infty d\sigma \\ &\leq \int_{\partial B(x,\epsilon)} |\Phi|\nabla u|_\infty d\sigma \\ &\leq \varphi(\epsilon)|\nabla u|_\infty \epsilon^{n-1}\int_{\partial B(x,1)} d\sigma\end{aligned}$$

$$\leq \varphi(\epsilon)|\nabla u|_\infty \epsilon^{n-1} \int_{\partial B(x,1)} d\sigma$$
$$\to 0 \text{ as } \epsilon \to 0.$$

Also,

$$\int_{\partial B(x,\epsilon)} u \nabla \Phi(x-y).\vec{n} d\sigma = \int_{\partial B(x,\epsilon)} \varphi'(\epsilon)||\vec{n}||^2 u(y) d\sigma$$
$$= \varphi'(\epsilon)|B(x,\epsilon)| \frac{1}{|B(x,\epsilon)|} \int_{\partial B(x,\epsilon)} u(y) d\sigma$$
$$\to u(y) \text{ as } \epsilon \to 0.$$

This proves the result.

Chapter 4

Analytic semigroups

4.1 Definitions

Let X be a Banach space, and let $\mathcal{L}(X)$ be the space of all bounded linear operators on X.

Definition 9 (Analytic Semigroup). *An $\mathcal{L}(X)$-valued function $U(z)$ defined in a sectorial domain*

$$\Sigma_\phi = \{z \in \mathbb{C}; |argz| \leq \phi;\} 0 < \phi < \frac{\pi}{2}$$

is called an analytic semigroup on X if $U(z)$ satisfies

1. *$U(z)$ is an analytic function in Σ_ϕ with values in $\mathcal{L}(X)$.*
2. *$U(z)$ satisfies the semigroup property $U(z)U(z') = U(z + z')$ for $z, z' \in \Sigma_\phi$.*
3. *$U(O) = I$ and $\lim_{z \to 0, z \in \Sigma_\phi} U(z)x = x$ for every $x \in X$.*

Definition 10 (Spectrum). *Let A be a densely defined, closed linear operator in X. The spectrum of A is defined as the complement the resolvent set $\rho(A)$. It is denoted by $\sigma(A)$.*

Definition 11 (Sectorial Operators). *Let A be a densely defined, closed linear operator in X. Then A is said to be sectorial if it satisfies:*

1. *the spectrum of A is contained in an open sectorialdomain,*

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C}; |\arg \lambda| < \omega\}, 0 < \omega \leq \pi, \text{ and}$$

2. *its resolvent satisfies the estimate*

$$||(\lambda I - A)^{-1}|| \leq \frac{M}{|\lambda|}, \lambda \notin \Sigma_\omega,$$

   with some constant $M \geq 1$.

Chapter 5

The Galerkin Method and some applications

See Allaire, J.L Lions, Daytray$_Lions, Magenes - Lions$....

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