Sharp interface limit of the Fisher-KPP equation

Matthieu Alfaro

I3M, Université de Montpellier 2,
CC051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France,

Arnaud Ducrot

UMR CNRS 5251 IMB and INRIA Sud-Ouest ANUBIS,
Université de Bordeaux, 3, Place de la Victoire, 33000 Bordeaux France.

Abstract

We investigate the singular limit, as $\varepsilon \to 0$, of the Fisher equation $\partial_t u = \varepsilon \Delta u + \varepsilon^{-1} u(1-u)$ in the whole space. We consider initial data with compact support plus, possibly, perturbations very small as $\|x\| \to \infty$. By proving both generation and motion of interface properties, we show that the sharp interface limit moves by a constant speed, which is the minimal speed of some related one-dimensional travelling waves. We obtain an estimate of the thickness of the transition layers. We also exhibit initial data "not so small" at infinity which do not allow the interface phenomena.

Key Words: population dynamics, Fisher equation, singular perturbation, generation of interface, motion of interface. [1]

1 Introduction

Reaction diffusion equations with logistic nonlinearity were introduced in the pioneer works of Fisher [9] or Kolmogorov, Petrovsky and Piskunov [12]. The equations read as

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u(t,x)(1-u(t,x))\,, \quad t>0\,, \quad x \in \mathbb{R}^N\,, \tag{1.1}$$

supplemented together with some suitable initial conditions. This kind of equation was widely used in the literature to model phenomena arising in population genetics, [9, 3], or in biological invasions, [15, 14, 13] and the references therein. We also refer to [6] for some extensions to biological invasion in nonhomogeneous media with logistic dynamics. The main property

[1]AMS Subject Classifications: 35K57, 35B25, 35R35, 35B50, 92D25.

of equation (1.1) is to admit (biologically relevant) travelling wave solutions with some semi-infinite interval of admissible wave speed. The aim of this work is to focus on the ability of equation (1.1) to generate some interfaces and to propagate them. Such a property is strongly related to these wave solutions. In order to observe such a property, we shall rescale equation (1.1) by putting

$$u^\varepsilon(t,x) = u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Therefore, we focus on the singular limit problem

$$(P^\varepsilon) \quad \begin{cases} \partial_t u = \varepsilon \Delta u + \dfrac{1}{\varepsilon} u(1-u) & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = u_{0,\varepsilon}(x) & \text{in } \mathbb{R}^N, \end{cases}$$

as the parameter $\varepsilon > 0$, related to the thickness of a diffuse interfacial layer, tends to zero.

We shall assume the following properties on the initial data.

Assumption 1.1. *We assume that $u_{0,\varepsilon} = g + h_\varepsilon$ where*

(i) g is a smooth (at least of the class C^2), nonnegative and compactly supported function. We define $\Omega_0 := \operatorname{supp} g$.

(ii) $0 \in \Omega_0$.

(iii) h_ε is a smooth and nonnegative function and there exist $\lambda \geq 1$ and $M > 0$ such that, for all $\varepsilon > 0$ small enough,

$$h_\varepsilon(x) \leq M e^{-\lambda \frac{\|x\|}{\varepsilon}}, \quad \forall x \in \mathbb{R}^N.$$

Assumption 1.2. *We assume that Ω_0 is convex.*

Assumption 1.3. *We assume the existence of $\delta > 0$ such that, if n denotes the Euclidian unit normal vector exterior to the "initial interface" $\Gamma_0 := \partial\Omega_0$, then*

$$\left|\frac{\partial g}{\partial n}(y)\right| \geq \delta \quad \textit{for all } y \in \Gamma_0. \tag{1.2}$$

Heuristics. In view of Assumption 1.1, it is standard that, for each $\varepsilon > 0$, Problem (P^ε) has a unique smooth solution $u^\varepsilon(t,x)$ on $[0,\infty) \times \mathbb{R}^N$. As $\varepsilon \to 0$, a formal asymptotic analysis shows the following: in the very early stage, the diffusion term $\varepsilon\Delta u^\varepsilon$ is negligible compared with the reaction term $\varepsilon^{-1}u^\varepsilon(1-u^\varepsilon)$ so that, in the rescaled time scale $\tau = t/\varepsilon$, the equation is well approximated by the ordinary differential equation $\partial_\tau u^\varepsilon = u^\varepsilon(1-u^\varepsilon)$. Hence the value of u^ε quickly becomes close to either 1 or 0 in most part of $\mathbb{R}^N$, creating a steep interface (transition layer) between the regions $\{u^\varepsilon \approx 1\}$

and $\{u^\varepsilon \approx 0\}$. Once such an interface develops, the diffusion term becomes large near the interface, and comes to balance with the reaction term. As a result, the interface ceases rapid development and starts to propagate in a much slower time scale.

The limit free boundary problem. To study this interfacial behavior, we consider the asymptotic limit of (P^ε) as $\varepsilon \to 0$. Then the limit solution $\tilde{u}(t,x)$ will be a step function taking the value 1 on one side of the moving interface, and 0 on the other side. This sharp interface, which we will denote by Γ_t^*, obeys the law of motion

$$(P^*) \qquad \begin{cases} V_n = c^* & \text{on } \Gamma_t^* \\ \Gamma_t^*\big|_{t=0} = \Gamma_0 \,, \end{cases}$$

where V_n denotes the normal velocity of Γ_t^* in the exterior direction and c^* the minimal speed of some related one-dimensional travelling waves (see subsection 2.2 for details).

Since the region enclosed by Γ_0, namely Ω_0, is smooth and convex, Problem (P^*), possesses a unique smooth solution on $[0,\infty)$, which we denote by $\Gamma^* = \bigcup_{t\geq 0}(\{t\} \times \Gamma_t^*)$. Hereafter, we fix $T > 0$ and work on $(0,T]$.

For each $t \in (0,T]$, we denote by Ω_t^* the region enclosed by the hypersurface Γ_t^*. We define a step function $\tilde{u}(t,x)$ by

$$\tilde{u}(t,x) = \begin{cases} 1 & \text{in } \Omega_t^* \\ 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega_t^*} \end{cases} \quad \text{for } t \in (0,T]\,, \tag{1.3}$$

which represents the asymptotic limit of u^ε (or the *sharp interface limit*) as $\varepsilon \to 0$.

Known related results. The question of the convergence of Problem (P^ε) to (P^*) has been addressed when the initial data $u_{0,\varepsilon}$ does not depend on ε and is compactly supported : first by Freidlin [10] using probabilistic methods and later by Evans and Souganidis [8] using Hamilton Jacobi technics (in this framework we also refer to [4, 5]). The purpose of the present work is to provide a new proof of convergence for Problem (P^ε) by using specific reaction-diffusion tools such as the comparison principle. These technics were recently used by Hilhorst *et al.* in [11] to consider the generation and propagation of interfaces for a degenerated Fisher equation. Degenerated Fisher equation have some semi-compactly supported travelling wave solutions, which is essential in the proof given in [11]. However Equation (1.1) does not possess solution with such a property. We adapt these technics to study the singular limit of (P^ε). For bistable nonlinearities, we refer to [1] where an optimal estimate of the transition layers is provided for the Allen-Cahn equation whose singular limit is a motion by mean curvature.

Let us precise that we improve the convergence of (P^{ε}) in two directions. On the one hand, we extend the set of initial data for which Problem (P^{ε}) has a singular limit, by allowing positive initial data with a suitable behavior at infinity. Moreover, we also exhibit initial data "not so small" at infinity which do not allow the interface phenomena. On the other hand we provide an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the thickness of the transition layers of the solutions u^{ε}.

Results. Our main result, Theorem 1.4, describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial data, the solution u^ε quickly becomes close to 1 or 0, except in a small neighborhood of the initial interface Γ_0, creating a steep transition layer around Γ_0 (*generation of interface*). The time needed to develop such a transition layer, which we will denote by t^ε, is of order $\varepsilon|\ln\varepsilon|$. The theorem then states that the solution u^ε remains close to the step function $\tilde{u}$ on the time interval $[t^\varepsilon, T]$ (*motion of interface*); in other words, the motion of the transition layer is well approximated by the limit interface equation (P^*).

Theorem 1.4 (Generation, motion and thickness of interface)**.** *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then there exist positive constants α and $\mathcal{C}$ such that, for all $\varepsilon > 0$ small enough and for all $t^\varepsilon \leq t \leq T$, where*

$$t^\varepsilon := \alpha\varepsilon|\ln\varepsilon|\,,$$

we have

$$u^\varepsilon(t,x) \in \begin{cases} [0, 1+\varepsilon] & \text{if} \quad x \in \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^*) \\ [1-2\varepsilon, 1+\varepsilon] & \text{if} \quad x \in \Omega_t^* \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^*) \\ [0,\varepsilon] & \text{if} \quad x \in (\mathbb{R}^N \setminus \overline{\Omega_t^*}) \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^*)\,, \end{cases} \tag{1.4}$$

where $\mathcal{N}_r(\Gamma_t^*) := \{x \in \mathbb{R}^N : \quad dist(x, \Gamma_t^*) < r\}$ denotes the tubular r-neighborhood of Γ_t^*.

As a immediate consequence of the above Theorem, we collect the convergence result.

Corollary 1.5 (Convergence)**.** *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then, as $\varepsilon \to 0$, u^ε converges to $\tilde{u}$ everywhere in $\bigcup_{0<t\leq T}(\{t\}\times\Omega_t^*)$ and $\bigcup_{0<t\leq T}\big(\{t\}\times(\mathbb{R}^N\setminus\overline{\Omega_t^*})\big)$.*

Next, it is intuitively clear that the limit problem depends dramatically on the initial data. In order to underline this fact, we show that for initial data "not so small" at infinity, the solution u^ε tends to 1 everywhere, as $\varepsilon \to 0$. Therefore, the interface phenomena does not occur in this case.

Assumption 1.6. *We assume that there exist $n > 0$, $m > 0$, $M > 0$ and $C_0 > 0$ such that, for all $\varepsilon > 0$ small enough,*

$$\frac{m}{1 + \dfrac{\|x\|^n}{\varepsilon^n}} \leq u_{0,\varepsilon}(x) \leq M\,, \quad \forall x \in \mathbb{R}^N\,,$$

$$\|u_{0,\varepsilon}\|_\infty + \|\nabla u_{0,\varepsilon}\|_\infty + \|\Delta u_{0,\varepsilon}\|_\infty \leq C_0\,.$$

Theorem 1.7 (There is no interface)**.** *Let Assumption 1.6 be satisfied. Then*

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(t,x) = 1\,, \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^N\,.$$

Plan. The organization of this paper is as follows. In Section 2, we present the basic tools that will be used in later sections for the construction of sub- and super-solutions. In Section 3 we construct sub-solutions for very small times and super-solutions for all times. They enable to prove a generation of interface property. Section 4 is devoted to the construction of sub-solutions for all times; their role is to control the solution u^ε from below, while the sharp interface limit propagates. In Section 5, by using our different sub- and super-solutions we prove the main result, Theorem 1.4. Last, in Section 6 we prove Theorem 1.7.

2 Materials

Let us recall that, in the classical works of Fisher [9] and Kolmogorov, Petrovsky and Piskunov [12], the authors consider a monostable nonlinearity f which is smooth and such that $f(u) > 0$ if $u \in (0,1)$, $f(u) < 0$ if $u \in (-\infty,0) \cup (1,\infty)$ and $f'(0) > 0$. For the sake of simplicity we select $f(u) = u(1-u)$ through this work.

2.1 A monostable ODE

The generation of interface is initiated by the dynamics of the corresponding ordinary differential equation. Therefore, we gather here well-known facts about the logistic dynamics that will be extensively used in the sequel.

To be more precise, the generation of interface is strongly related to the dynamical properties of the non-rescaled corresponding ordinary differential equation associated to (P^ε), that is

$$\frac{dz(t)}{dt} = z(t)(1 - z(t))\,, \quad t > 0\,.$$

In the sequel, for technical reasons we shall apply the semiflow generated by the above dynamical system to negative initial data. In order to have some good dynamical properties, let us modify the monostable nonlinearity

$u \to u(1-u)$ on $(-\infty, 0)$ so that the modified function, we call it $\bar{f} : \mathbb{R} \to \mathbb{R}$, is of the class C^2 and enjoys the bistable assumptions. More precisely, $\bar{f}$ has exactly three zeros $-1 < 0 < 1$ and

$$\bar{f}'(-1) < 0\,, \qquad \bar{f}'(0) = 1 > 0\,, \qquad \bar{f}'(1) = -1 < 0\,. \tag{2.1}$$

Note that $\bar{f}(u) = u(1-u)$ if $u \geq 0$. As done in Chen [7], we consider $\bar{f}_\varepsilon$ a slight modification of $\bar{f}$ defined by

$$\bar{f}_\varepsilon(u) := \psi(u)\frac{u - \varepsilon|\ln\varepsilon|}{|\ln\varepsilon|} + (1-\psi(u))\bar{f}(u)\,,$$

with ψ a smooth cut-off function satisfying conditions (29)—(32) as they appear in [11]. As explained in [11],

$$\bar{f}_\varepsilon(u) \leq \bar{f}(u) \quad \text{ for all } u \in \mathbb{R}\,. \tag{2.2}$$

Then we defined $w(s,\xi)$ as the semiflow generated by the ordinary differential equation

$$\begin{cases} \dfrac{dw}{ds}(s,\xi) = \bar{f}_\varepsilon(w(s,\xi))\,, \quad s > 0\,, \\ w(0,\xi) = \xi\,. \end{cases} \tag{2.3}$$

Here ξ ranges over the interval $[-\|g\|_\infty - M - 1, \|g\|_\infty + M + 1]$. We claim that $w(s,\xi)$ has the following properties (for proofs, see [7] or [11]).

Lemma 2.1 (Behavior of w)**.** *The following holds for all $\xi \in [-\|g\|_\infty - M - 1, \|g\|_\infty + M + 1]$.*

(i) If $\xi \geq \varepsilon|\ln\varepsilon|$ *then* $w(s,\xi) \geq \varepsilon|\ln\varepsilon| > 0$ *for all* $s > 0$.
If $\xi < 0$ *then* $w(s,\xi) < 0$ *for all* $s > 0$.
If $\xi \in (0, \varepsilon|\ln\varepsilon|)$ *then* $w(s,\xi) > 0$ *for all* $s \in (0, s_\varepsilon(\xi))$, *with*

$$s_\varepsilon(\xi) := |\ln\varepsilon| \left|\ln\left(1 - \frac{\xi}{\varepsilon|\ln\varepsilon|}\right)\right|\,.$$

(ii) $w(s,\xi) \in (-\|g\|_\infty - M - 1, \|g\|_\infty + M + 1)$ *for all* $s > 0$.

(iii) w *is of the class* C^2 *with respect to* ξ *and*

$$w_\xi(s,\xi) > 0 \quad \textit{for all } s > 0\,.$$

(iv) For all $a > 0$, *there exists a constant* $C(a)$ *such that*

$$\left|\frac{w_{\xi\xi}}{w_\xi}(s,\xi)\right| \leq \frac{C(a)}{\varepsilon} \quad \textit{for all } 0 < s \leq a|\ln\varepsilon|\,.$$

(v) *There exists a positive constant α such that, for all $s \geq \alpha\varepsilon|\ln\varepsilon|$, we have*

$$\textit{if} \quad \xi \in [\varepsilon|\ln\varepsilon|, \|g\|_\infty + M + 1] \quad \textit{then} \quad 0 < w(s,\xi) \leq 1 + \varepsilon\,,$$

and

$$\textit{if} \quad \xi \in [3\varepsilon|\ln\varepsilon|, \|g\|_\infty + M + 1] \quad \textit{then} \quad 1 - \varepsilon \leq w(s,\xi)\,.$$

2.2 Travelling waves

For the self-containedness of the present paper, we recall here well-known facts concerning one dimensional travelling waves related to our problem. We refer the reader to [2, 16] and the references therein.

A travelling wave is a couple (c, U) with $c > 0$ and $U \in C^2(\mathbb{R},\mathbb{R})$ a function such that

$$\begin{cases} U''(z) + cU'(z) + U(z)(1 - U(z)) = 0 \quad \text{for all } z \in \mathbb{R} \\ U(-\infty) = 1 \\ U(\infty) = 0\,. \end{cases} \tag{2.4}$$

Define $c^* := 2$. Then the following holds.

(i) For all $c \geq c^*$ there exists a unique (up to a translation in z) travelling wave denoted by (c, U) or (c^*, U^*). It is positive and monotone.

(ii) For all $0 < c < c^*$, there exists a unique (up to a translation in z) and non monotone travelling wave (c, U). It changes sign. In the sequel, for each $c \in (0, c^*)$ we select U as the solution of

$$\begin{cases} U''(z) + cU'(z) + U(z)(1 - U(z)) = 0 \quad \text{for all } z \in \mathbb{R} \\ U(-\infty) = 1 \\ U(\infty) = 0 \\ U(z) > 0 \quad \text{for all } z < 0 \\ U(0) = 0\,. \end{cases} \tag{2.5}$$

Lemma 2.2 (Behavior of U)**.** *Let $c > 0$ be arbitrary and consider the associated travelling wave (c, U). Then there exist constants $C > 0$ and $\mu > 0$ such that*

$$0 < 1 - U(z) \leq Ce^{-\mu|z|} \quad \textit{for } z \leq 0\,, \tag{2.6}$$

$$|U(z)| \leq Ce^{-\mu|z|} \quad \textit{for } z \geq 0\,, \tag{2.7}$$

$$|U'(z)| + |U''(z)| \leq Ce^{-\mu|z|} \quad \textit{for all } z \in \mathbb{R}\,. \tag{2.8}$$

Note that, as easily seen from the standard proof, the constants $C > 0$ and $\mu > 0$ depend continuously on c. This fact shall be used in Lemma 4.2.

At last, since a precise behavior of the wave with the minimal speed shall be necessary, let us recall that the travelling wave solution U^* associated to $c = c^*$ satisfies

$$\gamma^- z e^{-z} \le U^*(z) \le \gamma^+ z e^{-z}\,, \quad \forall z \ge 1\,, \tag{2.9}$$

for two constants $0 < \gamma^- < \gamma^+$.

2.3 Cut-off signed distance functions

For $c > 0$ we denote by $\Gamma^c = \bigcup_{t \ge 0}(\{t\} \times \Gamma_t^c)$ the smooth solution of the free boundary problem

$$(P^c) \quad \begin{cases} V_n = c & \text{on } \Gamma_t^c \\ \Gamma_t^c\big|_{t=0} = \Gamma_0\,. \end{cases}$$

If $c = c^*$, we naturally use the notations $\Gamma^* = \bigcup_{t \ge 0}(\{t\} \times \Gamma_t^*)$ and (P^*). Note that since the region enclosed by Γ_0, namely Ω_0, is convex, these solutions do exist for all $t \ge 0$. For each $t \ge 0$, we denote by Ω_t^c the region enclosed by the hypersurface Γ_t^c.

Let $\widetilde{d}$ be the signed distance function to Γ^c defined by

$$\widetilde{d}(t,x) = \begin{cases} -\operatorname{dist}(x, \Gamma_t^c) & \text{for } x \in \Omega_t^c \\ \operatorname{dist}(x, \Gamma_t^c) & \text{for } x \in \mathbb{R}^N \setminus \Omega_t^c\,, \end{cases} \tag{2.10}$$

where $\operatorname{dist}(x, \Gamma_t^c)$ is the distance from x to the hypersurface Γ_t^c. We remark that $\widetilde{d} = 0$ on Γ^c and that $|\nabla \widetilde{d}| = 1$ in a neighborhood of Γ^c.

We now introduce the "cut-off signed distance function" d, which is defined as follows. Recall that $T > 0$ is fixed. First, choose $d_0 > 0$ small enough so that $\widetilde{d}$ is smooth in the tubular neighborhood of Γ^c

$$\{(t,x) \in [0,T] \times \mathbb{R}^N : \ |\widetilde{d}(t,x)| < 3d_0\}\,.$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \le d_0 \\ -2d_0 & \text{if } s \le -2d_0 \\ 2d_0 & \text{if } s \ge 2d_0\,. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(t,x) = \zeta\left(\tilde{d}(t,x)\right)\,. \tag{2.11}$$

If $c = c^*$, we naturally use the notation d^*.

Note that

$$\text{if} \quad |d(t,x)| < d_0 \quad \text{then} \quad |\nabla d(t,x)| = 1\,, \tag{2.12}$$

and that the equation of motion (P^c) is now written as

$$\partial_t d(t,x) + c = 0 \qquad \text{on } \ \Gamma_t^c = \{x \in \mathbb{R}^N : \ d(t,x) = 0\}\,. \tag{2.13}$$

Then the mean value theorem provides a constant $N > 0$ such that

$$|\partial_t d(t,x) + c| \le N|d(t,x)| \qquad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N\,. \tag{2.14}$$

Moreover, there exists a constant $C > 0$ such that

$$|\nabla d(t,x)| + |\Delta d(t,x)| \le C \qquad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N\,. \tag{2.15}$$

At least, note that the constants $d_0 > 0$, $C > 0$ and $N > 0$ depend continuously on c. This fact shall be used in Lemma 4.2.

3 Generation of interface

The aim of this section is to prove a generation of interface property for Problem (P^ε). We shall prove that after a small time t^ε of order $\varepsilon|\ln\varepsilon|$ the solution u^ε becomes well prepared and looks like a rescaled wave solution. We shall more precisely prove the following result.

Theorem 3.1 (Generation of interface)**.** *Let Assumption 1.1 be satisfied. Then there exist $k > 0$, $\alpha > 0$ such that, for all $\varepsilon > 0$ small enough, the following holds.*

(i) For all $x \in \Omega_0$ such that $g(x) \ge k\varepsilon|\ln\varepsilon|$ we have

$$u^\varepsilon(t^\varepsilon, x) \ge 1 - \varepsilon\,,$$

wherein $t^\varepsilon := \alpha\varepsilon|\ln\varepsilon|$.

(ii) For all $x \in \mathbb{R}^N$ and all $t \ge 0$ we have

$$0 \le u^\varepsilon(t + t^\varepsilon, x) \le 1 + \varepsilon\,.$$

Moreover if Assumption 1.2 is also satisfied then there exists a constant $\widehat{K} > 1$ such that

$$u^\varepsilon(t,x) \le \widehat{K} U^* \left(\frac{d^*(0,x) - c^* t}{\varepsilon} \right)\,, \tag{3.1}$$

for all $t \ge 0$ and all $x \in \mathbb{R}^N$.

In order to prove Theorem 3.1, let us define the map

$$\underline{u}(t,x) := \max\left\{0, w\left(\frac{t}{\varepsilon}, g(x) - Kt\right)\right\},$$

where $w(s,\xi)$ is the solution of the ordinary differential equation (2.3) and where $K > 0$ is some constant to be specified below. Then one will show the following result.

Lemma 3.2 (Sub-solutions for the generation)**.** *Let Assumption 1.1 be satisfied. Then for all $a>0$, there exists $K>0$ such that, for all $\varepsilon>0$ small enough, we have*

$$\underline{u}(t,x)\leq u^\varepsilon(t,x)\,,\quad \forall t\in[0,a\varepsilon|\ln\varepsilon|]\,,\ \forall x\in\mathbb{R}^N\,.$$

Proof. Let us first notice that

$$\underline{u}(0,x)=g(x)\leq u^\varepsilon(0,x)\,,\quad \forall x\in\mathbb{R}^N\,.$$

Then we shall show that $\underline{u}$ is a sub-solution of Problem (P^ε). Note that if $x\notin\Omega_0$ then $g(x)=0$ and $\underline{u}(t,x)=0$. Let us consider the operator

$$\mathcal{L}^\varepsilon[v]:=\partial_t v-\varepsilon\Delta v-\frac{1}{\varepsilon}v(1-v)\,.$$

Let $a>0$ be arbitrary. We show below that, if $K>0$ is sufficiently large then, for all $\varepsilon>0$ small enough, $\mathcal{L}^\varepsilon[\underline{u}]\leq 0$ in the support of $\underline{u}$. In this support we have

$$\begin{aligned}\partial_t\underline{u}&=\tfrac{1}{\varepsilon}w_s-Kw_\xi\\ \Delta\underline{u}&=w_{\xi\xi}|\nabla g|^2+w_\xi\Delta g\,.\end{aligned}$$

Then we get

$$\begin{aligned}\mathcal{L}^\varepsilon[\underline{u}]\ &=\frac{1}{\varepsilon}w_s-Kw_\xi-\varepsilon\left(w_{\xi\xi}|\nabla g|^2+w_\xi\Delta g\right)-\frac{1}{\varepsilon}w(1-w)\\ &\leq\frac{1}{\varepsilon}w_s-Kw_\xi-\varepsilon\left(w_{\xi\xi}|\nabla g|^2+w_\xi\Delta g\right)-\frac{1}{\varepsilon}\bar{f}_\varepsilon(w)\\ &=-w_\xi\left[K+\varepsilon\left(\tfrac{w_{\xi\xi}}{w_\xi}|\nabla g|^2+\Delta g\right)\right]\,,\end{aligned}$$

where we have successively used (2.2) and (2.3). In view of Lemma 2.1 (v), there exists a constant $C(a)>0$ such that, for all (t,x) in the support of $\underline{u}$ with $0\leq t\leq a\varepsilon|\ln\varepsilon|$, we have

$$\left|\frac{w_{\xi\xi}}{w_\xi}|\nabla g|^2+\Delta g\right|\leq\frac{C(a)}{\varepsilon}\,.$$

Therefore, choosing $K>C(a)$ implies

$$\mathcal{L}^\varepsilon[\underline{u}]\leq -w_\xi\left(K-C(a)\right)\leq 0\,,$$

since $w_\xi>0$.

This completes the proof of Lemma 3.2. ◻

Next we prove the following result.

Lemma 3.3 (Super-solutions)**.** *Let Assumptions 1.1 and 1.2 be satisfied. Then there exists a constant $K_0 > 0$ such that, for all $\widehat{K} \geq K_0$, the following holds. For all $x_0 \in \Gamma_0 = \partial\Omega_0$, for all $\varepsilon > 0$ small enough, we have*

$$u^\varepsilon(t,x) \leq \widehat{K} U^* \left(\frac{(x-x_0)\cdot n_0 - c^* t}{\varepsilon} \right) \qquad \textit{for all } (t,x) \in [0,\infty) \times \mathbb{R}^N \,,$$

wherein n_0 is the outward normal vector to $\Gamma_0 = \partial\Omega_0$ at x_0.

Proof. We recall that λ and M were defined in Assumption 1.1 (iii) and that, due to (2.9), there exist two constants $0 < \gamma^- < \gamma^-$ such that

$$\gamma^- z e^{-z} \leq U^*(z) \leq \gamma^+ z e^{-z} \,, \ \ \forall z \geq 1 \,.$$

Thus since $\lambda \geq 1$, there exists some constant $m^- > 0$ such that

$$U^*(z) \geq m^- e^{-\lambda z} \,, \ \ \forall z \geq 0 \,.$$

Then we define

$$K_0 := \max\left(1, \frac{M}{m^-}, \ \frac{\|g\|_\infty + M}{U^*(0)}\right) \,.$$

Next, let $\widehat{K} \geq K_0$ and $x_0 \in \Gamma_0 = \partial\Omega_0$ be given. We consider the map

$$u_*^+(t,x) := \widehat{K} U^* \left(\frac{(x-x_0)\cdot n_0 - c^* t}{\varepsilon} \right) \,.$$

Straightforward computations yield

$$\varepsilon \mathcal{L}^\varepsilon[u_*^+] = \widehat{K}(\widehat{K}-1) U^{*2} \,,$$

and therefore $\mathcal{L}^\varepsilon[u_*^+] \geq 0$ in $(0,\infty)\times\mathbb{R}^N$. Hence, by the comparison principle, to complete the proof of the lemma it is enough to prove

$$u_{0,\varepsilon}(x) \leq u_*^+(0,x) = \widehat{K} U^* \left(\frac{(x-x_0)\cdot n_0}{\varepsilon} \right) \,, \ \ \forall x \in \mathbb{R}^N \,. \tag{3.2}$$

First, assume that x in the half plane $\{y \in \mathbb{R}^N : \ \ (y-x_0)\cdot n_0 \leq 0\}$. Then since U^* is decreasing we have

$$U^* \left(\frac{(x-x_0)\cdot n_0}{\varepsilon} \right) \geq U^*(0) \,.$$

Therefore we obtain that

$$u_{0,\varepsilon}(x) \leq \frac{\|g\|_\infty + M}{U^*(0)} U^* \left(\frac{(x-x_0)\cdot n_0}{\varepsilon} \right) \,,$$

which, thanks to the choice of K_0, yields $u_{0,\varepsilon}(x) \leq u_*^+(0,x)$.

Now, we assume that x is in the half plane $\{y \in \mathbb{R}^N : \ (y - x_0) \cdot n_0 > 0\}$. Since Ω_0 is convex, we have $x \notin \Omega_0$. Thus $g(x) = 0$ and

$$u_{0,\varepsilon}(x) = h_\varepsilon(x) \leq M e^{-\lambda \frac{\|x\|}{\varepsilon}} .$$

Moreover since $0 \in \Omega_0$ we have, for x such that $(x - x_0) \cdot n_0 > 0$,

$$(x - x_0) \cdot n_0 \leq \|x\| .$$

Finally since U^* is decreasing, we obtain

$$m^- e^{-\lambda \frac{\|x\|}{\varepsilon}} \leq U^* \left(\frac{\|x\|}{\varepsilon} \right) \leq U^* \left(\frac{(x - x_0) \cdot n_0}{\varepsilon} \right) .$$

Therefore we get

$$u_{0,\varepsilon}(x) \leq \frac{M}{m^-} U^* \left(\frac{(x - x_0) \cdot n_0}{\varepsilon} \right) ,$$

which, thanks to the choice of K_0, yields $u_{0,\varepsilon}(x) \leq u_*^+(0, x)$.

This completes the proof of Lemma 3.3. □

We now complete the proof of Theorem 3.1.

Proof. The proof of (i) directly follows from Lemma 2.1 (v) together with Lemma 3.2. In order to prove (ii) let us notice that the map

$$\overline{u}(t, x) := w \left(\frac{t}{\varepsilon}, \|g\|_\infty + M \right) , \ \ t \geq 0 ,$$

satisfies

$$u^\varepsilon(0, x) \leq \|g\|_\infty + M = \overline{u}(0, x) \quad \text{ for all } x \in \mathbb{R}^N ,$$
$$\mathcal{L}^\varepsilon[\overline{u}] = 0 \quad \text{ in } (0, \infty) \times \mathbb{R}^N .$$

Thus we obtain that

$$u^\varepsilon(t, x) \leq w \left(\frac{t}{\varepsilon}, \|g\|_\infty + M \right) , \ \ \forall t \geq 0 , \ \forall x \in \mathbb{R}^N .$$

Thus Lemma 2.1 (v) applies and completes the proof of (ii).

Finally, under the additional Assumption 1.2, the last point of Theorem 3.1 follows from Lemma 3.3. Indeed from this lemma we know that there exists $\widehat{K} > 1$ such that, for each $x_0 \in \partial\Omega_0$, we have

$$u^\varepsilon(t, x) \leq \widehat{K} U^* \left(\frac{(x - x_0) \cdot n_0 - c^* t}{\varepsilon} \right) \quad \text{ for all } t \geq 0, \text{ all } x \in \mathbb{R}^N .$$

Let $x \in \mathbb{R}^N$ be given and choose $x_0 \in \partial\Omega_0$ as the projection of x on the convex Ω_0. For such a choice we have

$$(x - x_0) \cdot n_0 = d^*(0, x) ,$$

and the result follows.

This completes the proof of Theorem 3.1. □

4 Motion of interface

We have proved in the previous section that, as $\varepsilon \to 0$, the solution u^ε develops, after a very short time $t^\varepsilon = \mathcal{O}(\varepsilon|\ln\varepsilon|)$, steep transition layers that separate the region where $\{u^\varepsilon \approx 0\}$ from the one where $\{u^\varepsilon \approx 1\}$. It is the goal of this section to study the motion of interface that then occurs in a much slower time range. Since Lemma 3.3 controls u^ε from above for all $t \geq 0$, this will be enough to construct sub-solutions.

For all $c \in (0, c^*)$, we define U as in (2.5) and V by

$$V(z) := \begin{cases} U(z) & \text{if } z < 0 \\ 0 & \text{if } z \geq 0 \,. \end{cases}$$

Next, we put

$$u_c^-(t,x) := (1-\varepsilon)V\left(\frac{d(t,x) + \varepsilon|\ln\varepsilon|m_1 e^{m_2 t}}{\varepsilon}\right), \tag{4.1}$$

where d denotes the cut-off signed distance function to the solution of the free boundary problem (P^c), as defined in subsection 2.3.

Lemma 4.1 (Ordering initial data)**.** *Let Assumptions 1.1 and 1.3 be satisfied. Then there exists $\tilde{m}_1 > 0$ such that for all $c \in (0, c^*)$, all $m_1 \geq \tilde{m}_1$, all $m_2 > 0$, all $\varepsilon > 0$ small enough, we have*

$$u_c^-(0,x) \leq u^\varepsilon(t^\varepsilon, x)\,, \tag{4.2}$$

for all $x \in \mathbb{R}^N$.

Proof. What we have to show is

$$u_c^-(0,x) = (1-\varepsilon)V\left(\frac{d(0,x) + m_1\varepsilon|\ln\varepsilon|}{\varepsilon}\right) \leq u^\varepsilon(t^\varepsilon, x)\,. \tag{4.3}$$

If x is such that $d(0,x) \geq -m_1\varepsilon|\ln\varepsilon|$ then this is obvious since the definition of V implies $u_c^-(0,x) = 0$. Next assume that x is such that $d(0,x) < -m_1\varepsilon|\ln\varepsilon|$. Note that, in view of hypothesis (1.2), the mean value theorem provides the existence of a constant $\tilde{m}_1 > 0$ such that

$$\text{if} \quad d(0,x) \leq -\tilde{m}_1\varepsilon|\ln\varepsilon| \quad \text{then} \quad g(x) \geq k\varepsilon|\ln\varepsilon|\,, \tag{4.4}$$

where k is as in Theorem 3.1. If we choose $m_1 \geq \tilde{m}_1$ and $m_2 > 0$, then inequality (4.3) follows from Theorem 3.1 (i) and the fact that $V \leq 1$. □

Lemma 4.2 (Sub-solutions for the motion)**.** *Choose $\eta > 0$ such that the constants $C > 0$, $\mu > 0$, $d_0 > 0$, $N > 0$, that appear in* (2.8), (2.12), (2.14), (2.15) *are independent of $c \in [c^* - \eta, c^*)$.*

Then there exists $\tilde{m}_2 > 0$ such that for all $c \in [c^* - \eta, c^*)$, all $m_1 \geq \tilde{m}_1$, all $m_2 \geq \tilde{m}_2$, all $\varepsilon > 0$ small enough, we have

$$\mathcal{L}^\varepsilon[u_c^-] := \partial_t u_c^- - \varepsilon \Delta u_c^- - \frac{1}{\varepsilon} u_c^-(1 - u_c^-) \leq 0 \quad \text{in } (0,\infty) \times \mathbb{R}^N . \tag{4.5}$$

Proof. In the set $\{(t,x) : \ d(t,x) \geq -\varepsilon |\ln \varepsilon| m_1 e^{m_2 t}\}$ this is obvious since $u_c^-(t,x) = 0$.

We now work in the set $\{(t,x) : \ d(t,x) < -\varepsilon |\ln \varepsilon| m_1 e^{m_2 t}\}$. By using straightforward computations we get

$$\begin{aligned}
\partial_t u_c^- &= (1-\varepsilon)\left(\frac{\partial_t d}{\varepsilon} + m_2 \varepsilon |\ln \varepsilon| m_1 e^{m_2 t}\right) U'(\theta) \\
\Delta u_c^- &= \frac{|\nabla d|^2}{\varepsilon^2}(1-\varepsilon) U''(\theta) + \frac{\Delta d}{\varepsilon}(1-\varepsilon) U'(\theta) \\
u_c^-(1-u_c^-) &= (1-\varepsilon) U(\theta)(1 - U(\theta)) + \varepsilon(1-\varepsilon) U^2(\theta) ,
\end{aligned}$$

where

$$\theta := \frac{d(t,x) + \varepsilon |\ln \varepsilon| m_1 e^{m_2 t}}{\varepsilon} . \tag{4.6}$$

Now, the ordinary differential equation $U'' + cU' + U(1-U) = 0$ yields

$$\varepsilon \mathcal{L}^\varepsilon[u_c^-] = (1-\varepsilon)(E_1 + \cdots + E_3) ,$$

with

$$\begin{aligned}
&E_1 := U'(\theta)\left(\partial_t d + c + m_2 \varepsilon |\ln \varepsilon| m_1 e^{m_2 t} - \varepsilon \Delta d\right) =: U'(\theta) F_1 \\
&E_2 := U''(\theta)(1 - |\nabla d|^2) \\
&E_3 := -\varepsilon U^2(\theta) .
\end{aligned}$$

We show below that the choice $\tilde{m}_2 := 2N(\dfrac{2}{\tilde{m}_1 \mu} + 1)$ is enough to prove the lemma. To that purpose we distinguish two cases, namely (4.7) and (4.8).

First assume that

$$-\frac{m_2}{2N} \varepsilon |\ln \varepsilon| m_1 e^{m_2 t} \leq d(t,x) < -\varepsilon |\ln \varepsilon| m_1 e^{m_2 t} . \tag{4.7}$$

It follows from (2.14) and (2.15) that, for $\varepsilon > 0$ small enough,

$$\begin{aligned}
F_1 &\geq Nd + m_2 \varepsilon |\ln \varepsilon| m_1 e^{m_2 t} - \varepsilon C \\
&\geq Nd + \tfrac{m_2}{2} \varepsilon |\ln \varepsilon| m_1 e^{m_2 t} \\
&\geq 0 ,
\end{aligned}$$

which implies that $E_1 \leq 0$. Moreover, in view of (2.12), we have, for $\varepsilon > 0$ small enough, $E_2 = 0$ and, obviously, $E_3 \leq 0$. Hence, $\mathcal{L}^\varepsilon[u_c^-] \leq 0$.

Now assume that

$$d(t,x) < -\frac{m_2}{2N}\varepsilon|\ln\varepsilon|m_1 e^{m_2 t}\,. \tag{4.8}$$

This implies

$$\theta \le -|\ln\varepsilon|m_1(\frac{m_2}{2N}-1)\,. \tag{4.9}$$

Using (2.8), (2.14) and (2.15) we see that, for $\varepsilon > 0$ small enough,

$$\begin{aligned} E_1 &\le Ce^{-\mu|\theta|}(N|d| + \varepsilon|\ln\varepsilon|m_1 m_2 e^{m_2 T} + \varepsilon C) \\ &\le Ce^{-\mu|\theta|}(N2d_0 + o(1)) \\ &\le C_1 \varepsilon^{m_1\mu(\frac{m_2}{2N}-1)}\,, \end{aligned}$$

where $C_1 := 3CNd_0$. The choice of $\tilde{m}_2$ then forces $E_1 \le C_1\varepsilon^2$. Using very similar arguments we see that there exists $C_2 > 0$ such that $E_2 \le C_2\varepsilon^2$. At least note that $\theta \to -\infty$ as $\varepsilon \to 0$. Hence, if $\varepsilon > 0$ is small enough then $U(\theta) \ge b$ for some $b > 0$, which in turn implies $E_3 \le -b^2\varepsilon$. Collecting these estimates yields

$$\mathcal{L}^\varepsilon[u_c^-] \le -b^2\varepsilon + (C_1 + C_2)\varepsilon^2 \le 0\,,$$

for $\varepsilon > 0$ small enough.

The lemma is proved. $\square$

5 Proof of Theorem 1.4

We are now ready to prove our main result which includes both generation and motion of interface properties, but also provides an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the transition layers. Roughly speaking, we will fit the sub- and super-solutions for the generation into the ones for the motion.

Proof. Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Choose $k > 0$, $\alpha > 0$ and $\widehat{K} > 1$ as in Theorem 3.1. As in Lemma 4.2, choose $\eta > 0$ such that the constants $C > 0$, $\mu > 0$, $d_0 > 0$, $N > 0$, that appear in (2.8), (2.12), (2.14), (2.15) are independent of $c \in [c^* - \eta, c^*)$. According to Lemma 4.1, Lemma 4.2 and the comparison principle, there exist $m_1 > 0$, $m_2 > 0$ such that, for all $c \in [c^* - \eta, c^*]$, we have, for $\varepsilon > 0$ small enough,

$$u_c^-(t - t^\varepsilon, x) \le u^\varepsilon(t,x)\,, \tag{5.1}$$

for all $(t,x) \in [t^\varepsilon, T] \times \mathbb{R}^N$. We choose $\mathcal{C}$ such that

$$\mathcal{C} > \max\left(1, 2(2T + m_1 e^{m_2 T}), \frac{2}{\mu}\right)\,. \tag{5.2}$$

Obviously, from Theorem 3.1 (ii), we have $u^\varepsilon(t,x) \in [0, 1+\varepsilon]$ for all $t^\varepsilon \le t \le T$, all $x \in \mathbb{R}^N$.

Next we take $x \in (\mathbb{R}^N \setminus \overline{\Omega_t^*}) \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^*)$, i.e.

$$d^*(t,x) \geq \mathcal{C}\varepsilon|\ln\varepsilon|\,, \tag{5.3}$$

and prove below that $u^\varepsilon(t,x) \leq \varepsilon$, for $t^\varepsilon \leq t \leq T$. Since

$$d^*(t,x) = d^*(0,x) - c^* t\,,$$

we deduce from (3.1), the decrease of U^* and (2.9) that, for $\varepsilon > 0$ small enough, for $0 \leq t \leq T$,

$$\begin{aligned} u^\varepsilon(t,x) &\leq \widehat{K} U^*(\mathcal{C}|\ln\varepsilon|) \\ &\leq \widehat{K}\gamma^+\mathcal{C}|\ln\varepsilon|\varepsilon^{\mathcal{C}} \\ &\leq \varepsilon\,, \end{aligned}$$

since $\mathcal{C} > 1$.

At last we take $x \in \Omega_t^* \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^*)$, i.e.

$$d^*(t,x) \leq -\mathcal{C}\varepsilon|\ln\varepsilon|\,, \tag{5.4}$$

and prove below that $u^\varepsilon(t,x) \geq 1 - 2\varepsilon$, for $t^\varepsilon \leq t \leq T$. Note that

$$d(t - t^\varepsilon, x) = d^*(t,x) + (c^* - c)t + ct^\varepsilon\,. \tag{5.5}$$

We define

$$c(\varepsilon) := c^* - \varepsilon|\ln\varepsilon|\,. \tag{5.6}$$

Combining (5.1), with $c(\varepsilon)$ playing the role of c, and (5.5), we see that

$$u^\varepsilon(t,x) \geq (1-\varepsilon) V\left(\frac{d^*(t,x) + \varepsilon|\ln\varepsilon|t + \varepsilon|\ln\varepsilon|t^\varepsilon + \varepsilon|\ln\varepsilon|m_1 e^{m_2 t}}{\varepsilon}\right),$$

for $t^\varepsilon \leq t \leq T$. In view of (5.4), the choice of $\mathcal{C}$ in (5.2) and (2.6), we get, for $\varepsilon > 0$ small enough,

$$\begin{aligned} u^\varepsilon(t,x) &\geq (1-\varepsilon) U\left(-\tfrac{\mathcal{C}}{2}|\ln\varepsilon|\right) \\ &\geq (1-\varepsilon)(1 - Ce^{-\mu\frac{\mathcal{C}}{2}|\ln\varepsilon|}) \\ &\geq 1 - \varepsilon - C\varepsilon^{\mu\frac{\mathcal{C}}{2}} \\ &\geq 1 - 2\varepsilon\,, \end{aligned}$$

since $\mathcal{C} > \dfrac{2}{\mu}$.

This completes the proof of Theorem 1.4. $\square$

6 When initial data are "not so small"

In this section we investigate the singular limit of (P^ε) when initial data $u_{0,\varepsilon}$ satisfy Assumption 1.6. We prove below Theorem 1.7.

We start with the following lemma. The proof is omitted since it follows the same arguments as those used in Theorem 3.1.

Lemma 6.1 (Generation of interface)**.** *Let Assumption 1.6 be satisfied. Then there exist some constants $\alpha > 0$ and $k > 0$ such that, for all $\varepsilon > 0$ small enough,*

$$u^\varepsilon(t^\varepsilon + t, x) \leq 1 + \varepsilon\,, \quad \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}^N\,,$$

$$u^\varepsilon(t^\varepsilon, x) \geq 1 - \varepsilon\,, \quad \text{for all } x \in \mathbb{R}^N \text{ such that } u_{0,\varepsilon}(x) \geq k\varepsilon|\ln\varepsilon|\,,$$

wherein $t^\varepsilon := \alpha\varepsilon|\ln\varepsilon|$.

Define

$$\xi_\varepsilon := \varepsilon \left\{ \frac{m}{k\varepsilon|\ln\varepsilon|} - 1 \right\}^{1/n}\,.$$

In view of Assumption 1.6, we see that the condition $u_{0,\varepsilon}(x) \geq k\varepsilon|\ln\varepsilon|$ is satisfied when $\|x\| \leq \xi_\varepsilon$. Therefore, Lemma 6.1 implies that

$$u^\varepsilon(t^\varepsilon, x) \geq 1 - \varepsilon\,, \quad \text{for all } x \in \mathbb{R}^N \text{ such that } \|x\| \leq \xi_\varepsilon\,. \tag{6.1}$$

Now, for each $c > 2$, we consider a travelling wave (c, U) solution of (2.4). Then let us recall that there exist some constants $0 < m_c < M_c$ such that

$$m_c e^{-\lambda_c z} \leq U(z) \leq M_c e^{-\lambda_c z}\,, \quad \forall z \geq 0\,,$$

wherein $\lambda_c > 0$ is the smallest root of the equation

$$\lambda^2 - c\lambda + 1 = 0\,. \tag{6.2}$$

Then we will show the following lemma.

Lemma 6.2 (Sub-solutions)**.** *Let Assumption 1.6 be satisfied. Let $c > 2$ be given. Then there exist some constants $\widetilde{M} > 0$, $\widetilde{k} > 0$ and $\widehat{k} > 0$ such that*

$$u^\varepsilon(t^\varepsilon, x) \geq \widetilde{M} \exp\left\{ -\lambda_c \frac{\|x\| - \widetilde{k}\varepsilon|\ln\varepsilon|}{\varepsilon} \right\}\,, \quad \text{if } \|x\| \geq \widehat{k}\varepsilon|\ln\varepsilon|\,.$$

Proof. Let (c, U) be a given travelling wave associated with the given speed $c > 2$. Let $c_1 \in (0, c)$ be given and fixed. Let $\rho > 0$ be chosen large enough such that

$$\rho \geq \max\left(\frac{N-1}{c - c_1}, \frac{n}{\lambda_c} \right)$$
$$\frac{m}{1 + \rho^n} \geq M_c e^{-\lambda_c \rho}\,.$$

Then we consider the map $v_0 = v_0(s)$ defined by

$$v_0(s) := \begin{cases} U(\rho) & \text{if } |s| \le \rho \\ U(|s|) & \text{if } |s| \ge \rho\,, \end{cases}$$

and define the function $W(t,x)$ by

$$W(t,x) := v_0\left(\frac{\|x\| - c_1 t}{\varepsilon}\right).$$

Note that due to the choice of ρ and the definition of v_0 we have

$$\frac{m}{1+\frac{\|x\|^n}{\varepsilon^n}} \ge W(0,x)\,, \quad \text{if } \|x\| \le \varepsilon\rho\,.$$

Moreover, since $\rho \ge n/\lambda_c$, we see that

$$\frac{m}{1+\frac{\|x\|^n}{\varepsilon^n}} \ge M_c e^{-\lambda_c \frac{\|x\|}{\varepsilon}}\,, \quad \text{if } \|x\| \ge \varepsilon\rho\,.$$

Then we get that

$$u_{0,\varepsilon}(x) \ge \frac{m}{1+\frac{\|x\|^n}{\varepsilon^n}} \ge W(0,x)\,, \quad \forall x \in \mathbb{R}^N\,.$$

On the other hand straightforward computations yield

$$\mathcal{L}^\varepsilon[W](t,x) = \begin{cases} -\frac{1}{\varepsilon}U(\rho)(1-U(\rho)) & \text{if } |\|x\| - c_1 t| \le \varepsilon\rho \\ \frac{1}{\varepsilon}U'\left(\frac{\|x\|-c_1 t}{\varepsilon}\right)\left(c - c_1 + \varepsilon\frac{N-1}{\|x\|}\right) & \text{if } |\|x\| - c_1 t| \ge \varepsilon\rho\,. \end{cases}$$

Thus $\mathcal{L}^\varepsilon[W](t,x) \le 0$ for all $(t,x) \in (0,\infty)\times\mathbb{R}^N$.

From the comparison principle, we then deduce that

$$W(t,x) \le u^\varepsilon(t,x)\,,$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}^N$. As a consequence we obtain that

$$u^\varepsilon(t^\varepsilon,x) \ge m_c e^{-\lambda_c \frac{\|x\| - c_1 t^\varepsilon}{\varepsilon}}\,, \quad \text{if } \|x\| \ge c_1 t^\varepsilon + \varepsilon\rho\,.$$

Using this, one then easily proves that the choices $\widetilde{M} = m_c$, $\widetilde{k} = c_1\alpha$ and $\widehat{k} = 2c_1\alpha$ are enough to conclude.

Lemma 6.2 is proved. □

We are now in the position to prove Theorem 1.7.

Proof. Let $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^N$ be given. Let $c > \max(\frac{\|x_0\|}{t_0}, 2)$ be given. Fix $c_1 \in (0, c)$ such that

$$\|x_0\| - c_1 t_0 < 0\,.$$

Let $\widehat{\varepsilon} \in (0,1)$ be given and fixed. We choose $\rho > 0$ large enough such that

$$c_1 + \frac{N-1}{\rho} \leq c\,.$$

Let U be the travelling wave solution of (2.4) associated with the wave speed c such that

$$U(0) = 1 - \widehat{\varepsilon}. \tag{6.3}$$

Next, for each $\rho > 0$, we consider the map

$$q(s) := \begin{cases} U(0) & \text{if } \; s \leq \rho \\ U(s-\rho) & \text{if } \; s \geq \rho\,, \end{cases}$$

and define the function $\widetilde{W}(t,x)$ by

$$\widetilde{W}(t,x) := q\left(\frac{\|x\| - c_1 t}{\varepsilon}\right)\,.$$

Using similar arguments to those used in the proof of Lemma 6.2, we see that

$$\mathcal{L}^\varepsilon[\widetilde{W}] \leq 0 \quad \text{ in } (0,\infty) \times \mathbb{R}^N\,.$$

Next, we prove below that there exists $\varepsilon_1 \in (0, \widehat{\varepsilon})$ such that, for all $\varepsilon \in (0, \varepsilon_1)$,

$$\widetilde{W}(0,x) \leq u^\varepsilon(t^\varepsilon, x)\,, \;\; \forall x \in \mathbb{R}^N\,. \tag{6.4}$$

Indeed, it directly follows from (6.1) and (6.3) that (6.4) holds true if $\|x\| \leq \xi_\varepsilon$ and $\varepsilon \leq \widehat{\varepsilon}$. Let us now assume that $\|x\| \geq \xi_\varepsilon$. Note that, for $\varepsilon > 0$ small enough,

$$\varepsilon\rho \leq \xi_\varepsilon\,, \;\; \widehat{k}\varepsilon|\ln\varepsilon| \leq \xi_\varepsilon\,.$$

Therefore, we have

$$\widetilde{W}(0,x) \leq M_c e^{-\lambda_c \frac{\|x\| - \rho\varepsilon}{\varepsilon}}\,.$$

Since, by Lemma 6.2, we have

$$u(t^\varepsilon, x) \geq \widetilde{M} e^{-\lambda_c \frac{\|x\| - \widetilde{k}\varepsilon|\ln\varepsilon|}{\varepsilon}}\,,$$

it follows that (6.4) holds true as well in the case $\|x\| \geq \xi_\varepsilon$, for $\varepsilon > 0$ small enough.

The comparison principle then applies and yields, for $\varepsilon > 0$ small enough,

$$\widetilde{W}(t - t^\varepsilon, x) \leq u^\varepsilon(t,x)\,, \;\; \forall (t,x) \in [t^\varepsilon, \infty) \times \mathbb{R}^N\,.$$

Since $\|x_0\| - c_1 t_0 < 0$ and $\lim_{\varepsilon \to 0^+} \rho\varepsilon - c_1 t^\varepsilon = 0$, we see that, for $\varepsilon > 0$ small enough,

$$\frac{\|x_0\| - c_1(t_0 - t^\varepsilon)}{\varepsilon} \leq \rho\,,$$

which in turn implies that

$$u^\varepsilon(t_0, x_0) \geq \widetilde{W}(t_0 - t^\varepsilon, x_0) = U(0)\,.$$

Thus we get

$$U(0) \leq \liminf_{\varepsilon \to 0} u^\varepsilon(t_0, x_0)\,.$$

Since $U(0) = 1 - \widehat{\varepsilon}$, with $\widehat{\varepsilon} > 0$ arbitrary small, we obtain

$$1 \leq \liminf_{\varepsilon \to 0} u^\varepsilon(t_0, x_0)\,.$$

Finally, due to the first part of Lemma 6.1, we get that

$$\limsup_{\varepsilon \to 0} u^\varepsilon(t_0, x_0) \leq 1\,,$$

which completes the proof of Theorem 1.7.

$\square$

References

[1] M. Alfaro, D. Hilhorst and H. Matano, *The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system*, J. Differential Equations **245** (2008), 505–565.

[2] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), 5–49. Lecture Notes in Math., Vol. 446, Springer, Berlin, 1975.

[3] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math. **30** (1978), 33-76.

[4] G. Barles, L. C. Evans and P. E. Souganidis, *Wavefront propagation for reaction-diffusion systems of PDE*, Duke Math. J. **61** (1990), no. 3, 835–858.

[5] G. Barles and P. E. Souganidis, *A remark on the asymptotic behavior of the solution of the KPP equation*, C. R. Acad. Sci. Paris Série I **319** (1994), 679–684.

[6] H. Berestycki, F. Hamel and L. Roques, *Équations de réaction-diffusion et modèles d'invasions biologiques dans les milieux périodiques*, C.R. Acad. Sci. Paris **33** (2009), 549–554.

[7] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential equations **96** (1992), 116–141.

[8] L. C. Evans and P. E. Souganidis, *A PDE approach to geometric optics for certain semilinear parabolic equations*, Indiana Univ. Math. J. **38** (1989), 141–172.

[9] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. of Eugenics **7** (1937), 355–369.

[10] M. I. Freidlin, *Limit theorems for large deviations and reaction-diffusion equations*, Ann. Probab. **13** (1985), 639–675.

[11] D. Hilhorst, R. Kersner, E. Logak and M. Mimura, *Interface dynamics of the Fisher equation with degenerate diffusion*, J. Differential Equations **244** (2008), 2872–2889.

[12] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat Moscou, Bjul. Moskowskogo Gos. Univ., 1937, 1–26.

[13] H. Malchow, S.V. Petrovskii and E. Venturino, *Spatiotemporal Patterns in Ecology and Epidemiology. Theory, Models, and Simulations*, Mathematical and Computational Biology Series, Chapman & Hall/CRC Press, Boca Raton, FL, 2008.

[14] S.V. Petrovskii and H. Malchow, eds. (2005), *Biological Invasions in a Mathematical Perspective*, (A special issue of Biological Invasions: Proceedings of Computational and Mathematical Population Dynamics, Trento, June 21-25, 2004), Springer, Dordrecht, 128 p.

[15] N. Shigesada and K. Kawasaki, *Biological Invasion: Theory and Practise*, Oxford University Press, 1997.

[16] A. Volpert, V. Volpert, V. Volpert, *Travelling Wave Solutions of Parabolic Systems*, Translations of Mathematical Monographs, vol. 140, AMS Providence, RI, 1994.