Sharp interface limit of the Fisher-KPP equation when initial data have slow exponential decay

Matthieu Alfaro

I3M, Université de Montpellier 2,
CC051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France,

Arnaud Ducrot

UMR CNRS 5251 IMB and INRIA Sud-Ouest ANUBIS,
Université de Bordeaux, 3, Place de la Victoire, 33000 Bordeaux France.

Abstract

We investigate the singular limit, as $\varepsilon \to 0$, of the Fisher equation $\partial_t u = \varepsilon \Delta u + \varepsilon^{-1} u(1-u)$ in the whole space. We consider initial data with compact support plus perturbations with *slow exponential decay*. We prove that the sharp interface limit moves by a constant speed, which dramatically depends on the tails of the initial data. By performing a fine analysis of both the generation and motion of interface, we provide a new estimate of the thickness of the transition layers.

Key Words: Fisher equation, singular perturbation, generation of interface, motion of interface, travelling waves, tails of the initial data. [1]

1 Introduction

In this work, we consider $u^\varepsilon = u^\varepsilon(t,x)$ the solution of the rescaled Fisher-KPP equation

$$(P^\varepsilon) \quad \begin{cases} \partial_t u^\varepsilon = \varepsilon \Delta u^\varepsilon + \dfrac{1}{\varepsilon} u^\varepsilon(1-u^\varepsilon) & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,x) = u_{0,\varepsilon}(x) & \text{in } \mathbb{R}^N\,, \end{cases}$$

with $\varepsilon > 0$ a small parameter, related to the thickness of a diffuse interfacial layer. Let us recall that, in the classical works of Fisher [8] and Kolmogorov, Petrovsky and Piskunov [12], the authors consider a smooth monostable nonlinearity $f : [0,1] \to \mathbb{R}$ such that

$$f(0) = f(1) = 0\,, \quad 0 < f(u) \leq f'(0)u \ \text{ for all } u \in (0,1)\,.$$

[1] AMS Subject Classifications: 35K57, 35B25, 35R35, 35B50, 92D25.

Our results would hold for such nonlinearities but, for the sake of clarity, we restrict ourselves to the case where $f(u) = u(1-u)$.

In [1] we have investigated the singular limit, as $\varepsilon \to 0$, of (P^ε) when initial data have compact support plus, possibly, perturbations with a *fast exponential decay*. We proved that the sharp interface limit moves by constant speed which is the minimal speed $c^* = 2$ of the underlying travelling waves. We also obtained a new $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate for the thickness of the transition layers of the solutions u^ε.

The present paper is a completion of [1]: we are concerned with initial data with a *slow exponential decay*. In this case, it turns out that the limit interface moves by a speed which dramatically depends on the tails of the initial data. We prove the convergence and again obtain a new $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate for the thickness of the transition layers of the solutions u^ε.

We shall assume the following properties on the initial data.

Assumption 1.1. *We assume that* $u_{0,\varepsilon} = g + h_\varepsilon$ *where*

(i) *g is a bounded, nonnegative and compactly supported function. We define* $\Omega_0 := \operatorname{supp} g$.

(ii) *We define $\widetilde{g}$ as the restriction of g on $\overline{\Omega_0}$ and assume that $\widetilde{g}$ is of the class C^2.*

(iii) *h_ε is a nonnegative function and there exist $\lambda \in (0,1)$ and $0 < m < M$ such that, for all $\varepsilon > 0$ small enough,*

$$me^{-\lambda\frac{|d(0,x)|}{\varepsilon}} \leq h_\varepsilon(x) \leq Me^{-\lambda\frac{|d(0,x)|}{\varepsilon}}\,, \quad \forall x \in \mathbb{R}^N\,,$$

where $d(0,\cdot)$ denotes the cut-off signed distance function to the "initial interface" $\Gamma_0 := \partial\Omega_0$ (see subsection 2.3).

Remark 1.2 (Fast/Slow exponential decay). In the fast exponential decay case considered in [1], (iii) is replaced by $h_\varepsilon(x) \leq Me^{-\lambda\frac{\|x\|}{\varepsilon}}$ for some $\lambda \geq 1$. In some sense, see subsection 2.2 for details, the exponential decay of the initial data is faster than the exponential decay of the underlying travelling wave of minimal speed c^*. In this case, $\lambda \geq 1$ does not affect the asymptotic speed of the limit interface which is always c^*.

In the present case $\lambda \in (0,1)$ we consider, (3.6) indicates that the exponential decay of the initial data and that of the underlying travelling wave of speed $\lambda + \lambda^{-1}$ are the same; it follows that the construction of efficient sub-solutions is more involved than in [1]. Here, $\lambda \in (0,1)$ does affect the asymptotic speed of the limit interface, which turns out to be $\lambda + \lambda^{-1}$.

Note that the regularity assumption (ii) can be relaxed. We refer to Remark 1.8. in [1].

Assumption 1.3. *We assume that* Ω_0 *is convex.*

Assumption 1.4. *We assume the existence of $\delta > 0$ such that, if n denotes the Euclidian unit normal vector exterior to the initial interface Γ_0, then*

$$\left|\frac{\partial \widetilde{g}}{\partial n}(y)\right| \geq \delta \quad \textit{for all } y \in \Gamma_0 \,. \tag{1.1}$$

Assumption 1.1 gives the structure of our allowed initial data. Note that Assumption 1.3 is used to find upper bounds for the solutions u^ε (see Lemma 4.1) whereas Assumption 1.4 is only used to derive the correspondence (3.4).

Before going into much details, let us comment on related known results. It is long known that the tails of the initial data play a key role in the study of the long time behavior of $u = u(t,x)$ the solution of the Fisher-KPP equation $\partial_t u = \Delta u + u(1-u)$. As far as initial data with exponential decay are concerned, we refer among others to [15], [5], [14] for a probabilistic framework and to [13], [16], [17] for a reaction-diffusion framework. More recently, Hamel and Roques [10] studied the case where the initial data decays more slowly than any exponentially decaying function.

In a singular limit framework, the question of the convergence of Problem (P^ε) has been addressed when the initial data $u_{0,\varepsilon}$ does not depend on ε and is compactly supported : first by Freidlin [9] using probabilistic methods and later by Evans and Souganidis [7] using Hamilton Jacobi technics (in this framework we also refer to [3, 4]). In [1], we provide a new proof of convergence for Problem (P^ε) with fast exponentially decaying initial data, by using specific reaction-diffusion tools such as the comparison principle. Moreover, we obtain an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ estimate of the thickness of the transition layers of the solutions u^ε. To the best of our knowledge, no such fine estimate of the thickness of the transition layers existed for the Fisher-KPP equation (in contrast with the Allen-Cahn equation).

As $\varepsilon \to 0$, by formally neglecting the diffusion term, we see that, in the very early stage, the value of u^ε quickly becomes close to either 1 or 0 in most part of $\mathbb{R}^N$, creating a steep interface (transition layer) between the regions $\{u^\varepsilon \approx 1\}$ and $\{u^\varepsilon \approx 0\}$. Once such an interface develops, the diffusion term is large near the interface comes to balance with the reaction term. As a result, the interface ceases rapid development and starts to propagate in a much slower time scale. Therefore the limit solution $\tilde{u}(t,x)$ will be a step function taking the value 1 on one side of the moving interface, and 0 on the other side.

We shall prove that this sharp interface, which we will denote by $\Gamma_t^{c_\lambda}$, obeys the law of motion

$$(P^{c_\lambda}) \qquad \begin{cases} V_n = c_\lambda := \lambda + \lambda^{-1} & \text{on } \Gamma_t^{c_\lambda} \\ \Gamma_t^{c_\lambda}\big|_{t=0} = \Gamma_0 \,, \end{cases}$$

where V_n denotes the normal velocity of $\Gamma_t^{c_\lambda}$ in the exterior direction. Note that $c_\lambda = \lambda + \lambda^{-1} > 2 = c^*$, with $c^* = 2$ the minimal speed of some related one-dimensional travelling waves (see subsection 2.2 for details). Therefore, as expected, the slower the initial data decay, the larger is the the speed of the sharp interface limit.

Since the region enclosed by Γ_0, namely Ω_0, is smooth and convex, Problem (P^{c_λ}), possesses a unique smooth solution on $[0, \infty)$, which we denote by $\Gamma^{c_\lambda} = \bigcup_{t \geq 0}(\{t\} \times \Gamma_t^{c_\lambda})$. Hereafter, we fix $T > 0$ and work on $(0, T]$.

For each $t \in (0, T]$, we denote by $\Omega_t^{c_\lambda}$ the region enclosed by the hypersurface $\Gamma_t^{c_\lambda}$. We define a step function $\tilde{u}(t, x)$ by

$$\tilde{u}(t,x) = \begin{cases} 1 & \text{in } \Omega_t^{c_\lambda} \\ 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega_t^{c_\lambda}} \end{cases} \quad \text{for } t \in (0,T], \tag{1.2}$$

which represents the asymptotic limit of u^ε as $\varepsilon \to 0$.

Our main result, Theorem 1.5, shows that, after a short time of order $\mathcal{O}(\varepsilon|\ln\varepsilon|)$, the solution u^ε quickly becomes close to 1 or 0, except in a small neighborhood of the initial interface Γ_0, creating a steep transition layer around Γ_0 (*generation of interface*). The theorem then states that the solution u^ε remains close to the step function $\tilde{u}$ on the time interval $[t^\varepsilon, T]$ (*motion of interface*). Last, (1.3) shows that, for any $0 < a < 1$, for all $t^\varepsilon \leq t \leq T$, the level-set $\Gamma_t^\varepsilon(a) := \{x \in \mathbb{R}^N : u^\varepsilon(t,x) = a\}$ lives in an $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ tubular neighborhood of the limit interface $\Gamma_t^{c_\lambda}$. In other words, we provide a new estimate of the *thickness of the transition layers* of the solutions u^ε.

Theorem 1.5 (Generation, motion and thickness of interface)**.** *Let Assumptions 1.1, 1.3 and 1.4 be satisfied. Then there exist positive constants α and $\mathcal{C}$ such that, for all $\varepsilon > 0$ small enough, for all $t^\varepsilon \leq t \leq T$, where*

$$t^\varepsilon := \alpha\varepsilon|\ln\varepsilon|,$$

we have

$$u^\varepsilon(t,x) \in \begin{cases} [0, 1+\varepsilon] & \text{if} \quad x \in \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^{c_\lambda}) \\ [1-2\varepsilon, 1+\varepsilon] & \text{if} \quad x \in \Omega_t^{c_\lambda} \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^{c_\lambda}) \\ [0, \varepsilon] & \text{if} \quad x \in (\mathbb{R}^N \setminus \overline{\Omega_t^{c_\lambda}}) \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^{c_\lambda}), \end{cases} \tag{1.3}$$

where $\mathcal{N}_r(\Gamma_t^{c_\lambda}) := \{x \in \mathbb{R}^N : dist(x, \Gamma_t^{c_\lambda}) < r\}$ denotes the tubular r-neighborhood of $\Gamma_t^{c_\lambda}$.

As a immediate consequence of the above Theorem, we collect the convergence result.

Corollary 1.6 (Convergence)**.** *Let Assumptions 1.1, 1.3 and 1.4 be satisfied. Then, as* $\varepsilon \to 0$*,* u^ε *converges to* $\tilde{u}$ *everywhere in* $\bigcup_{0<t\leq T} (\{t\} \times \Omega_t^{c_\lambda})$ *and* $\bigcup_{0<t\leq T} \Big(\{t\} \times (\mathbb{R}^N \setminus \overline{\Omega_t^{c_\lambda}})\Big)$.

The organization of this paper is as follows. In Section 2, we present the basic tools that will be used in later sections for the construction of sub- and super-solutions. In Section 3 we construct two sub-solutions, one for small times (during the generation of interface) and one for later times (during the motion of interface). Section 4 is devoted to the construction of a single super-solution which is efficient to study both the generation and the motion of interface. Last, in Section 5, by using our different sub- and super-solutions we prove Theorem 1.5.

2 Materials

The needed tools are the same than in [1]. For the self-containedness of the present paper we recall them here. Let us note than the non monotone travelling waves of speed $0 < c < c^*$ used in [1] are useless here.

2.1 A monostable ODE

The generation of interface is strongly related to the dynamical properties of the ordinary differential equation associated to (P^ε), that is

$$\frac{dz(t)}{dt} = z(t)(1 - z(t)) \,, \quad t > 0 \,.$$

In the sequel, for technical reasons we shall apply the semiflow generated by the above dynamical system to negative initial data. In order to have some good dynamical properties, let us modify the monostable nonlinearity $u \to u(1-u)$ on $(-\infty, 0)$ so that the modified function, we call it $\bar{f} : \mathbb{R} \to \mathbb{R}$, is of the class C^2 and enjoys the bistable assumptions. More precisely, $\bar{f}$ has exactly three zeros $-1 < 0 < 1$ and

$$\bar{f}'(-1) < 0 \,, \qquad \bar{f}'(0) = 1 > 0 \,, \qquad \bar{f}'(1) = -1 < 0 \,. \tag{2.1}$$

Note that $\bar{f}(u) = u(1-u)$ if $u \geq 0$. As done in Chen [6], we consider $\bar{f}_\varepsilon$ a slight modification of $\bar{f}$ defined by

$$\bar{f}_\varepsilon(u) := \psi(u) \frac{u - \varepsilon |\ln \varepsilon|}{|\ln \varepsilon|} + (1 - \psi(u)) \bar{f}(u) \,,$$

with ψ a smooth cut-off function satisfying conditions (29)—(32) as they appear in [11]. As explained in [11],

$$\bar{f}_\varepsilon(u) \leq \bar{f}(u) \quad \text{ for all } u \in \mathbb{R} \,. \tag{2.2}$$

Then we defined $w(s,\xi)$ as the semiflow generated by the ordinary differential equation

$$\begin{cases} \dfrac{dw}{ds}(s,\xi) = \bar{f}_\varepsilon(w(s,\xi))\,, \quad s > 0\,, \\ w(0,\xi) = \xi\,. \end{cases} \tag{2.3}$$

Here ξ ranges over the interval $[-\|g\|_\infty - M - 1, \|g\|_\infty + M + 1]$. We claim that $w(s,\xi)$ has the following properties (for proofs, see [6] or [11]).

Lemma 2.1 (Behavior of w)**.** *The following holds for all $\xi \in [-\|g\|_\infty - M - 1, \|g\|_\infty + M + 1]$.*

(i) If $\xi \geq \varepsilon|\ln\varepsilon|$ then $w(s,\xi) \geq \varepsilon|\ln\varepsilon| > 0$ for all $s > 0$.
If $\xi < 0$ then $w(s,\xi) < 0$ for all $s > 0$.
If $\xi \in (0, \varepsilon|\ln\varepsilon|)$ then $w(s,\xi) > 0$ for all $s \in (0, s_\varepsilon(\xi))$, with

$$s_\varepsilon(\xi) := |\ln\varepsilon| \left| \ln\left(1 - \frac{\xi}{\varepsilon|\ln\varepsilon|}\right) \right|.$$

(ii) $w(s,\xi) \in (-\|g\|_\infty - M - 1, \|g\|_\infty + M + 1)$ for all $s > 0$.

(iii) w is of the class C^2 with respect to ξ and

$$w_\xi(s,\xi) > 0 \quad \text{for all } s > 0\,.$$

(iv) For all $a > 0$, there exists a constant $C(a)$ such that

$$\left| \frac{w_{\xi\xi}}{w_\xi}(s,\xi) \right| \leq \frac{C(a)}{\varepsilon} \quad \text{for all } 0 < s \leq a|\ln\varepsilon|\,.$$

(v) There exists a positive constant α such that, for all $s \geq \alpha|\ln\varepsilon|$, we have

$$\text{if} \quad \xi \in [\varepsilon|\ln\varepsilon|, \|g\|_\infty + M + 1] \quad \text{then} \quad 0 < w(s,\xi) \leq 1 + \varepsilon\,,$$

and

$$\text{if} \quad \xi \in [3\varepsilon|\ln\varepsilon|, \|g\|_\infty + M + 1] \quad \text{then} \quad 1 - \varepsilon \leq w(s,\xi)\,.$$

2.2 Travelling waves

A travelling wave is a couple (c, U) with $c > 0$ and $U \in C^2(\mathbb{R},\mathbb{R})$ a function such that

$$\begin{cases} U''(z) + cU'(z) + U(z)(1 - U(z)) = 0 \quad \text{for all } z \in \mathbb{R} \\ U(-\infty) = 1 \\ U(\infty) = 0\,. \end{cases} \tag{2.4}$$

Define $c^* := 2$. Then, for all $c \geq c^*$ there exists a unique (up to a translation in z) travelling wave denoted by (c, U). It is positive and monotone.

Lemma 2.2 (Behavior of U)**.** *Let $c > 2 = c^*$ be arbitrary and consider the associated travelling wave (c, U). Then there exist constants $C > 0$ and $0 < r < R$ such that*

$$re^{-\eta|z|} \leq 1 - U(z) \leq Re^{-\eta|z|} \quad \text{for } z \leq 0\,, \tag{2.5}$$

$$re^{-\mu z} \leq U(z) \leq Re^{-\mu z} \quad \text{for } z \geq 0\,, \tag{2.6}$$

$$re^{-\eta|z|} \leq |U'(z)| + |U''(z)| \leq Re^{-\eta|z|} \quad \text{for } z \leq 0\,, \tag{2.7}$$

$$re^{-\mu|z|} \leq |U'(z)| + |U''(z)| \leq Re^{-\mu|z|} \quad \text{for } z \geq 0\,, \tag{2.8}$$

with $\eta > 0$ the positive root of equation $\eta^2 + c\eta - 1 = 0$ and $\mu > 0$ the smallest root of equation $\mu^2 - c\mu + 1 = 0$.

We refer the reader to [2, 18] and the references therein for more details.

2.3 Cut-off signed distance functions

Recall that $\Gamma^{c_\lambda} = \bigcup_{t\geq 0}(\{t\}\times\Gamma_t^{c_\lambda})$ is the smooth solution of the free boundary problem (P^{c_λ}) and that, for each $t > 0$, $\Omega_t^{c_\lambda}$ is the region enclosed by the hypersurface $\Gamma_t^{c_\lambda}$.

Let $\widetilde{d}(t,\cdot)$ be the signed distance function to $\Gamma_t^{c_\lambda}$ defined by

$$\widetilde{d}(t,x) = \begin{cases} -\operatorname{dist}(x, \Gamma_t^{c_\lambda}) & \text{for } x \in \Omega_t^{c_\lambda} \\ \operatorname{dist}(x, \Gamma_t^{c_\lambda}) & \text{for } x \in \mathbb{R}^N \setminus \Omega_t^{c_\lambda}\,. \end{cases} \tag{2.9}$$

We remark that $\widetilde{d} = 0$ on Γ^{c_λ} and that $|\nabla\widetilde{d}| = 1$ in a neighborhood of Γ^{c_λ}.

We now introduce the "cut-off signed distance function" d, which is defined as follows. Recall that $T > 0$ is fixed. First, choose $d_0 > 0$ small enough so that $\widetilde{d}$ is smooth in the tubular neighborhood of Γ^{c_λ}

$$\{(t,x) \in [0,T] \times \mathbb{R}^N : \ |\widetilde{d}(t,x)| < 4d_0\}\,.$$

Next let $\zeta(s)$ be a smooth function satisfying

$$0 \leq \zeta'(s) \leq 1 \quad \text{for all } s \in \mathbb{R}\,, \tag{2.10}$$

such that

$$\zeta(s) = \begin{cases} s & \text{if } \ |s| \leq d_0 \\ -2d_0 & \text{if } \ s \leq -3d_0 \\ 2d_0 & \text{if } \ s \geq 3d_0\,. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(t,x) := \zeta\left(\tilde{d}(t,x)\right)\,. \tag{2.11}$$

Note that

$$\text{if} \quad |d(t,x)| < d_0 \quad \text{then} \quad |\nabla d(t,x)| = 1\,, \tag{2.12}$$

and that the equation of motion (P^{c_λ}) yields

$$\text{if} \quad |d(t,x)| < d_0 \quad \text{then} \quad \partial_t d(t,x) + c_\lambda = 0\,. \tag{2.13}$$

Then the mean value theorem provides a constant $A > 0$ such that

$$|\partial_t d(t,x) + c_\lambda| \le A|d(t,x)| \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N\,. \tag{2.14}$$

Moreover, there exists a constant $C > 0$ such that

$$|\nabla d(t,x)| + |\Delta d(t,x)| \le C \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N\,. \tag{2.15}$$

3 Sub-solutions

As explained before, the construction of efficient sub-solutions is more involved than in the fast exponential decay case considered in [1]. We start by constructing refined sub-solutions for small times.

3.1 Generation of interface

Let us recall that $me^{-\lambda\frac{|d(0,x)|}{\varepsilon}} \le h_\varepsilon(x) \le Me^{-\lambda\frac{|d(0,x)|}{\varepsilon}}$ for all $x \in \mathbb{R}^N$. We then define the map

$$\underline{u}(t,x) := \max\left\{\tilde{m}e^{-\lambda\frac{|d(0,x)|}{\varepsilon}}, w\left(\frac{t}{\varepsilon}, g(x) - Kt\right)\right\}, \tag{3.1}$$

where $w(s,\xi)$ is the solution of the ordinary differential equation (2.3) and where $\tilde{m} = \min(\frac{1}{2}, m)$. The constant $K > 0$ is to be specified below.

Lemma 3.1 (Sub-solutions for small times)**.** *Let Assumption 1.1 be satisfied. Then for all $a > 0$, there exists $K > 0$ such that, for all $\varepsilon > 0$ small enough, we have*

$$\underline{u}(t,x) \le u^\varepsilon(t,x)\,, \quad \forall t \in [0, a\varepsilon|\ln\varepsilon|]\,,\ \forall x \in \mathbb{R}^N\,. \tag{3.2}$$

Proof. Let us first notice that, for all $x \in \mathbb{R}^N$,

$$\underline{u}(0,x) = \max\left(\tilde{m}e^{-\lambda\frac{|d(0,x)|}{\varepsilon}}, g(x)\right) \le me^{-\lambda\frac{|d(0,x)|}{\varepsilon}} + g(x) = u^\varepsilon(0,x)\,.$$

Then it remains to show that $\underline{u}$ is a sub-solution of Problem (P^ε).

Let us consider the operator

$$\mathcal{L}^\varepsilon[v] := \partial_t v - \varepsilon\Delta v - \frac{1}{\varepsilon}v(1-v)\,.$$

Let $a > 0$ be arbitrary. We show below that, if $K > 0$ is sufficiently large then, for all $\varepsilon > 0$ small enough, $\mathcal{L}^\varepsilon[\underline{u}] \leq 0$. We distinguish two cases.

We first consider the points (t, x) where $\underline{u}(t, x) = \tilde{m}e^{-\lambda\frac{|d(0,x)|}{\varepsilon}}$. Assume further that $d(0, x) \geq 0$ so that $\underline{u}(t, x) = \tilde{m}e^{-\lambda\frac{d(0,x)}{\varepsilon}} =: \varphi_\varepsilon(x)$. We compute

$$\partial_t \underline{u} = 0$$

$$\Delta \underline{u} = \lambda^2 \frac{|\nabla d|^2}{\varepsilon^2} \varphi_\varepsilon - \lambda \frac{\Delta d}{\varepsilon} \varphi_\varepsilon \,.$$

Therefore, we get

$$\varepsilon \mathcal{L}^\varepsilon[\underline{u}](t, x) = \varphi_\varepsilon \left(-\lambda^2 |\nabla d|^2 + \varepsilon \lambda \Delta d - 1 + \varphi_\varepsilon\right) \,.$$

Since $0 \leq \varphi_\varepsilon \leq 1/2$, we get

$$\varepsilon \mathcal{L}^\varepsilon[\underline{u}](t, x) \leq \varphi_\varepsilon \left(\varepsilon \lambda \Delta d - \frac{1}{2}\right) \leq 0 \,,$$

for $\varepsilon > 0$ sufficiently small. The case where $d(0, x) \leq 0$ is very similar and omitted.

Next we consider the points (t, x) where $\underline{u}(t, x) = w\left(\frac{t}{\varepsilon}, g(x) - Kt\right)$. We have

$$\partial_t \underline{u} = \frac{1}{\varepsilon} w_s - K w_\xi$$

$$\Delta \underline{u} = w_{\xi\xi} |\nabla g|^2 + w_\xi \Delta g \,.$$

Then, we get

$$\begin{aligned}
\mathcal{L}^\varepsilon[\underline{u}](t, x) \; &= \frac{1}{\varepsilon} w_s - K w_\xi - \varepsilon \left(w_{\xi\xi} |\nabla g|^2 + w_\xi \Delta g\right) - \frac{1}{\varepsilon} w(1 - w) \\
&\leq \frac{1}{\varepsilon} w_s - K w_\xi - \varepsilon \left(w_{\xi\xi} |\nabla g|^2 + w_\xi \Delta g\right) - \frac{1}{\varepsilon} \bar{f}_\varepsilon(w) \\
&= -w_\xi \left[K + \varepsilon \left(\tfrac{w_{\xi\xi}}{w_\xi} |\nabla g|^2 + \Delta g\right)\right] \,,
\end{aligned}$$

where we have successively used (2.2) and (2.3). In view of Lemma 2.1 (iv), there exists a constant $C(a) > 0$ such that, for all (t, x) with $0 \leq t \leq a\varepsilon |\ln \varepsilon|$, we have

$$\left| \frac{w_{\xi\xi}}{w_\xi} |\nabla g|^2 + \Delta g \right| \leq \frac{C(a)}{\varepsilon} \,.$$

Therefore, choosing $K > C(a)$ implies

$$\mathcal{L}^\varepsilon[\underline{u}](t, x) \leq -w_\xi \left(K - C(a)\right) \leq 0 \,,$$

since $w_\xi > 0$.

This completes the proof of Lemma 3.1. $\square$

As a consequence of the above construction of sub-solutions for small times, we deduce that, after a very short time, the solution u^ε approaches 1 in most part of the support of the initial data.

Corollary 3.2 (Generation of interface "from the inside")**.** *Let Assumption 1.1 be satisfied. Then there exist $k>0$, $\alpha>0$ such that, for all $\varepsilon>0$ small enough,*

$$d(0,x)\leq -k\varepsilon|\ln\varepsilon| \implies 1-\varepsilon\leq u^\varepsilon(t^\varepsilon,x)\leq 1+\varepsilon\,, \tag{3.3}$$

wherein $t^\varepsilon:=\alpha\varepsilon|\ln\varepsilon|$.

Proof. In Lemma 3.1, we select $a=\alpha$ where $\alpha>0$ is as in Lemma 2.1 (v). Note that, in view of (1.1), the mean value theorem provides the existence of a constant $k>0$ such that

$$\text{if} \quad d(0,x)\leq -k\varepsilon|\ln\varepsilon| \quad \text{then} \quad g(x)\geq (3+K\alpha)\varepsilon|\ln\varepsilon|\,. \tag{3.4}$$

Then, using (3.2), we see that, for all x satisfying $d(0,x)\leq -k\varepsilon|\ln\varepsilon|$,

$$u^\varepsilon(t^\varepsilon,x)\geq \underline{u}(t^\varepsilon,x)\geq w(\alpha|\ln\varepsilon|,3\varepsilon|\ln\varepsilon|)\geq 1-\varepsilon\,.$$

The last inequality follows from Lemma 2.1 (v).

Note that the upper bound $1+\varepsilon$ is actually valid for all x and all $t\geq t^\varepsilon$, as seen in (5.1), and will be proved in Section 5.

The corollary is proved. □

3.2 Motion of interface

In the fast exponential decay case, the limit speed of the sharp interface limit is c^*. Therefore sub-solutions for the motion of interface are constructed in [1] by using the travelling waves associated with the speeds $c^*-o(\varepsilon)<c^*$. Since they are changing sign, a slight modification make the sub-solutions compactly supported from one side. In the slow exponential decay case we consider, the limit interface moves with speed $c_\lambda>c^*$. Since the travelling waves with speeds $c_\lambda-o(\varepsilon)>c^*$ are not changing sign we are not able to construct compactly supported sub-solutions.

In the sequel, we denote by U the travelling wave associated with the speed of the limit interface $c_\lambda=\lambda+\lambda^{-1}$ and by $d(t,x)$ the cut-off signed distance function associated with Γ^{c_λ} the solution of the free boundary problem (P^{c_λ}). We also consider V the travelling wave associated with the speed $c_\varepsilon:=c_\lambda-\varepsilon|\ln\varepsilon|$.

From Lemma 2.2, we see that for U (whose speed is $c_\lambda=\lambda+\lambda^{-1}$), the μ that appears in (2.6) is actually equal to λ. Therefore, we collect

$$re^{-\eta|z|}\leq 1-U(z)\leq Re^{-\eta|z|} \quad \text{for } z\leq 0\,, \tag{3.5}$$

$$re^{-\lambda z}\leq U(z)\leq Re^{-\lambda z} \quad \text{for } z\geq 0\,. \tag{3.6}$$

Moreover, since V which is slightly slower it will decay faster. More precisely, we deduce from Lemma 2.2 that

$$re^{-\eta_\varepsilon|z|} \leq 1 - V(z) \leq Re^{-\eta_\varepsilon|z|} \quad \text{ for } z \leq 0\,, \tag{3.7}$$

$$re^{-\mu_\varepsilon z} \leq V(z) \leq Re^{-\mu_\varepsilon z} \quad \text{ for } z \geq 0\,, \tag{3.8}$$

with $\eta_\varepsilon \geq \eta + \gamma\varepsilon|\ln\varepsilon|$ and $\mu_\varepsilon \geq \lambda + \gamma\varepsilon|\ln\varepsilon|$, for some $\gamma > 0$. The estimates on the derivatives of U and V corresponding to (2.7) and (2.8) also hold.

We are looking for sub-solutions in the form

$$u^-(t,x) := U\left(\frac{d(t,x)+\varepsilon|\ln\varepsilon|m_1e^{m_2t}}{\varepsilon}\right) - \varepsilon V\left(\frac{d(t,x)+\varepsilon|\ln\varepsilon|m_1e^{m_2t}}{\varepsilon}\right). \tag{3.9}$$

In the sequel we set

$$z(t,x) := \frac{d(t,x)+\varepsilon|\ln\varepsilon|m_1e^{m_2t}}{\varepsilon}\,. \tag{3.10}$$

Lemma 3.3 (Ordering initial data)**.** *Let Assumptions 1.1 and 1.4 be satisfied. Then there exists $\tilde{m}_1 > 0$ such that for all $m_1 \geq \tilde{m}_1$, all $m_2 > 0$, all $\varepsilon > 0$ small enough, we have*

$$u^-(0,x) \leq u^\varepsilon(t^\varepsilon, x)\,, \tag{3.11}$$

for all $x \in \mathbb{R}^N$, with $t^\varepsilon = \alpha\varepsilon|\ln\varepsilon|$.

Proof. Choose $k > 0$ so that (3.3) holds and $m_1 \geq \tilde{m}_1 := 3k$. Note that to prove Lemma 3.3, it is sufficient to check that

$$u^-(0,x) = U\left(z(0,x)\right) - \varepsilon V\left(z(0,x)\right) \leq \underline{u}(t^\varepsilon, x)\,, \tag{3.12}$$

where $\underline{u}$ is the sub-solution for small times defined in (3.1). To prove this inequality we shall split our arguments into three parts according to the value of $d(0,x)$.

If x is such that $d(0,x) \geq 0$. Since $z(0,x) \geq 0$, we deduce from (3.6) that, for $\varepsilon >$ small enough,

$$u^-(0,x) \leq U(z(0,x)) \leq Re^{-\lambda\frac{d(0,x)}{\varepsilon}}e^{-\lambda m_1|\ln\varepsilon|} \leq \tilde{m}e^{-\lambda\frac{|d(0,x)|}{\varepsilon}} \leq \underline{u}(t^\varepsilon, x)\,.$$

If x is such that $-k\varepsilon|\ln\varepsilon| \leq d(0,x) \leq 0$. The choice of m_1 implies that $z(0,x) \geq 2k|\ln\varepsilon|$ so that, for $\varepsilon >$ small enough,

$$u^-(0,x) \leq Re^{-\lambda k|\ln\varepsilon|}e^{-\lambda k|\ln\varepsilon|} \leq \tilde{m}e^{-\lambda k|\ln\varepsilon|} \leq \tilde{m}e^{-\lambda\frac{|d(0,x)|}{\varepsilon}} \leq \underline{u}(t^\varepsilon, x)\,.$$

If x is such that $-2d_0 \leq d(0,x) \leq -k\varepsilon|\ln\varepsilon|$, which implies that $z(0,x) \geq -d_0/\varepsilon$. We claim that (see proof below), for $\varepsilon > 0$ small enough,

$$U'(z) - \varepsilon V'(z) < 0 \quad \text{ for all } z \in \mathbb{R}\,. \tag{3.13}$$

Therefore, by using (3.5) and (3.7) we get

$$\begin{aligned} u^-(0,x) &\leq (U - \varepsilon V)\,(-d_0/\varepsilon) \\ &\leq 1 - re^{-\eta d_0/\varepsilon} - \varepsilon(1 - Re^{-\eta_\varepsilon d_0/\varepsilon}) \\ &= 1 - \varepsilon - e^{-\eta d_0/\varepsilon}(r - \varepsilon Re^{-(\eta_\varepsilon - \eta)d_0/\varepsilon}) \\ &\leq 1 - \varepsilon - e^{-\eta d_0/\varepsilon}(r - \varepsilon R) \\ &\leq 1 - \varepsilon\,, \end{aligned}$$

for $\varepsilon > 0$ small enough. In view of Corollary 3.2, this completes the proof of (3.11).

It remains to prove (3.13). Note that $(U - \varepsilon V)(-\infty) = 1 - \varepsilon$ and $(U - \varepsilon V)(\infty) = 0$ and assume, by contradiction, that there exist $\varepsilon_0 > 0$ and a family $\{z_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ such that

$$U'(z_\varepsilon) - \varepsilon V'(z_\varepsilon) = 0 \quad \text{ for each } \varepsilon \in (0, \varepsilon_0)\,.$$

First assume that $\{z_\varepsilon\}$ is bounded. Then there exists a sequence $\{\varepsilon_n\}_{n\geq 0}$ tending to zero such that $z_{\varepsilon_n} \to z_0 \in \mathbb{R}$ when $n \to \infty$ for some z_0. Recall that V depends on $\varepsilon > 0$ and is uniformly bounded with respect to ε up to its second derivative. Passing to the limit $n \to \infty$ leads us to $U'(z_0) = 0$, a contradiction. Next, assume that $\{z_\varepsilon\}$ is unbounded. Then there exists a sequence $\{\varepsilon_n\}_{n\geq 0}$ tending to zero such that $z_{\varepsilon_n} \to \infty$ or $z_{\varepsilon_n} \to -\infty$ when $n \to \infty$. Consider the case where $z_{\varepsilon_n} \to \infty$ then we obtain that

$$\varepsilon_n = \frac{U'(z_{\varepsilon_n})}{V'(z_{\varepsilon_n})} \geq \frac{re^{-\lambda z_{\varepsilon_n}}}{Re^{-\mu_\varepsilon z_{\varepsilon_n}}} \geq \frac{r}{R} e^{(\mu_{\varepsilon_n} - \lambda)z_{\varepsilon_n}} \geq \frac{r}{R}\,,$$

a contradiction with the behavior of $\{\varepsilon_n\}$ when $n \to \infty$. The case where $z_{\varepsilon_n} \to -\infty$ is similar. The claim (3.13) is proved. □

Lemma 3.4 (Sub-solutions for later times)**.** *Recall that u^- was defined in (3.9) and assume that $U(0) > \frac{1}{2}$. Then there exists $\tilde m_2 > 0$ such that for all $m_1 \geq \tilde m_1$, all $m_2 \geq \tilde m_2$, all $\varepsilon > 0$ small enough, we have*

$$\varepsilon \mathcal{L}^\varepsilon[u^-] = \varepsilon \partial_t u^- - \varepsilon^2 \Delta u^- - u^-(1 - u^-) \leq 0 \quad \text{ in } (0,\infty) \times \mathbb{R}^N\,. \tag{3.14}$$

Proof. By using straightforward computations we get

$$\begin{aligned} \varepsilon \partial_t u^- &= (U' - \varepsilon V')(z)\left(\partial_t d + m_2 \varepsilon |\ln\varepsilon| m_1 e^{m_2 t}\right) \\ \varepsilon^2 \Delta u^- &= (U'' - \varepsilon V'')(z)|\nabla d|^2 + \varepsilon(U' - \varepsilon V')(z)\Delta d \\ u^-(1 - u^-) &= U(1 - U) + \varepsilon UV - \varepsilon V(1 - U) - \varepsilon^2 V^2\,, \end{aligned}$$

where $z = z(t,x)$ was defined in (3.10). Now, the ordinary differential equations $U'' + c_\lambda U' + U(1-U) = 0$ and $V'' + (c_\lambda - \varepsilon|\ln\varepsilon|)V' + V(1-V) = 0$ yield

$$\varepsilon\mathcal{L}^\varepsilon[u^-] = E_1 + \cdots + E_4\,,$$

with

$$E_1 := (U' - \varepsilon V')(z)\left(\partial_t d + c_\lambda + m_2\varepsilon|\ln\varepsilon|m_1 e^{m_2 t} - \varepsilon\Delta d\right)$$

$$E_2 := \varepsilon^2|\ln\varepsilon|V'(z)$$

$$E_3 := (U'' - \varepsilon V'')(z)(1 - |\nabla d|^2)$$

$$E_4 := -\varepsilon V(2U - V)(z) + \varepsilon^2 V^2(z)\,.$$

Note that $E_2 \leq 0$. We show below that the choice

$$\tilde{m}_2 := 2A\left(\frac{2}{\tilde{m}_1\eta} + 1\right)\,,$$

is enough to prove the lemma, with $A > 0$ the constant that appears in (2.14). To that purpose we distinguish four cases, namely (3.15), (3.17), (3.18) and (3.20). In the sequel we denote by C positive constants which do not depend on ε (and may change from places to places).

Assume that

$$-2d_0 \leq d(t,x) \leq -\frac{m_2}{2A}\varepsilon|\ln\varepsilon|m_1 e^{m_2 t}\,. \tag{3.15}$$

This implies

$$z \leq -|\ln\varepsilon|m_1\left(\frac{m_2}{2A} - 1\right) \leq \frac{2}{\eta}\ln\varepsilon\,. \tag{3.16}$$

Using the estimates for U' and V' we see that, for $\varepsilon > 0$ small enough,

$$|E_1| \leq C(e^{-\eta|z|} + \varepsilon e^{-\eta_\varepsilon|z|}) \leq Ce^{-\eta|z|} \leq C\varepsilon^2\,,$$

thanks to (3.16). Using similar arguments we see that $|E_3| \leq C\varepsilon^2$. At least note that $z \to -\infty$ as $\varepsilon \to 0$ and that $V(2U - V)(-\infty) = 1$. Therefore, if $\varepsilon > 0$ is small enough then $E_4 \leq -\frac{1}{2}\varepsilon + C\varepsilon^2$. It follows that $\varepsilon\mathcal{L}^\varepsilon[u^-] \leq -\frac{1}{2}\varepsilon + C\varepsilon^2 \leq 0$.

Assume that

$$-\frac{m_2}{2A}\varepsilon|\ln\varepsilon|m_1 e^{m_2 t} \leq d(t,x) \leq -\varepsilon|\ln\varepsilon|m_1 e^{m_2 t}\,. \tag{3.17}$$

From (2.12) we deduce that $E_3 = 0$. From (2.13), we deduce that $\partial_t d + c_\lambda = 0$. Since, for $\varepsilon > 0$ small enough, $m_2\varepsilon|\ln\varepsilon|m_1 e^{m_2 t} - \varepsilon\Delta d \geq 0$ we deduce from (3.13) that $E_1 \leq 0$. Next, since (3.17) implies that $z \leq 0$ we get

$$2U(z) - V(z) - \varepsilon V(z) \geq 2U(0) - 1 - \varepsilon \geq 0\,,$$

since $U(0) > \frac{1}{2}$. Hence we obtain that $E_4 \leq 0$, so that we have $\varepsilon\mathcal{L}^\varepsilon[u^-] \leq 0$.

Assume that

$$- \varepsilon|\ln\varepsilon|m_1 e^{m_2 t} \leq d(t,x) \leq d_0\,. \tag{3.18}$$

Here, $E_3 = 0$ and $\partial_t d + c_\lambda = 0$ also hold true. Note that, for $\varepsilon > 0$ small enough,

$$m_2\varepsilon|\ln\varepsilon|m_1 e^{m_2 t} - \varepsilon\Delta d \geq \frac{m_1 m_2}{2}\varepsilon|\ln\varepsilon|\,. \tag{3.19}$$

Using that $U'(z) - \varepsilon V'(z) < \frac{1}{2}U'(z)$ for all $z \in \mathbb{R}$, whose proof is similar to that of (3.13), we see that

$$E_1 \leq \frac{1}{2}U'(z)\frac{m_1 m_2}{2}\varepsilon|\ln\varepsilon| \leq -\frac{r m_1 m_2}{4}\varepsilon|\ln\varepsilon|e^{-\lambda z}\,,$$

since (3.18) implies $z \geq 0$. Next, we see that $E_4 \leq C\varepsilon V(z) \leq C\varepsilon e^{-\mu_\varepsilon z} \leq C\varepsilon e^{-\lambda z}$. As a consequence we get

$$\varepsilon\mathcal{L}^\varepsilon[u^-] \leq \varepsilon e^{-\lambda z}\left(-\frac{r m_1 m_2}{4}|\ln\varepsilon| + C\right) \leq 0\,,$$

for $\varepsilon > 0$ small enough.

Assume that

$$d_0 \leq d(t,x) \leq 2d_0\,. \tag{3.20}$$

We rewrite $\varepsilon\mathcal{L}^\varepsilon[u^-] \leq F_1 + F_2 + F_3$ where

$F_1 := U'(z)\,(\partial_t d + c_\lambda) + U''(z)(1 - |\nabla d|^2)$

$F_2 := U'(z)(m_2\varepsilon|\ln\varepsilon|m_1 e^{m_2 t} - \varepsilon\Delta d)$

$F_3 := -\varepsilon V'(z)(\partial_t d + c_\lambda + m_2\varepsilon|\ln\varepsilon|m_1 e^{m_2 t} - \varepsilon\Delta d) - \varepsilon V''(z)(1 - |\nabla d|^2) - \varepsilon V(2U - V)(z) + \varepsilon^2 V^2(z)\,.$

Since $|F_3| \leq C\varepsilon e^{-\mu_\varepsilon z} \leq C\varepsilon e^{-\lambda z}$, we deduce from (3.19) that, for $\varepsilon > 0$ small enough,

$$F_2 + F_3 \leq \varepsilon e^{-\lambda z}\left(-\frac{r m_1 m_2}{2}|\ln\varepsilon| + C\right) \leq 0\,.$$

It remains to estimate the term F_1. Since (3.20) implies that $z \geq d_0/\varepsilon$ we have, for $\varepsilon > 0$ small enough, $U(z) \leq \alpha$ with $0 < \alpha < 1$ to be selected below. Therefore it holds that

$$U''(z) \leq -c_\lambda U'(z) - (1-\alpha)U(z).$$

Recall that (see subsection 2.3)

$$\partial_t d = (\partial_t \tilde{d})\zeta' = -c_\lambda\zeta' \quad \text{ and } \quad |\nabla d|^2 = |\nabla\tilde{d}|^2(\zeta')^2 = (\zeta')^2\,,$$

where the function ζ' is evaluated at point $\widetilde{d}$. It follows that

$$\begin{aligned} F_1 &\leq U'(z)(-c_\lambda \zeta' + c_\lambda) - c_\lambda U'(z)(1-(\zeta')^2) - (1-\alpha)U(z)(1-(\zeta')^2) \\ &\leq -(1-\zeta')\left[(1-\alpha)(1+\zeta')U(z) + c_\lambda \zeta' U'(z)\right] . \end{aligned}$$

Since, as $\varepsilon \to 0$, $U'(z) = -\beta e^{-\lambda z}(1+o(1))$ and $U(z) = \frac{\beta}{\lambda} e^{-\lambda z}(1+o(1))$ for some $\beta > 0$, we see that

$$(1-\alpha)(1+\zeta')U(z) + c_\lambda \zeta' U'(z) = \left[\frac{1-\alpha}{\lambda}(1+\zeta') - c\zeta'\right] \beta e^{-\lambda z}(1+o(1)) .$$

Recall that $0 \leq \zeta' \leq 1$ so that by selecting $\alpha \in (0, \frac{1-\lambda^2}{2})$ we see that $\frac{1-\alpha}{\lambda}(1+\zeta') - c\zeta' \geq 0$ so that, for $\varepsilon > 0$ small enough, $F_1 \leq 0$. Hence $\varepsilon \mathcal{L}^\varepsilon[u^-] \leq 0$.

The lemma is proved. □

4 Super-solutions

This section is devoted to the construction of super-solutions which are efficient for the study of both the generation and the motion of interface.

Lemma 4.1 (Super-solutions)**.** *Let Assumptions 1.1 and 1.3 be satisfied. Then there exists a constant $K_0 > 1$ such that, for all $\widehat{K} \geq K_0$, the following holds. For all $x_0 \in \Gamma_0 = \partial\Omega_0$, for all $\varepsilon > 0$ small enough, we have*

$$u^\varepsilon(t,x) \leq \widehat{K} U\left(\frac{(x-x_0)\cdot n_0 - c_\lambda t}{\varepsilon}\right) \qquad \textit{for all } (t,x) \in [0,\infty) \times \mathbb{R}^N ,$$

wherein n_0 is the outward normal vector to $\Gamma_0 = \partial\Omega_0$ at x_0.

Before proving the lemma, for a given $x \in \mathbb{R}^N$, choose $x_0 \in \partial\Omega_0$ as the projection of x on the convex Ω_0. For such a choice we have

$$(x-x_0)\cdot n_0 = d(0,x) ,$$

and the lemma yields, for some $\widehat{K} > 1$,

$$u^\varepsilon(t,x) \leq \widehat{K} U\left(\frac{d(0,x) - c_\lambda t}{\varepsilon}\right) , \tag{4.1}$$

for all $t \geq 0$ and all $x \in \mathbb{R}^N$. Next, we prove Lemma 4.1.

Proof. We recall that λ and M were defined in Assumption 1.1 (iii) and that $U(z) \geq re^{-\lambda z}$ for all $z \geq 0$. Then we define

$$K_0 := \max\left(1, \frac{M}{r}, \frac{\|g\|_\infty + M}{U(0)}\right) .$$

Next, let $\widehat{K} \geq K_0$ and $x_0 \in \Gamma_0 = \partial\Omega_0$ be given. We consider the map

$$u^+(t,x) := \widehat{K} U\left(\frac{(x-x_0)\cdot n_0 - c_\lambda t}{\varepsilon}\right).$$

Straightforward computations and equation $U'' + c_\lambda U' + U(1-U) = 0$ for the travelling wave yield

$$\varepsilon \mathcal{L}^\varepsilon[u^+](t,x) = \widehat{K}(\widehat{K}-1)(U)^2\left(\frac{(x-x_0)\cdot n_0 - c_\lambda t}{\varepsilon}\right),$$

and therefore $\mathcal{L}^\varepsilon[u^+] \geq 0$ in $(0,\infty)\times\mathbb{R}^N$. Hence, to complete the proof of the lemma it remains to order the initial data, i.e.

$$u_{0,\varepsilon}(x) \leq u^+(0,x) = \widehat{K} U\left(\frac{(x-x_0)\cdot n_0}{\varepsilon}\right), \quad \forall x \in \mathbb{R}^N. \tag{4.2}$$

First, assume that x in the half plane $\{y \in \mathbb{R}^N : \quad (y-x_0)\cdot n_0 \leq 0\}$. Then since U is decreasing we have

$$U\left(\frac{(x-x_0)\cdot n_0}{\varepsilon}\right) \geq U(0).$$

Therefore we obtain that

$$u_{0,\varepsilon}(x) \leq \frac{\|g\|_\infty + M}{U(0)} U\left(\frac{(x-x_0)\cdot n_0}{\varepsilon}\right) \leq \widehat{K} U\left(\frac{(x-x_0)\cdot n_0}{\varepsilon}\right),$$

in view of the choice of K_0.

Next, we assume that x is in the half plane $\{y \in \mathbb{R}^N : \quad (y-x_0)\cdot n_0 > 0\}$. Since Ω_0 is convex, we have $x \notin \Omega_0$ and $d(0,x) \geq (x-x_0)\cdot n_0 > 0$ so that, in view of the choice of K_0,

$$\begin{aligned} u_{0,\varepsilon}(x) = h_\varepsilon(x) \quad &\leq M e^{-\lambda d(0,x)/\varepsilon} \\ &\leq \widehat{K} r e^{-\lambda (x-x_0)\cdot n_0/\varepsilon} \\ &\leq \widehat{K} U\left(\frac{(x-x_0)\cdot n_0}{\varepsilon}\right). \end{aligned}$$

This completes the proof of Lemma 4.1. □

As a consequence of the above construction of super-solutions, we deduce that, after a very short time, the solution u^ε approaches 0 in most part of the complementary of the support of the initial data. The proof is an easy consequence of the upper bound (4.1) combined with the exponential decay (3.6) of U. Details are omitted.

Corollary 4.2 (Generation of interface "from the outside")**.** *Let Assumptions 1.1 and 1.3 be satisfied. For any $p > 0$, there exists $k_p > 0$ such that, for all $\varepsilon > 0$ small enough,*

$$d(0,x) \geq k_p \varepsilon |\ln \varepsilon| \implies 0 \leq u^\varepsilon(t^\varepsilon, x) \leq \varepsilon^p,$$

where we recall that $t^\varepsilon = \alpha\varepsilon|\ln\varepsilon|$.

5 Proof of Theorem 1.5

We first claim that

$$0 \le u^\varepsilon(t+t^\varepsilon, x) \le 1+\varepsilon\,, \tag{5.1}$$

for all $x \in \mathbb{R}^N$ and all $t \ge 0$. Indeed, the map

$$\overline{u}(t,x) := w\left(\frac{t}{\varepsilon}, \|g\|_\infty + M\right)\,, \quad t \ge 0\,,$$

satisfies $\mathcal{L}^\varepsilon[\overline{u}] = 0$ in $(0,\infty)\times\mathbb{R}^N$ and $u^\varepsilon(0,x) \le \|g\|_\infty + M = \overline{u}(0,x)$ for all $x \in \mathbb{R}^N$. Therefore the comparison principle yields

$$u^\varepsilon(t,x) \le w\left(\frac{t}{\varepsilon}, \|g\|_\infty + M\right)\,, \quad \forall t \ge 0\,,\ \forall x \in \mathbb{R}^N\,.$$

Thus Lemma 2.1 (v) applies and completes the proof of (5.1).

Now, let Assumptions 1.1, 1.3 and 1.4 be satisfied. Choose $k>0$, $\alpha>0$ as in Corollary 3.2 and $\widehat{K}>0$ as in Lemma 4.1. According to Lemma 3.3, Lemma 3.4 and the comparison principle, there exist $m_1>0$ and $m_2>0$ such that, for $\varepsilon>0$ small enough,

$$u^-(t-t^\varepsilon, x) \le u^\varepsilon(t,x)\,, \tag{5.2}$$

for all $(t,x) \in [t^\varepsilon, T]\times\mathbb{R}^N$. We choose $\mathcal{C}$ such that

$$\mathcal{C} > \max\left(\lambda^{-1}, 2(c_\lambda\alpha + m_1 e^{m_2 T}), 2\eta^{-1}\right)\,, \tag{5.3}$$

with $\eta>0$ the constant that appears in (3.5).

First, we take $x \in (\mathbb{R}^N \setminus \overline{\Omega_t^{c_\lambda}}) \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^{c_\lambda})$, i.e.

$$d(t,x) \ge \mathcal{C}\varepsilon|\ln\varepsilon|\,, \tag{5.4}$$

and prove below that $u^\varepsilon(t,x) \le \varepsilon$, for $t^\varepsilon \le t \le T$. Since

$$d(t,x) = d(0,x) - c_\lambda t\,,$$

we deduce from (4.1), the decrease of U and (3.6) that, for $\varepsilon>0$ small enough, for $0 \le t \le T$,

$$u^\varepsilon(t,x) \le \widehat{K}U(\mathcal{C}|\ln\varepsilon|) \le \widehat{K}R\varepsilon^{\lambda\mathcal{C}} \le \varepsilon\,,$$

since $\mathcal{C} > \lambda^{-1}$.

Next, we take $x \in \Omega_t^{c_\lambda} \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln\varepsilon|}(\Gamma_t^{c_\lambda})$, i.e.

$$d(t,x) \le -\mathcal{C}\varepsilon|\ln\varepsilon|\,, \tag{5.5}$$

and prove below that $u^\varepsilon(t,x) \ge 1-2\varepsilon$, for $t^\varepsilon \le t \le T$. In view of (5.2) we have $u^\varepsilon(t,x) \ge (U-\varepsilon V)(z(t-t^\varepsilon,x))$. Note that $d(t-t^\varepsilon,x) = d(t,x) + c_\lambda t^\varepsilon$.

Hence the choice of $\mathcal{C}$ implies that $z(t-t^\varepsilon,x)\leq -\frac{\mathcal{C}}{2}|\ln\varepsilon|$. Therefore, using (3.9) and (3.5), we see that, for $\varepsilon>0$ small enough,

$$\begin{aligned} u^\varepsilon(t,x) &\geq U\left(-\tfrac{\mathcal{C}}{2}|\ln\varepsilon|\right)-\varepsilon V\left(-\tfrac{\mathcal{C}}{2}|\ln\varepsilon|\right)\\ &\geq 1-Re^{-\eta\frac{\mathcal{C}}{2}|\ln\varepsilon|}-\varepsilon\\ &\geq 1-2\varepsilon\,, \end{aligned}$$

since $\mathcal{C}>\dfrac{2}{\eta}$.

This completes the proof of Theorem 1.5.

References

[1] M. Alfaro and A. Ducrot, *Sharp interface limit of the Fisher-KPP equation*, submitted.

[2] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), 5–49. Lecture Notes in Math., Vol. 446, Springer, Berlin, 1975.

[3] G. Barles, L. C. Evans and P. E. Souganidis, *Wavefront propagation for reaction-diffusion systems of PDE*, Duke Math. J. **61** (1990), no. 3, 835–858.

[4] G. Barles and P. E. Souganidis, *A remark on the asymptotic behavior of the solution of the KPP equation*, C. R. Acad. Sci. Paris Série I **319** (1994), 679–684.

[5] M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc. **44** (1983), no. 285, iv+190 pp.

[6] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential equations **96** (1992), 116–141.

[7] L. C. Evans and P. E. Souganidis, *A PDE approach to geometric optics for certain semilinear parabolic equations*, Indiana Univ. Math. J. **38** (1989), 141–172.

[8] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. of Eugenics **7** (1937), 355–369.

[9] M. I. Freidlin, *Limit theorems for large deviations and reaction-diffusion equations*, Ann. Probab. **13** (1985), 639–675.

[10] F. Hamel and L. Roques, *Fast propagation for KPP equations with slowly decaying initial conditions*, in preparation.

[11] D. Hilhorst, R. Kersner, E. Logak and M. Mimura, *Interface dynamics of the Fisher equation with degenerate diffusion*, J. Differential Equations **244** (2008), 2872–2889.

[12] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat Moscou, Bjul. Moskowskogo Gos. Univ., 1937, 1–26.

[13] D. A. Larson, *Transient bounds and time-asymptotic behavior of solutions to nonlinear equations of Fisher type*, SIAM J. Appl. Math. **34** (1978), 93–103.

[14] K.-S. Lau, *On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov*, J. Differential Equations **59** (1985), no. 1, 44–70.

[15] H. P. McKean, *Application of Brownian motion to the equation ofKolmogorov-Petrovskii-Piskunov*, Comm. Pure Appl. Math. **28** (1975), no. 3, 323-331.

[16] F. Rothe, *Convergence to travelling fronts in semilinear parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A **80** (1978), no. 3-4, 213–234.

[17] K. Uchiyama, *The behavior of solutions of some nonlinear diffusion equations for large time*, J. Math. Kyoto Univ. **18** (1978), no. 3, 453–508.

[18] A. Volpert, V. Volpert, V. Volpert, *Travelling Wave Solutions of Parabolic Systems*, Translations of Mathematical Monographs, vol. 140, AMS Providence, RI, 1994.