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# Global weak solution for a singular two component reaction-diffusion system

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Abstract

We study a singular Reaction-Diffusion system motivated by a dedicated diffusive predator-prey model system devised in the spatially homogeneous case by Courchamp and Sugihara [**2**]. The reactive part features a functional response to predation and a singular numerical functional response to predation specifically designed for modeling the introduction of greedy predators into a fragile or insular environment. Under some circumstances this may lead to finite time quenching of the solution, that is finite time extinction for both species. The aim of this work is to derive a suitable notion of global (in time) weak solution and to prove that such global weak solutions do exist. The existence part is achieved by approximating the reactive part by a more classical and non singular one and then passing to the limit in the resulting Reaction-Diffusion system. Our first result shows that this limiting process supply global weak solutions. In the case of equidiffusivities such global weak solutions satisfy a suitable free boundary value problem.

## 1. *Introduction*

We consider the following singular Reaction-Diffusion system

$$\begin{cases} \partial_t u - D\Delta u = ku\left(1-u\right) - v, \\ \partial_t v - \Delta v = rv\left(1-\frac{v}{u}\right), \end{cases} \tag{1.1}$$

wherein the above dimensionless problem is posed for time $t > 0$ and spatial location $x \in \Omega$, a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Here $D > 0$ and $r > 0$ are given constants while $k \geq 0$. This problem is supplemented with homogeneous Neumann boundary conditions

$$\nabla u(t,x)\cdot\eta(x) = \nabla v(t,x)\cdot\eta(x) = 0, \;\; t > 0 \text{ and } x \in \partial\Omega, \tag{1.2}$$

$\eta(x)$ being the outward unit normal vector to $\Omega$ at $x \in \partial\Omega$, and initial conditions

$$u(0,\cdot) = u_0, \;\; v(0,\cdot) = v_0, \tag{1.3}$$

the initial data satisfying $(u_0, v_0) \in \left(L^\infty(\Omega)\right)^2$ and

$$0 \leq u_0(x), \; v_0(x) \leq 1, \;\; x \in \Omega.$$

The underlying Ordinary Differential Equation system has been proposed by Courchamp and Sugihara [**2**] in order to model a strong predator-prey interaction within some fragile or insular environment. The spatially structured system (1.1) has been introduced by Gaucel et al. [**8**] and Gaucel and Langlais in [**7**] in order to take into account the spatial motion of both species, $u(t,x)$ (resp. $v(t,x)$) denoting the density of a prey species (resp. predator) at time $t$ and spatial location $x \in \Omega$. It is shown in [**7**] that under suitable conditions on the parameter set as well as on the initial data set System (1.1)-(1.2)-(1.3) may exhibit a finite time quenching, that means that the density $u$ may reach the singular value $u = 0$ at some finite time. A

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simplified system, with $k = 0$, has been considered by Ducrot and Guo [**4**] where the authors studied the quenching rate of the solutions as well as some properties of the quenching set. One point quenching with non-self-similar type singularity may occur for some specific initial data. From obvious biological reasons, the population dynamics continues to evolve in time beyond a possible finite and spatially localized extinction, especially for one point quenching.

The aim of this work is to understand how to extend the notion of solution to (1.1)-(1.2)-(1.3) to get a globally defined nonnegative semiflow. In this direction, some hints has been recently provided by Ducrot and Langlais in [**5**] who deal with the existence of weak travelling wave solutions for a singular system close to (1.1). Roughly speaking, System (1.1) with $k = 0$ admits a family of (weak) travelling waves describing the predator invasion process within an homogeneous population of prey. Under some conditions on the parameter set the tail of the invasion front corresponds to the zero level-set of both species.

In order to deal with the continuation of the solutions to (1.1)-(1.2)-(1.3), we introduce the following notion of weak solutions.

Definition 1 Weak solution. A pair of nonnegative, measurable and bounded functions $(u, v) : (0, \infty) \times \Omega \to [0, \infty) \times [0, \infty)$ is a global weak solution to (1.1)-(1.2)-(1.3) if the following properties are fulfilled:

(i) For each $T > 0$ the following regularity holds true

$$(\partial_t u, \nabla u) \in L^2(Q_T)^2, \quad \nabla v \in L^2(Q_T),$$

wherein we have set $Q_T = (0, T) \times \Omega$.

(ii) For each $T > 0$ and each pair of test functions $(\varphi, \psi) \in \left(C^1\left([0, T] \times \overline{\Omega}\right)\right)^2$ with $\varphi(T, \cdot) = \psi(T, \cdot) \equiv 0$ the pair $(u, v)$ satisfies

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \varphi(0, x)u_0^2(x)dx - \int_{Q_T} \frac{1}{2}\partial_t \varphi u^2 dtdx \\ &\quad = -D\int_{Q_T} \nabla u \nabla (u\varphi)\, dtdx + \int_{Q_T} \varphi\left(ku^2(1-u) - uv\right) dtdx, \end{aligned}$$

and

$$\begin{aligned} &-\int_{Q_T} \psi(0, x)u_0(x)v_0(x)dx - \int_{Q_T} \partial_t(\psi u)vdtdx \\ &\quad = -\int_{Q_T} \nabla (u\psi) \nabla v dtdx + r\int_{Q_T} \psi v\,(u - v)\, dtdx. \end{aligned}$$

In order to prove the existence of such weak solutions, we shall consider the following $\varepsilon-$perturbed Holling-Tanner reaction-diffusion system (see for instance [**12**], [**15**] and references therein)

$$\begin{cases} \partial_t u_\varepsilon - D\Delta u_\varepsilon = ku_\varepsilon(1 - u_\varepsilon) - \frac{v_\varepsilon u_\varepsilon}{u_\varepsilon + \varepsilon}, \ t > 0, \ x \in \Omega \\ \partial_t v_\varepsilon - \Delta v_\varepsilon = rv_\varepsilon\left(1 - \frac{v_\varepsilon}{u_\varepsilon + \varepsilon}\right), \end{cases} \tag{1.4}$$

supplemented with the boundary and initial conditions in (1.2)-(1.3).
From a formal point of view, System (1.1) is derived from (1.4) by taking $\varepsilon = 0$. The aim of this work is to pass to the limit $\varepsilon \searrow 0^+$ into the globally defined solutions to (1.4) to show the existence of a global weak solution to (1.1).

Similar regularization processes have been used in the literature in order to deal with the continuation of solutions to parabolic equations beyond finite time blow-up or quenching. We refer for instance to Quirós et al [**13**], the survey paper of Galaktionov and Vázquez [**6**] and to the references cited therein. We also refer to Levine [**11**], Chan et al [**1**] and Dávila and Montenegro [**3**] as well as to the survey chapter of Hernández and Mancebo [**9**] concerning singular parabolic problems. Here note that the $\varepsilon-$modification of the singularity appearing into the $v-$equation has a regularisation effect provided that $u \geq 0$. This latter condition is ensured by the $\varepsilon-$modification into the $u-$equation.

In order to explain this limiting procedure let us consider the simplified $\varepsilon-$perturbed ODE system

$$\begin{cases} u_\varepsilon'(t) = -\frac{u_\varepsilon(t)v_\varepsilon(t)}{u_\varepsilon(t)+\varepsilon}, \\ v_\varepsilon'(t) = rv_\varepsilon(t) - r\frac{v_\varepsilon(t)^2}{u_\varepsilon(t)+\varepsilon}, \\ u_\varepsilon(0) = u_0 > 0, \;\; v_\varepsilon(0) = v_0 > 0. \end{cases} \tag{1.5}$$

For this ODE system one can check that the following relation holds true

$$u_\varepsilon(t)^r = \frac{u_0^r}{v_0} v_\varepsilon(t) e^{-rt}, \;\; \forall t \geq 0.$$

This allows to solve (1.5) in closed form to find

$$\frac{1}{1-r}\left(u^{1-r}(t) - u_0^{1-r}\right) - \frac{\varepsilon}{r}\left(u^{-r}(t) - u_0^{-r}\right) = -\frac{v_0}{u_0^r}\frac{e^{rt}-1}{r}, \;\; \forall t \geq 0.$$

If one assumes that $r \in (0,1)$ then by letting $\varepsilon \to 0$ one gets that $u_\varepsilon$ converges locally uniformly to a nonnegative function $u$

$$u(t) = \max\left[0, \left(u_0^{1-r} - \frac{1-r}{r}\frac{v_0}{u_0^r}\left(e^{rt}-1\right)\right)\right]^{\frac{1}{1-r}},$$

while $v_\varepsilon$ converges locally uniformly a function $v$ satisfying

$$v(t) = \frac{v_0 e^{rt}}{u_0^r} u^r(t), \;\; \forall t \geq 0.$$

Since $r \in (0,1)$, the latter function $u$ vanishes at the finite time $T$ defined by

$$T = \frac{1}{r}\ln\left(1 + \frac{r}{1-r}\frac{u_0}{v_0}\right).$$

Therefore this implies that the pair $(u,v)$ satisfies $v = v\chi_{\{u>0\}}$ as well as the following system of equations

$$\begin{cases} \frac{du(t)}{dt} = -v(t)\chi_{\{u>0\}} \text{ in } \mathcal{D}'(0,\infty), \\ \frac{dv(t)}{dt} = rv(t) - r\frac{v^2}{u}\chi_{\{u>0\}} \text{ in } \mathcal{D}'(0,\infty). \end{cases} \tag{1.6}$$

In the above system and in the sequel, for each set $A$, $\chi_A$ denotes the characteristic function of $A$.

We now come back to System (1.4)-(1.2)-(1.3). Before stating our main result let us introduce a first generic set of assumptions

Assumption 1.1. *Let $r > 0$, $D > 0$ and $k \geq 0$ be given.*
*The initial pair $(u_0, v_0)$ satisfies $u_0 \in L^\infty(\Omega) \setminus \{0\}$, $v_0 \in L^\infty(\Omega)$ and*

$$0 \leq u_0(x) \leq 1, \;\; 0 \leq v_0(x) \leq 1, \;\; a.e.\ x \in \Omega.$$

Then we shall first prove

Theorem 1.2. *Let Assumption 1.1 be satisfied. Assume furthermore that $u_0 \in H^1(\Omega)$. Then for each sequence $\{\varepsilon_n\}_{n\geq 0} \subset (0,\infty)$ such that $\varepsilon_n \searrow 0$, there exists a subsequence still denoted by $\{\varepsilon_n\}_{n\geq 0}$ such that the sequence $\{(u_{\varepsilon_n}, v_{\varepsilon_n})\}_{n\geq 0}$, solution to (1.4)-(1.2)-(1.3) with*

*$\varepsilon = \varepsilon_n$, converges almost everywhere and locally uniformly on $(0,\infty)\times\overline{\Omega}$ to some nonnegative pair of functions $(u,v)$, a weak solution to (1.1)-(1.2)-(1.3) according to Definition 1.*

When $D=1$ in (1.1) one shall show the limit solutions constructed above using the limit procedure $\varepsilon \searrow 0$ leads to a problem similar to the one obtained for the underlying ordinary differential equation, namely (1.6).

ASSUMPTION 1.3. *Assume that $D=1$. In addition to Assumption 1.1, let us assume that there exists some constant $M_r>0$ such that*

$$0 \le v_0(x) \le M_r u_0(x)^{\min(1,r)} \quad a.e.\ x\in\Omega.$$

Let us introduce $\{T_\Delta(t)\}_{t\ge 0}$ the strongly continuous semigroup on $L^1(\Omega)$ generated by the Laplace operator $\Delta : D(\Delta) : L^1(\Omega)\to L^1(\Omega)$ together with homogeneous Neumann boundary conditions on $\partial\Omega$.
Then one shall prove the following results

THEOREM 1.4 Case $r\ge 1$. *Let Assumption 1.3 be satisfied. Assume $r\ge 1$.*
*There exists a sequence $\{\varepsilon_n\}_{n\ge 0}$ such that $\varepsilon_n\searrow 0$ when $n\to\infty$ and such that*

(i) *$(u_{\varepsilon_n}, v_{\varepsilon_n})$ converges locally uniformly in $(0,\infty)\times\Omega$ to some positive pair of functions $(u,v)$, that is $u(t,x)>0$ and $v(t,x)>0$ for all $(t,x)\in(0,\infty)\times\overline{\Omega}$.*
(ii) *For each $T\ge 0$ one has $\frac{v^2}{u}\in L^\infty(Q_T)$.*
(iii) *For all $t>0$ one has*

$$u(t) = T_\Delta(t)u_0 + \int_0^t T_\Delta(t-s)\left[ku(s)\left(1-u(s)\right)-v(s)\right]ds,$$
$$v(t) = T_\Delta(t)v_0 + r\int_0^t T_\Delta(t-s)\left[v(s)-\frac{v^2(s)}{u(s)}\right]ds.$$

THEOREM 1.5 Case $r<1$. *Let Assumption 1.3 be satisfied. Assume $r<1$.*
*There exists a sequence $\{\varepsilon_n\}_{n\ge 0}$ such that $\varepsilon_n\searrow 0$ when $n\to\infty$, and a pair of nonnegative functions $(u,v)\in\left[C\left((0,\infty)\times\overline{\Omega}\right)\cap L^\infty\left((0,\infty)\times\Omega\right)\right]^2$ such that*

(i) *For all $(t,x)\in(0,\infty)\times\Omega$, $v(t,x)\le M_r e^{rt}u(t,x)^r$.*
(ii) *The sequence $\{u_{\varepsilon_n}, v_{\varepsilon_n}\}_{n\ge 0}$ satisfies*

$$\begin{cases} u_{\varepsilon_n}\to u \text{ locally uniformly on } (0,\infty)\times\overline{\Omega},\\ v_{\varepsilon_n}\to v \text{ in } L^1_{loc}\left([0,\infty)\times\Omega\right).\end{cases}$$

(iii) *For each $p\in\left[1,\frac{1}{1-r}\right)$ and each $T>0$, the function $\frac{v^2}{u}\chi_{\{u>0\}}$ belongs to $L^p(Q_T)$ and one has for each $t>0$*

$$u(t) = T_\Delta(t)u_0 + \int_0^t T_\Delta(t-s)\left[ku(s)\left(1-u(s)\right)-v(s)\chi_{\{u(s)>0\}}\right]ds,$$
$$v(t) = T_\Delta(t)v_0 + r\int_0^t T_\Delta(t-s)\left[v(s)-\frac{v(s)^2}{u(s)}\chi_{\{u(s)>0\}}\right]ds.$$

*Furthermore when $r\ge\frac{1}{2}$ then function $\frac{v^2}{u}\chi_{\{u>0\}}$ belongs to $L^\infty(Q_T)$ for each $T>0$.*

REMARK 1. If there exists $T > 0$ such that $u(T, .) \equiv 0$ then one has

$$0 \leq u(T+t) \leq k \int_0^t T_\Delta(t-s)u(T+s)ds, \quad t \geq 0.$$

The latter implies that $u(T+t, .) \equiv 0$ for all $t \geq 0$.

Let us comment Theorem 1.5. The integrability property of $\frac{v^2}{u}\chi_{\{u>0\}}$, namely $\frac{v^2}{u}\chi_{\{u>0\}} \in L^p(Q_T)$ for each $p \in \left[1, \frac{1}{1-r}\right)$, shows that function $v$ enjoys the usual $L^p-$parabolic regularity and the solution pair $(u, v)$ provided by Theorem 1.5 turns out to be a classical solution. This actually implies that $(u, v)$ satisfies the following free boundary value problem

$$\begin{cases} \partial_t u - \Delta u = ku(1-u) - v \\ \partial_t v - \Delta v = rv\left(1 - \frac{v}{u}\right) \end{cases} \quad \text{in } \{u > 0\} \text{ and } v = 0 \text{ in } \{u = 0\},$$

without any jump condition for the gradient of $v$ at the free-boundary $\partial\{u = 0\}$.

This manuscript is organized as follows. Section 2 is devoted to deriving basic estimates independent of $\varepsilon > 0$ for the solution of System (1.4)-(1.2)-(1.3). This will allows us to prove Theorem 1.2 and Theorem 1.4. Section 3 is concerned with the proof of Theorem 1.5. The proof of such a result will require more refined $\varepsilon-$independent estimates on the solutions of (1.4)-(1.2)-(1.3).

## 2. *Proofs of Theorem 1.2 and Theorem 1.4*

Let us first introduce some notations: for each $\tau \geq 0$ and $T > 0$ set

$$Q_{\tau,T} = (\tau, T) \times \Omega, \quad Q_T = Q_{0,T}.$$

The following three lemmas hold true. Their proofs are rather straightforward using parabolic estimates and usual integrations by parts.

LEMMA 2.1. *Let Assumption 1.1 be satisfied. Let $r > 0$ be given. Then for each $\varepsilon > 0$ one has*

$$0 \leq u_\varepsilon(t, x) \leq 1, \quad 0 \leq v_\varepsilon(t, x) \leq 1 + \varepsilon, \quad t > 0, \ x \in \Omega.$$

*For each $T > 0$ there exists some constant $M_T > 0$ such that for each $\varepsilon \in (0, 1]$ one has*

$$\int_{Q_T} |\nabla v_\varepsilon|^2(t, x)dxdt + \int_{Q_T} \frac{v_\varepsilon^2}{u_\varepsilon + \varepsilon}(t, x)dtdx \leq M_T.$$

LEMMA 2.2 Compactness. *Let Assumption 1.1 be satisfied. Let $r > 0$ be given. Then the family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0,1]}$ satisfies the following estimates:*

(i) *Let $\alpha \in (1, 2)$ be given. For each $0 < \tau < T$ there exists some constant $K_{\tau,T} > 0$ such that for each $\varepsilon \in (0, 1]$*

$$\|u_\varepsilon\|_{C^{\frac{\alpha}{2},\alpha}\left([\tau,T]\times\overline{\Omega}\right)} \leq K_{\tau,T}.$$

(ii) *The family $\{\partial_t v_\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded for the topology of $L^1 + L^2\left(\left(H^1\right)'\right)$, $(H^1)'$ being the dual space of $H^1(\Omega)$. More precisely, for each $T > 0$ there exists some constant $C_T > 0$ such that for each $\varepsilon \in (0, 1]$ one has*

$$\|\partial_t v_\varepsilon\|_{L^1(Q_T)+L^2(0,T;H^1(\Omega)')} \leq C_T.$$

(iii) *If moreover $u_0 \in H^1(\Omega)$ then the family $\{\nabla u_\varepsilon\}_{\varepsilon\in(0,1]}$ in compact for the topology of $L^2_{loc}\left([0,\infty);L^2(\Omega)\right)$ while for each $T>0$ there exists some constant $M(T)>0$ such that for each $\varepsilon>0$*

$$\int_{Q_T}\left(|\partial_t u_\varepsilon|^2+|\Delta u_\varepsilon|^2\right)(t,x)dxdt \leq M(T).$$

*Proof.* The proof of $(i)$ relies on usual maximal parabolic regularity. We refer for instance to Lamberton [**10**], Pruss [**14**] and the references therein for more details on this topic.
Let us first note that due to Lemma 2.1 the family of functions $\{f_\varepsilon\}_{\varepsilon\in(0,1]}$ defined by

$$f_\varepsilon := ku_\varepsilon\left(1-u_\varepsilon\right)-\frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+\varepsilon},$$

is uniformly bounded in $L^\infty\left((0,\infty)\times\Omega\right)$. Next note that one has

$$u_\varepsilon(t,.)=T_\Delta(t)u_0+\int_0^t T_\Delta(s)f_\varepsilon(t-s)ds,\ \ \forall t\geq 0, \tag{2.1}$$

wherein $\{T_\Delta(t)\}_{t\geq 0}$ denotes the usual heat semigroup associated to Neumann boundary conditions. Next let recall (see [**14**]) that the heat semigroup $\{T_\Delta(t)\}_{t\geq 0}$ satisfies the so-called maximal parabolic regularity, that reads as for each $p\in(1,\infty)$ and each $T>0$ there exists some constant $K_p(T)>0$ such that for each $f\in L^p\left(0,T;L^p(\Omega)\right)$ one has:

$$\left\|\int_0^{.} T_\Delta(s)f(.-s)ds\right\|_{W^{2,1}_p(Q_T)} \leq \|f\|_{L^p(Q_T)}.$$

Let $\alpha\in(0,2)$ be given and let us select $p\in(1,\infty)$ such that $\alpha<2-(N+2)/p$, that implies the following Sobolev embedding $W^{1,2}_p\left(Q_T\right)\hookrightarrow C^{\frac{\alpha}{2},\alpha}\left(\overline{Q_T}\right)$. Finally let us recall that $T_\Delta(.)u_0\in C^\infty\left((0,\infty)\times\overline{\Omega}\right)$. Hence the above stated parabolic regularity applied with this choice of $p$ implies that $(i)$ holds true.

Statement $(ii)$ directly follows from the estimates for $v_\varepsilon$ stated in Lemma 2.1.

It remains to prove $(iii)$. To do so let us first notice that if $u_0\in H^1(\Omega)\cap L^\infty(\Omega)$ then $T_\Delta(.)u_0\in L^2_{loc}\left([0,\infty);H^1(\Omega)\right)$ so that the compactness of $\{\nabla u_\varepsilon\}_{\varepsilon\in(0,1]}$ follows from the above stated parabolic regularity. Next the estimates for $\partial_t u_\varepsilon$ and $\Delta u_\varepsilon$ are also a consequence of parabolic regularity since $T_\Delta(.)u_0\in W^{1,2}_2(Q_T)$ as soon as $u_0\in H^1(\Omega)$. This can also be more directly obtained by multiplying the $u-$equation in (1.4) by $\Delta u_\varepsilon$ and integrating over $\Omega$ that yields to for each $t>0$

$$\frac{d}{dt}\int_\Omega|\nabla u_\varepsilon|^2dx+D\int_\Omega|\Delta u_\varepsilon|^2dx=-\int_\Omega\Delta u_\varepsilon f_\varepsilon dx.$$

This completes the proof of the result. □

Using these basic estimates, we are now able to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $\{\varepsilon_n\}_{n\geq 0}\subset(0,\infty)$ be a given sequence such that $\varepsilon_n\searrow 0$ as $n\to\infty$. Recalling that $u_0\in H^1(\Omega)$, using Lemma 2.1 and 2.2 combined together with Aubin's like compactness arguments (see for instance [**16**]), up to a subsequence, one may assume that

$\{(u_{\varepsilon_n}, v_{\varepsilon_n})\}_{n\geq 0}$ converges to a nonnegative pair of functions $(u, v)$ for the following topologies

$$\begin{cases} u_n := u_{\varepsilon_n} \to u \text{ locally uniformly in } (0,\infty)\times\overline{\Omega}, \\ \nabla u_n \to \nabla u \text{ strongly in } L^2(0,T;L^2(\Omega) \text{ for each } T>0, \\ \partial_t u_n \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ for each } T>0, \\ v_n := v_{\varepsilon_n} \to v \ a.e. \text{ for } (t,x)\in(0,\infty)\times\Omega, \\ \nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ for each } T>0. \end{cases} \tag{2.2}$$

Let $T>0$ be given and $(\varphi,\psi)\in C^1\left([0,T]\times\overline{\Omega}\right)^2$ be given such that $(\varphi,\psi)(T,.)\equiv(0,0)$. Let $n\geq 0$ be given, then multiplying the $u-$equation and the $v-$equation respectively by $(u_n+\varepsilon_n)\varphi$ and $(u_n+\varepsilon_n)\psi$ and integrating over $Q_T$ one obtains

$$\begin{aligned} &-\frac{1}{2}\int_\Omega \varphi(0,x)\left(u_0(x)+\varepsilon_n\right)^2dx - \int_{Q_T}\frac{1}{2}\partial_t\varphi\left(u_n+\varepsilon_n\right)^2dtdx \\ &= -D\int_{Q_T}\nabla u_n\nabla\left(\left(u_n+\varepsilon_n\right)\varphi\right)dtdx + \int_{Q_T}\varphi\left(ku_n(u_n+\varepsilon_n)(1-u_n)-u_nv_n\right)dtdx. \end{aligned}$$

and

$$\begin{aligned} &-\int_{Q_T}\psi(0,x)\left(u_0(x)+\varepsilon_n\right)v_0(x)dx - \int_{Q_T}\partial_t\left(\psi\left(u_n+\varepsilon_n\right)\right)v_ndtdx \\ &= -\int_{Q_T}\nabla\left(\left(u_n+\varepsilon_n\right)\psi\right)\nabla v_ndtdx + r\int_{Q_T}\psi v_n\left(u_n+\varepsilon_n-v_n\right)dtdx. \end{aligned}$$

Letting $n\to\infty$ into the above equalities and using the convergence properties described in (2.2), the result follows. □

Before going to the proof of Theorem 1.4, we first derive some preliminary estimates that will be needed in the sequel. When Assumption 1.1 holds and using Aubin's like compactness arguments (see for instance [**16**]), as a direct consequence of the above estimates provided by Lemma 2.1 and 2.2 (*i*) and (*ii*), one gets that for each sequence $\{\varepsilon_n\}_{n\geq 0}\subset(0,\infty)$ tending to zero as $n\to\infty$, there exists a subsequence, still denoted by $\{\varepsilon_n\}_{n\geq 0}$ such that $\varepsilon_n\searrow 0^+$ as $n\to\infty$ and

$$\begin{aligned} &u_n := u_{\varepsilon_n} \to u \text{ locally uniformly in } (0,\infty)\times\overline{\Omega}, \\ &v_n := v_{\varepsilon_n} \to v \ a.e. \text{ for } (t,x)\in(0,\infty)\times\Omega. \end{aligned} \tag{2.3}$$

Lemma 2.3. *Let Assumption 1.3 be satisfied. Let $r>0$ be given. Then for each $\varepsilon>0$ one has*

$$v_\varepsilon(t,x)\leq M_re^{rt}u_\varepsilon(t,x)^{\min(r,1)},\ \ \forall t\geq 0,\ x\in\overline{\Omega},$$

*wherein the constant $M_r$ is defined in Assumption 1.3.*

*Proof.* First consider the cases $r\geq 1$. Let $\varepsilon>0$ be given. Let $\eta>0$ be given. Consider $u^\eta$ and $v^\eta$ the solution of (1.4) with initial data

$$u^\eta(0,x)=u_0(x)+\eta,\ \ v^\eta(x)=v_0(x)\ a.e.\ x\in\Omega.$$

Note that from the comparison principle, one has

$$u^\eta\leq 1+\eta \text{ and } v^\eta\leq 1+\varepsilon+\eta.$$

Next let us notice that when $\eta\searrow 0$, using the continuity of the semiflow with respect to its initial data, one obtains that $u^\eta\to u_\varepsilon$ and $v^\eta\to v_\varepsilon$ at least almost everywhere for $(t,x)\in$

$(0,\infty)\times\Omega$. Consider now the map $P^\eta = P = \frac{v^\eta}{u^\eta}$ that satisfies the following equation

$$\partial_t P - \Delta P - 2\nabla P\frac{\nabla u_\varepsilon}{u_\varepsilon} = (1-r)\frac{P^2 u}{u+\varepsilon} + rP - kP(1-u). \tag{2.4}$$

Since $r \geq 1$ and $u \leq 1+\eta$ then

$$\partial_t P - \Delta P - 2\nabla P\frac{\nabla u_\varepsilon}{u_\varepsilon} - (r+k\eta)\,P \leq 0.$$

The comparison principle implies that

$$P^\eta(t,x) \leq \left\|\frac{v_0}{u_0+\eta}\right\|_\infty e^{(r+k\eta)t}, \ \ \forall x\in\overline{\Omega}.$$

Due to Assumption 1.3, one obtains that for each $\eta>0$:

$$v^\eta(t,x) \leq M_r e^{(r+k\eta)t} u^\eta(t,x),$$

and the first part of the result follows for $r\geq 1$ by letting $\eta\searrow 0$.

Consider now the case $r\in(0,1)$. Let $\varepsilon>0$ be given. Consider the function $J(t,x) = e^{-rt}v_\varepsilon(t,x) - M_r u_\varepsilon^r(t,x)$ where the constant $M_r$ is defined in Assumption 1.3. Then one has

$$\begin{aligned}
\partial_t J &= e^{-rt}\partial_t v - re^{-rt}v - M_r r u^{r-1}\partial_t u,\\
\nabla J &= e^{-rt}\nabla v - M_r r u^{r-1}\nabla u,\\
\Delta J &= e^{-rt}\Delta v - M_r r(r-1)u^{r-2}|\nabla u|^2 - M_r r u^{r-1}\Delta u.
\end{aligned}$$

Thus one obtains that $J$ satisfies

$$\begin{aligned}
\partial_t J - \Delta J &= e^{-rt}(v_t - \Delta v - rv) + M_r r(r-1)u^{r-2}|\nabla u|^2 + M_r r u^{r-1}\frac{uv}{u+\varepsilon} - rM_r k u^r(1-u)\\
&= -re^{-rt}v\frac{v}{u+\varepsilon} + M_r r(r-1)u^{r-2}|\nabla u|^2 + M_r r u^r\frac{v}{u+\varepsilon} - rM_r k u^r(1-u)\\
&= Kr(r-1)u^{r-2}|\nabla u|^2 - rM_r k u^r(1-u) - \frac{rv}{u+\varepsilon}J.
\end{aligned}$$

Since $r\in(0,1)$ and $u\leq 1$, this leads us to

$$J_t - \Delta J + \frac{rv_\varepsilon}{u_\varepsilon+\varepsilon}J \leq 0.$$

Due to the definition of constant $M_r$ one gets $J(0,.)\leq 0$. The parabolic comparison principle applies and completes the proof of the result. □

LEMMA 2.4. *Let Assumption 1.3 be satisfied. Let $r>0$ be given. Then the following holds true*

$$\frac{v_n(t,x)u_n(t,x)}{u_n(t,x)+\varepsilon_n} \to v(t,x)\chi_{\{u(t,x)>0\}} \text{ a.e. } (t,x)\in(0,\infty)\times\Omega.$$

*Proof.* Let $(t,x)\in(0,\infty)\times\overline{\Omega}$ such that $u(t,x)>0$. Then we obtain that $\frac{1}{u_n(t,x)+\varepsilon_n}\to\frac{1}{u(t,x)}$. As a consequence we get

$$\frac{v_n(t,x)u_n(t,x)}{u_n(t,x)+\varepsilon_n} \to v(t,x) \text{ a.e. for } (t,x)\in\{u>0\}.$$

Next for each $(t,x)\in\{u=0\}$ one has from Lemma 2.3 that

$$\frac{v_n(t,x)u_n(t,x)}{u_n(t,x)+\varepsilon_n} \leq M_r e^{rt}u_n^{\min(1,r)}(t,x).$$

This implies that

$$\frac{v_n(t,x)u_n(t,x)}{u_n(t,x)+\varepsilon_n} \to 0 \text{ a.e. for } (t,x) \in \{u=0\}.$$

This completes the proof of the result. □

Before completing the proof of Theorem 1.4 let us first derive a lower estimate for the $u-$component that prevent from finite time quenching in the case $r \geq 1$:

LEMMA 2.5. *Let Assumption 1.3 be satisfied. Let $r \geq 1$ be given. Consider the map $\underline{u}$ defined by the resolution of the linear heat equation*

$$\begin{aligned}
&\partial_t \underline{u} - \Delta \underline{u} + M_r e^{rt} \underline{u} = 0,\ t>0,\ x \in \Omega,\\
&\nabla \underline{u}(t,x).\eta(x) = 0,\ t>0,\ x \in \partial\Omega,\\
&\underline{u}(0,.) = u_0.
\end{aligned}$$

*Then for all $\varepsilon > 0$ one has*

$$u_\varepsilon(t,x) \geq \underline{u}(t,x),\ \ \forall (t,x) \in (0,\infty)\times\overline{\Omega}.$$

*Proof.* Since $r \geq 1$ then from Lemma 2.3, one has for all $\varepsilon > 0$ that

$$v_\varepsilon(t,x) \leq M_r e^{rt} u_\varepsilon(t,x),\ \ \forall (t,x) \in [0,\infty)\times\overline{\Omega}.$$

Then function $u_\varepsilon$ satisfies

$$\partial_t u_\varepsilon - \Delta u_\varepsilon \geq k u_\varepsilon (1-u_\varepsilon) - M_r e^{rt} u_\varepsilon \geq -M_r e^{rt} u_\varepsilon.$$

The comparison principle applies and provides that

$$u_\varepsilon(t,x) \geq \underline{u}(t,x)\ \ \forall (t,x) \in [0,\infty)\times\overline{\Omega},$$

and the result follows. □

*Proof of Theorem 1.4.* Let us first notice that the sequence $\left\{\frac{v_n u_n}{u_n+\varepsilon_n}\right\}_{n\geq 0}$ almost everywhere converges to $v$. In fact, it converges to $v\chi_{\{u>0\}}$ by using Lemma 2.4 but $u>0$ by Lemma 2.5 since $r \geq 1$ and $\underline{u}(t,x) > 0$ for all $t>0$ and $x \in \overline{\Omega}$. Using once again Lemma 2.3, we obtain that the sequence $\left\{\frac{v_n u_n}{u_n+\varepsilon_n}\right\}_{n\geq 0}$ is bounded on each $Q_T$ for $T>0$. Indeed we have

$$\frac{v_n u_n}{u_n+\varepsilon_n} \leq K_r e^{rt} u_n \leq K_r e^{rt},\ \ \forall n \geq 0.$$

Therefore Lebesgue convergence theorem applies and provides that for each $p \in [1,\infty)$ and each $T>0$

$$\lim_{n\to\infty} \left\| \frac{v_n u_n}{u_n+\varepsilon_n} - v \right\|_{L^p(Q_T)} = 0.$$

This remark implies that function $u$ satisfies for each $t \geq 0$:

$$u(t) = T_\Delta(t)u_0 + \int_0^t T_\Delta(t-s)\left[ku(s)\left(1-u(s)\right) - v(s)\right] ds.$$

In the same way, since $u>0$ then

$$\frac{v_n^2}{u_n+\varepsilon_n} \to \frac{v^2}{u} \text{ a.e. for } (t,x) \in (0,\infty)\times\Omega.$$

One the other hand, using Lemma 2.3, one has for each $n \geq 0$

$$\frac{v_n^2}{u_n + \varepsilon_n} \leq M_r^2 e^{2rt} u_n \leq M_r^2 e^{2rt}.$$

Therefore Lebesgue convergence theorem applies and provides that for each $p \in [1, \infty)$ and each $T > 0$:

$$\lim_{n\to\infty} \left\| \frac{v_n^2}{u_n + \varepsilon_n} - \frac{v^2}{u} \right\|_{L^p(Q_T)} = 0.$$

We are now able to pass to the limit $n \to \infty$ in the weak formulation of $v_n$, that reads

$$v_n(t) = T_\Delta(t)v_0 + r\int_0^t T_\Delta(t-s)\left[v_n(s) - \frac{v_n^2(s)}{u_n(s)+\varepsilon_n}\right] ds, \quad \forall t \geq 0$$

Thus function $v$ satisfies for all $t \geq 0$:

$$v(t) = T_\Delta(t)v_0 + r\int_0^t T_\Delta(t-s)\left[v(s) - \frac{v^2(s)}{u(s)}\right] ds.$$

This completes the proof of Theorem 1.4. □

## 3. *Proof of Theorem 1.5*

It requires much more refined estimates given in the following statement

PROPOSITION 3.1. *Let Assumption 1.3 be satisfied. Let $r \in (0,1)$ be given. Then for each $p \in [1, \infty)$ and each $T > 0$ one has*

$$\lim_{n\to\infty} \left\| \frac{v_n u_n}{u_n + \varepsilon} - v\chi_{\{u>0\}} \right\|_{L^p(Q_T)} = 0, \tag{3.1}$$

*for each $\beta \in \left[0, \frac{r}{1-r}\right)$, one has*

$$\begin{aligned} &\frac{v^2}{u}\chi_{\{u>0\}} \in L^{1+\beta}(Q_T), \\ &\text{the sequence } \left\{\frac{v_n^2}{u_n+\varepsilon}\right\}_{n\geq 0} \text{ is bounded in } L^{1+\beta}(Q_T), \end{aligned} \tag{3.2}$$

*and one has possibly along a subsequence:*

$$\lim_{n\to\infty} \left\| \frac{v_n^2}{u_n + \varepsilon} - \frac{v^2}{u}\chi_{\{u>0\}} \right\|_{L^1(Q_T)} = 0. \tag{3.3}$$

The proof of Theorem 1.5 thus becomes a direct consequence of the above proposition. Indeed let us recall that for each $T > 0$, each $p \in [1, \infty)$ and each sequence $\{f_n\}_{n\geq 0} \subset L^p(Q_T)$ then if $f_n \to f$ in $L^p(Q_T)$ then uniformly for $t \in [0, T]$ one has

$$\lim_{n\to\infty} \left\| \int_0^t T_\Delta(t-s)f_n(s)ds - \int_0^t T_\Delta(t-s)f(s)ds \right\|_{L^p(\Omega)} = 0.$$

Hence it remains to prove prove Proposition 3.1.
We will split the proof into two parts. We first investigate estimate (3.1).

*Proof of (3.1).* The proof of this result is a direct consequence of the Lebesgue convergence theorem. Indeed from Lemma 2.4, one has the following almost everywhere convergence

$$\frac{v_n u_n}{u_n+\varepsilon_n} \to v\chi_{\{u>0\}}.$$

Next from Lemma 2.3, one has

$$\frac{v_n u_n}{u_n+\varepsilon_n} \le K_r e^{rt} u_n^r \le K_r e^{rt}, \quad \forall n \ge 0.$$

Then (3.1) follows from Lebesgue convergence theorem. □

We shall now prove (3.2). This result relies on the following estimates:

LEMMA 3.2. *Let $r \in (0,1)$ be given. Let Assumption 1.3 be satisfied. Then for each $T>0$ and each $\beta \in \left[0, \frac{r}{1-r}\right)$, there exists $C = C(\beta, T) > 0$ such that for each $\varepsilon \in (0,1]$*

$$\int_{Q_T} \frac{v_\varepsilon^{2+\beta}}{u_\varepsilon^\beta (u_\varepsilon+\varepsilon)} dxdt \le C(\beta, T).$$

*Proof.* Let $\beta \in \left(0, \frac{r}{1-r}\right)$ be given. (Note that the case $\beta = 0$ is already known). One shall use a regularization procedure similar to the one given in the proof of Lemma 2.3. Let $\varepsilon \in (0,1]$ be given. Let $\eta > 0$ be given. Consider the map $w^\eta = \frac{(v^\eta)^{1+\beta}}{(u^\eta)^\beta}$, where function $u^\eta$ and $v^\eta$ are similar to the ones introduced in the proof of Lemma 2.3. For notational simplicity, in the sequel, one shall write $w$, $u$ and $v$ for $w^\eta$, $u^\eta$ and $v^\eta$. Then function $w$ satisfies

$$\begin{aligned}\partial_t w - \Delta w + (r+\beta(r-1))\frac{v}{u+\varepsilon} w =& (\beta+1) r w - \beta k w(1-u) \\ & - \left(\beta(\beta+1)\frac{v^{\beta-1}}{u^\beta}|\nabla v|^2 - 2(\beta+1)\beta \frac{v^\beta}{u^{\beta+1}}\nabla u \nabla v + \beta(\beta+1)\frac{v^{\beta+1}}{u^{\beta+2}}|\nabla u|^2\right).\end{aligned}$$

Therefore, since $u \le 1+\eta$, function $w$ satisfies

$$\partial_t w - \Delta w + (r+\beta(r-1))\frac{v}{u+\varepsilon} w \le ((\beta+1)r + \beta k \eta)\, w - \beta(\beta+1)\left(\frac{v^{\frac{\beta-1}{2}}}{u^{\frac{\beta}{2}}}\nabla v - \frac{v^{\frac{\beta+1}{2}}}{u^{\frac{\beta}{2}+1}}\nabla u\right)^2,$$

supplemented together with the homogeneous Neumann boundary condition on $\partial\Omega$. This leads us to

$$\begin{cases} \partial_t w - \Delta w + (r+\beta(r-1))\frac{v}{u+\varepsilon} w \le ((\beta+1)r + \beta k\eta)\, w, \\ \nabla w(t,x).\eta(x) = 0, \ t>0, \ x \in \partial\Omega, \\ w(0,x) = \frac{v_0(x)^{1+\beta}}{(u_0(x)+\eta)^\beta}. \end{cases} \tag{3.4}$$

Let us now notice that one has

$$w = \left(\frac{v^{\frac{1}{r}}}{u}\right)^\beta v^{1-\beta\frac{1-r}{r}}.$$

Since $\beta \in \left(0, \frac{r}{1-r}\right)$, $1-\beta\frac{1-r}{r} > 0$ and we get due to Lemma 2.3 that

$$w \le K_r^{\frac{1}{r}} e^t (1+\eta+\varepsilon)^{1-\beta\frac{1-r}{r}}.$$

(Note that $v^\eta \le 1+\eta+\varepsilon$) As a consequence for each $T>0$, there exists some constant $\widehat{M}(T) > 0$ such that for each $\varepsilon \in (0,1]$ and $\eta \in (0,1)$:

$$w^\eta(t,x) \le \widehat{M}(T), \quad \forall (t,x) \in \overline{Q_T}.$$

Let $T>0$ be given. Integrating (3.4) over $Q_T$ yields

$$(r+\beta(r-1))\int_{Q_T}\frac{v^\eta}{u^\eta+\varepsilon}wdxdt\leq\int_\Omega\frac{v_0(x)^{1+\beta}}{(u_0(x)+\eta)^\beta}dx+((\beta+1)r+\eta\beta k)\,T|\Omega|\widehat{M}(T).$$

Using Assumption 1.3 and letting $\eta\searrow 0$, the result follows by using Fatou Lemma. $\square$

We are now able to complete the proof of (3.2). Let $\beta\in\left(0,\frac{r}{1-r}\right)$ be given. Let $T>0$ be given. Then one has for each $n\geq 0$:

$$\int_{Q_T}\left(\frac{v_n^2}{u_n+\varepsilon_n}\right)^{1+\beta}dxdt\leq\int_{Q_T}\frac{v_n^{2\beta+2}}{u_n^\beta(u_n+\varepsilon_n)}dxdt$$
$$\leq(1+\varepsilon_n)^\beta\int_{Q_T}\frac{v_n^{2+\beta}}{u_n^\beta(u_n+\varepsilon_n)}dxdt.$$

This above estimates show that for each $T>0$ and $\beta\in\left(0,\frac{r}{1-r}\right)$, the sequence $\left\{\frac{v_n^2}{u_n+\varepsilon_n}\right\}_{n\geq 0}$ is bounded in $L^{1+\beta}(Q_T)$. Next Fatou Lemma implies that for each $T>0$

$$\frac{v^2}{u}\chi_{\{u>0\}}\in L^{1+\beta}(Q_T),\ \ \forall\beta\in\left(0,\frac{r}{1-r}\right).$$

This completes the proof of (3.2).

Let us now prove the convergence result stated in (3.3). To do so, let us first notice that

$$\frac{v_n^2}{u_n+\varepsilon_n}\to\frac{v^2}{u}\text{ a.e. on }Q_T\cap\{u>0\}.$$

Now let $\beta\in\left(0,\frac{r}{1-r}\right)$ be given. Note that for each $n\geq 0$ one has

$$\left(\frac{v_n^2}{u_n+\varepsilon_n}\right)^{1+\beta}\leq\frac{v_n^{2+\beta}}{u_n^\beta(u_n+\varepsilon_n)}v_n^\beta.$$

Hence the estimate provided in Lemma 2.3 yields to

$$\left(\frac{v_n^2}{u_n+\varepsilon_n}\right)^{1+\beta}\leq K_r^\beta e^{r\beta t}\frac{v_n^{2+\beta}}{u_n^\beta(u_n+\varepsilon_n)}u_n^{r\beta}. \tag{3.5}$$

Let $0<\tau<T$ be given. Integrating (3.5) over $Q_{\tau,T}\cap\{u=0\}$ leads us to

$$\int_{Q_{\tau,T}\cap\{u=0\}}\left(\frac{v_n^2}{u_n+\varepsilon_n}\right)^{1+\beta}dxdt\leq K_r^\beta\int_{Q_T}e^{r\beta t}\frac{v_n^{2+\beta}}{u_n^\beta(u_n+\varepsilon_n)}dxdt\left(\sup_{Q_{\tau,T}\cap\{u=0\}}u_n\right)^{r\beta}. \tag{3.6}$$

Recalling that $\{u_n\}$ converges uniformly towards $u$ on $Q_{\tau,T}$ for each $\tau\in(0,T)$ the above inequality ensures that for each $0<\tau<T$:

$$\lim_{n\to\infty}\int_{Q_{\tau,T}\cap\{u=0\}}\left(\frac{v_n^2}{u_n+\varepsilon_n}\right)^{1+\beta}dxdt=0.$$

Hence possibly along a subsequence, one obtains that $\frac{v_n^2}{u_n+\varepsilon_n}\to 0$ a.e. in $Q_T\cap\{u=0\}$. Finally note that we have obtained that

$$\frac{v_n^2}{u_n+\varepsilon_n}\to\frac{v^2}{u}\chi_{\{u>0\}}\text{ a.e. in }Q_T.$$

Due to the uniform bound in $L^{1+\beta}(Q_T)$ for the sequence $\left\{\frac{v_n^2}{u_n+\varepsilon_n}\right\}_{n\geq 0}$ stated in (3.2), one obtains that $\left\{\frac{v_n^2}{u_n+\varepsilon_n}\right\}_{n\geq 0}$ is uniformly integrable so that Vitali's theorem applies and completes the proof of (3.3).

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