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Two-dimensional travelling wave solutions of a system modelling near equidiffusional flames

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Abstract

We consider a semi-linear elliptic system in a strip arising in combustion theory. The model describes the propagation of two-dimensional near-equidiffusional flames. We prove the existence of travelling wave solutions for high activation energy.

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1. Introduction

In this paper we study the existence of a travelling wave solution for the system

$$\frac{\partial u}{\partial t} = \Delta u + f(u, y)v, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Lambda \Delta v - f(u, y)v, \tag{1.2}$$

set in the infinite domain $\Omega = \{(x, y) \in \mathbb{R} \times (0, 1)\}$.

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This system stems from the theory of combustion. It describes the propagation of a premixed flame in the infinite tube Ω in the framework of the classical thermo-diffusive model, see Williams [16,12,13,17]. Here, the mixture is assumed to be at rest and we consider a one-step reaction $R \to P$. The unknowns are the normalized temperature u and the concentration of the reactant v. The term $f(u,y)v$ corresponds to the chemical reaction and depends on a small parameter ε that is the inverse of the activation energy. The limit $\varepsilon \to 0$ (high activation energy asymptotics) is of great physical interest. Formal analytic methods based on the small parameter ε lead to some classical free boundary problems in combustion theory that are constantly used, see [16–18,13] for instance. Finally $\Lambda > 0$ denotes the inverse of the Lewis number.

The travelling wave solutions $u(x+ct, y)$, $v(x+ct, y)$ of (1.1) and (1.2) satisfy the equations

$$-\Delta u + c\,\frac{\partial u}{\partial x} = f(u,y)v \quad \text{in } \Omega \tag{1.3}$$

$$-\Lambda\Delta v + c\,\frac{\partial v}{\partial x} = -f(u,y)v \quad \text{in } \Omega \tag{1.4}$$

and the following boundary conditions (which are classical in combustion theory):

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0 \quad \text{on } \Gamma = \mathbb{R} \times \{0,1\}, \tag{1.5}$$

$$u(-\infty, y) = 0, \quad v(-\infty, y) = 1 \quad \text{for } y \in (0,1), \tag{1.6}$$

$$u(+\infty, y) = 1, \quad v(+\infty, y) = 0 \quad \text{for } y \in (0,1). \tag{1.7}$$

The constant c, the wave velocity, is unknown and should be found together with the functions u and v as a solution of (1.3)–(1.7). Multi-dimensional solutions correspond to curved front propagating in the tube Ω. Such fronts are evidenced experimentally, analytically and numerically in particular when the one-dimensional planar flame becomes unstable, see [12,13,16,17].

For $\Lambda = 1$, the above system reduces to a scalar equation ($v = 1 - u$). This equation has been extensively studied from a mathematical point of view: existence, uniqueness and qualitative properties of multi-dimensional solutions are investigated in [3,4,6,14] (see also the references therein.) The asymptotic limit $\varepsilon \to 0$ is studied in [2]. The arguments heavily rely on the maximum principle and monotonicity properties (with respect to x) of the temperature that do not hold anymore for $\Lambda \neq 1$.

Travelling wave solutions for the system ($\Lambda \neq 1$) and their singular limits were only previously considered in the one-dimensional setting (planar flames), see [5,11,8,15]. Let us also mention the work of Langlois and Marion [10,9] that deals with the parabolic system (1.1)–(1.2) for $\Lambda \neq 1$. Existence results in the case $\Lambda \neq 1$ were also discussed in [1].

In this paper we investigate two-dimensional travelling wave solutions for $\Lambda \neq 1$. In particular, we aim to show the existence of solutions of problem (1.3)–(1.7) that are relevant in the limit $\varepsilon \to 0$ (high activation energy asymptotics.) In that context we will assume that

$f = f_\varepsilon$ takes the form

$$f_\varepsilon(u, y) = \frac{1}{\varepsilon^2}\,\psi\left(\frac{u-1}{\varepsilon}\right)\chi(u, y), \tag{1.8}$$

where $\psi = \psi(s)$ is positive, increasing and decays sufficiently fast at $s = -\infty$. We do not assume any growth condition at $s = +\infty$. Note that the Arrhenius term, arising in the theory of combustion, is bounded by its upper bound depends exponentially on ε^{-1}. It is usually modelled by (1.8) with $\psi(s) = e^s$ (see [16,13] for instance). Also the function $\chi : \mathbb{R} \times (0, 1) \to [0, 1]$ is increasing with respect to u and vanishes for $u \leqslant \theta$ (existence of an ignition temperature.)

The models in the physical literature require the Lewis number to depend on the parameter ε, that is $\Lambda = \Lambda_\varepsilon$, and to satisfy the so-called near equidiffusional assumption

$$\Lambda_\varepsilon - 1 = O(\varepsilon^\gamma) \tag{1.9}$$

for a convenient value of γ (see [16,13]). Our main objective in this paper is to investigate problem (1.3)–(1.7) with $f = f_\varepsilon$ given by (1.8) and $\Lambda = \Lambda_\varepsilon$ satisfying (1.9).

Let us now describe our main results and the contents of the paper. The first step in our study consists in studying (1.3) and (1.4) for some *fixed bounded* nonlinear term f. The following hypotheses on f will be imposed:

$$f \in L^\infty(\Omega) \cap C^0(\bar{\Omega}), \tag{1.10}$$

$$\exists \theta \in (0, 1), \quad f(s, y) = 0 \quad \text{if } s \leqslant \theta \text{ and } f(s, y) > 0 \quad \text{if } s > \theta. \tag{1.11}$$

We first consider some problem analogous to (1.3)–(1.7) posed on the bounded rectangle $R_a = (-a, a) \times (0, 1)$ with $a > 0$. This allows the reduction to a fixed point formulation. Then the usual Leray–Schauder degree gives the existence of a solution (u_a, v_a, c_a) in the bounded domain R_a (Section 2).

Taking the limit $a \to +\infty$ requires some estimates on (u_a, v_a, c_a) that are independent of a. They are derived in Section 3. The crucial step consists in obtaining a positive lower bound of the velocity c_a. For that purpose an essential tool consists in introducing the functions $H = u + v - 1$, $G = u + \Lambda v - 1$ as well as the averaged quantities with respect to the y variable.

Then, Section 4 deals with the limit procedure $a \to +\infty$. For any $\Lambda > 0$, we derive the existence of a triplet (u, v, c) satisfying Eqs. (1.3) and (1.4) together with (1.5) and (1.6). Also, at $x = +\infty$, we show that $u(+\infty, y) = 1 - v(+\infty, y) = u^+$, where $u^+ \in \{\theta, 1\}$. We also prove that $u^+ = 1$ if Λ is sufficiently close to one. This is our first existence result for (1.3)–(1.7) but the condition on Λ is far too restrictive for high activation energy (see Remark 4.1).

Section 5 is concerned with travelling waves solutions in the context of large activation energy asymptotics, that is under assumptions (1.8) and (1.9). We derive existence provided that ε is small enough and (1.9) holds with $\gamma > \frac{5}{4}$. As earlier mentioned in [9,10], the study of near-equidiffusional flames is tightly related to precise upper bounds for the temperature of the form $u(x, y) \leqslant 1 + \varepsilon$. In order to study (1.3)–(1.7), we start by truncating $f_\varepsilon(x, y)$ for $u \geqslant 1 + \varepsilon$. The corresponding problem (1.3)–(1.7) involves a bounded nonlinearity and we first investigate that problem (Theorem 5.1). Next, estimates of $\|H\|_{L^\infty(\Omega)}$ allow us to prove the existence of travelling waves solutions for the initial problem (Theorem 5.2).

Our methods apply to more general situations: travelling waves solutions in higher space dimensions, complex chemistry, problems including some y dependence in the convective term such as $c\alpha(y)$ or $c+\alpha(y)$. These questions will be studied in a subsequent paper.

2. A problem in a bounded rectangle

Let f be a positive, continuous function satisfying (1.10) and (1.11). In this section we investigate a problem analogous to (1.3)–(1.7) posed in a bounded domain. More precisely, for $a>0$, we consider the following bounded rectangles:

$$R_a=(-a,a)\times(0,1), \quad R_a^-=(-a,0)\times(0,1).$$

We introduce the problem: find $u, v:\bar{R}_a\to\mathbb{R}$ and $c\in\mathbb{R}$ such that

$$-\Delta u+c\,\frac{\partial u}{\partial x}=f(u,y)v, \tag{2.1}$$

$$-\Lambda\Delta v+c\,\frac{\partial v}{\partial x}=-f(u,y)v \tag{2.2}$$

together with the conditions:

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}=0 \quad \text{on } \Gamma_a=(-a,a)\times\{0,1\}, \tag{2.3}$$

$$-u_x(-a,y)+cu(-a,y)=0, \quad u(a,y)=1, \tag{2.4}$$

$$-\Lambda v_x(-a,y)+cv(-a,y)=c, \quad v(a,y)=0, \tag{2.5}$$

$$\max_{(x,y)\in R_a^-} u(x,y)=\theta, \tag{2.6}$$

$$c\geqslant 0. \tag{2.7}$$

Conditions (2.3)–(2.5) are boundary conditions. Condition (2.6) was first introduced in [3]. It allows to take care of the translation invariance in problem (1.3)–(1.7) and to avoid trivial solutions when later considering the limit $a\to+\infty$.

We will derive the following existence result.

Proposition 2.1. *Under assumptions* (1.10) *and* (1.11)*, let* $\Lambda>0$ *be given. Then, for all* $a>0$*, problem* (2.1)–(2.7) *possesses a solution* (u_a,v_a,c_a) *in* $C^1(\bar{R}_a)\times C^1(\bar{R}_a)\times\mathbb{R}_+^*$*. In addition,* u_a *and* v_a *satisfy*

$$u_a\geqslant 0 \quad \textit{and} \quad 0\leqslant v_a\leqslant 1.$$

The proof of this proposition is based on a topological degree argument. As usual, the argument relies on a priori estimates on the solutions of (2.1)–(2.7) that we first derive.

Lemma 2.1. *Let* (u, v, c) *be a solution of* (2.1)–(2.7). *Then the functions* u *and* v *satisfy*

$$(u, v) \in W^{2,p}(R_a) \quad \forall p \in [1, +\infty),$$
$$u \geqslant 0 \quad \textit{and} \quad 0 \leqslant v \leqslant 1 \quad \textit{in } R_a. \tag{2.8}$$

Proof. Since the function f takes positive values, the maximum principle applies to the unknown v and it easily follows that $0 \leqslant v \leqslant 1$. Consequently, we have $f(u, y)v \geqslant 0$. Applying again the maximum principle we see that $u \geqslant 0$. Finally, since $f(u, y)v$ belong to $L^\infty(R_a)$, some classical elliptic estimates yield that u and v are in $W^{2,p}(R_a)$ for all $p \geqslant 1$. □

Next, we derive some estimates on the velocity c. Let us set

$$M = \sup_{\substack{s>0 \\ y \in (0,1)}} f(s, y). \tag{2.9}$$

Lemma 2.2. *There exists some constant* K_a *depending only on* M, θ *and* $a > 0$ *such that*

$$0 < c \leqslant K_a.$$

Furthermore, the choice of K_a *can be made independent of* $a \geqslant 1$.

Proof. Let us first show that $c > 0$. Assume that $c = 0$. Then by taking the average with respect to y of Eq. (2.1) for u we see that the function

$$u_0(x) = \int_0^1 u(x, y)\,\mathrm{d}y$$

satisfies

$$-u_0'' \geqslant 0,$$
$$u_0'(-a) = 0 \quad \text{and} \quad u_0(a) = 1.$$

It easily follows that $u_0 \geqslant 1$ on $[-a, a]$, that contradicts (2.6). Therefore, we have $c > 0$.

Let us now find an upper bound for c. We consider a C^1-function θ_1 satisfying

$$-\theta_1'' + c\theta_1' = M\mathbb{1}_{x \geqslant 0} \quad \text{on } (-a, a),$$
$$-\theta_1'(-a) + c\theta_1(-a) = 0,$$
$$\theta_1(a) = 1,$$

where M is given by (2.9). Thanks to the maximum principle we easily see that

$$u(x, y) \leqslant \theta_1(x) \quad \forall\ (x, y) \in R_a.$$

In particular, $u(0, y) \leqslant \theta_1(0)$, $\forall y \in (0, 1)$, and in view of (2.6)

$$\theta \leqslant \theta_1(0).$$

Now, a simple computation gives

$$\theta_1(0) = M \int_0^a x\mathrm{e}^{-cx}\,\mathrm{d}x + \mathrm{e}^{-ca} \leqslant \mathrm{e}^{-ca} + \frac{M}{c^2}.$$

Combining these inequalities we conclude that

$$c \leqslant \max\left(\frac{1}{a}\log\frac{2}{\theta}, \sqrt{\frac{2M}{\theta}}\right). \tag{2.10}$$

Note that the right-hand side of this last inequality can be majorized by a constant independent of $a \geqslant 1$. □

Proof of Proposition 2.1. It is convenient to formulate the equations as a fixed point problem. We set $E = C^1(\bar{R}_a) \times C^1(\bar{R}_a) \times \mathbb{R}$ and define the following mapping $T : E \to E$:

$$T(u, v, c) = \left(U, V, c + \theta - \max_{R_a^-} U\right),$$

where U and V are given by the resolution of

$$-\Delta U + c\frac{\partial U}{\partial x} = f(u, y)v, \tag{2.11}$$

$$-\Lambda\Delta V + c\frac{\partial V}{\partial x} = -f(u, y)v \tag{2.12}$$

together with the following conditions:

$$\frac{\partial U}{\partial y} = 0, \quad \frac{\partial V}{\partial y} = 0 \quad \text{on } \Gamma_a, \tag{2.13}$$

$$-U_x(-a, y) + cU(-a, y) = 0, \quad U(a, y) = 1, \tag{2.14}$$

$$-\Lambda V_x(-a, y) + cV(-a, y) = c, \quad V(a, y) = 0. \tag{2.15}$$

Then, problem (2.1)–(2.7) is equivalent to finding a fixed point for the mapping T. We note that the mapping $T : E \to E$ is a compact operator. Hence, for any open and bounded set $S \subset E$ such that T has no fixed point on ∂S, the Leray–Schauder degree, $d(Id_E - T, S, 0)$ is well defined. Thanks to the preceding lemmas and classical elliptic estimates, for all $a > 0$, we can find a constant $k_a > 0$ (depending on a) such that; if $(u, v, c) \in E$ is a fixed point of T, then $\|(u, v, c)\|_E < k_a$, where $\| \|_E$ is the norm on E defined by: $\|(u, v, c)\|_E = \|u\|_{C^1(\bar{R}_a)} + \|v\|_{C^1(\bar{R}_a)} + |c|$.

Therefore let us consider

$$S = \{(u, v, c) \in E \mid \|(u, v, c)\|_E < k_a\}.$$

Then, $d(Id_E - T, S, 0)$ is well defined. In order to compute this number, we use the homotopy invariance. For $\tau \in [0, 1]$, we consider the mapping $T_\tau : E \to E$ given by

$$T_\tau(u, v, c) = \left(U, V, c + \theta - \max_{R_a^-} U\right),$$

where U and V satisfy

$$- \Delta U + c \frac{\partial U}{\partial x} = \tau f(u, y)v,$$
$$- \Lambda \Delta V + c \frac{\partial V}{\partial x} = -\tau f(u, y)v$$

together with the boundary conditions (2.13)–(2.15). Then for all $\tau \in [0, 1]$, T_τ is a compact operator in E. Furthermore, the estimates obtained for the fixed points of T are also valid for the fixed points of T_τ, for $0 \leqslant \tau \leqslant 1$. Therefore, we conclude that $Id_E - T_\tau$ does not vanish on ∂S and the homotopy invariance of the topological degree provides that

$$d(Id_E - T, S, 0) = d(Id_E - T_1, S, 0) = d(Id_E - T_0, S, 0).$$

Now $T_0(u, v, c)$ is independent of u and v and can be explicitly written as

$$T_0(u, v, c) = \left(\bar{u}, \bar{v}, c + \theta - \max_{R_a^-} \bar{u} \right)$$

with

$$\bar{u}(x) = \mathrm{e}^{c(x-a)}, \quad \bar{v}(x) = 1 - \mathrm{e}^{(c/\Lambda)(x-a)}.$$

Finally in order to compute $d(Id_E - T_0, S, 0)$, we consider the following homotopy: for $s \in [0, 1]$, $\Phi_s : E \to E$ is given by

$$\Phi_s(u, v, c) = (u - s\bar{u}, v - s\bar{v}, \theta - \bar{u}(0)).$$

Then $Id_E - T_0 = \Phi_1$ and, for all s, Φ_s is a compact perturbation of the identity. Therefore,

$$d(Id_E - T_0, S, 0) = d(\Phi_0, S, 0).$$

We have $\Phi_0(u, v, c) = (u, v, \theta - \mathrm{e}^{-ca})$. This yields that $d(\Phi_0, S, 0) = 1$ since the function $c \to \theta - \mathrm{e}^{-ca}$ is increasing. This computation concludes the proof of Proposition 2.1. □

3. Estimates independent of the bounded rectangle

In this section, we derive estimates for the solution (u_a, v_a, c_a) of (2.1)–(2.7) that are independent of $a \geqslant 1$.

3.1. Preliminary remarks

In the sequel of the paper, it will be very useful to consider the functions

$$H_a = u_a + v_a - 1 \quad \text{and} \quad G_a = u_a + \Lambda v_a - 1. \tag{3.1}$$

Note that H_a satisfies

$$-\Delta H_a + c_a \frac{\partial H_a}{\partial x} = (\Lambda - 1)\Delta v_a, \tag{3.2}$$

$$-\frac{\partial H_a}{\partial x}(-a,y)+c_a H_a(-a,y)=(\Lambda-1)\frac{\partial v_a}{\partial x}(-a,y), \tag{3.3}$$

$$H_a(a,y)=0, \tag{3.4}$$

together with homogeneous Neumann conditions at Γ_a, while G_a is a solution of

$$-\Delta G_a+c_a\frac{\partial G_a}{\partial x}=(\Lambda-1)c_a\frac{\partial v_a}{\partial x}, \tag{3.5}$$

$$-\frac{\partial G_a}{\partial x}(-a,y)+c_a G_a(-a,y)=(\Lambda-1)c_a v_a(-a,y), \tag{3.6}$$

$$G_a(a,y)=0, \tag{3.7}$$

supplemented with homogeneous Neumann conditions at Γ_a. We also introduce the functions that are averaged quantities with respect to y:

$$u_{a,0}(x)=\int_0^1 u_a(x,y)\,\mathrm{d}y,\quad v_{a,0}(x)=\int_0^1 v_a(x,y)\,\mathrm{d}y, \tag{3.8}$$

$$h_{a,0}(x)=\int_0^1 H_a(x,y)\,\mathrm{d}y,\quad g_{a,0}(x)=\int_0^1 G_a(x,y)\,\mathrm{d}y. \tag{3.9}$$

The following monotonicity properties hold:

Lemma 3.1. *We have*

$$u'_{a,0}\geqslant 0\quad\textit{and}\quad v'_{a,0}\leqslant 0\quad\textit{on } [-a,a], \tag{3.10}$$

$$g'_{a,0}\leqslant 0\quad\textit{if } \Lambda>1 \textit{ and } g'_{a,0}\geqslant 0 \textit{ if } \Lambda<1. \tag{3.11}$$

Proof. In the proof, as often below, we omit to write down the dependence with respect to a of the different quantities. Let us check that $v_0'\leqslant 0$. We take the average with respect to y of Eq. (2.2). Since $f(u,y)v\geqslant 0$, this provides

$$-v_0''+\frac{c}{\Lambda}v_0'\leqslant 0$$

and by integrating from x to a

$$v_0'(x)\,\mathrm{e}^{-(c/\Lambda)x}\leqslant v_0'(a)\mathrm{e}^{-(c/\Lambda)a}. \tag{3.12}$$

Now, $v_0\geqslant 0$ and, by (2.5), $v_0(a)=0$. Therefore $v_0'(a)\leqslant 0$ and (3.12) gives $v_0'\leqslant 0$. The proof of the other inequality in (3.10) is similar.

Let us now derive (3.11). Taking the averages of (3.5)–(3.7), we have that

$$-g_0''+cg_0'=(\Lambda-1)cv_0', \tag{3.13}$$

$$-g_0'(-a)+cg_0(-a)=(\Lambda-1)cv_0(-a), \tag{3.14}$$

$$g_0(a)=0. \tag{3.15}$$

Note that $g_0'(a)=0$. Indeed by integrating Eq. (3.13) from $-a$ to a and using (3.14), we see that

$$-g_0'(a)+cg_0(a)-(\Lambda-1)cv_0(-a)=(\Lambda-1)c(v_0(a)-v_0(-a))$$

that re-writes $g_0'(a)=0$ since $g_0(a)=v_0(a)=0$.

Assume now that $\Lambda>1$ (the other case is similar). Then, since $v_0'\leqslant 0$, (3.13) yields

$$-g_0''+cg_0'\leqslant 0$$

and by integrating from x to a:

$$g_0'(x)\,\mathrm{e}^{-cx}\leqslant g_0'(a)\mathrm{e}^{-ca}=0,$$

which is (3.11). □

3.2. *First estimates independent of a*

Let us recall that Lemma 2.2 provides the existence of a constant $\overline{c}$ independent of $a\geqslant 1$ such that

$$c_a\leqslant\overline{c}\quad\forall a\geqslant 1.$$

The following lemma provides various estimates on u_a and v_a. For later use, it is important to emphasize the dependence of the different estimates with respect to c_a.

Lemma 3.2. *The following estimates hold*:

$$\|\nabla v_a\|_{L^2(R_a)}\leqslant\sqrt{\frac{c_a}{\Lambda}}, \tag{3.16}$$

$$\|\nabla u_a\|_{L^2(R_a)}\leqslant(1+|\Lambda-1|)\sqrt{\frac{c_a}{\Lambda}}, \tag{3.17}$$

$$\iint_{R_a} f(u_a,y)v_a\,\mathrm{d}x\,\mathrm{d}y\leqslant c_a, \tag{3.18}$$

$$\|\Delta v_a\|_{L^2(R_a)}\leqslant\frac{1}{\Lambda}\left(\sqrt{Mc_a}+c_a\sqrt{\frac{c_a}{\Lambda}}\right). \tag{3.19}$$

Proof. Let us first derive (3.16). We multiply Eq. (2.2) by v and we integrate on R_a. Using the Green's formula and the positivity of v, we have

$$\Lambda\int_{R_a}|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y+\Lambda\int_0^1\frac{\partial v}{\partial x}(-a,y)v(-a,y)\,\mathrm{d}y-\frac{c}{2}\int_0^1 v^2(-a,y)\,\mathrm{d}y\leqslant 0.$$

Now thanks to the flux condition (2.5) and Lemma 2.1, we can write

$$\Lambda\int_{R_a}|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant c\int_0^1 v(-a,y)\,\mathrm{d}y\leqslant c,$$

that provides (3.16).

In order to prove the estimate for u, we use the function H given by (3.1). More precisely, we aim to show the following estimate:

$$\begin{aligned}\|\nabla H\|_{L^2(R_a)} &\leqslant |\Lambda - 1| \|\nabla v\|_{L^2(R_a)} \\ &\leqslant |\Lambda - 1| \sqrt{\frac{c}{\Lambda}}.\end{aligned} \tag{3.20}$$

In order to derive this inequality we multiply Eq. (3.2) by H and we integrate on R_a. This gives

$$\begin{aligned}&\iint_{R_a} |\nabla H|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \frac{\partial H}{\partial x}(-a, y) H(-a, y)\,\mathrm{d}y - \frac{c}{2}\int_0^1 H^2(-a, y)\,\mathrm{d}y \\ &\quad = (1-\Lambda)\int_0^1 \frac{\partial v}{\partial x}(-a, y) H(-a, y)\,\mathrm{d}y + (1-\Lambda)\iint_{R_a} \nabla v \nabla H \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Hence,

$$\iint_{R_a} |\nabla H|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant (1-\Lambda)\iint_{R_a} \nabla v \nabla H \,\mathrm{d}x\,\mathrm{d}y,$$

that provides (3.20) thanks to the Cauchy–Schwarz inequality.

Finally, writing

$$\nabla u = \nabla H - \nabla v,$$

(3.16) and (3.20) give (3.17). Next, by integrating (2.2) on R_a, we find that

$$\iint_{R_a} f(u, y) v \,\mathrm{d}x\,\mathrm{d}y = c(v_0(-a) - v_0(a)) + \Lambda(v_0'(a) - v_0'(-a)).$$

Since v_0 is decreasing, using also the boundary condition (2.5), we see that

$$\iint_{R_a} f(u, y) v \,\mathrm{d}x\,\mathrm{d}y = c + \Lambda v_0'(a).$$

Finally, Eq. (2.2) yields

$$\Lambda \|\Delta v\|_{L^2(R_a)} \leqslant c \left\| \frac{\partial v}{\partial x} \right\|_{L^2(R_a)} + \sqrt{M} \|f(u, y) v\|_{L^1(R_a)}, \tag{3.21}$$

where M is given by (2.9). Combining (3.18), (3.21) and (3.16) provides (3.19) and concludes the proof of lemma 3.2. □

Next, the crucial step consists in obtaining some lower bound for c_a.

3.3. An estimate from below for the velocity

Recalling the average function $h_{a,0}$ and $g_{a,0}$ given by (3.9), we introduce the decompositions

$$H_a(x, y) = h_{a,0}(x) + \tilde{H}_a(x, y) \quad G_a(x, y) = g_{a,0}(x) + \tilde{G}_a(x, y). \tag{3.22}$$

We first give various estimates of $h_{a,0}$, $\tilde{H}_a$ and $\tilde{G}_a$. Again it will be important to emphasize the dependence of these estimates with respect to c_a.

Lemma 3.3. *The function $h_{a,0}$ is positive if $\Lambda > 1$ and negative if $\Lambda < 1$; moreover, we have*

$$\|h_{a,0}\|_{L^1(-a,a)} \leqslant \frac{|\Lambda - 1|}{c_a}, \tag{3.23}$$

$$\|h_{a,0}\|_{L^2(-a,a)} \leqslant \frac{|\Lambda - 1|}{\sqrt{\Lambda c_a}}, \tag{3.24}$$

$$\|h_{a,0}\|_{L^\infty(-a,a)} \leqslant |\Lambda - 1|. \tag{3.25}$$

Furthermore, the following estimates holds:

$$\|\tilde{H}_a\|_{L^2(R_a)} \leqslant |\Lambda - 1| \sqrt{\frac{c_a}{\Lambda}}, \tag{3.26}$$

$$\|\tilde{G}_a\|_{L^2(R_a)} \leqslant |\Lambda - 1| \frac{c_a}{\sqrt{\pi \Lambda}}. \tag{3.27}$$

Proof. Let us first check the estimates for h_0. Taking the average of (3.2)–(3.4), we see that h_0 satisfies

$$\begin{cases} -h_0'' + c h_0' = (\Lambda - 1) v_0'', \\ -h_0'(-a) + c h_0(-a) = (\Lambda - 1) v_0'(-a), \\ h_0(a) = 0. \end{cases}$$

It follows that h_0 is given by

$$h_0(x) = (\Lambda - 1) \int_x^a v_0'(t) \mathrm{e}^{c(x-t)} \, \mathrm{d}t. \tag{3.28}$$

Consequently, since $v_0' \leqslant 0$, the function h_0 is positive if $\Lambda > 1$ and negative if $\Lambda < 1$. Next, bounds (3.23) and (3.24) follow from formula (3.28) and Young 's inequality for the convolution product. Indeed we have

$$\begin{aligned} \|h_0\|_{L^1(-a,a)} &\leqslant |\Lambda - 1| \|(v_0' 1_{(-a,a)}) * (\mathrm{e}^{cx} \mathbb{1}_{x \leqslant 0})\|_{L^1(\mathbb{R})} \\ &\leqslant |\Lambda - 1| \|v_0' 1_{(-a,a)}\|_{L^1(\mathbb{R})} \|\mathrm{e}^{cx} \mathbb{1}_{x \leqslant 0}\|_{L^1(\mathbb{R})} \end{aligned}$$

which yields (3.23) in view of the monotony of v_0'. Also for bound (3.24), we write

$$\begin{aligned} \|h_0\|_{L^2(-a,a)} &\leqslant |\Lambda - 1| \|(v_0' 1_{(-a,a)}) * (\mathrm{e}^{cx} \mathbb{1}_{x \leqslant 0})\|_{L^2(\mathbb{R})} \\ &\leqslant |\Lambda - 1| \|v_0' 1_{(-a,a)}\|_{L^2(\mathbb{R})} \|\mathrm{e}^{cx} \mathbb{1}_{x \leqslant 0}\|_{L^1(\mathbb{R})} \end{aligned}$$

and we obtain (3.24) thanks to (3.17). Finally, for the L^∞ bound, (3.28) implies that

$$\forall x \in (-a, a), \quad |h_0(x)| \leqslant |\Lambda - 1| \int_x^a |v_0'(t)| \mathrm{e}^{c(x-t)} \, \mathrm{d}t,$$

Now, since v_0 is decreasing, $|v_0'(x)| = -v_0'(x)$ and the above inequality gives

$$|h_0(x)| \leqslant |\Lambda - 1| \int_x^a -v_0'(t)\,\mathrm{d}t$$

which yields (3.25) since $0 \leqslant v_0 \leqslant 1$.

We aim now to derive estimate (3.26). For that purpose, for $n \geqslant 1$, let us consider

$$h_n(x) = \int_0^1 H(x,y)\cos(\pi n y)\,\mathrm{d}y,$$

$$v_n(x) = \int_0^1 v(x,y)\cos(\pi n y)\,\mathrm{d}y,$$

$$\phi_n(x) = \int_0^1 \Delta v(x,y)\cos(\pi n y)\,\mathrm{d}y. \tag{3.29}$$

Then, we have:

$$\|\tilde{H}\|^2_{L^2(R_a)} = \sum_{n=1}^{\infty} \|h_n\|^2_{L^2(-a,a)}. \tag{3.30}$$

We aim to estimate the quantities $\|h_n\|^2_{L^2(-a,a)}$ for $n \geqslant 1$. Clearly, in view of (3.2), the function h_n satisfies

$$\begin{cases} -h_n'' + ch_n' + \pi^2 n^2 h_n = (\Lambda - 1)\phi_n, \\ -h_n'(-a) + ch_n(-a) = (\Lambda - 1)v_n'(-a), \\ h_n(a) = 0. \end{cases}$$

Multiplying this equation by h_n and integrating on $(-a,a)$, we obtain that

$$\pi^2 n^2 \int_{-a}^a h_n^2\,\mathrm{d}x + \int_{-a}^a (h_n')^2\,\mathrm{d}x - [h_n' h_n]_{-a}^a + \frac{c}{2}[h_n^2]_{-a}^a = (\Lambda - 1)\int_{-a}^a h_n \phi_n\,\mathrm{d}x.$$

In view of the boundary conditions, this expression reduces to

$$\pi^2 n^2 \int_{-a}^a h_n^2\,\mathrm{d}x + \int_{-a}^a (h_n')^2\,\mathrm{d}x + \frac{c}{2} h_n^2(-a)$$
$$= (\Lambda - 1)v_n'(-a)h_n(-a) + (\Lambda - 1)\int_{-a}^a h_n\phi_n\,\mathrm{d}x. \tag{3.31}$$

Also, in (3.31), ϕ_n is given by (3.29) and we see that

$$\phi_n(x) = \int_0^1 \Delta v(x,y)\cos(\pi n y)\,\mathrm{d}y = \varphi_n'(x) + n\pi\varpi_n(x), \tag{3.32}$$

where

$$\varphi_n(x) = \int_0^1 \frac{\partial v}{\partial x}(x,y)\cos(\pi n y)\,\mathrm{d}y, \quad \varpi_n(x) = \int_0^1 \frac{\partial v}{\partial y}(x,y)\sin(\pi n y)\,\mathrm{d}y.$$

Combining decomposition (3.32) and identity (3.31), we easily derive that

$$\begin{aligned} n^2\pi^2 \int_{-a}^{a} h_n^2 \,\mathrm{d}x + \int_{-a}^{a} (h_n')^2 \,\mathrm{d}x + \frac{c}{2} h_n^2(-a) \\ = (\Lambda - 1)\left(- \int_{-a}^{a} h_n' \varphi_n \,\mathrm{d}x + n\pi \int_{-a}^{a} h_n \varpi_n \,\mathrm{d}x \right). \end{aligned}$$

Therefore, thanks to the Cauchy–Schwarz inequality

$$n^2\pi^2 \|h_n\|_{L^2}^2 \leqslant |\Lambda - 1|(\|\varphi_n\|_{L^2}\|h_n'\|_{L^2} + n\pi\|h_n\|_{L^2}\|\varpi_n\|_{L^2}).$$

Now the following inequalities hold:

$$\|\varphi_n\|_{L^2} \leqslant \|\nabla v\|_{L^2}, \quad \|\varpi_n\|_{L^2} \leqslant \|\nabla v\|_{L^2}, \quad \|h_n'\|_{L^2} \leqslant \|\nabla H\|_{L^2}.$$

Using also (3.16) and (3.20), we infer from (3.3) that

$$\|h_n\|_{L^2(-a,a)}^2 \leqslant 6|\Lambda - 1|^2 \frac{c_a}{\Lambda\pi^2 n^2}.$$

Consequently, in view of (3.30), we conclude that

$$\|\tilde{H}_a\|_{L^2(R_a)}^2 \leqslant |\Lambda - 1|^2 \frac{c_a}{\Lambda}.$$

The proof of (3.27) is similar and is omitted. $\square$

We now derive some estimates of the norms of $\tilde{H}$ and $\tilde{G}$ in $L^\infty(R_a)$.

Lemma 3.4. *The following estimates holds*:

$$\|\tilde{H}_a\|_{L^\infty(R_a)} \leqslant \Phi_\infty(\Lambda, c_a, M), \tag{3.33}$$

$$\|H_a\|_{L^\infty(R_a)} \leqslant |\Lambda - 1| + \Phi_\infty(\Lambda, c_a, M), \tag{3.34}$$

$$\|u_a\|_{L^\infty(R_a)} \leqslant 1 + |\Lambda - 1| + \Phi_\infty(\Lambda, c_a, M), \tag{3.35}$$

where M is the L^∞ bound of the nonlinearity given by (2.9) *and*

$$\Phi_\infty(\Lambda, c, M) = k|\Lambda - 1|\sqrt{\frac{2c}{\Lambda}}\sqrt{\log\left(1 + \left(1 + \frac{1}{\Lambda}\right)c + \sqrt{M\Lambda}\right)} \tag{3.36}$$

and k is some absolute constant. Furthermore, there is some constant N depending on M, θ and Λ such that

$$\|\tilde{G}_a\|_{L^\infty(R_a)} \leqslant N\sqrt{c_a}. \tag{3.37}$$

Proof. Bound (3.33) follows from some local estimates in H^2-norm together with the Brezis–Gallouet inequality [7]. More precisely, we consider the domain $Q_x = (x, x+1) \times (0, 1)$ for $x \in (-a, a-1)$. Due to (3.20), the function $\tilde{H}_a$ satisfies

$$\|\tilde{H}_a\|_{H^1(Q_x)} \leqslant |\Lambda - 1|\sqrt{\frac{2c_a}{\Lambda}}.$$

Then thanks to Brezis–Gallouet inequality, there exists a constant k independent of $x \in (-a, a-1)$ such that

$$\|\tilde{H}_a\|_{L^\infty(Q_x)} \leqslant k|\Lambda - 1|\sqrt{\frac{c_a}{\Lambda}}\left(\log\left(1 + \frac{\|\tilde{H}_a\|_{H^2(Q_x)}}{|\Lambda - 1|}\sqrt{\frac{\Lambda}{c_a}}\right)\right)^{1/2}.$$

Clearly, in view of (3.19), we have

$$\Lambda\|\Delta v\|_{L^2(Q_x)} \leqslant c_a\sqrt{\frac{c_a}{\Lambda}} + \sqrt{Mc_a}. \tag{3.38}$$

Now, the function $\tilde{H}$ satisfies:

$$-\Delta\tilde{H} + c\,\frac{\partial\tilde{H}}{\partial x} = (\Lambda - 1)(\Delta v - v_0'').$$

Therefore, for $x \in (-a, a-1)$, thanks to (3.20), (3.38), we see that

$$\begin{aligned}\|\Delta\tilde{H}\|_{L^2(Q_x)} &\leqslant c\left\|\frac{\partial\tilde{H}}{\partial x}\right\|_{L^2(Q_x)} + |\Lambda - 1|\|\Delta v - v_0''\|_{L^2(Q_x)} \\ &\leqslant |\Lambda - 1|\left(\left(1 + \frac{1}{\Lambda}\right)\frac{c_a}{\sqrt{\Lambda}} + \sqrt{M}\right)\sqrt{c_a}\end{aligned} \tag{3.39}$$

while

$$\iint_{Q_x} \tilde{H}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Now, there is some constant k depending only on the diameter of Q_x such that

$$\|\tilde{H}\|_{H^2(Q_x)} \leqslant k\|\Delta\tilde{H}\|_{L^2(Q_x)}. \tag{3.40}$$

Then, thanks to (3.39) and (3.40), we obtain the following bound:

$$\|\tilde{H}_a\|_{L^\infty(R_a)} \leqslant \Phi_\infty(\Lambda, c_a, M).$$

Finally, thanks to decomposition (3.22) and bound (3.25), we easily obtain bound (3.34). The proof of the L^∞-bound (3.37) for the function $\tilde{G}$ is similar and is omitted. □

We can now find a lower bound for the velocity that is independent of the rectangle.

Proposition 3.1. *There exist $\underline{c} > 0$ and $a_0 > 1$ such that for all $a \geqslant a_0$*

$$c_a > \underline{c}. \tag{3.41}$$

Proof. The argument relies on upper and lower bounds of the quantity

$$\iint_{R_a} f(u_a, y)v_a u_{a,0}'\,\mathrm{d}x\,\mathrm{d}y,$$

where $u_{a,0}$ is given (3.8). Note that since $u_{a,0}$ is increasing, this quantity is positive.

The upper bound is easily obtained. Indeed thanks to Cauchy–Schwarz inequality, (3.18) and (3.17) we have

$$\begin{aligned}\int\int_{R_a} f(u,y)vu_0'\,\mathrm{d}x\,\mathrm{d}y &\leqslant \|f(u,y)v\|_{L^2(R_a)}\|u_0'\|_{L^2(R_a)}\\ &\leqslant (1+|\Lambda-1|)\sqrt{\frac{M}{\Lambda}}\,c_a. \qquad (3.42)\end{aligned}$$

For the lower bound, we first note that definition (3.1) and (3.22) provide the following decomposition:

$$u = u_0 + \frac{\tilde{G}-\Lambda\tilde{H}}{1-\Lambda}. \qquad (3.43)$$

Consequently, in view of bounds (3.33), (3.37) and Lemma 2.2, we can find some constant K_1 independent of $a>1$ such that

$$u \geqslant u_0 - K_1\sqrt{c_a}.$$

Therefore since f is increasing, we see that

$$\int\int_{R_a} f(u,y)vu_0'\,\mathrm{d}x\,\mathrm{d}y \geqslant \int\int_{R_a} f(u_0-K_1\sqrt{c_a},y)vu_0'\,\mathrm{d}x\,\mathrm{d}y.$$

Next, we introduce some appropriate decomposition of v. At this point we need to distinguish the cases $\Lambda<1$ and $\Lambda>1$. Let us first assume $\Lambda<1$. We write

$$v = v_0 + \frac{1}{1-\Lambda}(\tilde{H}-\tilde{G}) = h_0 + 1 - u_0 + \tilde{V}, \quad \text{where } \tilde{V} = \frac{\tilde{H}-\tilde{G}}{1-\Lambda}. \qquad (3.44)$$

Using this decomposition, (3.44) yields that

$$\begin{aligned}\int\int_{R_a} f(u,y)vu_0'\,\mathrm{d}x\,\mathrm{d}y \geqslant &\int\int_{R_a} f(u_0-K_1\sqrt{c_a},y)(1-u_0)u_0'\,\mathrm{d}x\,\mathrm{d}y\\ &+\int\int_{R_a} f(u_0-K_1\sqrt{c_a},y)h_0u_0'\,\mathrm{d}x\,\mathrm{d}y\\ &+\int\int_{R_a} f(u_0-K_1\sqrt{c_a},y)\tilde{V}u_0'\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.45)\end{aligned}$$

We aim to estimate the different terms in the right-hand side of (3.45). For the first term, the positivity of f together with (2.6) guarantee that

$$\begin{aligned}&\int\int_{R_a} f(u_0-K_1\sqrt{c_a},y)(1-u_0)u_0'\,\mathrm{d}x\,\mathrm{d}y\\ &\quad = \int_0^1\int_{u_0(-a)}^1 f(s-K_1\sqrt{c_a},y)(1-s)\,\mathrm{d}s\,\mathrm{d}y\\ &\quad \geqslant \int_0^1\int_\theta^1 f(s-K_1\sqrt{c_a},y)(1-s)\,\mathrm{d}s\,\mathrm{d}y \equiv I(c_a). \qquad (3.46)\end{aligned}$$

Also the last term is majorized by $\mu_1 c_a$ where μ_1 is some constant. Indeed in view of (3.26), (3.27) and (3.17), we have

$$\left| \int\int_{R_a} f(u_0 - K_1\sqrt{c_a}, y)\tilde{V}u_0' \,\mathrm{d}x\,\mathrm{d}y \right| \leqslant M\|\tilde{V}\|_{L^2(R_a)}\|u_0'\|_{L^2(R_a)} \leqslant \mu_1 c_a, \tag{3.47}$$

where μ_1 is a constant depending on M, Λ, θ.

A similar property does not hold for the second term but since $\Lambda < 1$, we have $h_0 \geqslant 0$ (see Lemma 3.1) and this term is positive. Combining the above estimates, we infer from (3.45) that

$$\int\int_{R_a} f(u, y)vu_0' \,\mathrm{d}x\,\mathrm{d}y \geqslant I(c_a) - \mu_1 c_a.$$

Now this lower bound together with (3.42) enable us to say that

$$I(c_a) \leqslant (1 + |\Lambda - 1|)\sqrt{\frac{M}{\Lambda}}\, c_a + \mu_1 c_a. \tag{3.48}$$

But, thanks to Lebesgue's theorem, the function $I(c)$ satisfies

$$\lim_{c\to 0} I(c) = \int_0^1 \int_\theta^1 f(s, y)(1 - s)\,\mathrm{d}s\,\mathrm{d}y > 0.$$

Consequently, inequality (3.48) yields the existence of some constants $\underline{c}$ and $a_0 > 1$ satisfying (3.41).

In the case $\Lambda > 1$, we use a decomposition of the function v different from (3.44). Indeed we now write

$$v = \frac{1}{\Lambda}(G + 1 - u) = \frac{1}{\Lambda}(g_0 + 1 - u_0) + \frac{\tilde{H} - \tilde{G}}{1 - \Lambda}.$$

Due to (3.11), we can easily see that the function g_0 is positive. Hence, computations similar to the ones above yield (3.41) in that case. The proof of Proposition 3.1 is complete. $\square$

3.4. Other useful estimates for the velocity

Before investigating the limit $a \to +\infty$, we derive some further estimates for the velocity that will be very important in the sequel.

Proposition 3.2. *Let $a > 1$. For $\Lambda > 1$ the following inequality holds*:

$$\frac{m}{\Lambda} \leqslant \frac{c_a^2\theta^2}{2} + \frac{c_a^2}{\Lambda}\Lambda'^2 + M|\Lambda - 1|\Lambda' \frac{c_a\sqrt{c_a}}{\sqrt{\pi}\Lambda} \tag{3.49}$$

while for $\Lambda < 1$, we have

$$\begin{aligned} m \leqslant{} & \frac{c_a^2\theta^2}{2} + \frac{\Lambda^2}{\Lambda}c_a^2 + |\Lambda - 1|M\,\frac{\Lambda' c_a}{\Lambda} \\ & + (1 - \Lambda)\int_0^1 F(2 - \Lambda + \Phi_\infty(\Lambda, c_a, M), y)\,\mathrm{d}y. \end{aligned} \tag{3.50}$$

Here, M is given by (2.9), Φ_∞ *by* (3.36), $\Lambda' = 1 + |\Lambda - 1|$,

$$m = \iint_{(0,1)^2} f(s, y)(1 - s)\,\mathrm{d}s\,\mathrm{d}y \tag{3.51}$$

and

$$F(s, y) = \int_0^s f(t, y)\,\mathrm{d}t. \tag{3.52}$$

Remark 3.1. For $\Lambda > 1$, Proposition 3.2 provides a different proof of (3.41) thanks to (3.49). However this is not true for $\Lambda < 1$. Also, even for $\Lambda > 1$, the techniques of Proposition 3.2 would not provide (3.41) for more general chemistry (for example an order n reaction: $nR \to P$ with $n > 1$) while the proof of Proposition 3.1 can be extended to that case.

Proof. Let $a > 1$. Let us first consider the case $\Lambda > 1$. By multiplying Eq. (2.1) by $\partial u_a/\partial x$ and integrating over R_a, we obtain

$$\frac{c_a^2}{2}\int_0^1 u_a(-a, y)^2\,\mathrm{d}y + c_a\iint_{R_a}\left|\frac{\partial u_a}{\partial x}\right|^2 \mathrm{d}x\,\mathrm{d}y \geqslant \iint_{R_a} f(u_a, y)v_a\,\frac{\partial u_a}{\partial x}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.53}$$

Here, due to (2.6), we have $u(-a, y) \leqslant \theta$. Also, the second term in the left-hand side of (3.53) can be estimated thanks to (3.17). For the right-hand side, recalling (3.1) and (3.22) we introduce the decomposition

$$v = \frac{1}{\Lambda}(g_0 + \tilde{G}) + \frac{1}{\Lambda}(1 - u). \tag{3.54}$$

Combining these remarks we infer from (3.53)

$$\begin{aligned}\frac{c_a^2\theta^2}{2} + \frac{c_a^2}{\Lambda}\Lambda'^2 \geqslant \frac{m}{\Lambda} + \frac{1}{\Lambda}\iint_{R_a} f(u, y)g_0 u_x\,\mathrm{d}x\,\mathrm{d}y \\ + \frac{1}{\Lambda}\iint_{R_a} f(u, y)\tilde{G}u_x\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{3.55}$$

Next, estimates (3.27) and (3.17) yield

$$\left|\iint_{R_a} f(u, y)\tilde{G}u_x\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant M\|\tilde{G}\|_{L^2(R_a)}\|u_x\|_{L^2(R_a)} \leqslant M|\Lambda - 1|\Lambda'\,\frac{c_a\sqrt{c_a}}{\sqrt{\pi}\Lambda}. \tag{3.56}$$

Also an integration by parts provides

$$\iint_{R_a} f(u, y)g_0 u_x\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 [F(u, y)g_0]_{-a}^{a}\,\mathrm{d}y - \iint_{R_a} F(u, y)g_0'\,\mathrm{d}x\,\mathrm{d}y, \tag{3.57}$$

where F is defined by (3.52). Since $u(-a, y) \leqslant \theta$ and $g_0(a) = 0$, the first term in the right-hand side of (3.57) vanishes while the second one is positive since $\Lambda > 1$ implies $g_0' \leqslant 0$ (see (3.11)). Therefore

$$\int_{R_a} f(u, y)g_0 u_x \geqslant 0. \tag{3.58}$$

Combining (3.58), (3.56) and (3.55) enables us to say that

$$\frac{c_a^2\theta^2}{2} + \frac{c_a^2}{\Lambda}\Lambda'^2 + M|\Lambda - 1|\Lambda' \frac{c_a\sqrt{c_a}}{\sqrt{\pi}\Lambda} \geqslant \frac{m}{\Lambda}.$$

That is (3.49).

Let us now assume $\Lambda < 1$. Instead of (3.54), we consider the decomposition

$$v = h_0 + \tilde{H} + 1 - u.$$

Therefore, as above, we infer from (3.53) that

$$\frac{c_a^2\theta^2}{2} + \frac{c_a^2}{\Lambda}\Lambda'^2 \geqslant m + \int_{R_a} f(u, y)h_0 u_x \,\mathrm{d}x\,\mathrm{d}y + \int_{R_a} f(u, y)\tilde{H} u_x \,\mathrm{d}x\,\mathrm{d}y. \tag{3.59}$$

Then, thanks to (3.26) and (3.17)

$$\left|\int_{R_a} f(u, y)\tilde{H} u_x \,\mathrm{d}x\,\mathrm{d}y\right| \leqslant M\|\tilde{H}\|_{L^2(R_a)}\|u_x\|_{L^2(R_a)} \leqslant M\Lambda'|\Lambda - 1|\frac{c_a}{\Lambda}. \tag{3.60}$$

Also,

$$\begin{aligned}\iint_{R_a} f(u, y)h_0 u_x \,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 [F(u, y)h_0]_{-a}^{a}\,\mathrm{d}y - \iint_{R_a} F(u, y)h_0' \,\mathrm{d}x\,\mathrm{d}y\\ &= 0 + \iint_{R_a} F(u, y)((\Lambda - 1)v_0' - ch_0)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Then, as v_0 is decreasing, we see that

$$\iint_{R_a} f(u, y)h_0 u_x \,\mathrm{d}x\,\mathrm{d}y \geqslant c_a \iint_{R_a} F(u, y)h_0 \,\mathrm{d}x\,\mathrm{d}y.$$

Thanks to the positivity of the function f and the L^1-bound for the function h_0 (3.23), we derive

$$\begin{aligned}\iint_{R_a} f(u, y)h_0 u_x \,\mathrm{d}x\,\mathrm{d}y &\geqslant -c_a \int_0^1 F(\|u\|_\infty, y)\|h_0\|_{L^1}\,\mathrm{d}y\\ &\geqslant -(1-\Lambda)\int_0^1 F(2-\Lambda+\Phi_\infty(\Lambda, c_a, M), y)\,\mathrm{d}y.\end{aligned} \tag{3.61}$$

Gathering (3.59)–(3.61) lead us to estimate (3.50). □

4. Problem in the cylinder

In this section, we investigate the limit $a \to +\infty$ of the solutions of (2.1)– (2.7) in R_a. We first prove the following result that holds for any $\Lambda > 0$.

Theorem 4.1. *Assume that* (1.10) *and* (1.11) *hold. Let* $\Lambda > 0$ *be given. Then there exists an increasing sequence* $(a_n)_{n\in\mathbb{N}}$ *with* $\lim_{n\to+\infty} a_n = +\infty$ *such that the solution* $(u_{a_n}, v_{a_n}, c_{a_n})$

of (2.1)–(2.7) *in* R_{a_n} *given by Proposition* 2.1 *converges for the topology of* $C^1_{\mathrm{loc}}(\bar{\Omega}) \times C^1_{\mathrm{loc}}(\bar{\Omega}) \times \mathbb{R}$ *to* (u, v, c) *satisfying*

$$-\Delta u + c\,\frac{\partial u}{\partial x} = f(u, y)v \quad \text{in } \Omega, \tag{4.1}$$

$$-\Lambda\Delta v + c\,\frac{\partial v}{\partial x} = -f(u, y)v \quad \text{in } \Omega, \tag{4.2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0 \quad \text{on } \Gamma, \tag{4.3}$$

$$\max_{(x,y)\in\Omega^-} u(x, y) = \theta. \tag{4.4}$$

In addition,

$$\lim_{x\to-\infty} u(x, y) = 0, \qquad \lim_{x\to-\infty} v(x, y) = 1, \tag{4.5}$$

$$\lim_{x\to+\infty} u(x, y) = u^+, \qquad \lim_{x\to+\infty} v(x, y) = 1 - u^+, \ \text{where } u^+ \in \{\theta, 1\}, \tag{4.6}$$

$$0 < c \leqslant \sqrt{\frac{2M}{\theta}}. \tag{4.7}$$

The convergences in (4.5) *and* (4.6) *hold uniformly with respect to* $y \in [0, 1]$.

Proof. Let us take a solution (u_a, v_a, c_a) of (2.1)–(2.7). In view of Lemma 2.2, (2.10) and (3.1), for a large enough we have

$$\underline{c} \leqslant c_a \leqslant \sqrt{\frac{2M}{\theta}}. \tag{4.8}$$

Moreover, since $0 \leqslant v_a \leqslant 1$, $f(u_a, y)v_a$ is bounded in $L^\infty(\Omega)$ whereas in view of (3.35), u_a is bounded in $L^\infty(\Omega)$. It easily follows that u_a and v_a are bounded $W^{2,p}(R_A)$ for all p, $1 \leqslant p < +\infty$ and $A > 0$, independently of a sufficiently large. As a consequence, there exists an increasing sequence a_n with $\lim_{n\to+\infty} a_n = +\infty$ such that $(u_{a_n}, v_{a_n}, c_{a_n})$ converges to (u, v, c) in the topology of $C^1_{\mathrm{loc}}(\bar{\Omega}) \times C^1_{\mathrm{loc}}(\bar{\Omega}) \times \mathbb{R}$. Next, it is easily seen that (u, v, c) satisfy (4.1)–(4.4) while (4.7) follows from (4.8).

There remains to investigate the limits at $x = \pm\infty$. The behavior at $-\infty$ follows easily from some exponential estimates for the functions u and v.

Lemma 4.1. *For all* $(x, y) \in \mathbb{R}_- \times (0, 1)$, *the following properties hold*:

$$u(x, y) \leqslant \theta \mathrm{e}^{cx}, \tag{4.9}$$

$$1 - v(x, y) \leqslant \mathrm{e}^{(c/\Lambda)x}. \tag{4.10}$$

In particular,

$$\lim_{x\to-\infty} u(x, y) = 0, \qquad \lim_{x\to-\infty} v(x, y) = 1,$$

where the limits are uniform with respect to $y \in [0, 1]$.

Proof. Let $(x, y) \in \mathbb{R}_- \times (0, 1)$. Then for n large enough, we have $(x, y) \in R^-_{a_n}$. Due to condition (2.6) and assumption (1.11), u_{a_n} satisfy

$$-\Delta u + c\,\frac{\partial u}{\partial x} = 0 \quad \text{in } R^-_{a_n},$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{on } (-a_n, 0) \times \{0, 1\},$$

$$u(0, y) \leqslant \theta,$$

$$-u_x(-a_n, y) + cu(a_n, y) = 0.$$

Therefore the maximum principle yields

$$u_{a_n}(x, y) \leqslant \theta \mathrm{e}^{c_{a_n} x}.$$

Similarly, we easily see that we have

$$1 - v_{a_n}(x, y) \leqslant \mathrm{e}^{(c_{a_n}/\Lambda)x}.$$

Taking the limit $n \to +\infty$ yields the expected inequalities. □

We now aim to investigate the behavior at $x = +\infty$. The arguments are mainly classical and close to the ones in [3]. Let (t_m) be sequence of positive numbers going to infinity as $m \to +\infty$. We define the sequences of functions $\bar{u}_m$ and $\bar{v}_m$ as follows:

$$\bar{u}_m(x, y) = u(x + t_m, y) \quad \bar{v}_m(x, y) = v(x + t_m, y).$$

Then, as in [3], it is easily seen that the sequences $\bar{u}_m$ and $\bar{v}_m$ are relatively compact in $C^1([0, 1]^2)$.

Next, let us consider two sub-sequences of $(\bar{u}_m)$ and $(\bar{v}_m)$ still denoted by $(\bar{u}_m)$ and $(\bar{v}_m)$ that converge in $C^1([0, 1]^2)$ towards some functions l and k. Taking the limit $m \to +\infty$, we find that

$$\lim_{m\to+\infty} \iint_{(0,1)^2} |H(x + t_m, y)|^2 \,\mathrm{d}x\,\mathrm{d}y = \iint_{(0,1)^2} |l + k - 1|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

We can suppose (up to the extraction of some sub-sequences) that $t_{m+1} > t_m + 1$. Hence, the Fatou lemma together with estimates (3.24) and (3.26) provide

$$+\infty > \iint_{\Omega} |u + v - 1|^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant \sum_{p=1}^{+\infty} \iint_{(0,1)^2} |H(x + t_p, y)|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, we necessarily have

$$\iint_{(0,1)^2} |l + k - 1|^2 \,\mathrm{d}x\,\mathrm{d}y = 0.$$

A similar argument yields that the functions l and k are constant functions. Indeed, we can write

$$+\infty > \iint_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant \sum_{p=1}^{+\infty} \iint_{(0,1)^2} |\nabla \bar{u}_p|^2 \,\mathrm{d}x\,\mathrm{d}y$$

and

$$+\infty > \iint_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant \sum_{p=1}^{+\infty} \iint_{(0,1)^2} |\nabla \bar{v}_p|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Therefore the convergence of $\bar{u}_m$ and $\bar{v}_m$ in $C^1([0,1]^2)$ provides that the functions l and k are constant and satisfy $l + k - 1 = 0$.

To conclude, we have to show that the constant function l does not depend on the chosen sequence $(t_m)_m$. Before giving an explicit expression for l, we need the following lemma.

Lemma 4.2. *The function u satisfies*

$$\lim_{x\to\pm\infty} |\nabla u(x,y)| = 0, \tag{4.11}$$

where the convergence is uniform with respect to $y \in (0,1)$.

The proof is similar to arguments in [3] and is omitted.

Now let (t_{n_k}) be a sub-sequence of (t_n) such that the sequences $\bar{u}_{n_k}$ and $\bar{v}_{n_k}$ converge in $C^1([0,1]^2)$ towards the constants l and $1-l$, respectively. Integrating Eq. (4.1) over $T_{n_k} = (-t_{n_k}, t_{n_k}) \times (0,1)$, we obtain

$$C(t_{n_k}) - C(-t_{n_k}) = \int_{-t_{n_k}}^{t_{n_k}} \int_0^1 f(u,y)v \,\mathrm{d}x\,\mathrm{d}y$$

with

$$C(z) = -\int_0^1 \frac{\partial u}{\partial x}(z,y)\,\mathrm{d}y + c\int_0^1 u(z,y)\,\mathrm{d}y.$$

Using (4.9) and (4.11) and letting k go to $+\infty$, we derive

$$l = \frac{1}{c}\iint_\Omega f(u,y)v \,\mathrm{d}x\,\mathrm{d}y. \tag{4.12}$$

The quantity in the right-hand side of (4.12) does not depend on the sequence (t_n) and we will denote it by u^+.

At this point we have obtained the following convergence results:

$$\lim_{x\to+\infty} u(x,y) = u^+, \qquad \lim_{x\to+\infty} v(x,y) = 1 - u^+.$$

There remains to show that $u^+ \in \{\theta, 1\}$. We easily see that u^+ satisfies

$$f(u^+, y)(1-u^+) = 0$$

so that $u^+ \in [0,\theta] \cup \{1\}$. Next, the averaged functions

$$u_0(x) = \int_0^1 u(x,y)\,\mathrm{d}y, \quad v_0(x) = \int_0^1 v(x,y)\,\mathrm{d}y,$$

satisfy $u_0(-\infty)=1-v_0(-\infty)=0$ and $u_0(+\infty)=1-v_0(+\infty)=u^+$. Since u_0 is increasing (recall (3.10)) and $u_0(0)>0$, necessarily we have $u^+>0$. Now suppose that $u^+ \in]0,\theta[$. Then there exists $X \in \mathbb{R}$ such that

$$\forall x \geqslant X, \quad \forall y \in [0,1], \quad u(x,y) < \theta.$$

Recalling property (1.11) of the ignition temperature, v_0 satisfies $-\Lambda v_0'' + cv_0' = 0$ on $(X,+\infty)$. Hence, v_0 is constant on $(X,+\infty)$. Applying the maximum principle, it easily follows that v_0 is constant on $\mathbb{R}$ which is impossible since $v_0(+\infty) < 1 = v_0(-\infty)$. The proof of Theorem 4.1 is now complete. □

Theorem 4.1 provides a solution (u,v,c) of (4.1)–(4.7) obtained as the limit of the solution (u_a,v_a,c_a) of the problem in the bounded rectangle R_a. Since the convergences of u_a and v_a are not uniform on Ω, estimates on u_a and v_a do not readily yield estimates on u and v. Nevertheless, some further information on (u,v,c) can be obtained. We now state some results on (u,v,c) that will be important in the sequel. First taking the limit $a \to +\infty$ in (3.34), we see that $H = u+v-1$ satisfies

$$\|H\|_{L^\infty(\Omega)} \leqslant |\Lambda - 1| \left(1 + k\sqrt{\frac{2c}{\Lambda}}\sqrt{\log\left(1+\left(1+\frac{1}{\Lambda}\right)c + \sqrt{M\Lambda}\right)}\right). \tag{4.13}$$

Also we can take the limit $a \to +\infty$ in the results of Proposition 3.2. We obtain that, for $\Lambda > 1$, c satisfies

$$\frac{m}{\Lambda} \leqslant \frac{c^2\theta^2}{2} + \frac{c^2}{\Lambda}\Lambda'^2 + M|\Lambda-1|\Lambda' \frac{c\sqrt{c}}{\sqrt{\pi}\Lambda} \tag{4.14}$$

while, for $\Lambda < 1$, we have

$$\begin{aligned} m \leqslant\; & \frac{c^2\theta^2}{2} + \frac{\Lambda'^2}{\Lambda}c^2 + |\Lambda-1|M\,\frac{\Lambda' c}{\Lambda} \\ & + (1-\Lambda)\int_0^1 F(2-\Lambda+\Phi_\infty(\Lambda,c,M),y)\,\mathrm{d}y, \end{aligned} \tag{4.15}$$

where Φ_∞ is given by (3.36) and F by (3.52).

Let us now derive an upper bound for c in the strip that improves (4.7).

Lemma 4.3. *The following estimate holds*:

$$c^2 \leqslant \frac{1}{\theta}\,\|f(u,.)v\|_{L^\infty(\Omega)}. \tag{4.16}$$

Proof. Estimate (4.16) follows from the comparison of the function u with $\hat{u}$ defined by

$$\hat{u}(x) = \begin{cases} \dfrac{S}{c^2}\,\mathrm{e}^{cx} & \text{if } x \leqslant 0, \\ \dfrac{S}{c^2} + \dfrac{S}{c}\,x & \text{if } x \geqslant 0, \end{cases}$$

where $S = \| f(u, .)v\|_{L^\infty(\Omega)}$. Indeed $\hat{u}$ satisfies

$$-\hat{u}'' + c\hat{u}' = S\mathbb{1}_{x \geqslant 0}, \quad \hat{u}(-\infty) = 0, \quad \hat{u}(+\infty) = +\infty,$$

so that the maximum principle yields that $u \leqslant \hat{u}$. We conclude the proof of Lemma 4.3 by expressing that $\theta \leqslant \hat{u}(0)$. $\square$

Let us conclude this section by showing that the solution of (4.1)–(4.7) satisfies $u(+\infty, y) = 1$ if Λ is sufficiently close to one. This property guarantees that (u, v, c) is a solution of (1.3)–(1.7).

Theorem 4.2. *Assume that* (1.10) *and* (1.11) *hold. There exists* $\delta > 0$ *such that if* $\Lambda > 0$ *satisfies* $|\Lambda - 1| < \delta$, *then the triplet* (u, v, c) *given by Theorem* 4.1 *is a solution of problem* (1.3)–(1.7)

Proof. The proof is based on inequalities (4.14) for $\Lambda > 1$ and (4.15) for $\Lambda < 1$. We will only consider the case $\Lambda > 1$. The other case is similar.

We argue by contradiction. Let us suppose that $u(+\infty, y) = u^+ = \theta$. Then, by (4.6), $v(+\infty, y) = 1 - \theta$. Applying the maximum principle to v, we easily see that

$$1 - \theta \leqslant v \leqslant 1.$$

Consequently, $u = H + 1 - v$ satisfies

$$\|u\|_{L^\infty(\Omega)} \leqslant \theta + \|H\|_{L^\infty(\Omega)}$$

and, thanks to (4.13)

$$\|u\|_{L^\infty(\Omega)} \leqslant \theta + (\Lambda - 1) + \Phi_\infty(\Lambda, c, M). \tag{4.17}$$

But, in view of (3.36), Φ_∞ is an increasing function with respect to c. Therefore, bound (4.7) combined with (4.17) enable us to say that

$$\|u\|_{L^\infty(\Omega)} \leqslant \theta + (\Lambda - 1) + \tilde{\Phi}_\infty(\Lambda) \quad \text{with} \quad \tilde{\Phi}_\infty(\Lambda) = \Phi_\infty\left(\Lambda, \sqrt{\frac{2M}{\theta}}, M\right). \tag{4.18}$$

Estimate (4.18) allows us to obtain an improved upper bound on c. Thanks to (4.16) we find that

$$c^2 \leqslant \frac{1}{\theta} \sup_{y \in (0,1)} f(\theta + (\Lambda - 1) + \tilde{\Phi}_\infty(\Lambda), y) \equiv \hat{c}(\Lambda).$$

Next, we combine this estimate with (4.14) and obtain that

$$\frac{m}{\Lambda} \leqslant \frac{\hat{c}(\Lambda)\theta^2}{2} + \frac{\hat{c}(\Lambda)}{\Lambda}\Lambda'^2 + M|\Lambda - 1|\Lambda' \frac{\hat{c}(\Lambda)^{3/4}}{\sqrt{\pi}\Lambda}. \tag{4.19}$$

We can now conclude the proof of Theorem 4.2. Indeed, since $f(\theta, y) = 0$, we have

$$\lim_{\Lambda \to 1} \hat{c}(\Lambda) = 0.$$

Consequently, the right-hand side of (4.19) goes to zero as $\Lambda \to 1$ whereas $m > 0$ is independent of Λ. It follows that (4.19) can not hold true for Λ sufficiently close to 1 and necessarily, for such Λ, $u(+\infty, y) = 1$. Theorem 4.2 is proved. □

Remark 4.1. The Arrhenius term arising in combustion theory is given by

$$f(u) = \frac{\Lambda}{2\varepsilon^2} \exp\left(\frac{1}{\varepsilon} \frac{u-1}{1+\gamma(u-1)}\right),$$

where ε is the inverse of the reduced activation energy and $\gamma \in]0, 1[$ is a heat release parameter. We see that $M = (\Lambda/2\varepsilon^2)\exp(1/\gamma\varepsilon)$ so that the constant δ, provided by Theorem 4.2, depends of ε through $\varepsilon^2 \exp(-1/\gamma\varepsilon)$. Such a dependence is irrelevant for $\varepsilon \ll 1$. The aim of the following section is to obtain improved estimates that do not assume any bound on f.

5. Travelling waves for high activation energy

In this section we assume that f and Λ depend on a parameter $\varepsilon > 0$ and we set $f = f_\varepsilon$, $\Lambda = \Lambda_\varepsilon$. We consider the problem

$$-\Delta u + c\frac{\partial u}{\partial x} = f_\varepsilon(u, y)v \quad \text{in } \Omega = \mathbb{R} \times (0, 1), \tag{5.1}$$

$$-\Lambda_\varepsilon \Delta v + c\frac{\partial v}{\partial x} = -f_\varepsilon(u, y)v \quad \text{in } \Omega \tag{5.2}$$

together with the following boundary conditions:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0 \quad \text{on } \Gamma = \mathbb{R} \times \{0, 1\}, \tag{5.3}$$

$$u(-\infty, y) = 0, \quad v(-\infty, y) = 1 \quad \text{for } y \in (0, 1), \tag{5.4}$$

$$u(+\infty, y) = 1, \quad v(+\infty, y) = 0 \quad \text{for } y \in (0, 1). \tag{5.5}$$

We assume that the function f_ε takes the form

$$f_\varepsilon(u, y) = \frac{1}{\varepsilon^2}\psi\left(\frac{u-1}{\varepsilon}\right)\chi(u, y), \tag{5.6}$$

where $\psi : \mathbb{R} \to \mathbb{R}^+$ is locally Lipschitz continuous on $\mathbb{R}$ and satisfies

$$\begin{cases} \psi \text{ increasing and } \psi(s) = \circ\left(\frac{1}{s^6}\right) \text{ when } s \to -\infty, \\ \int_{-\infty}^0 \psi(s)\,\mathrm{d}s > 0 \end{cases} \tag{5.7}$$

while $\chi : \mathbb{R}^+ \times (0,1) \to \mathbb{R}$ is Lipschitz continuous and such that

$$\begin{cases} \exists\, \theta \in (0,1) \chi(s,y) = 0 \text{ if } s \leqslant \theta \text{ and } 0 < \chi(s,y) \leqslant 1 \text{ if } s > \theta, \\ s \to \chi(s,y) \text{ is increasing, for all } y \in (0,1). \end{cases} \tag{5.8}$$

As already mentioned, the high activation energy asymptotics, i.e. the limit $\varepsilon \to 0$, is a major tool in combustion theory. When the Lewis number differs from one ($\Lambda_\varepsilon \neq 1$), the first step in the asymptotics methods used by physicists consists in assuming a suitable (heuristic) bound on the temperature of the form

$$u(x,t) \leqslant 1 + O(\varepsilon).$$

Note that such a bound was rigorously derived in [10,9] for the parabolic problem (1.1)–(1.2).

In order to study problem (5.1)–(5.5), we first truncate f_ε for large values of u. More precisely, we introduce $(\tilde{f}_\varepsilon)$ given by

$$\tilde{f}_\varepsilon(s,y) = \begin{cases} f_\varepsilon(s,y) & \text{if } s \leqslant 1+\varepsilon, \\ f_\varepsilon(1+\varepsilon, y) & \text{if } s \geqslant 1+\varepsilon. \end{cases} \tag{5.9}$$

Then the first step of our study consists in investigating problem (5.1)–(5.5) with f_ε replaced by $\tilde{f}_\varepsilon$ (Theorem 5.1). Next, we show how existence results for this problem allows to obtain results for the initial problem (5.1)–(5.5) (Theorem 5.2).

For all $\varepsilon > 0$, the function $\tilde{f}_\varepsilon$ satisfies (1.10) and (1.11). Therefore Theorem 4.1 applies and provides the existence of a solution of (4.1)–(4.7) for $\Lambda = \Lambda_\varepsilon$ and $f = \tilde{f}_\varepsilon$. Let us denote by $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ this solution. The limits at $x = +\infty$ are such that $u_\varepsilon(+\infty, y) = 1 - v_\varepsilon(+\infty, y) = u_\varepsilon^+$. We aim to show that $u_\varepsilon^+ = 1$ under appropriate assumptions on Λ_ε and for ε sufficiently small.

Theorem 5.1. *Assume that* (5.7) *and* (5.8) *hold. Assume furthermore that* Λ_ε *satisfies either* $1 - l_2\varepsilon^{r_2} \leqslant \Lambda_\varepsilon < 1$ *with* $r_2 > 1$, *or* $1 < \Lambda_\varepsilon \leqslant 1 + l_1\varepsilon^{r_1}$ *with* $r_1 > \frac{1}{2}$, *where* l_1 *and* l_2 *denote positive constants independent of* ε. *Then, for* ε *sufficiently small*, $\varepsilon \leqslant \varepsilon_0$, *we have*

$$u_\varepsilon^+ = \lim_{x \to +\infty} u_\varepsilon(x,y) = 1.$$

In particular, $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ *is a solution of* (5.1)–(5.5) *with* f_ε *replaced by* $\tilde{f}_\varepsilon$.

Proof. The definition of $\tilde{f}_\varepsilon$, together with expression (5.6) for f_ε yield the existence of a constant K independent of ε such that

$$\|\tilde{f}_\varepsilon\|_{L^\infty} \leqslant \frac{K}{\varepsilon^2}. \tag{5.10}$$

Note that $\tilde{f}_\varepsilon$ is not uniformly bounded with respect to ε. Therefore, we need to carefully keep track of the dependence with respect to ε of the various estimates. We first note that estimate (4.7) of the velocity provides

$$c_\varepsilon \leqslant \frac{1}{\varepsilon}\sqrt{\frac{2K}{\theta}}. \tag{5.11}$$

Let us assume that

$$1 < \Lambda_\varepsilon \leqslant 1 + l_1\varepsilon^{r_1} \quad \text{for some } r_1 > \tfrac{1}{2}. \tag{5.12}$$

This assumption guarantees a convenient upper bound for the enthalpy $H_\varepsilon = u_\varepsilon + v_\varepsilon - 1$. Indeed recall estimate (4.13) for H_ε. There, $\Lambda = \Lambda_\varepsilon$, $c = c_\varepsilon$ and $M = M_\varepsilon = \sup \tilde{f}_\varepsilon$. Using (5.10) and (5.11), we infer from (4.13) that

$$\|H_\varepsilon\|_{L^\infty(\Omega)} \leqslant |\Lambda_\varepsilon - 1| \left(1 + \frac{k}{\sqrt{\varepsilon\Lambda_\varepsilon}}\left(\frac{2K}{\theta}\right)^{1/4}\sqrt{\log\left(1 + \frac{\lambda_\varepsilon}{\varepsilon}\right)}\right), \tag{5.13}$$

where $\lambda_\varepsilon = \sqrt{(2K/\theta)}(1 + \Lambda_\varepsilon^{-1}) + \sqrt{K\Lambda_\varepsilon}$. Now, due to assumption (5.12), we can find $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$ we have

$$|\Lambda_\varepsilon - 1| \left(1 + \frac{k}{\sqrt{\varepsilon\Lambda_\varepsilon}}\left(\frac{2K}{\theta}\right)^{1/4}\sqrt{\log\left(1 + \frac{\lambda_\varepsilon}{\varepsilon}\right)}\right) \leqslant \frac{1-\theta}{2}.$$

Therefore bound (5.13) yields

$$\|H_\varepsilon\|_{L^\infty(\Omega)} \leqslant \frac{1-\theta}{2} \quad \text{for } \varepsilon < \varepsilon_1. \tag{5.14}$$

Next, we argue by contradiction to prove that $u_\varepsilon^+ = 1$ for sufficiently small ε. Assume that $u_\varepsilon^+ = \theta$. Then $v_\varepsilon(+\infty, y) = 1 - \theta$ and the maximum principle provides

$$1 - \theta \leqslant v_\varepsilon \leqslant 1. \tag{5.15}$$

Therefore, combining (5.14) and (5.15), we see that $u_\varepsilon = H_\varepsilon + 1 - v_\varepsilon$ satisfies

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant \frac{1+\theta}{2}. \tag{5.16}$$

Now, due to (5.6)–(5.8), the function $s \to f_\varepsilon(s, y)$ is increasing. Consequently, since $0 \leqslant \chi \leqslant 1$, (5.16) enables us to say that

$$\sup_{(x,y)\in\bar{\Omega}} \tilde{f}_\varepsilon(u_\varepsilon(x, y), y)v_\varepsilon(x, y) \leqslant \frac{1}{\varepsilon^2}\,\psi\left(\frac{\theta - 1}{2\varepsilon}\right).$$

Then, we majorize the velocity thanks to (4.16) and we obtain

$$c_\varepsilon \leqslant \frac{1}{\varepsilon}\sqrt{\frac{1}{\theta}\,\psi\left(\frac{\theta-1}{2\varepsilon}\right)}.$$

In view of assumption (5.7) on ψ, we conclude

$$c_\varepsilon = o(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{5.17}$$

We now use inequality (4.14) that reads here

$$m_\varepsilon \leqslant \frac{c_\varepsilon^2\theta^2}{2}\Lambda_\varepsilon + c_\varepsilon^2\frac{\Lambda_\varepsilon'^2}{\Lambda_\varepsilon} + |\Lambda_\varepsilon - 1|\Lambda_\varepsilon' K\frac{c_\varepsilon^{3/2}}{\varepsilon^2\sqrt{\pi}}. \tag{5.18}$$

Due to (5.17), the right-hand side of (5.18) goes to 0, as $\varepsilon \to 0$. On the other hand, m_ε given by (3.51) with $f = f_\varepsilon$ satisfies

$$\varliminf_{\varepsilon\to 0} m_\varepsilon = \int_{-\infty}^0 (-s)\psi(s)\,\mathrm{d}s \int_0^1 \chi(1, y)\,\mathrm{d}y > 0.$$

Consequently inequality (5.18) cannot hold true for sufficiently small ε and, for such ε, we have $u_\varepsilon(+\infty, y) = 1$.

Let us now consider the other case, that is

$$1 - l_2\varepsilon^{r_2} \leqslant \Lambda_\varepsilon < 1 \quad \text{for some } r_2 > 1. \tag{5.19}$$

Since $r_2 > 1/2$, computations similar to the ones above guarantee that

$$\|H_\varepsilon\|_{L^\infty(\Omega)} \leqslant \frac{1-\theta}{2} \quad \text{for } \varepsilon < \varepsilon_1.$$

Next, we again argue by contradiction and assume $u_\varepsilon^+ = \theta$. As above, we see that $c_\varepsilon = o(\varepsilon^2)$ as $\varepsilon \to 0$. We now use (4.15) instead of (4.14). We see that

$$m_\varepsilon \leqslant \frac{c_\varepsilon^2\theta^2}{2} + \frac{\Lambda_\varepsilon'^2}{\Lambda_\varepsilon}c_\varepsilon^2 + |\Lambda_\varepsilon - 1|\frac{K\Lambda_\varepsilon' c_\varepsilon}{\varepsilon^2\Lambda_\varepsilon} + (1-\Lambda_\varepsilon)\int_0^1 F_\varepsilon(1+\gamma_\varepsilon, y)\,\mathrm{d}y, \tag{5.20}$$

where we have set

$$\gamma_\varepsilon = 1 - \Lambda_\varepsilon + \Phi_\infty\left(\Lambda_\varepsilon, c_\varepsilon, \frac{K}{\varepsilon^2}\right).$$

As above, as $r_2 > \frac{1}{2}$, the first three terms in the right-hand side of (5.20) go to zero as $\varepsilon \to 0$. For the last term, since $0 \leqslant \chi \leqslant 1$, we have

$$F_\varepsilon(s, y) \leqslant \frac{1}{\varepsilon^2}\int_0^s \psi\left(\frac{t-1}{\varepsilon}\right)\mathrm{d}t = \frac{1}{\varepsilon}\int_{-1/\varepsilon}^{(s-1)/\varepsilon} \psi(u)\,\mathrm{d}u.$$

Consequently, the last term in (5.20) is majorized as follows:

$$(1-\Lambda_\varepsilon)\int_0^1 F_\varepsilon(1+\gamma_\varepsilon, y)\,\mathrm{d}y \leqslant \frac{1-\Lambda_\varepsilon}{\varepsilon}\int_{-1/\varepsilon}^{\gamma_\varepsilon/\varepsilon} \psi(u)\,\mathrm{d}u. \tag{5.21}$$

Since $r_2 > 1$, we easily see that $\gamma_\varepsilon/\varepsilon$ goes to 0 as $\varepsilon \to 0$ and we have

$$\frac{1-\Lambda_\varepsilon}{\varepsilon} \to 0.$$

Therefore quantity (5.21) goes to zero as $\varepsilon \to 0$ and as above we conclude that (5.20) cannot hold for sufficiently small ε. This completes the proof of Theorem 5.1. □

Theorem 5.1 allows us to derive the following existence result for problem (5.1)–(5.5) with f_ε.

Theorem 5.2. *Assume that* (5.7) *and* (5.8) *hold. Assume furthermore that*

$$|\Lambda_\varepsilon - 1| \leqslant l\varepsilon^\gamma \quad \textit{for } \gamma > \tfrac{5}{4}, \quad l > 0. \tag{5.22}$$

Then, for sufficiently small ε, *there exists a solution* $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ *of* (5.1)–(5.5).

Proof. Let $\varepsilon > 0$ be small enough so that problem (1.3)–(1.7) for $\tilde{f}_\varepsilon$ and $|\Lambda_\varepsilon - 1| \leqslant l\varepsilon^\gamma$ possesses a solution given by Theorem 5.1. The proof consists in deriving an estimate of the velocity involving the L^∞-bound of the function H_ε, and then by expressing this estimate in (4.13) to obtain a suitable bound for the temperature.

We first note that

$$\tilde{f}_\varepsilon(u_\varepsilon, y)v_\varepsilon \leqslant \frac{K}{\varepsilon^2}\,\|H_\varepsilon\|_{L^\infty(\Omega)} + \frac{\beta}{\varepsilon} \quad \text{for } (x, y) \in \Omega, \tag{5.23}$$

where

$$K = \psi(1) \quad \text{and} \quad \beta = \sup_{s \in (-\infty, 0)} |s|\psi(s).$$

Indeed in order to derive this estimate, we distinguish the regions $\{(x, y), u_\varepsilon(x, y) \leqslant 1 + \varepsilon\}$ and $\{(x, y), u_\varepsilon(x, y) > 1 + \varepsilon\}$. In $\{u_\varepsilon \leqslant 1 + \varepsilon\}$, we write that

$$\tilde{f}_\varepsilon(u_\varepsilon, y)v_\varepsilon = \frac{1}{\varepsilon^2}\,\psi\left(\frac{u_\varepsilon - 1}{\varepsilon}\right)\chi(u_\varepsilon, y)(H_\varepsilon + 1 - u_\varepsilon)$$

which provides

$$\begin{aligned}
\tilde{f}_\varepsilon(u_\varepsilon, y)v_\varepsilon &\leqslant \frac{\psi(1)}{\varepsilon^2}\,\|H_\varepsilon\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\,\psi\left(\frac{u_\varepsilon - 1}{\varepsilon}\right)(1 - u_\varepsilon)\mathbb{1}_{u_\varepsilon \leqslant 1} \\
&\leqslant \frac{\psi(1)}{\varepsilon^2}\,\|H_\varepsilon\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\,\sup_{s \leqslant 0}\,(-s)\psi(s)
\end{aligned}$$

that is (5.23) in $\{u_\varepsilon \leqslant 1 + \varepsilon\}$. Next, in $\{u_\varepsilon \geqslant 1 + \varepsilon\}$, we have

$$\tilde{f}_\varepsilon(u_\varepsilon, y)v_\varepsilon = \frac{1}{\varepsilon^2}\,\psi(1)\chi(u_\varepsilon, y)v_\varepsilon. \tag{5.24}$$

But

$$v_\varepsilon \leqslant \varepsilon + v_\varepsilon \leqslant u_\varepsilon + v_\varepsilon - 1 = H_\varepsilon$$

so that (5.24) yields

$$\tilde{f}_\varepsilon(u_\varepsilon, y)v_\varepsilon \leqslant \frac{K}{\varepsilon^2}\,\|H_\varepsilon\|_{L^\infty(\Omega)}$$

which implies (5.23) in that region.

Note that by combining (5.23) and (4.16), we obtain the following bound of the velocity:

$$c_\varepsilon^2 \leqslant \frac{1}{\theta}\left(\frac{K}{\varepsilon^2}\,\|H_\varepsilon\|_{L^\infty(\Omega)} + \frac{\beta}{\varepsilon}\right). \tag{5.25}$$

We now aim to derive an estimate of $\|H_\varepsilon\|_{L^\infty}$ thanks to (4.13). We first majorize the logarithmic term in (4.13) thanks to (5.11). We obtain the existence of some constant still denoted by k such that, for sufficiently small ε, we have

$$\log\left(1 + (1+\Lambda_\varepsilon^{-1})c_\varepsilon + \sqrt{\frac{K\Lambda_\varepsilon}{\varepsilon^2}}\right) \leqslant k\,\log\frac{1}{\varepsilon}.$$

Therefore (4.13) provides

$$\|H_\varepsilon\|_{L^\infty(\Omega)} \leqslant |\Lambda_\varepsilon - 1| + k|\Lambda_\varepsilon - 1|\sqrt{c_\varepsilon}\sqrt{\log\frac{1}{\varepsilon}}. \tag{5.26}$$

Next, we make use of (5.25) and infer from (5.26) that, for small enough ε,

$$\|H_\varepsilon\|_{L^\infty(\Omega)} \leqslant |\Lambda_\varepsilon - 1| + k|\Lambda_\varepsilon - 1|\sqrt{\log\frac{1}{\varepsilon}}\left[\frac{K}{\theta\varepsilon^2}\,\|H_\varepsilon\|_{L^\infty(\Omega)} + \frac{\beta}{\theta\varepsilon}\right]^{1/4}. \tag{5.27}$$

We can now conclude the proof of Theorem 5.2. Inequality (5.27) together with assumption (5.22) guarantees the existence of $\varepsilon_0 > 0$ such that

$$\|H_\varepsilon\|_{L^\infty(\Omega)} \leqslant \varepsilon \quad \forall \varepsilon \leqslant \varepsilon_0.$$

Since $u_\varepsilon = H_\varepsilon + 1 - v_\varepsilon$, we obtain the following upper bound on u_ε:

$$\forall \varepsilon \leqslant \varepsilon_0, \quad \|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant 1 + \varepsilon.$$

Now, recalling definition (5.9) of $\tilde{f}_\varepsilon$, we see that

$$\tilde{f}_\varepsilon(u_\varepsilon(x,y),y) \equiv f_\varepsilon(u_\varepsilon(x,y),y)$$

so that $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ is a solution of (5.1)–(5.5). The proof of Theorem 5.2 is complete. □

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