A center manifold for second order semi-linear differential equations on the real line and applications to the existence of wave trains for the Gurtin-McCamy equation

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Abstract

This work is mainly motivated by the study of periodic wave train solutions for the so-called Gurtin-McCamy equation. To that aim we construct a smooth center manifold for a rather general class of abstract second order semi-linear differential equations involving non-densely defined operators. We revisit results on commutative sums of linear operators using the integrated semigroup theory. These results are used to reformulate the notion of the weak solutions of the problem. We also derive a suitable fixed point formulation for the graph of the local center manifold that allows us to conclude to the existence and smoothness of such a local invariant manifold. Then we derive a Hopf bifurcation theorem for second order semi-linear equations. This result is applied to study the existence of periodic wave trains for the Gurtin-McCamy problem, that is for a class of non-local age structured equations with diffusion.

Key-words: Center manifold; second order semi-linear equations; integrated semigroups; Hopf bifurcation; periodic wave trains; Gurtin-McCamy equation.

MSC: 37L10, 35J61, 35C07, 34C23, 47D62

Contents

1 Introduction

In this work we are concerned with the existence of periodic wave train solutions for the following age-structured equation with diffusion

$$\begin{cases} (\partial_t + \partial_a - \Delta_z)\, u(t,a,z) = -\mu u(t,a,z),\ t \in \mathbb{R},\ a > 0 \text{ and } z \in \mathbb{R}^N, \\ u(t,0,z) = f\left(\int_0^\infty \beta(a) u(t,a,z) \mathrm{d}a\right),\ t \in \mathbb{R},\ z \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here Δ_z denotes the Laplace operator for the variable $z \in \mathbb{R}^N$, for some given integer $N \geq 1$. This equation is refereed as the *Gurtin-McCamy equation* and was introduced in its nonlinear form in [35, 36] to study the interactions between age and spatial motion in the spatio-temporal evolution of biological populations. Such questions were already addressed in the early 50's by Skellman in [70] using a related linear model.

During the last decades there has been considerable interest in this problem. Indeed various biological applications, included population invasions, can be handled by using such models. The mathematical analysis of this problem and related equations has given rise to a huge literature. In addition to the works quoted above, and without being exhaustive, one can mention the works of Chipot [12], Di Blasio [23], Kubo and Langlais [47], Langlais [48] and Walker [82]. One also refers to the following monographs and book chapters [1, 7, 19, 43, 85] and the references cited therein.

Similarly to delay differential equations, in the spatially homogeneous case and under suitable circumstances, Problem (1.1) may undergo Hopf bifurcation leading to the existence of – spatially homogeneous – periodic orbits. Such results have been obtained by Magal and Ruan in [54] (see also the references therein). In this aforementioned work the authors construct a smooth center manifold for a rather general class of abstract Cauchy problems that allows them to obtain finite dimensional ODE reduction of the problem and to derive Hopf bifurcation results.

However, as far as spatially heterogeneous solutions are concerned, the interplay between temporal oscillations and spatial motion remains largely open. This is more particularly true when the spatial domain is unbounded as the situation we consider in this work. Indeed, posed in the whole space, (1.1) may admit propagating pattern solutions such as travelling wave solutions. We refer for instance to the work of Fang and Zhao in [31] where the authors derived existence results for travelling wave solutions for monostable integral equations. Under suitable assumptions on the nonlinear function f these results can be applied to (1.1) to ensure the existence of travelling wave solutions.

In the slightly different context of delay differential equations, the coupling between temporal oscillations and spatial diffusion may lead to the existence of wave train solutions. The existence of such solutions have been proved for some specific examples of reaction-diffusion with time delay. For instance, by coupling the results of Hasik and Trofimcuk [38] and Ducrot and Nadin [27] proves that periodic wave train solutions do exist for the so-called Hutchinson equation (if the delay is large enough). In that case, the existence of periodic wave trains does not follow from a local bifurcation analysis but they arise as the limit behaviour of travelling wave solutions of invasion. More specifically these propagating periodic patterns describe the state of the population behind the front of invasion. The proof is based on a phase plane analysis that makes use of the specific structure of these second order delay differential equations. Such an oscillating behaviour was already observed by So, Wu and Zou [71] (see also the references therein) for other monsotable reaction-diffusion equations with time delay. We also refer to Duehring and Huang [29] who proved the existence of periodic wave train solutions with large wave speed for some non-local reaction-diffusion equations by using singular perturbation analysis.

In the context of the Gurtin-McCamy equation, namely (1.1), we expect that the temporal oscillations generated by the age-structured part would interact with the spatial diffusion to lead to the existence of periodic wave train solutions. Here recall that a couple $(\gamma, U \equiv U(x,a))$ is said to be a periodic wave train profile with speed $\gamma \in \mathbb{R}$ if the function U is periodic with respect to its variable $x \in \mathbb{R}$, namely there exists a period $T > 0$ such that $U(T + \cdot, \cdot) = U(\cdot, \cdot)$, and such that for each direction $e \in \mathbb{S}^{N-1}$ the function $u(t,a,z) := U(z \cdot e + \gamma t, a)$ is an entire solution of (1.1). In other words, the profile (γ, U) is a periodic – in x – solution of the following second order problem

$$\begin{cases} \partial_x^2 U(x,a) - \gamma \partial_x U(x,a) - \partial_a U(x,a) - \mu U(x,a) = 0, \quad x \in \mathbb{R},\ a > 0, \\ U(x,0) = f\left(\displaystyle\int_0^\infty \beta(a) U(x,a) \mathrm{d}a \right). \end{cases} \tag{1.2}$$

In this work we shall prove the existence of wave train solutions for (1.1) (or periodic profiles for (1.2)) by developing bifurcation techniques for the above problem. As it will be discussed in Section 8 below, Problem (1.2) can be rewritten as a second order abstract semilinear problem of the form

$$\varepsilon \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + F\left(u(x)\right) = 0, \quad \text{for } x \in \mathbb{R}, \tag{1.3}$$

where $\varepsilon \geq 0$ and $\gamma \in \mathbb{R}$ are two given constants, $A : D(A) \subset X \to X$ is a weak Hille-Yosida linear operator (see Assumption 2.1 below) acting on a real Banach space $(X, \|.\|)$ while $F : \overline{D(A)} \to X$ is a given smooth nonlinear map. Because of the (weak) Hille-Yosida assumption for the linear operator A, Problem (1.3) is not hyperbolic but shares similarities with vector valued elliptic equations. As mentioned above, Problem (1.3) contains (1.2) as a special case (see Section 8 for more details). In that case the corresponding operator A turns out to be a non-densely defined Hille-Yosida linear operator. However, Problem (1.3) consists in a more general class of equations. As an other example, one may also think at semi-linear elliptic equation on infinite straight cylinder of the form

$$\varepsilon \partial_x^2 u(x,y) - \gamma \partial_x u(x,y) + \Delta_y u(x,y) + F\left(u(x,y)\right) = 0 \text{ for } (x,y) \in \mathbb{R} \times \Omega,$$

with $\varepsilon > 0$ and wherein Ω denotes a bounded smooth domain. When supplemented by appropriate boundary conditions on $\mathbb{R} \times \partial\Omega$, such as homogeneous Dirichlet, Neumann or Robin conditions, this equation enters the general framework of (1.3). To illustrate this, assume for instance that it is equipped with the homogeneous Neumann boundary conditions. Then, when posed in the space of continuous functions on $\overline{\Omega}$ for instance, this elliptic problem becomes a special case of (1.3) with the densely defined linear operator $A : D(A) \subset C(\overline{\Omega}) \to C(\overline{\Omega})$ given by (see Stewart [72, 73])

$$A\varphi := \Delta_y \varphi, \ \forall \varphi \in D(A),$$

with the domain

$$D(A) := \left\{ \varphi \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega) : \Delta\varphi \in C(\overline{\Omega}) \text{ and } \vec{n} \cdot \nabla\varphi = 0 \text{ on } \partial\Omega \right\}.$$

Here $\vec{n}(y)$ denotes the outward unit normal vector to the boundary at $y \in \partial\Omega$.

As already mentioned, in this work, we shall develop bifurcation methods to study the existence of periodic solutions for Problem (1.2). We shall more generally focus on the class of second order equations of the form (1.3). Our aim is to construct a smooth center manifold for Problem (1.3) and use it to prove the existence of periodic solutions emanating from Hopf bifurcation, before coming back to the special case of Problem (1.2).

The first main question addressed is how to solve an abstract second order equation of the form

$$\frac{d^2 u(x)}{dx^2} - \gamma \frac{du(x)}{dx} + Au(x) = f(x), \text{ with } x \in \mathbb{R}.$$

The usual approach to solve such a problem is to re-write the system as a first order evolution equation involving the fractional power A^θ of A. We refer to [8, 30, 63, 65] and the references therein for more results on this subject. This kind of technic requires A to be sectorial. Here we will not assume A to be sectorial. Instead we will use some ideas developed by Da Prato and Grisvard [21] and Thieme [75] to study the commutative sum of linear operators. When A is non-densely defined, we will combine these ideas with integrated semigroup theory to extend the existing results.

Now by using the state space decomposition associated to the spectral properties of the weak Hille-Yosida operator A (see [54] whenever A is non-densely defined), we can re-write the problem into a more tractable formulation that consists in an ordinary differential equation coupled with a Banach valued advanced and retarded difference equation of the form

$$\begin{cases} \dfrac{\mathrm{d}u(x)}{\mathrm{d}x} = F_1(u(x), v(x)) \\ v(x) = F_2\left(u(x+.), v(x+.)\right) \end{cases}, \ x \in \mathbb{R},$$

wherein F_2 is a Banach valued integral operator with some nice properties. This re-formulation will be finally used to develop Lyapunov-Perron like arguments to complete the construction of a smooth local center manifold. Finally a finite dimensional reduction will be performed to derive a Hopf bifurcation theorem for Problem (1.3).

Let us furthermore mention that the existence of a smooth center manifold for Problem (1.3) has been obtained in [54] for the special case $\varepsilon = 0$. Here our approach allows us to cover both cases $\varepsilon = 0$ and $\varepsilon > 0$. And as a consequence, we retrieve some of the results obtained in [54] for first order problems. We expect that the methodology developed in this work may be extended to study higher order differential equations.

The classical center manifold theory was first established by Pliss [64] and Kelley [45] and was further developed by Carr [9], Sijbrand [69], Vanderbauwhede [79], etc. The finite dimensional center manifold theorem roughly states the dynamical behaviour in the neighbourhood of a nonhyperbolic equilibrium reduces to the one of an ODE on the lower dimensional invariant center manifold. This center manifold around such a nonhyperbolic equilibrium is tangent to the generalized eigenspace associated to the corresponding eigenvalues with zero real parts. The center manifold theory has significant applications in studying problems in dynamical systems, such as bifurcation, stability, perturbation, and so on. It has in particular been used to study various applied problems arising for instance in biology, engineering, physics.

The classical center manifold theory around a nonhyperbolic equilibrium has been extended for various types of invariant sets. One may refer to Chow and Lu [16] for center manifolds for an invariant torus, to Fenichel [33] for invariant set of equilibria, to Homburg [42] and Sandstede [66] for homoclinic orbits, to Chow and Yi [17] for skew-product flows, to Hirsch *et al.* [41] for pieces of trajectories. We also refer to Chow *et al.* [13, 14] where center manifolds for smooth invariant manifolds and compact invariant sets are considered.

Recently, great attention has been paid to the study of center manifolds in infinite dimensional systems. The center manifold theory has been developed for various infinite dimensional systems such as partial differential equations (Bates and Jones [5], Da Prato and Lunardi [22], Henry [40]), semiflows in Banach spaces (Bates *et al.* [6], Chow and Lu [15], Gallay [34], Scarpellini [67], Vanderbauwhede [78], Vanderbauwhede and van Gils [80]), delay differential equations (Hale [37], Diekmann and van Gils [24, 25], Diekmann *et al.* [26], Walther [83]), infinite dimensional nonautonomous differential equations (Mielke [57, 58], Chicone and Latushkin [11]), partial functional differential equations (Lin *et al.* [49], Faria *et al.* [32], Minh and Wu [60], Wu [86]), etc. Several additional difficulties have to be overcome when dealing with infinite dimensional systems. Indeed these problems usually do not have some of the nice properties the finite dimensional systems have. For example, the initial value problem may be ill posed, the solutions may not be extended backward in time, the solutions may not be smooth enough, the domain of operators may not be dense in the state space, etc. Here let us emphasis that the center manifold reduction of the infinite dimensional systems plays a very important role in their mathematical analysis since it allows us to study ODE on the finite dimensional center manifolds. Vanderbauwhede and Iooss in [81] described some minimal conditions which allow to generalize the approach of Vanderbauwhede [79] to the case of infinite dimensional systems.

The case of semi-linear second order differential equations seems to be scarcely studied. Construction of suitable manifolds and finite dimensional reduction have been obtained in some particular cases, including a large class of elliptic equations. We refer to the work Kirchgässner [46] based on Green functions. We also refer to Mielke [57], to Chapter 4 in [20] and the references therein. Note that such a method has allowed these authors to construct bounded solutions for some elliptic problems, including wave train solutions and spiral waves (see also [68]). We also refer to the notion of essential manifold, that roughly speaking corresponds to the set of all bounded orbits, that has been developed by Mielke [59] for elliptic equations posed on infinite cylinders. However such manifolds are in general non-smooth. This does not allow the application of Crandall-Rabinowitz Hopf bifurcation theorem (see [18]).

Here let us observe that aforementioned works do not apply to perform a finite dimensional reduction for Problem (1.1). In that spirit we also refer the reader to the work of Chen, Matano and Véron in [10] and to the book chapter of Matano [56]. In these works, the authors observed that the solutions of some semi-linear elliptic equations can be reformulated as the entire solutions of a suitable semiflow coming from the resolution of a pseudo-differential equation of parabolic type and involving suitable fractional powers of the Laplace operator. The study of the solutions of the elliptic problem is then handled using infinite dimensional dynamical system tools, such as invariant manifolds theory. Here again let us observe that such a factorisation argument does not apply to (1.3) when the linear operator A is neither sectorial nor almost sectorial (see [28]) as in the case of Problem (1.1).

This paper is organized as follows: Section 2 is devoted to the statements of the main results obtained in this work. Section 3 recalls some known notions and results on integrated semigroups. Section 4 is concerned with the theory of commutative sums of operators in weighted spaces while section 5 provides applications of these theoretical results to first and second order Banach valued differential operators. Section 6 deals with the construction and smoothness of a local center manifold for (1.3) around some nonhyperbolic equilibrium point. Finally Section 7 provides a Hopf bifurcation theorem that is followed by Section 8, which is concerned with an application of these results to study the existence of periodic wave trains for the Gurtin-McCamy equation, namely (1.1).

2 Main results

The goal of this article is to obtain a center manifold theorem for the (exponentially bounded) weak solutions of (1.3). In order to deal with (1.2), the linear operator A is not assumed to be densely defined, this means that, in general, one has

$$\overline{D(A)} \neq X.$$

Throughout this article, the linear operator A will satisfy the following set of assumptions.

Assumption 2.1 (Weak Hille-Yosida property) *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|.\|)$. We assume that there exist two constants $\omega_A \in \mathbb{R}$ and $M_A \geq 1$ such that the following properties hold true:*

(a) $(\omega_A, +\infty) \subset \rho(A)$, *where $\rho(A)$ is the resolvent set of A;*

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \forall x \in X$;

(c) *For each $\lambda > \omega_A$ and each $n \geq 1$ the following resolvent estimate*

$$\left\| (\lambda I - A)^{-n} \right\|_{\mathcal{L}\left(\overline{D(A)}\right)} \leq \frac{M_A}{(\lambda - \omega_A)^n}.$$

We will need further assumptions on the linear operator A that are related to the first order abstract Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), \text{ for } t \in [0, \tau], \ u(0) = 0, \tag{2.1}$$

where $f : [0, \tau] \to X$ is a continuous function.

Before going to our assumptions, we recall the following notion of a weak (or mild) solution for Problem (2.1).

Definition 2.2 *We will say that a function $u \in C\left([0, \tau], \overline{D(A)}\right)$ such that $u(0) = 0$ is **a weak solution** of the Cauchy problem (2.1) if for each $\lambda \in \rho(A)$, the resolvent set of A, one has*

$$(\lambda I - A)^{-1} u(.) \in C^1([0, \tau], X),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\lambda I - A)^{-1} u(t)\right] = -u(t) + (\lambda I - A)^{-1}\left[f(t) + \lambda u(t)\right].$$

Using the above definition we will assume that (2.1) satisfies the following properties.

Assumption 2.3 (First order solvability) *Let $\tau > 0$ be fixed. We assume that there exists a map $\delta : [0, \tau] \to [0, +\infty)$ such that*

$$\lim_{t \to 0} \delta(t) = 0,$$

and such that for each continuous function $f : [0, \tau] \to X$, there exists $u_f \in C\left([0, \tau], \overline{D(A)}\right)$ a weak solution of the Cauchy problem (2.1) satisfying the following estimate

$$\|u_f(t)\| \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \forall t \in [0, \tau].$$

We now turn to Problem (1.3) and define a notion of a weak solution for such a second order semi-linear equation. To that aim let us first introduce for each interval $I \subset \mathbb{R}$, each Banach space $(Y, \|.\|)$ and each weight $\eta \in \mathbb{R}$, the weighted space $BC^0_\eta(I, Y)$ defined by

$$BC^0_\eta(I, Y) = \left\{\varphi \in C(I, Y) : \sup_{x \in I} e^{-\eta|x|}\|\varphi(x)\| < \infty\right\}. \tag{2.2}$$

It becomes a Banach space when endowed with the norm

$$\|\varphi\|_{0,\eta} := \sup_{x \in I} e^{-\eta|x|}\|\varphi(x)\|.$$

We also define for each integer $k \geq 1$ the space $BC^k_\eta(I, Y)$ by

$$BC^k_\eta(I, Y) = \left\{\varphi \in C^k(I, Y) : \frac{\mathrm{d}^l\varphi}{\mathrm{d}x^l} \in BC^0_\eta(I, Y),\ l = 0, .., k\right\}. \tag{2.3}$$

These spaces is a Banach space endowed with the usual weighted uniform norm

$$\|\varphi\|_{k,\eta} = \sum_{m=0}^{k} \left\|\frac{\mathrm{d}^m\varphi}{\mathrm{d}x^m}\right\|_{0,\eta}.$$

Using these notations, we propose the following notions of solutions for (1.3).

Definition 2.4 (Weak and classical solution of (1.3)) *Let $\eta > 0$ be given. We define different types of solutions for (1.3).*

(i) *We will say that* $u \in BC_\eta^0\left(\mathbb{R}, \overline{D(A)}\right)$ *is* ***a weak solution*** *of (1.3) if for each* $\lambda \in \rho(A)$, *we have*

$$(\lambda I - A)^{-1} u \in BC_\eta^2\left(\mathbb{R}, \overline{D(A)}\right)$$

and

$$u = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}}{\mathrm{d}x}\right)\left[(\lambda I - A)^{-1} u\right] + (\lambda I - A)^{-1}\left[F(u) + \lambda u\right].$$

(ii) *We will say that* $u \in BC_\eta^0\left(\mathbb{R}, \overline{D(A)}\right)$ *is* ***a classical solution*** *of (1.3) if*

$$u \in BC_\eta^2\left(\mathbb{R}, \overline{D(A)}\right)$$
$$u(x) \in D(A),\ \forall x \in \mathbb{R} \text{ and } x \mapsto Au(x) \in BC_\eta^0(\mathbb{R}, X)$$

and

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}}{\mathrm{d}x}\right) u(x) + Au(x) + F(u(x)), \forall x \in \mathbb{R}.$$

Before stating our main center manifold theorem and related bifurcation results one need to introduce more notation. Let us introduce $X_0 := \overline{D(A)}$ and let $A_0 : D(A_0) \subset X_0 \to X_0$ denotes the part of A in $\overline{D(A)}$, namely

$$D(A_0) := \{x \in D(A) : Ax \in X_0\} \text{ and } A_0 x = Ax, \forall x \in D(A_0).$$

Note that Assumption 2.1 implies that A_0 is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X_0, denoted by $\{T_{A_0}(t)\}_{t \geq 0}$, and we will assume that

Assumption 2.5 *We assume that the essential growth rate of* A_0

$$\omega_{0,ess}(A_0) = \lim_{t \to +\infty} \frac{\ln\left(\|T_{A_0}(t)\|_{ess}\right)}{t} < 0.$$

Remark 2.6 *In the above assumption,* $\|L\|_{ess}$ *denotes the essential norm of a bounded linear operator* L *on the Banach space* X_0. *Recall that it is defined by*

$$\|L\|_{ess} = \kappa\left(L\left(B_{X_0}(0,1)\right)\right),$$

wherein $B_{X_0}(0,1) = \{x \in X_0 : \|x\|_X \leq 1\}$ *is the ball of radius* 1 *in* X_0 *and while* $\kappa(B)$ *denotes the Kuratowski's measure of non-compactness of* B, *a bounded subset of* X_0, *defined by*

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}.$$

Next we set

$$\sigma_{cu}(A) := \{\lambda \in \sigma(A) : \operatorname{Re}(\lambda) \geq 0\},$$

and

$$\sigma_s(A) := \{\lambda \in \sigma(A) : \operatorname{Re}(\lambda) < 0\}.$$

Now, recalling Assumption 2.1, 2.3 and the above Assumption 2.5 and using the results of Magal and Ruan [54, Proposition 3.5 p.13], we obtain that there exists a uniquely determined finite rank bounded linear projector $\Pi_{cu} \in \mathcal{L}(X)$ satisfying the following set of properties:

(a) $\Pi_{cu}(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi_{cu}, \forall \lambda \in \rho(A)$;

(b) $A_{cu} \in \mathcal{L}(X_{cu})$ the part of A in X_{cu} satisfies $\sigma(A_{cu}) = \sigma_{cu}(A)$.

(c) $A_s \in \mathcal{L}(X_s)$ the part of A in X_s satisfies $\sigma(A_s) = \sigma_s(A)$.

Therefore this leads us to the following splitting of the state spaces X_0 and X

$$X_0 = X_{cu} \oplus X_{0s} \text{ and } X = X_{cu} \oplus X_s,$$

wherein we have set $\Pi_s := I - \Pi_{cu}$ and

$$X_{0s} := \Pi_s(X_0) \subset X_0 \text{ and } X_s := \Pi_s(X).$$

Next let us denote by $\mathcal{P}$ the parabola of the complex plane

$$\mathcal{P} = \left\{\omega^2 + \gamma\omega i;\ \omega \in \mathbb{R}\right\},$$

and let us set

$$\sigma_{\mathcal{P}}(A) := \sigma(A) \cap \mathcal{P} = \sigma_{cu}(A) \cap \mathcal{P}.$$

We denote by $\Pi_{\mathcal{P}} \in \mathcal{L}(X)$ the projector on the generalized eigenspace of A associated to the set of eigenvalues $\sigma_{\mathcal{P}}$. Set

$$X_{\mathcal{P}} := \Pi_{\mathcal{P}}(X_{cu}) = \Pi_{\mathcal{P}}(X), \text{ and } X_{\mathcal{Q}} := (I - P_{\mathcal{P}})(X_{cu}), \tag{2.4}$$

so that

$$X_{cu} = X_{\mathcal{P}} \oplus X_{\mathcal{Q}}.$$

Finally define $A_{\mathcal{P}} : X_{\mathcal{P}} \to X_{\mathcal{P}}$ the part of A in $X_{\mathcal{P}}$ by

$$A_{\mathcal{P}} = A \text{ on } X_{\mathcal{P}}.$$

As usual, to speak about a center manifold, one needs to assume that the "center" space is not empty. In our context of second order differential equations this assumption reads as follows.

Assumption 2.7 *We assume that*

$$\sigma_{\mathcal{P}}(A) \neq \varnothing.$$

Remark 2.8 *Note that up to change the parameter γ in (2.4), the above assumption is satisfied as soon as*

$$\sigma_{cu}(A) \bigcap [\{0\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0 \text{ and } \operatorname{Im}(\lambda) \neq 0\}] \neq \emptyset.$$

Indeed let $\lambda^* \in \sigma_{cu}(A)$ be given such that either

$$\lambda^* = 0 \text{ or, } \operatorname{Re}(\lambda^*) \neq 0 \text{ and } \operatorname{Im}(\lambda^*) \neq 0.$$

Then one can find $\gamma > 0$ such that $\sigma_{\mathcal{P}}(A) \neq \varnothing$. Indeed, since A is a linear operator on a real Banach space, one deduces that

$$\lambda \in \sigma_{cu}(A) \Leftrightarrow \overline{\lambda} \in \sigma_{cu}(A).$$

If $\lambda^* = 0$ we fix $\omega = 0$ and $\lambda^* \in \sigma_{\mathcal{P}}(A)$. Otherwise we can assume that

$$\operatorname{Re}(\lambda^*) > 0 \text{ and } \operatorname{Im}(\lambda^*) > 0,$$

and setting

$$\omega = \sqrt{\operatorname{Re}(\lambda^*)} \text{ and } \gamma = \frac{\operatorname{Im}(\lambda^*)}{\omega},$$

one obtains that $\sigma_{\mathcal{P}}(A) \neq \emptyset$.

Keeping in mind the above notation, we are now able to state our first main result. It is concerned with the existence of a global Lipschitz continuous center manifold for Problem (1.3).

Theorem 2.9 (Existence of a global center manifold) *Let Assumptions 2.1, 2.3, 2.5 and 2.7 be satisfied. Let $\eta > 0$ be given small enough. Assume that $F : X_0 \to X$ is a Lipschitz continuous function such that $F(0) = 0$. Then, there exists some constant $\kappa > 0$ small enough such that if*

$$\|F\|_{\operatorname{Lip}(X_0, X)} \leq \kappa,$$

then there exists a unique map $\Psi \in \operatorname{Lip}(X_{\mathcal{P}} \times X_{\mathcal{P}}, X_{\mathcal{Q}} \oplus X_{0s})$ with

$$\Psi(0) = 0,$$

satisfying the following properties:

(i) *If $v_c : \mathbb{R} \to X_{\mathcal{P}}$ is an entire solution of the reduced ordinary differential equation, for all $x \in \mathbb{R}$,*

$$v_c''(x) - \gamma v_c'(x) + A_{\mathcal{P}} v_c(x) + \Pi_{\mathcal{P}} F [v_c(x) + \Psi(v_c(x), v_c'(x))] = 0, \quad (2.5)$$

then $v_c \in BC_\eta^2(\mathbb{R}, X_{\mathcal{P}})$ and the function $u : \mathbb{R} \to X_0$ defined by

$$u(x) := v_c(x) + \Psi(v_c(x), v_c'(x)),$$

is a weak solution of (1.3) on $\mathbb{R}$.

(ii) *If* $u \in BC_\eta(\mathbb{R}, X_0)$ *is a weak solution of (1.3), then the function* $v_c : \mathbb{R} \to X_{\mathcal{P}}$ *defined by*

$$v_c = \Pi_{\mathcal{P}} u,$$

is a solution of the reduced equation (2.5).

The idea of the proof of this result is based on a suitable splitting of problem (1.3). Formally, when u is a solution of (1.3) then we set $u_s(x) = \Pi_s u(x)$ and $u_{cu}(x) = \Pi_{cu} u(x)$ and, projecting (1.3) respectively on X_s and X_{cu} yields

$$\begin{cases} u_s''(x) - \gamma u_s'(x) + A_s u_s(x) + \Pi_s F\left(u_s(x) + u_{cu}(x)\right) = 0, \ x \in \mathbb{R}, \\ u_{cu}''(x) - \gamma u_{cu}'(x) + A_{cu} u_{cu}(x) + \Pi_{cu} F\left(u_s(x) + u_{cu}(x)\right) = 0, \ x \in \mathbb{R}. \end{cases}$$

Still from a formal point of view, we invert the operator $\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}}{\mathrm{d}x} + A_s\right)$ and reformulate the solutions of the above system of equations as the following finite dimensional ordinary differential equation coupled with a neutral equation, that is, for $x \in \mathbb{R}$

$$\begin{cases} u_s(x) = -\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}}{\mathrm{d}x} + A_s\right)^{-1} \left[\Pi_s F\left(u_s(.) + u_{cu}(.)\right)\right](x), \\ u_{cu}''(x) - \gamma u_{cu}'(x) + A_{cu} u_{cu}(x) + \Pi_{cu} F\left(u_s(x) + u_{cu}(x)\right) = 0. \end{cases} \tag{2.6}$$

Then we shall use this formulation to derivable a suitable fixed point problem for the graph of the center manifold to finally conclude to the proof of the above theorem. The invertibility of the operator $\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}}{\mathrm{d}x} + A_s\right)$ will be fully justified in Proposition 5.8 by using suitable weighted spaces of functions. Basically we shall give a precise sense of the above formal transformation by making use of suitable weighted spaces, commutative sums of linear operators and the definition of weak solutions.

Thus, based on the above global center manifold result, by using a truncation argument for the nonlinear function F, we obtain the following local theorem.

Theorem 2.10 (Existence and smoothness of a local center manifold) *Let Assumptions 2.1, 2.3, 2.5 and 2.7 be satisfied. Let* $\eta > 0$ *be small enough. Assume that* $F : X_0 \to X$ *is a k-time continuously differentiable map for some* $k \geq 1$ *such that*

$$F(0) = 0 \text{ and } DF(0) = 0.$$

Then there exist a map $\Psi \in C^k\left(X_{\mathcal{P}} \times X_{\mathcal{P}}, X_{\mathcal{Q}} \oplus X_{0s}\right)$ *and* Ω*, a bounded neighbourhood of* 0 *in* X_0*, such that*

$$\Psi(0) = 0 \text{ and } D\Psi(0) = 0,$$

and such that the following properties hold:

(i) *If* $v_c : \mathbb{R} \to X_{\mathcal{P}}$ *is a solution of the reduced equation*

$$0 = v_c''(x) - \gamma v_c'(x) + A_{\mathcal{P}} v_c(x) + \Pi_{\mathcal{P}} F(v_c(x) + \Psi\left(v_c(x), v_c'(x)\right)), \tag{2.7}$$

then

$$v_c \in BC^2_\eta \left(\mathbb{R}, X_{\mathcal{P}}\right).$$

Moreover if

$$v_c(x) + \Psi\left(v_c(x), v_c'(x)\right) \in \Omega, \forall x \in \mathbb{R},$$

then

$$u(x) = v_c(x) + \Psi\left(v_c(x), v_c'(x)\right)$$

is a classical solution of (1.3) on $\mathbb{R}$.

(ii) *If* $u \in BC^2_\eta\left(\mathbb{R}, X_0\right)$ *is a weak solution of (1.3) such that*

$$u(x) \in \Omega, \forall x \in \mathbb{R},$$

then

$$v_c = \Pi_{\mathcal{P}} u$$

is a solution of the reduced equation (2.7).

We now consider system (1.3) depending on some parameter $\mu \in \mathbb{R}$. To be more precise we consider the following second order equation

$$0 = \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + F(\mu, u(x)), \text{ for } x \in \mathbb{R}, \tag{2.8}$$

where

$$u \in BC^0_\eta\left(\mathbb{R}, \overline{D(A)}\right) \text{ and } \mu \in \mathbb{R}.$$

Here $F : \mathbb{R} \times \overline{D(A)} \to X$ is a k-time continuously differentiable map for some $k \geq 1$. Next our Hopf bifurcation theorem reads as follows.

Theorem 2.11 (Hopf bifurcation theorem) *Let Assumptions 2.1, 2.3 and 2.5 be satisfied. Assume that* $F : \mathbb{R} \times \overline{D(A)} \to X$ *is of the class* C^k *for some* $k \geq 4$ *and satisfies*

(a) $F(\mu, 0) = 0$ *for all* $\mu \in \mathbb{R}$ *and* $\partial_u F(0, 0) = 0$.

(b) *For each* μ *in some neighbourhood of* $\mu = 0$, *there exists a pair of conjugated simple eigenvalues of* $\left(A + \partial_u F(\mu, 0)\right)_{cu}$, *denoted by* $\lambda^{(1)}(\mu)$ *and* $\overline{\lambda^{(1)}(\mu)}$, *such that*

$$\lambda^{(1)}(0) = \omega_0^2 + i\gamma\omega_0 \text{ for some } \omega_0 > 0,$$

and

$$\sigma\left(A_0\right) \cap \mathcal{P} = \left\{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\right\} \text{ with } \mathcal{P} = \{\xi^2 + i\gamma\xi;\ \xi \in \mathbb{R}\}. \tag{2.9}$$

We furthermore assume that the map $\mu \mapsto \lambda^{(1)}(\mu)$ *is continuously differentiable and satisfies*

$$\operatorname{Re}\left[\frac{1}{\gamma - 2i\omega_0} \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu}\right] \neq 0. \tag{2.10}$$

Then, there exist a constant $\eta^* > 0$ and two C^{k-1} maps, $\eta \mapsto \mu(\eta)$ from $(0,\eta^*)$ into $\mathbb{R}$ and $\eta \mapsto \omega(\eta)$ from $(0,\eta^*)$ into $\mathbb{R}$, such that for each $\eta \in (0,\eta^*)$ there exists a non-trivial $\omega(\eta)-$periodic function $u_\eta \in C(\mathbb{R}, X_0)$, which is a weak solution of (2.8) with the parameter value $\mu = \mu(\eta)$. Moreover for $\eta = 0$

$$\mu(0) = 0 \text{ and } u_0 = 0. \tag{2.11}$$

Remark 2.12 *As a consequence of the two-dimensional reduction used to prove the above Hopf bifurcation theorem, another type of entire solution do exist. More precisely, for each parameter $\mu(\eta)$ we can find either 1) a solution connecting the equilibrium 0 at $x = -\infty$ to the periodic orbit u_η at $x = +\infty$ or 2) a solution connecting the periodic orbit u_η to the equilibrium 0 respectively at $x = -\infty$ and $x = +\infty$. The distinction between these two situations relies on the stability analysis of the stationary equilibrium, that is by distinguishing between sub and super critical Hopf bifurcation. One can also observe that such solutions will approach – forward or backward – from 0 by spiralling around the trivial equilibrium. To decide which case occurs, one would need to extend the normal form theory recently developed by Liu, Magal and Ruan in [51].*

We now deal with the persistence of non-degenerate Hopf bifurcation for Problem (2.8) for large speed $\gamma \gg 1$. Observe that if u is a solution of (2.8) for some $\gamma \neq 0$ then the function $v(x) := u(\gamma x)$ satisfies the problem

$$0 = \frac{1}{\gamma^2}\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}v(x)}{\mathrm{d}x} + Av(x) + F(\mu, v(x)), \text{ for } x \in \mathbb{R}. \tag{2.12}$$

For $|\gamma| \gg 1$ large enough, the above equation becomes a singular perturbation of the following first order evolution equation

$$\frac{\mathrm{d}v(x)}{\mathrm{d}x} = Av(x) + F(\mu, v(x)). \tag{2.13}$$

Our next result will show that non-degenerate Hopf bifurcation for (2.13) persists for (2.12) when γ is large enough. Our detailed result reads as follows.

Theorem 2.13 (Persistence of Hopf bifurcation) *Let Assumption 2.1, 2.3 and 2.5 be satisfied. Assume that $F : \mathbb{R} \times \overline{D(A)} \to X$ is of the class C^k for some $k \geq 4$ and satisfies*

(a) *$F(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$ and $\partial_u F(0,0) = 0$.*

(b) *for each μ in some neighbourhood of $\mu = 0$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_u F(\mu,0))_0$, denoted by $\lambda^{(1)}(\mu)$ and $\overline{\lambda^{(1)}(\mu)}$, such that*

$$\lambda^{(1)}(0) = i\omega_0 \text{ for some } \omega_0 > 0,$$

and

$$\sigma(A_0) \cap i\mathbb{R} = \left\{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\right\}.$$

We furthermore assume that the map $\mu \to \lambda^{(1)}(\mu)$ is continuously differentiable and satisfies

$$\mathrm{Re}\,\frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} > 0 \text{ respectively} < 0.$$

Then, there exist $\delta > 0$ and a map $\gamma \equiv \gamma(\mu)$ defined from $(0, \delta)$ (resp. on $(-\delta, 0)$) into $(0, \infty)$ such that for each $\mu_0 \in (0, \delta)$ (respectively for each $\mu_0 \in (-\delta, 0)$, if we fix $\gamma = \gamma(\mu_0)$, then the system (2.12) has a Hopf bifurcation around μ_0 (in other words the conclusion of Theorem 2.11 holds true).

This result is finally applied to investigate the existence of periodic wave trains with large wave speed γ for system (1.1). Here recall that a wave train profile with speed γ is an entire solution of (1.1) of the form $u(t, z, a) = U(x, a)$ with $x = z - \gamma t$ and where the function U is periodic with respect to the variable $x \in \mathbb{R}$, namely there exists a period $T > 0$ such that for all $x \in \mathbb{R}$ and $a > 0$ one has $U(x + T, a) = U(x, a)$. This leads us to the following problem: find an $x-$periodic profile $U \equiv U(x, a)$ and a speed $\gamma \in \mathbb{R}$ solution of the problem

$$\begin{cases} \partial_a U(x,a) = \partial_x^2 U(x,a) - \gamma \partial_x U(x,a) - \mu U(x,a), & x \in \mathbb{R},\ a > 0, \\ U(x,0) = \alpha f\left(\int_0^\infty \beta(a) U(x,a)\mathrm{d}a\right), \end{cases} \tag{2.14}$$

where $\mu > 0$ is a given and fixed parameter while $\alpha > 0$ is a parameter that will be used as a bifurcation parameter. We also assume that the function f takes the form of the so-called Ricker nonlinearity. This assumption reads as follows.

Assumption 2.14 *The function f is assumed to be of Ricker's type, that is*

$$f(u) = ue^{-u}.$$

The function β is assumed to be a delayed $\Gamma-$distribution, namely

$$\beta(a) = \begin{cases} 0 \text{ if } a \in (0, \tau), \\ \delta(a - \tau)^n e^{-\zeta(a-\tau)} \text{ if } a \geq \tau, \end{cases}$$

for some integer $n \geq 1$ while $\tau \geq 0$, $\zeta > 0$ and $\delta > 0$ are given constants such that the following normalisation condition holds true

$$\int_0^\infty \beta(a) e^{-\mu a} \mathrm{d}a = 1.$$

Before stating our bifurcation result, let us first observe that for each $\alpha > 1$, the function

$$\overline{U}_\alpha(a) = \ln \alpha e^{-\mu a}, \quad \forall a > 0,$$

is an $x-$independent solution of (2.14). Then our main result reads as follows.

Theorem 2.15 (Existence of periodic wave train solutions) *Let Assumption 2.14 be satisfied. Then there exist $\alpha^* > 1$, $\gamma^* > 0$ large enough and $\eta^* > 0$ such that for each $\gamma \in (\gamma^*, \infty)$, there exists $\alpha_\gamma \in (\alpha^* - \eta^*, \alpha^* + \eta^*)$ such that system (2.14) with $\gamma \in (\gamma^*, \infty)$ undergoes a Hopf bifurcation at $\alpha = \alpha_\gamma$ around the $x-$independent solution $\overline{U}_{\alpha_\gamma}$ and the bifurcated solution is a periodic wave train of (1.1) with speed γ.*

3 Integrated semigroups

In this section we recall some results about integrated semigroups that will be used in the following. We refer to Arendt [2, 3], Neubrander [61], Kellermann and Hieber [44], Thieme [74, 76], and Arendt *et al.* [4] for more detailed results on the subject. We also refer to Magal and Ruan [52, 53, 54, 55] for some of the results recalled below.

Let $(X, \|.\|)$ be a real Banach space. Let $A : D(A) \subset X \to X$ be a linear operator satisfying 2.1. Let us set $X_0 := \overline{D(A)}$ and let A_0 denotes the part of A in X_0, which is a linear operator on X_0 defined by

$$A_0x = Ax,\ \forall x \in D(A_0) := \{y \in D(A) : Ay \in X_0\}.$$

First recall that due to Lemma 2.1 in Magal and Ruan [52] we know that

$$\overline{D(A_0)} = X_0.$$

By Hille-Yosida theorem (see Pazy [62], Theorem 5.3 on p.20) and Magal and Ruan [54, Lemma 2.4] we obtain

Lemma 3.1 *Assumption 2.1 is satisfied if and only if $\rho(A) \neq \emptyset$, A_0 is the infinitesimal generator of a linear C_0-semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ on X_0, and*

$$\|T_{A_0}(t)\| \leq M_A e^{\omega_A t}, \forall t \geq 0.$$

We now recall the definition of an integrated semigroup.

Definition 3.2 *Let $(X, \|.\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on X is called an* ***integrated semigroup*** *if it satisfies*

(i) $S(0) = 0$.

(ii) *The map $t \to S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$.*

(iii) *$S(t)$ satisfies*

$$S(s)S(t) = \int_0^s (S(r+t) - S(r))\,\mathrm{d}r,\quad \forall t, s \geq 0. \tag{3.1}$$

An integrated semigroup $\{S(t)\}_{t\geq 0}$ is said to be *non-degenerate* if $S(t)x = 0,\ \forall t \geq 0$, then $x = 0$. According to Thieme [74], a linear operator $A : D(A) \subset X \to X$ is the *generator* of a non-degenerate integrated semigroup $\{S(t)\}_{t\geq 0}$ on X if and only if

$$x \in D(A),\ y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)y\mathrm{d}s,\ \forall t \geq 0. \tag{3.2}$$

From [74, Lemma 2.5], we know that if A generates $\{S_A(t)\}_{t\geq 0}$, then for each $x \in X$ and $t \geq 0$,

$$\int_0^t S_A(s)x\mathrm{d}s \in D(A) \text{ and } S_A(t)x = A\int_0^t S_A(s)x\mathrm{d}s + tx.$$

An integrated semigroup $\{S(t)\}_{t\geq 0}$ is said to be *exponentially bounded* if there exist two constants, $\widehat{M} > 0$ and $\widehat{\omega} > 0$, such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq \widehat{M}e^{\widehat{\omega}t}, \forall t \geq 0.$$

When we restrict ourselves to the class of non-degenerate exponentially bounded integrated semigroups, Thieme's notion of generator is equivalent the one introduced by Arendt in [3].
Then the following result is well known in the context of integrated semigroups.

Proposition 3.3 *Let Assumption 2.1 be satisfied. Then A generates a uniquely determined non-degenerate exponentially bounded integrated semigroup $\{S_A(t)\}_{t\geq 0}$. Moreover, for each $x \in X$, each $t \geq 0$, and each $\mu > \omega_A$, $S_A(t)x$ is given by*

$$S_A(t)x = (\mu I - A_0)\int_0^t T_{A_0}(s)\left(\mu I - A\right)^{-1}x\mathrm{d}s, \tag{3.3}$$

or equivalently

$$S_A(t)x = \mu\int_0^t T_{A_0}(s)\left(\mu I - A\right)^{-1}x\mathrm{d}s + \left[I - T_{A_0}(t)\right]\left(\mu I - A\right)^{-1}x.$$

Furthermore, the map $t \mapsto S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{\mathrm{d}S_A(t)x}{\mathrm{d}t} = T_{A_0}(t)x,\ \ \forall t \geq 0,\ \forall x \in X_0.$$

From now on we define for each $\tau > 0$

$$\left(S_A * f\right)(t) = \int_0^t S_A(t-s)f(s)\mathrm{d}s, \forall t \in [0,\tau],$$

whenever $f \in L^1\left((0,\tau), X\right)$.

We now consider the first order non-homogeneous Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t),\ t \in [0,\tau],\ \ u(0) = x \in X_0, \tag{3.4}$$

and assume that f belongs to some appropriate subspace of $L^1((0,\tau),X)$. In order to deal with the solutions of the above problem, let us recall the following definition.

Definition 3.4 *Let $\tau > 0$ be given and let $f \in L^1((0,\tau),X)$ be given. A continuous map $u \in C([0,\tau],X)$ is called a **weak solution** of (3.4) if*

$$\int_0^t u(s)\mathrm{d}s \in D(A), \quad \forall t \in [0,\tau]$$

and

$$u(t) = x + A\int_0^t u(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s, \ \forall t \in [0,\tau].$$

Remark 3.5 *It is clear that u is a weak solution of (3.4) if and only if $\forall \lambda \in \rho(A)$,*

$$(\lambda I - A)^{-1} u(t) \in C^1([0,\tau],X)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\lambda I - A)^{-1} u(t)\right] = -u(t) + (\lambda I - A)^{-1}[f(t) + \lambda u(t)], \ \forall t \in [0,\tau].$$

This shows that Definition 3.4 corresponds to the notion of a solution introduced in Definition 2.2 above.

Now in order to deal with the solvability of Equation (3.4), let us denote by

$$(S_A \diamond f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(S_A * f)(t),$$

whenever the convolution map $t \mapsto (S_A * f)(t)$ is continuously differentiable.

We will say that $\{S_A(t)\}_{t\geq 0}$ has **a bounded semi-variation** on $[0,t]$ if

$$V^\infty(S_A,0,t) := \sup\left\{\left\|\sum_{i=1}^n \big(S_A(t_i) - S_A(t_{i-1})\big)x_i\right\|\right\} < +\infty,$$

where the supremum is taken over all partitions $0 = t_0 < .. < t_n = t$ of the interval $[0,t]$ and over any $(x_1,..,x_n) \in X^n$ with $\|x_i\|_X \leq 1$, $\forall i = 1,..,n$.

We now deal with assumption 2.3 and we give an equivalent formulation in term of bounded semi-variation.

Theorem 3.6 *Let Assumption 2.1 be satisfied. Then Assumption 2.3 is satisfied if and only if $\{S_A(t)\}_{t\geq 0}$ has a bounded semi-variation on $[0,\tau]$ and*

$$\lim_{t\to 0} V^\infty(S_A,0,t) = 0.$$

Moreover for any such a function $\delta : [0,\tau] \to [0,+\infty)$ arising in Assumption 2.3 one has

$$V^\infty(S_A,0,t) \leq \delta(t), \forall t \in [0,\tau].$$

We refer to Magal and Ruan [52] and Ducrot *et al.* [28] for verifying Assumption 2.3 for age-structured models and parabolic equations as well. Note now that since we have

$$S_A(\tau+h) = S_A(\tau) + T_{A_0}(\tau)S_A(h), \forall h \geq 0,$$

by using Assumption 2.3, one deduces that $t \to S_A(t)$ has a bounded semi-variation on $[0, 2\tau]$. Now using induction arguments, we deduce that $t \to S_A(t)$ has a bounded semi-variation on $[0, \tau]$ for each $\tau \geq 0$.

Next, Assumption 2.1 and 2.3 lead us to the following solvability result.

Theorem 3.7 *Let Assumptions 2.1 and 2.3 be satisfied. Then for each $\tau > 0$, $t \to S_A(t)$ has a bounded semi-variation on $[0, \tau]$. Moreover, for each $f \in C([0,\tau], X)$, the map $t \mapsto (S_A * f)(t)$ is continuously differentiable, $(S_A * f)(t) \in D(A), \forall t \in [0,\tau]$, and $u(t) = (S_A \diamond f)(t)$ satisfies*

$$u(t) = A\int_0^t u(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s,\ \forall t \in [0,\tau],$$

and

$$\|u(t)\| \leq V^\infty(S_A, 0, t) \sup_{s\in[0,t]} \|f(s)\|,\ \forall t \in [0,\tau].$$

Furthermore, for each $\lambda \in (\omega, +\infty)$, we have

$$(\lambda I - A)^{-1} (S_A \diamond f)(t) = \int_0^t T_{A_0}(t-s) (\lambda I - A)^{-1} f(s)\mathrm{d}s. \tag{3.5}$$

As a corollary of the above result we obtain:

Corollary 3.8 *Let Assumptions 2.1 and 2.3 be satisfied. Then for each $x \in X_0$ and each $f \in C([0,\tau], X_0)$, the first order Cauchy problem (3.4) has a unique weak solution $u \in C([0,\tau], X_0)$ given by*

$$u(t) = T_{A_0}(t)x + (S_A \diamond f)(t),\ \forall t \in [0,\tau].$$

Moreover, we have

$$\|u(t)\| \leq M_A e^{\omega t} \|x\| + V^\infty(S_A, 0, t) \sup_{s\in[0,t]} \|f(s)\|,\ \forall t \in [0,\tau].$$

In the following we will also make use of the following proposition for which we refer to [53, see Proposition 2.14]. We refer to the next section and also to Magal and Ruan [53, 54] for interesting applications of this result.

Proposition 3.9 *Let Assumptions 2.1 and 2.3 be satisfied. Then the following estimate holds true*

$$\|(S_A \diamond f)(t)\| \leq C(\tau, \gamma) \sup_{s\in[0,t]} e^{\gamma(t-s)} \|f(s)\|,\quad \forall t \geq 0$$

for $\gamma \in (\omega_A, +\infty)$, $f \in C(\mathbb{R}_+, X)$ *and* $\tau > 0$, *while the constant* $C(\tau, \gamma)$ *is explicitly given by*

$$C(\tau, \gamma) := \frac{2M_A V^\infty(S_A, 0, \tau) \max(1, e^{-\gamma\tau})}{1 - e^{(\omega_A - \gamma)\tau}}.$$

We continue this section by some preparatory lemma that will be used in the next section to study the solvability of the sum of two commuting linear operators. For that purpose recall that $\omega_0(A_0)$, the **growth bound** of the C_0-semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ is defined by

$$\omega_0(A_0) := \lim_{t\to+\infty} \frac{\ln\left(\|T_{A_0}(t)\|_{\mathcal{L}(X)}\right)}{t} \in [-\infty, +\infty).$$

For the remaining of this section in addition to Assumption 2.1 and 2.3 we furthermore assume that

$$\omega_0(A_0) < 0. \tag{3.6}$$

Then the proof of the following lemma is similar to the proof of Lemma 4.6 p. 24 in Magal and Ruan [54].

Lemma 3.10 *Under the above mentioned assumptions and recalling the definition of the Banach spaces in (2.2) the following properties hold true:*

(i) *Let* $\eta \in (0, -\omega_0(A_0))$ *be given. Then for each* $f \in BC^0_\eta([0, +\infty), X)$ *the following limit exists in* X_0

$$\lim_{t\to+\infty} (S_A \diamond f(t - .))(t) \textit{ exists.}$$

(ii) *For each constant* $\beta \in (0, -\omega_0(A_0))$, *there exists some constant* $C = C(\beta) > 0$, *such that for each* $\eta \in [0, \beta]$ *the linear operator* $\mathbb{K}_A : BC^0_\eta(\mathbb{R}_+, X) \to X_0$ *defined by*

$$\mathbb{K}_A(f) := \lim_{t\to+\infty} (S_A \diamond f(t - .))(t)$$

is bounded and we have a uniform evaluation

$$\|\mathbb{K}_A\|_{\mathcal{L}\left(BC^0_\eta(\mathbb{R}_+, X), X_0\right)} \leq C(\beta).$$

Remark 3.11 *If* $f \in BC^0_\eta([0, +\infty), X_0)$ *one has*

$$\mathbb{K}_A(f) = \lim_{t\to+\infty} (S_A \diamond f(t - .))(t) = \lim_{t\to+\infty} \int_0^t T_{A_0}(t - s) f(t - s) \mathrm{d}s,$$

so that $\mathbb{K}_A(f)$ *can be rewritten as*

$$\mathbb{K}_A(f) = \int_0^{+\infty} T_{A_0}(l) f(l) \mathrm{d}l.$$

Lemma 3.12 *Under the same assumptions as in Lemma 3.10, let $\eta > 0$ and $f \in BC^1_{-\eta}([0,+\infty), X)$ be given. Then the following hold true*

$$\mathbb{K}_A(f) \in D(A), \text{ and } A\mathbb{K}_A(f) + \mathbb{K}_A(f') + f(0) = 0.$$

Proof. Let $\gamma \in (0, -\omega_0(A))$ be given. Applying Proposition 3.9 with any constant function $f(t) \equiv x$ for $x \in X$ and noticing that in such a case one has

$$(S_A \diamond f)(t) = S_A(t)x,$$

then there exists some constant $C > 0$ such that

$$\|S_A(t)x\| = \|(S_A \diamond f)(t)\| \leq C \sup_{s\in[0,t]} e^{-\gamma(t-s)} \|x\|, \quad \forall t \geq 0.$$

Thus

$$\|S_A(t)x\| \leq C \|x\|.$$

Now let $f \in BC^1_{-\eta}([0,+\infty), X)$ be given. Then we have

$$\|S_A(t)f(t)\| \leq C \ \|f(t)\| = Ce^{-\eta t} \|f\|_{0,-\eta} \to 0 \text{ as } t \to +\infty.$$

On the one hand note that since $f \in BC^1_{-\eta}([0,+\infty), X)$, we have

$$(S_A \diamond f)(t) = S_A(t)f(0) + (S_A * f')(t) = S_A(t)f(0) + \int_0^t (S_A \diamond f')(s)\mathrm{d}s.$$

Thus this yields

$$\begin{aligned}
(S_A \diamond f(t-.))(t) &= S_A(t)f(t) - (S_A * f'(t-.))(t) \\
&= S_A(t)f(t) - \int_0^t S_A(t-s)f'(t-s)\mathrm{d}s \\
&= S_A(t)f(t) - \int_0^t (S_A \diamond f'(t-.))(s)\mathrm{d}s,
\end{aligned}$$

hence, we get

$$\mathbb{K}_A(f) = -\lim_{t\to+\infty} \int_0^t (S_A \diamond f'(t-.))(s)\mathrm{d}s.$$

On the other hand note that the function $t \mapsto (S_A \diamond f'(t-.))(t)$ satisfies the equation

$$\begin{aligned}
(S_A \diamond f'(t-.))(t) &= A\int_0^t (S_A \diamond f'(t-.))(s)\mathrm{d}s + \int_0^t f'(t-s)\mathrm{d}s \\
&= A\int_0^t (S_A \diamond f'(t-.))(s)\mathrm{d}s + \int_0^t f'(s)\mathrm{d}s \\
&= A\int_0^t (S_A \diamond f'(t-.))(s)\mathrm{d}s + f(t) - f(0).
\end{aligned}$$

Thus, one gets

$$(-A)^{-1}\left[\left(S_A \diamond f'(t-.)\right)(t)+f(0)-f(t)\right]=-\int_0^t \left(S_A \diamond f'(t-.)\right)(s)\mathrm{d}s.$$

Taking the limit $t\to\infty$ ensures that

$$(-A)^{-1}\left[\mathbb{K}_A(f')+f(0)\right]=\mathbb{K}_A(f),$$

and the result follows. ■

We complete this section by the following lemma.

Lemma 3.13 *Under the same assumptions of Lemma 3.10 and let $\eta \in (0,-\omega_0(A_0))$. One has for each $f\in BC_\eta^0([0,+\infty),X)$*

$$\mathbb{K}_A(f):=\mathbb{K}_{A+\delta I}(e^{-\delta .}f),\forall\delta<-\omega_0(A_0).$$

Proof. Let $\lambda\in\rho(A)$ be given. Let $f\in BC_\eta^0([0,+\infty),X)$ be given and let us fix $\delta<-\omega_0(A_0)$. Then one has

$$\begin{aligned}(\lambda I-A)^{-1}\left(S_A\diamond f(t-.)\right)(t)&=\int_0^t T_{A_0}(t-s)(\lambda I-A)^{-1}f(t-s)\mathrm{d}s\\&=\int_0^t T_{A_0}(l)(\lambda I-A)^{-1}f(l)\mathrm{d}l\\&=\int_0^t T_{A_0+\delta I}(l)(\lambda I-A)^{-1}e^{-\delta l}f(l)\mathrm{d}l,\end{aligned}$$

hence, one obtains

$$(\lambda I-A)^{-1}\left(S_A\diamond f(t-.)\right)(t)=(\lambda I-A)^{-1}\left(S_{A+\delta I}\diamond e^{-\delta(t-.)}f(t-.)\right)(t),\quad\forall t\geq 0.$$

This yields

$$\left(S_A\diamond f(t-.)\right)(t)=\left(S_{A+\delta I}\diamond e^{-\delta(t-.)}f(t-.)\right)(t),\ \forall t\geq 0,$$

and the result follows by using Lemma 3.10 and letting $t\to\infty$. ■

4 Sum of commutating operators and applications

In this section we reconsider the problem of the closability and the invertibility of the sum of two commuting linear operators. Such results have been investigated in the context of densely defined linear operators by Da Prato and Grisvard [21]. More recently this problem has also been considered by Thieme [75, 76] by using integrated semigroups for non-densely defined linear operators. Basically the problem is concerned with the solvability of the equation

$$-\left[A+B\right]x=y, \tag{4.1}$$

wherein $A : D(A) \subset X \to X$ is non-densely defined satisfying Assumptions 2.1 and 2.3 and, $B : D(B) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of linear operators on X and, where A and B commute. The goal of this section is prove that $A + B$ is closable, and to derive some conditions ensuring that this closure $\overline{A+B}$ is invertible. In this section, we reconsider this question by using a slightly different approach compared to the one developed by Thieme [75, 76]. This approach will allow us to clarify the notion of a weak solution for the problem (1.3) (see Definition 2.4).

Throughout this section we will assume that the following hypothesis are satisfied.

Assumption 4.1 *Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumptions 2.1 and 2.3 and let $B : D(B) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T_B(t)\}_{t\geq 0}$ on X. We assume in addition that*

(i) *The linear operators A and B commute in the sense that one has*

$$(\lambda I - A)^{-1} (\mu I - B)^{-1} = (\mu I - B)^{-1} (\lambda I - A)^{-1}, \forall \lambda, \mu \in \rho(A) \cap \rho(B).$$

(ii) *The linear operator A_0 has a negative growth rate, namely*

$$\omega_0 (A_0) < 0. \tag{4.2}$$

Then our first lemma reads as follows.

Lemma 4.2 *Let Assumption 4.1 be satisfied. Let $\eta \in [0, -\omega_0 (A_0))$ be given. Then for each $f \in BC^0_\eta ([0, +\infty), D(B))$ we have*

$$\mathbb{K}_A(f) \in D(B),$$

and the following commutativity property holds

$$\mathbb{K}_A(Bf) = B\mathbb{K}_A(f).$$

Here, when dealing with the space $BC^0_\eta ([0, +\infty), D(B))$, $D(B)$ is endowed with the graph norm.

Proof. Let $\lambda \in \rho(B)$ be given. Then one has

$$\begin{aligned}
(\lambda I - B)^{-1} \mathbb{K}_A(Bf) &= (\lambda I - B)^{-1} \lim_{t\to\infty} (S_A \diamond Bf(t - .)) (t) \\
&= \lim_{t\to\infty} \left(S_A \diamond B (\lambda I - B)^{-1} f(t - .)\right) (t) \\
&= B (\lambda I - B)^{-1} \mathbb{K}_A(f).
\end{aligned}$$

Next since $B (\lambda I - B)^{-1} = -I + \lambda (\lambda I - B)^{-1}$, this implies that

$$\mathbb{K}_A(f) \in D(B),$$

and the result follows. ■

In order to simplify the presentation we consider the case where the closure of $A + B$ will be invertible, this means that we make the following assumption.

Assumption 4.3 *We assume that* $\omega_0(B) < -\omega_0(A_0)$.

Under Assumption 4.3, using 3.10 we define the linear operator $\mathcal{R}_{(A+B)} \in \mathcal{L}(X)$ by

$$\mathcal{R}_{(A+B)}x := \mathbb{K}_A(T_B(.)x), \forall x \in X.$$

Then by using Lemma 3.10-(ii) we obtain the following estimate

Proposition 4.4 (Uniform estimate) *Let Assumptions 4.1 and 4.3 be satisfied. Let* $\beta \in (\omega_0(B), -\omega_0(A_0))$ *be given. Then there exists some constant* $C = C(\beta) > 0$, *such that for each* $\eta \in (\omega_0(B), \beta]$ *the linear operator* $\mathcal{R}_{(A+B)}$ *satisfies the following estimate*

$$\left\|\mathcal{R}_{(A+B)}x\right\|_X \leq C(\beta) \left\|T_B(.)x\right\|_{0,\eta}, \ \forall x \in X.$$

We now derive some regularity lemma for the linear operator $\mathcal{R}_{(A+B)}$.

Lemma 4.5 *Let Assumptions 4.1 and 4.3 be satisfied. Then the following regularity holds true*

$$\mathcal{R}_{(A+B)}D(B) \subset D(A) \textit{ and } \mathcal{R}_{(A+B)}D(B) \subset D(B),$$

and

$$-(A+B)\mathcal{R}_{(A+B)}x = x, \ \ \forall x \in D(B).$$

Proof. Let $x \in D(B)$ be given. Let us fix $\delta \in (\omega_0(B), -\omega_0(A_0))$. Then by using Lemma 3.13, we obtain

$$\mathcal{R}_{(A+B)}x = \mathbb{K}_{A+\delta I}\left(T_{B-\delta I}(.)x\right), \forall x \in X.$$

Therefore applying Lemma 3.12 to $\mathbb{K}_{A+\delta I}(f)$ with $f(t) = T_{B-\delta I}(.)x$, we deduce that

$$\mathbb{K}_{A+\delta I}\left(T_{B-\delta I}(.)x\right) \in D(A),$$

and

$$(A+\delta I)\,\mathbb{K}_{A+\delta I}\left(T_{B-\delta I}(.)x\right) + \mathbb{K}_{A+\delta I}\left((B-\delta I)\,T_{B-\delta I}(.)x\right) + x = 0.$$

Now by using Lemma 4.2 we have

$$\mathbb{K}_{A+\delta I}\left(T_{B-\delta I}(.)x\right) \in D(B)$$

and

$$\mathbb{K}_{A+\delta I}\left((B-\delta I)\,T_{B-\delta I}(.)x\right) = (B-\delta I)\,\mathbb{K}_{A+\delta I}\left(T_{B-\delta I}(.)x\right),$$

and the result follows. ■

We now come back to equation (4.1) and define a suitable notion of solution for this problem. The definition below is mainly motivated by the following arguments. Let $y \in X$ be given and fixed. Let $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ be given and let us set

$$x = \mathcal{R}_{(A+B)}y.$$

Then one has

$$(\mu - B)^{-1} (\lambda - A)^{-1} x = \mathcal{R}_{(A+B)} (\mu - B)^{-1} (\lambda - A)^{-1} y.$$

By using Lemma 4.5, and since $(\mu - B)^{-1} (\lambda - A)^{-1} y \in D(B)$, we deduce that

$$\begin{aligned} (\mu - B)^{-1} (\lambda - A)^{-1} y &= -(A+B) \mathcal{R}_{(A+B)} (\mu - B)^{-1} (\lambda - A)^{-1} y \\ &= -(A+B)(\mu - B)^{-1} (\lambda - A)^{-1} x, \end{aligned}$$

therefore

$$(\mu - B)^{-1} (\lambda - A)^{-1} y = -(A+B)(\mu - B)^{-1} (\lambda - A)^{-1} x.$$

This leads us to the following relation

$$\left[(\mu - B)^{-1} + (\lambda - A)^{-1}\right] x = (\mu - B)^{-1} (\lambda - A)^{-1} [y + (\lambda + \mu) x].$$

The above computations motivate the introduction of the following definition.

Definition 4.6 (Weak and Classical solutions) *Let $y \in X$ be given. We will say that $x \in X$ is **a weak solution** of the problem*

$$-[A+B] x = y, \tag{4.3}$$

if x and y satisfy the following equality

$$\left[(\mu - B)^{-1} + (\lambda - A)^{-1}\right] x = (\mu - B)^{-1} (\lambda - A)^{-1} [y + (\lambda + \mu) x] \tag{4.4}$$

for some $\lambda \in \rho(A)$ and $\mu \in \rho(B)$.

*We will say that x is **a classical solution** of (4.3) if*

$$x \in D(A) \cap D(B) \text{ and } -[A+B] x = y.$$

Now let $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ be given. Define

$$G := \left\{(x,y) \in X \times X : \left[(\mu - B)^{-1} + (\lambda - A)^{-1}\right] x = (\mu - B)^{-1} (\lambda - A)^{-1} [y + (\lambda + \mu) x]\right\}.$$

The first main result of this section is the following lemma.

Lemma 4.7 *Let Assumptions 4.1 and 4.3 be satisfied. The following properties hold true:*

(i) $G = \left\{(x,y) \in X \times X : x = \mathcal{R}_{(A+B)} y\right\}$;

(ii) *G is independent of λ in $\rho(A)$ and μ in $\rho(B)$;*

(iii) *G is the graph of a linear operator $-\mathcal{L} : D(\mathcal{L}) \subset X \to X$ which is invertible and such that*

$$(-\mathcal{L})^{-1} = \mathcal{R}_{(A+B)}.$$

Proof. Proof of (i)*:* By defining the notion of weak solutions, we have already proved that

$$G \supset \left\{ (x,y) \in X \times X : x = \mathcal{R}_{(A+B)} y \right\}.$$

The converse inclusion uses the same arguments and (i) follows.
Proof of (ii)*:* Let $\widehat{\lambda} \in \rho(A)$ and $\widehat{\mu} \in \rho(B)$ be given. Then one has

$$\left[(\mu - B)^{-1} + (\lambda - A)^{-1} \right] x = (\mu - B)^{-1} (\lambda - A)^{-1} \left[y + (\lambda + \mu) x \right].$$

The latter identity is equivalent to

$$\begin{aligned} &\left[(\mu - B)^{-1} + (\lambda - A)^{-1} \right] (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} x \\ &= (\mu - B)^{-1} (\lambda - A)^{-1} (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} \left[y + (\lambda + \mu) x \right]. \end{aligned}$$

Hence applying $(\mu - B)$ and $(\lambda - A)$ on both sides, we deduce that this last equation is also equivalent to

$$-\left[A + B \right] (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} x = (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} y.$$

Therefore by adding $\left(\widehat{\mu} + \widehat{\lambda} \right) (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} x$, we deduce the equivalence together with

$$\left[\widehat{\lambda} I - B + \widehat{\mu} - A \right] (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} x = (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} \left[y + \left(\widehat{\lambda} + \widehat{\mu} \right) x \right],$$

which is also equivalent to

$$\left[\left(\widehat{\lambda} - A \right)^{-1} + (\widehat{\mu} - B)^{-1} \right] x = (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} \left[y + \left(\widehat{\lambda} + \widehat{\mu} \right) x \right].$$

The independence of G with respect to λ in $\rho(A)$ and μ in $\rho(B)$ follows.
Proof of (iii)*:* It is clear that G is a closed subspace. Moreover, if $(x,y) \in G$ and $(x, \widehat{y}) \in G$ then

$$0 = (\widehat{\mu} - B)^{-1} \left(\widehat{\lambda} - A \right)^{-1} (y - \widehat{y})$$

which implies that

$$y = \widehat{y}.$$

Therefore G is the graph of the closed linear operator denoted by $-\mathcal{L} : D(\mathcal{L}) \subset X \to X$ and by using (i) we deduce that (iii) holds true. ∎

We summarizes the results of this section in the following theorem.

Theorem 4.8 (Existence and uniqueness of the weak solution) *Let Assumptions 4.1 and 4.3 be satisfied. For each* $y \in X$ *there exists a unique weak solution* x *of (4.1) given by*

$$x = \mathcal{R}_{(A+B)} y.$$

Proof. The existence of the weak solution follows from Lemma 4.7. It remains to prove its uniqueness. Let $\delta \in (\omega_0(B), -\omega_0(A_0))$ be given. Then (4.4) with $\lambda = -\delta$ and $\mu = \delta$ re-writes as

$$\left[(\delta - B)^{-1} + (-\delta - A)^{-1}\right] x = (\delta - B)^{-1} (-\delta - A)^{-1} y. \tag{4.5}$$

To prove the uniqueness, let us assume that there exists $x \in X$, such that

$$\left[(\delta - B)^{-1} + (-\delta - A)^{-1}\right] x = 0,$$

and let us show that $x = 0$. To do so, let us apply $\mathcal{R}_{(A+B)}$ on both sides of the above equation. This yields

$$\begin{aligned} 0 &= \left[(\delta - B)^{-1} + (-\delta - A)^{-1}\right] \mathcal{R}_{(A+B)} x \\ &= [-A - B] (\delta - B)^{-1} (-\delta - A)^{-1} \mathcal{R}_{(A+B)} x \\ &= [-A - B] \mathcal{R}_{(A+B)} (\delta - B)^{-1} (-\delta - A)^{-1} x, \end{aligned}$$

and, by using Lemma 4.5 we obtain

$$(\delta - B)^{-1} (-\delta - A)^{-1} x = 0.$$

This implies that $x = 0$ and the uniqueness of weak solution follows. ■

The next lemma deals with the regularity of the solutions.

Lemma 4.9 *Let Assumptions 4.1 and 4.3 be satisfied. Let $x, y \in X$ be given such that x is a weak solution of (4.1). Then the following properties are satisfied:*

(i) *if either $x \in D(A)$ or $x \in D(B)$, then $x \in D(A) \cap D(B)$ and x is a classical solution of (4.1);*

(ii) *if $y \in D(B)$ then x is a classical solution of (4.1).*

Proof. Assume that x is a weak solution of (4.1) and assume for example that

$$x = \mathcal{R}_{(A+B)} y \in D(A).$$

Let $\delta \in (\omega_0(B), -\omega_0(A_0))$. Then by applying $(-\delta - A)$ on both sides of (4.5) yields

$$x = (\delta - B)^{-1} [y + (-\delta - A) x].$$

Therefore $x \in D(B)$ and, by using Lemma 4.5, we deduce that x is a classical solution. This completes the proof of the lemma. ■

Proposition 4.10 (Closability) *Let Assumptions 4.1 and 4.3 be satisfied. The linear operator $A+B : D(A) \cap D(B) \subset X \to X$ is closable, and its closure $\overline{A+B}$ satisfies*

$$\overline{A+B} = \mathcal{L}.$$

Remark 4.11 *From the property* (i) *of Lemma 4.9 we deduce that*

$$D(\overline{A+B}) \cap D(A) = D(\overline{A+B}) \cap D(B) = D(A) \cap D(B).$$

Proof. Set

$$G_0 = \{(x,y) : x \in D(A) \cap D(B) \text{ and } y = -(A+B)\,x\}.$$

By using the definition of the weak solution, we deduce that

$$G_0 \subset G.$$

Define also

$$G_1 = \left\{(x,y) : x = (-\mathcal{L})^{-1} y \text{ and } y \in D(B)\right\}.$$

By using Lemma 4.5, we have

$$G_1 \subset G_0,$$

and since $D(B)$ is dense in X, we have $\overline{G_1} = G$, so that $\overline{G_0} = G$. This completes the proof of the lemma. ∎

Resolvent formula: Let Assumptions 4.1 and 4.3 be satisfied. The resolvent of the linear operator $\left(\overline{A+B}\right)$ is given by

$$\left(\lambda - \left(\overline{A+B}\right)\right)^{-1} x := \mathbb{K}_A\left(T_{B-\lambda I}(.)x\right),$$

whenever $\lambda > \omega_0(A_0) + \omega_0(B)$ and $x \in X$.

Finally by combining Theorem 4.7 in Thieme [76, Theorem 4.7], and all the above results we obtain the following theorem

Theorem 4.12 ***(Integrated semigroup)*** *Let Assumption 4.1* (i) *be satisfied. Then the linear operator* $A+B : D(A) \cap D(B) \to X$ *is closable, and its closure* $\overline{A+B} : D(\overline{A+B}) \subset X \to X$ *satisfies Assumptions 2.1 and 2.3. More precisely the following properties hold true:*

(i) *The linear operator* $\left(\overline{A+B}\right)_0 : D(\left(\overline{A+B}\right)_0) \subset \overline{D(A)} \to \overline{D(A)}$ *defined as the part of* $\overline{A+B}$ *in* $X_0 := \overline{D(A)}$ *is the infinitesimal generator of a* C_0-*semigroup* $\left\{T_{\left(\overline{A+B}\right)_0}(t)\right\}_{t \geq 0}$ *on* X_0 *and*

$$T_{\left(\overline{A+B}\right)_0}(t)x = T_B(t)T_{A_0}(t)x, \forall x \in X_0, \forall t \geq 0.$$

In addition one has

$$\omega_0\left(\left(\overline{A+B}\right)_0\right) \leq \omega_0\left(A_0\right) + \omega_0\left(B\right).$$

(ii) *The linear operator* $\overline{A+B}$ *generates an exponential bounded (non degenerate) integrated semigroup* $\left\{S_{\overline{A+B}}(t)\right\}_{t \geq 0}$ *of bounded linear operators on* X*, given by*

$$S_{\overline{A+B}}(t)x = \left(S_A \diamond T_B(t-.)x\right)(t), \forall x \in X, \forall t \geq 0,$$

and

$$V^\infty(S_{\overline{A+B}}, 0, t) \leq \sup_{s \in [0,t]} \|T_B(s)\|\, V^\infty(S_A, 0, t), \forall t \geq 0.$$

(iii) *The following inclusions hold*

$$\begin{aligned} D(A_0) \cap D(B) &\subset D\left(\left(\overline{A+B}\right)_0\right) \subset \overline{D(A)} \\ \overline{D(A) \cap D(B)} &\subset \overline{D\left(\overline{A+B}\right)} \subset \overline{D(A)} \\ \overline{D\left(\left(\overline{A+B}\right)_0\right)} &= \overline{D\left(\overline{A+B}\right)} = \overline{D(A)} \end{aligned}$$

(iv) *The equality*

$$-\left(\overline{A+B}\right) x = y \text{ and } x \in D(\overline{A+B})$$

holds if and only if

$$\left[(\mu - B)^{-1} + (\lambda - A)^{-1}\right] x = (\mu - B)^{-1} (\lambda - A)^{-1} [y + (\lambda + \mu) x]$$

for some $\lambda \in \rho(A)$ *and* $\mu \in \rho(B)$.

5 Application to first and second order problems

In this section we will apply the results of the preceding section to first and second order differential equations on the real line. In order to perform our analysis we need to introduce further notations and additional weighted spaces. Recalling Definitions 2.2 and 2.3 let us define for each Banach space Y, each interval $I \subset \mathbb{R}$ and each $\eta \in \mathbb{R}$ the closed subspace of $BC^0_\eta(I, Y)$ given by

$$BUC_\eta(I, Y) := \left\{\varphi \in BC^0_\eta(I, Y) : x \to e^{-\eta |x|}\varphi(x) \text{ is uniformly continuous}\right\}.$$

Now observe that if $\eta \geq 0$ one has for all $x \in \mathbb{R}$

$$e^{-\eta|x|} = \min\left(e^{-\eta x}, e^{\eta x}\right) = \left(\max\left(e^{-\eta x}, e^{\eta x}\right)\right)^{-1}$$

while

$$\max\left(e^{-\eta x}, e^{\eta x}\right) \leq e^{-\eta x} + e^{\eta x} \leq 2 \max\left(e^{-\eta x}, e^{\eta x}\right).$$

Therefore when $\eta \geq 0$, we can use the following equivalent norm on $BC^0_\eta(I, Y)$

$$|\varphi|_\eta = \sup_{x \in I} \frac{\|\varphi(x)\|}{\cosh(\eta x)}, \tag{5.1}$$

that satisfies the following estimates

$$\|\varphi\|_{0,\eta} \leq |\varphi|_\eta \leq 2 \|\varphi\|_{0,\eta}, \forall \varphi \in BC^0_\eta(I, Y) \tag{5.2}$$

where

$$\|\varphi\|_{0,\eta} := \sup_{x \in \mathbb{R}} e^{-\eta|x|}\|\varphi(x)\|.$$

We also define for each $\eta \in \mathbb{R}$ the space

$$BC_{0,\eta}(\mathbb{R}, Y) := \left\{ \varphi \in BC^0_\eta(\mathbb{R}, Y) : \lim_{x \to \pm\infty} e^{-\eta|x|} \|\varphi(x)\| = 0 \right\}.$$

Then $BC_{0,\eta}(\mathbb{R}, Y)$ is a closed subspace of $BUC_\eta(\mathbb{R}, Y)$. Define for each integer $k \geq 1$,

$$BC^k_{0,\eta}(\mathbb{R}, Y) := \left\{ \varphi \in BC^k_\eta(\mathbb{R}, Y) : \frac{\mathrm{d}^m \varphi}{\mathrm{d}x^m} \in BC_{0,\eta}(\mathbb{R}, Y), \forall m = 0, 1, ..., k \right\}.$$

The space $BC^k_{0,\eta}(\mathbb{R}, Y)$ is a closed subspace of $BC^k_\eta(\mathbb{R}, Y)$ and $(BC^k_{0,\eta}(\mathbb{R}, Y), \|.\|_{0,\eta})$ is a Banach space. Moreover if $0 < \xi < \eta$ we have the following inclusion

$$BC_{0,\xi}(\mathbb{R}, Y) \subset BC_{0,\eta}(\mathbb{R}, Y)$$

and the embedding is continuous; more precisely we have

$$\|\varphi\|_{0,\eta} \leq \|\varphi\|_{0,\xi}, \forall \varphi \in BC_{0,\xi}(\mathbb{R}, Y).$$

5.1 Strongly continuous semigroups and integrated semigroups on spaces of continuous functions

Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumptions 2.1 and 2.3. Recalling the definition of $BC_{0,\eta}(\mathbb{R}, X)$ above let us define for each $\eta \in \mathbb{R}$ the space

$$BC_{0,\eta}(\mathbb{R}, D(A)) := \{\varphi \in BC_{0,\eta}(\mathbb{R}, X) : \varphi(x) \in D(A), \forall x \in \mathbb{R} \text{ and } A\varphi \in BC_{0,\eta}(\mathbb{R}, X)\}.$$

Let $\eta \geq 0$ be given. We consider the linear operator $\mathcal{A} : D(\mathcal{A}) \subset BC_{0,\eta}(\mathbb{R}, X) \to BC_{0,\eta}(\mathbb{R}, X)$ defined by

$$\begin{cases} D(\mathcal{A}) = BC_{0,\eta}(\mathbb{R}, D(A)), \\ \mathcal{A}(\varphi)(x) = A\varphi(x), \forall x \in \mathbb{R}. \end{cases}$$

In this section we precise some properties of the linear operator $\mathcal{A}$. We refer to the book by Chicone and Latushkin [11] for more results on this topic.
Let us first observe that

$$\overline{D(\mathcal{A})} = BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right).$$

Moreover one has $\rho(\mathcal{A}) = \rho(A)$ and for each $\varphi \in BC_{0,\eta}(\mathbb{R}, X)$

$$(\lambda - \mathcal{A})^{-1}(\varphi)(x) = (\lambda - A)^{-1}\varphi(x), \forall \lambda \in \rho(\mathcal{A}).$$

It follows that $\mathcal{A}$ satisfies Assumption 2.1 (a) and (c). Moreover, since

$$\lim_{x \to \pm\infty} \frac{\varphi(x)}{\cosh(\eta x)} = 0,$$

whenever $\varphi \in BC_{0,\eta}(\mathbb{R}, X)$, it follows that for each given $\varphi \in BC_{0,\eta}(\mathbb{R}, X)$ the subset

$$\left\{ \frac{\varphi(x)}{\cosh(\eta x)} \right\}_{x \in \mathbb{R}}$$

is relatively compact in X. Hence we obtain that

$$\lim_{\lambda \to \infty} \left\| (\lambda - \mathcal{A})^{-1} \varphi \right\|_{0,\eta} = 0, \ \forall \varphi \in BC_{0,\eta}(\mathbb{R}, X),$$

and the operator $\mathcal{A}$ also satisfies Assumption 2.1 (b).

Now by using the results recalled in Section 3, we obtain the following lemma.

Lemma 5.1 *The linear operator $\mathcal{A}_0$, the part of $\mathcal{A}$ in $\overline{D(\mathcal{A})}$ is the infinitesimal generator of a C_0- semigroup of bounded linear operators $\{T_{\mathcal{A}_0}(t)\}_{t \geq 0}$ which is defined for each $\varphi \in BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right)$ by*

$$T_{\mathcal{A}_0}(t)(\varphi)(x) = T_{A_0}(t)\varphi(x), \forall x \in \mathbb{R},$$

wherein A_0 denotes the part of A in $\overline{D(A)}$.

By using the above lemma we deduce that

$$\omega_0(\mathcal{A}_0) = \omega_0(A_0).$$

Lemma 5.2 *The linear operator $\mathcal{A}$ is the generator of the integrated semigroup $\{S_{\mathcal{A}}(t)\}_{t \geq 0}$ of bounded linear operators on $BC_{0,\eta}(\mathbb{R}, X)$ that is defined for each $\varphi \in BC_{0,\eta}(\mathbb{R}, X)$ by*

$$S_{\mathcal{A}}(t)(\varphi)(x) = S_A(t)\varphi(x).$$

We finally summarize in the next proposition, all the properties we shall need in the following.

Proposition 5.3 *Let $\eta \geq 0$ be given. The linear operator $\mathcal{A}$ satisfies Assumptions 2.1 and 2.3. For each $\tau > 0$, for each $f \in C([0, \tau], BC_{0,\eta}(\mathbb{R}, X))$, the map $t \to (S_{\mathcal{A}} * f)(t)(x)$ is continuously differentiable from $[0, \tau]$ into $BC_{0,\eta}(\mathbb{R}, X_0)$ and*

$$(S_{\mathcal{A}} \diamond f)(t)(x) = (S_A \diamond f(., x))(t),$$

and

$$\|(S_{\mathcal{A}} \diamond f)(t)\|_{0,\eta} \leq V^{\infty}(S_A, 0, t) \sup_{s \in [0,t]} \|f(s)\|_{0,\eta}, \forall t \geq 0.$$

Proof. Since for each $x \in \mathbb{R}$, the Dirac mass δ_x is a bounded linear functional on $C_{0,\eta}(\mathbb{R}, X)$, we deduce that

$$\begin{aligned} \delta_x \left(\int_0^t (S_{\mathcal{A}} \diamond f(.,.))(s)\mathrm{d}s \right) &= \int_0^t (S_A \diamond f(., x))(s)\mathrm{d}s = (S_A * f(., x))(s) \\ &= \delta_x (S_{\mathcal{A}} * f(.,.)), \end{aligned}$$

and the first part of the result follows. Next we observe that

$$\begin{aligned} e^{-\eta|x|}\left\|\left(S_{\mathcal{A}}\diamond f\right)(t)(x)\right\| &= e^{-\eta|x|}\left\|\left(S_{A}\diamond f(.,x)\right)(t)\right\| \\ &\leq V^{\infty}(S_A,0,t)\sup_{s\in[0,t]}e^{-\eta|x|}\left\|f(s,x)\right\|, \end{aligned}$$

and the second estimate follows. ∎

5.2 First order differential operators

Let $\eta \geq 0$ be given and let $(X, \|.\|)$ be a Banach space. Let us consider the first order differential operator $\partial : D(\partial) \subset BC_{0,\eta}(\mathbb{R}, X) \to BC_{0,\eta}(\mathbb{R}, X)$ defined by

$$\begin{cases} D(\partial) = BC^1_{0,\eta}(\mathbb{R}, X) \\ \partial\varphi = \dfrac{\mathrm{d}\varphi}{\mathrm{d}x}. \end{cases}$$

Then the following lemma holds true.

Lemma 5.4 *The linear operator ∂ is the infinitesimal generator of a C_0-group $\{T_\partial(t)\}_{t\in\mathbb{R}}$ of bounded linear operators on $BC_{0,\eta}(\mathbb{R}, X)$, defined by*

$$T_\partial(t)(\varphi)(x) = \varphi(x+t), \forall t, x \in \mathbb{R}.$$

Moreover the following estimate holds true

$$|T_\partial(t)|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},X))} \leq e^{\eta|t|}, \forall t \in \mathbb{R}.$$

Here $|.|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},X))}$ denotes the operator norm associated to the norm $|.|_\eta$ on $BC_{0,\eta}(\mathbb{R}, X)$.

Proof. Note that for each $\varphi \in BC_{0,\eta}(\mathbb{R}, X)$ we have

$$2\left\|T_\partial(t)(\varphi)(x)\right\| = 2\left\|\varphi(x+t)\right\| \leq \left(e^{-\eta(x+t)} + e^{\eta(x+t)}\right)|\varphi|_\eta, \ \ \forall t \geq 0,\ x \in \mathbb{R}.$$

Therefore one gets

$$\left\|T_\partial(t)(\varphi)(x)\right\| \leq e^{\eta|t|}\cosh(\eta x)\,|\varphi|_\eta,$$

and the result follows. ∎

5.3 Second order differential operators

Let $\eta \geq 0$ be given and let $(X, \|.\|)$ be a Banach space. We consider in this section the following second order differential operator $\partial^2 : D\left(\partial^2\right) \subset BC_{0,\eta}(\mathbb{R}, X) \to BC_{0,\eta}(\mathbb{R}, X)$ defined by

$$\begin{cases} D\left(\partial^2\right) = BC^2_{0,\eta}(\mathbb{R}, X), \\ \partial^2\varphi = \dfrac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}. \end{cases}$$

Then the following lemma holds true.

Lemma 5.5 *The linear operator ∂^2 is the infinitesimal generator of an analytic semigroup $\{T_{\partial^2}(t)\}_{t\geq 0}$ of bounded linear operators on $BC_{0,\eta}(\mathbb{R},X)$. Moreover one has*

$$T_{\partial^2}(t)(\varphi)(x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}}\varphi(x-y)\mathrm{d}y \text{ for } t>0,$$

and

$$|T_{\partial^2}(t)|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},X))} \leq e^{\eta^2 t}, \forall t\geq 0.$$

Proof. We have for each $x\in\mathbb{R}$ and $\varphi\in BC_{0,\eta}(\mathbb{R},X)$

$$\begin{aligned}
2\|T_{\partial^2}(t)(\varphi)(x)\| &= 2\left\|\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}}\varphi(x-y)\mathrm{d}y\right\| \\
&\leq \frac{|\varphi|_\eta}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}}\left(e^{-\eta(x-y)}+e^{\eta(x-y)}\right)\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi t}}|\varphi|_\eta\left[e^{-\eta x}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+\eta y}\mathrm{d}y + e^{\eta x}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}-\eta y}\mathrm{d}y\right]
\end{aligned}$$

By using the formula

$$\int_{\mathbb{R}} e^{-\frac{z^2}{a}+bz}\mathrm{d}z = \sqrt{\pi a}e^{\frac{b^2 a}{4}}, \ \ \forall a>0,\ b\in\mathbb{R},$$

we obtain

$$\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+\eta y}\mathrm{d}y = \int_{\mathbb{R}} e^{-\frac{y^2}{4t}-\eta y}\mathrm{d}y = \sqrt{4\pi t}e^{\eta^2 t}.$$

This leads us to

$$2\|T_{\partial^2}(t)(\varphi)(x)\| \leq e^{\eta^2 t}\left(e^{-\eta x}+e^{\eta x}\right)|\varphi|_\eta,$$

and the result follows. ■

Now let us define the map $\Psi : BC_{0,0}(\mathbb{R},X)\to BC_{0,\eta}(\mathbb{R},X)$ by

$$\Psi(\psi)(x) = \cosh(\eta x)\psi(x). \tag{5.3}$$

Since Ψ is an isomorphism with $\Psi^{-1} : BC_{0,\eta}(\mathbb{R},X)\to BC_{0,0}(\mathbb{R},X)$ given by

$$\Psi^{-1}(\varphi)(x) = \cosh(\eta x)^{-1}\varphi(x), \tag{5.4}$$

the linear operator ∂^2 can be identified together with the linear operator $\widetilde{\partial^2}$: $D\left(\widetilde{\partial^2}\right)\subset BC_{0,0}(\mathbb{R},X)\to BC_{0,0}(\mathbb{R},X)$ defined by

$$\widetilde{\partial^2}\psi = \cosh(\eta x)^{-1}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\cosh(\eta x)\psi\right) = \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+2\eta\frac{\sinh(\eta x)}{\cosh(\eta x)}\frac{\mathrm{d}\psi}{\mathrm{d}x}+\eta^2\psi.$$

This remark will used in the next section.

5.4 Both first and second order systems

Let $\varepsilon > 0$ and $\gamma \in \mathbb{R}$. Let $\eta \geq 0$ be given. Consider the linear operator $\mathcal{D} : D(\mathcal{D}) \subset BC_{0,\eta}(\mathbb{R}, X) \to BC_{0,\eta}(\mathbb{R}, X)$ defined by

$$\begin{cases} D(\mathcal{D}) = BC^2_{0,\eta}(\mathbb{R}, X), \\ \mathcal{D} = \varepsilon \partial^2 + \gamma \partial. \end{cases}$$

By using the theory of commutating sums of operators, we obtain that $\mathcal{D}$ is the inifitesimal generator of a C_0-semigroup on $BC_{0,\eta}(\mathbb{R}, X)$ and

$$T_{\mathcal{D}}(t) = T_{\gamma\partial}(t) T_{\varepsilon\partial^2}(t) = T_{\partial}(\gamma t) T_{\partial^2}(\varepsilon t), \ \forall t \geq 0.$$

Then we infer from all the above explicit formula that $T_{\mathcal{D}}(t)$ is given by

$$T_{\mathcal{D}}(t)(\varphi)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}} \varphi(x + \gamma t - y) \mathrm{d}y = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x+\gamma t - y)^2}{4t}} \varphi(y) \mathrm{d}y.$$

This allows us to obtain the following growth rate estimate.

Lemma 5.6 *The linear operator $\mathcal{D}$ is the infinitesimal generator of strongly continuous semigroup $\{T_{\mathcal{D}}(t)\}_{t \geq 0}$ of bounded linear operators on $BC_{0,\eta}(\mathbb{R}, X)$. Moreover the following estimate holds true*

$$|T_{\mathcal{D}}(t)|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},X))} \leq e^{\left[\varepsilon\eta^2 + |\gamma|\eta\right]t}, \ \forall t \geq 0,$$

therefore

$$\omega_0(\mathcal{D}) \leq \left[\varepsilon\eta^2 + |\gamma|\,\eta\right].$$

In the rest of this section we consider a linear operator $A : D(A) \subset X \to X$ satisfying Assumptions 2.1 and 2.3. We are now concerned with the linear operator $\mathcal{D} + \mathcal{A}$. Using the above results coupled with the results obtained in Section 4 lead us to the following important theorem.

Theorem 5.7 *Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumptions 2.1 and 2.3. Let $\eta \geq 0$ be given. Then the linear operator $\mathcal{D} + \mathcal{A} : D(\mathcal{D}) \cap D(\mathcal{A}) \subset BC_{0,\eta}(\mathbb{R}, X) \to BC_{0,\eta}(\mathbb{R}, X)$ is closable, and its closure $\overline{\mathcal{D} + \mathcal{A}} : D\left(\overline{\mathcal{D} + \mathcal{A}}\right) \subset BC_{0,\eta}(\mathbb{R}, X) \to BC_{0,\eta}(\mathbb{R}, X)$ satisfies Assumptions 2.1 and 2.3. More precisely it satisfies the following properties:*

(i) *The following inclusion holds true*

$$BC^2_{0,\eta}(\mathbb{R}, X) \cap BC_{0,\eta}(\mathbb{R}, D(A)) \subset \overline{D\left(\overline{\mathcal{D} + \mathcal{A}}\right)} = BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right).$$

(ii) *The part of $\overline{\mathcal{D} + \mathcal{A}}$ in $BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right)$, denoted by $\left(\overline{\mathcal{D} + \mathcal{A}}\right)_0$ is the infinitesimal generator of a C_0- semigroup $\left\{T_{\left(\overline{\mathcal{D}+\mathcal{A}}\right)_0}(t)\right\}_{t \geq 0}$ on $BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right)$, and one has*

$$T_{\left(\overline{\mathcal{D}+\mathcal{A}}\right)_0}(t) = T_{\mathcal{D}}(t) T_{\mathcal{A}_0}(t) = T_{\mathcal{A}_0}(t) T_{\mathcal{D}}(t), \forall t \geq 0,$$

so that the following growth rate estimate holds true

$$\omega_0\left(\left(\overline{\mathcal{D}+\mathcal{A}}\right)_0\right) \leq \omega_0\left(A_0\right) + \left[\varepsilon\eta^2 + |\gamma|\,\eta\right].$$

(iii) *The linear operator $\overline{\mathcal{D}+\mathcal{A}}$ generates an exponential bounded (non degenerate) integrated semigroup $\left\{S_{\overline{\mathcal{D}+\mathcal{A}}}(t)\right\}_{t\geq 0}$ of bounded linear operators on $BC_{0,\eta}\left(\mathbb{R},X\right)$ given by*

$$S_{\overline{\mathcal{D}+\mathcal{A}}}(t) = \left(S_{\mathcal{A}} \diamond T_{\mathcal{D}}(t-.)\right)(t),\ \forall t\geq 0,$$

and

$$V^{\infty}\left(S_{\overline{\mathcal{D}+\mathcal{A}}},0,t\right) \leq e^{\left[\varepsilon\eta^2+|\gamma|\eta\right]t}V^{\infty}(S_A,0,t),\ \forall t\geq 0.$$

(iv) *The equality*

$$-\left(\overline{\mathcal{D}+\mathcal{A}}\right)u = v \text{ and } u\in D(\overline{\mathcal{D}+\mathcal{A}})$$

holds if and only if

$$\left[(\lambda-\mathcal{D})^{-1} + (\mu-\mathcal{A})^{-1}\right]u = (\lambda-\mathcal{D})^{-1}(\mu-\mathcal{A})^{-1}\left[v+(\lambda+\mu)u\right].$$

for some $\lambda\in\rho\left(\mathcal{D}\right)$ and $\mu\in\rho\left(\mathcal{A}\right)$. This last formula is also equivalent to

$$(\mu-A)^{-1}u(.)\in BC^2_{0,\eta}\left(\mathbb{R},X\right),$$

and, for some $\mu\in\rho\left(A\right)$ and any $x\in\mathbb{R}$,

$$-\left(\varepsilon\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\gamma\frac{\mathrm{d}}{\mathrm{d}x}\right)(\mu-A)^{-1}u(x) = -u(x)+(\mu-A)^{-1}\left[v(x)+\mu u(x)\right].$$

We complete this section by deriving an important estimate for the resolvent of the operator $\mathcal{D}+\mathcal{A}$. Roughly speaking, one will show, that under suitable conditions, the norm of the resolvent operator is uniformly bounded with respect to η small enough, the size of the exponential weight. The following Proposition will be fundamental, in the sequel, to construct a second order center manifold theory.

Proposition 5.8 (Resolvent estimate) *Let $A: D(A)\subset X\to X$ be a linear operator satisfying Assumptions 2.1 and 2.3. Assume furthermore that the growth rate of A_0 satisfies*

$$\omega_0(A_0)<0.$$

Consider $\eta_0>0$ the unique solution of

$$\omega_0\left(A_0\right)+\varepsilon\eta_0^2+|\gamma|\,\eta_0 = 0.$$

Let $\beta\in(0,\eta_0)$ be given. Then the following holds true

(i) *Let*

$$\rho^* := \left[\omega_0\left(A_0\right) + \varepsilon\beta^2 + |\gamma|\,\beta\right] < 0$$

then for each $\eta \in [0, \beta]$, *the linear operator* $\overline{\mathcal{D}+\mathcal{A}} : D\left(\overline{\mathcal{D}+\mathcal{A}}\right) \subset BC_{0,\eta}\left(\mathbb{R}, X\right) \to BC_{0,\eta}\left(\mathbb{R}, X\right)$ *satisfies*

$$\left(\rho^*, \infty\right) \subset \rho\left(\overline{\mathcal{D}+\mathcal{A}}\right).$$

(ii) *There exists* $C = C(\beta) > 0$ *such that for each* $\eta \in [0, \beta]$

$$\left|\left(-\overline{(\mathcal{D}+\mathcal{A})}\right)^{-1}\right|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},X))} \leq C(\beta).$$

Proof. Proof of (i). Let us first recall that according to the resolvent formula derived in Section 4,

$$\left(\lambda - \overline{(\mathcal{D}+\mathcal{A})}\right)^{-1} u = \mathbb{K}_{\mathcal{A}}\left(e^{-\lambda \cdot} T_{\mathcal{D}}(.)u\right)$$

which is well defined when

$$\omega_0\left(A_0\right) + \omega_0\left(\mathcal{D} - \lambda\right) < 0.$$

Therefore according to Lemma 5.6, it follows that

$$\left(\omega_0\left(A_0\right) + \left[\varepsilon\eta^2 + |\gamma|\,\eta\right], +\infty\right) \subset \rho\left(\overline{\mathcal{D}+\mathcal{A}}\right),$$

and (i) follows.
Proof of (ii). By using the maps Ψ_η and Ψ_η^{-1} defined respectively in (5.3) and (5.4), we have for each $u \in BC_{0,\eta}(\mathbb{R}, X)$

$$\begin{aligned}
\left|\left(-\overline{(\mathcal{D}+\mathcal{A})}\right)^{-1} u\right|_\eta &= \left|\Psi_\eta^{-1}\left(-\overline{(\mathcal{D}+\mathcal{A})}\right)^{-1} u\right|_0 \\
&= \left|\Psi_\eta^{-1}\left(-\overline{(\mathcal{D}+\mathcal{A})}\right)^{-1} \Psi_\eta \Psi_\eta^{-1} u\right|_0 \\
&= \left|\Psi_\eta^{-1} \mathbb{K}_{\mathcal{A}}\left(T_{\mathcal{D}}(.)\Psi_\eta \Psi_\eta^{-1} u\right)\right|_0.
\end{aligned}$$

Since Ψ_η commutes with $\mathcal{A}$, we deduce that

$$\left|\left(-\overline{(\mathcal{D}+\mathcal{A})}\right)^{-1} u\right|_\eta = \left|\mathbb{K}_{\mathcal{A}}\left(\Psi_\eta^{-1} T_{\mathcal{D}}(.)\Psi_\eta \Psi_\eta^{-1} u\right)\right|_0, \ \forall u \in BC_{0,\eta}\left(\mathbb{R}, X\right).$$

Next note that one has for all $v \in BC_{0,0}\left(\mathbb{R}, X\right)$ and $t \geq 0$

$$\left|\Psi_\eta^{-1} T_{\mathcal{D}}(t) \Psi_\eta v\right|_0 = \left|T_{\mathcal{D}}(t)\Psi_\eta v\right|_\eta \leq e^{\left[\varepsilon\eta^2 + |\gamma|\eta\right]t} \left|\Psi_\eta v\right|_\eta \leq e^{\left[\varepsilon\eta^2 + |\gamma|\eta\right]t} \left|v\right|_0.$$

Now by applying Proposition 3.10 (ii) we deduce that there exists some constant $C = C(\beta) > 0$, such that for all $\eta \in [0, \beta]$ and all $u \in BC_{0,\eta}(\mathbb{R}, X)$

$$\begin{aligned}\left|\left(-\overline{(\mathcal{D}+\mathcal{A})}\right)^{-1} u\right|_{\eta} &= \left|\mathbb{K}_{\mathcal{A}}\left(\Psi_{\eta}^{-1} T_{\mathcal{D}}(t) \Psi_{\eta} \Psi_{\eta}^{-1} u\right)\right|_{0} \\ &\leq C(\beta) \sup_{t \geq 0} e^{-\left[\varepsilon \eta^{2}+|\gamma| \eta\right] t}\left|\Psi_{\eta}^{-1} T_{\mathcal{D}}(t) \Psi_{\eta} \Psi_{\eta}^{-1} u\right|_{0} \\ &\leq C(\beta)\left|\Psi_{\eta}^{-1} u\right|_{0} = C(\beta)|u|_{\eta}.\end{aligned}$$

Hence the result follows. ■

6 Center manifolds

The aim of this section is to prove Theorem 3.6 that states the existence of a global center manifold for (1.3) when F is Lipschitz small. The first step of this construction is to reformulate (1.3) into a more suitable form. The second step consists in reformulating the entire orbits of (1.3) under a suitable fixed point problem. Before going to our main step let us mention and emphasize that throughout this section Assumptions 2.1, 2.3 and 2.5 are satisfied. We moreover assume that $\varepsilon = 1$.

6.1 Reformulation of (1.3)

As mentioned in Section 2, the idea of this re-formulation is to project the solution of (1.3) according to the decomposition $X = X_s \oplus X_{cu}$. Formally this corresponds to consider the following system posed for $x \in \mathbb{R}$

$$\begin{cases}\dfrac{\mathrm{d}^2 u_{cu}(x)}{\mathrm{d}x^2} - \gamma \dfrac{\mathrm{d}u_{cu}(x)}{\mathrm{d}x} + A_{cu} u_{cu}(x) = \Pi_{cu} F\left(u_{cu}(x) + u_s(x)\right), \\ \dfrac{\mathrm{d}^2 u_s(x)}{\mathrm{d}x^2} - \gamma \dfrac{\mathrm{d}u_s(x)}{\mathrm{d}x} + A_s u_s(x) = \Pi_s F\left(u_{cu}(x) + u_s(x)\right),\end{cases}$$

wherein $u_h(x) = \Pi_h u(x)$ for $h \in \{s, cu\}$. Recall that $\dim X_{cu} < \infty$ so that the first equation is an ODE in X_{cu}. A rigorous meaning of the above splitting is described in the following proposition.

Proposition 6.1 *Let Assumptions 2.1, 2.3 and 2.5 be satisfied. Let $\eta > 0$ be given. Consider the second order differential operators for $h \in \{s, cu\}$ defined on $BC_{0,\eta}(\mathbb{R}, X_h)$ by $\mathcal{D}_h = \partial^2 - \gamma\partial$. Then $u \in BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right)$ is a weak solution of (1.3) (according to Definition 2.4) if and only if the maps*

$$u_{cu} = \Pi_{cu} u \in BC_{0,\eta}(\mathbb{R}, X_{cu}) \text{ and } u_s = \Pi_s u \in BC_{0,\eta}(\mathbb{R}, X_s),$$

defined by

$$u_h(x) = \Pi_h(u(x)), \forall x \in \mathbb{R}, \forall h \in \{s, cu\},$$

are respectively weak solutions of

$$-\left(\overline{\mathcal{D}_{cu}+\mathcal{A}_{cu}}\right)u_{cu}=\Pi_{cu}F(u_{cu}+u_s)$$

and

$$-\left(\overline{\mathcal{D}_{s}+\mathcal{A}_{s}}\right)u_{s}=\Pi_{s}F(u_{cu}+u_s).$$

Here and in the sequel, $\mathcal{A}$ denotes the linear operator associated to A as in Subsection 5.1 while for each $h\in\{s,cu,0\}$, $\mathcal{A}_h$ denotes the linear operator associated to A_h as above.

Proof. Let us consider the linear operator on $BC_{0,\eta}(\mathbb{R},X)$ defined by $\mathcal{D}=\partial^2-\gamma\partial$. Let us first notice that due to Definition 2.4 and its equivalent formulation in Theorem 5.7 (iv), $u\in BC_{0,\eta}(\mathbb{R},X)$ is a weak solution of (1.3) if and only if one has for each $\lambda\in\rho(\mathcal{D})$ and $\mu\in\rho(\mathcal{A})$

$$\left[(\lambda-\mathcal{D})^{-1}+(\mu-\mathcal{A})^{-1}\right]u=(\lambda-\ \mathcal{D})^{-1}(\mu-\mathcal{A})^{-1}\left[F(u)+(\lambda+\mu)u\right]. \quad (6.1)$$

Next let us notice that for each $\lambda\in\rho(A)$ and each $u\in BC_{0,\eta}(\mathbb{R},X)$, one has for all $x\in\mathbb{R}$ and $h\in\{s,cu\}$

$$\Pi_h\left((\lambda I-\mathcal{A})^{-1}u\right)(x)=(\lambda I-A_h)^{-1}\Pi_h u(x)=\left((\lambda I-\mathcal{A}_h)^{-1}\Pi_h u\right)(x).$$

On the other hand, one has for all $\lambda\in\rho(\mathcal{D})$ and each $u\in BC_{0,\eta}(\mathbb{R},X)$

$$\Pi_h\left((\lambda I-\mathcal{D})^{-1}u\right)(x)=(\lambda I-\mathcal{D}_h)^{-1}\left(\Pi_h u\right)(x).$$

Next for each $x\in\mathbb{R}$, projecting (6.1) on X_h for $h\in\{s,cu\}$ leads us to the following system

$$\Pi_h\left[(\lambda-\mathcal{D})^{-1}+(\mu-\mathcal{A})^{-1}\right](u)(x)=\Pi_h(\lambda-\mathcal{D})^{-1}(\mu-\mathcal{A})^{-1}\left[F(u)+(\lambda+\mu)u\right](x).$$

Using the aforementioned commutating property yields

$$\left[(\lambda-\mathcal{D}_h)^{-1}+(\mu-\mathcal{A}_h)^{-1}\right]u_h=\Pi_h(\lambda-\ \mathcal{D}_h)^{-1}(\mu-\mathcal{A}_h)^{-1}\left[\Pi_h F(u)+(\lambda+\mu)u_h\right].$$

This proves the first of the result. The converse part of the proof can be handled similarly. ■

Remark 6.2 *Let $\eta>0$ be given and let $u\in BC_{0,\eta}\left(\mathbb{R},\overline{D(A)}\right)$ be a weak solution of (1.3). Then according to Proposition 6.1 the map*

$$u_{cu}:=\Pi_{cu}u\in BC_{0,\eta}\left(\mathbb{R},X_{cu}\right),$$

is a weak solution of

$$-\left(\overline{\mathcal{D}_{cu}+\mathcal{A}_{cu}}\right)u_{cu}=\Pi_{cu}F(u_{cu}+u_s).$$

Next when $\lambda \in \rho(A_{cu})$, then according to Theorem 5.7 (iv) one has $(\lambda - A_{cu})^{-1} u_{cu} \in BC^2_{0,\eta}(\mathbb{R}, X_{cu})$. Since $\dim X_{cu} < \infty$ then $(\lambda - A_{cu}) \in \mathcal{L}(X_{cu})$ so that $u_{cu} \in BC^2_{0,\eta}(\mathbb{R}, X_{cu})$. It therefore satisfies the following finite dimensional second order ordinary differential equation for each $x \in \mathbb{R}$

$$0 = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}}{\mathrm{d}x}\right] u_{cu}(x) + A_{cu} u_{cu}(x) + \Pi_{cu} F\left(u_s(x) + u_{cu}(x)\right).$$

As a consequence of the above proposition and the above remark, one obtains the following equivalent formulation.

Lemma 6.3 *Let Assumptions 2.1, 2.3 and 2.5 be satisfied. Let $\eta > 0$ be given. Then $u \in BC_{0,\eta}\left(\mathbb{R}, \overline{D(A)}\right)$ is a weak solution of (1.3) if and only if the maps*

$$\begin{pmatrix} u'_{cu} \\ u_{cu} \end{pmatrix} \in BC^1_{0,\eta}(\mathbb{R}, X_{cu} \times X_{cu}) \text{ and } u_s = \Pi_s u \in BC_{0,\eta}(\mathbb{R}, X_s),$$

satisfy:

(i) *The function u_s is a weak solution of*

$$-\left(\overline{\mathcal{D}_s + \mathcal{A}_s}\right) u_s = \Pi_s F(u_{cu} + u_s);$$

(ii) *The function (u_{cu}, u'_{cu}) satisfies, for all $x \in \mathbb{R}$, the problem*

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_{cu} \\ u'_{cu} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_{cu} & \gamma I \end{pmatrix}\begin{pmatrix} u_{cu} \\ u'_{cu} \end{pmatrix} + \begin{pmatrix} 0 \\ -\Pi_{cu} F\left(u_s(x) + u_{cu}(x)\right) \end{pmatrix}.$$

According to the above remark we set

$$\mathcal{B} = \begin{pmatrix} 0 & I \\ -A_{cu} & \gamma I \end{pmatrix} \in \mathcal{L}(X_{cu} \times X_{cu}) \text{ and } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} u_{cu} \\ u'_{cu} \end{pmatrix} \in X_{cu} \times X_{cu}.$$

We also set $G : X_{0s} \times (X_{cu} \times X_{cu}) \to X_{cu} \times X_{cu}$ and $H : X_{0s} \times (X_{cu} \times X_{cu}) \to X_s$ defined by

$$G\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, u_s\right) = \begin{pmatrix} 0 \\ -\Pi_{cu} F(v_1 + u_s) \end{pmatrix} \text{ and } H\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, u_s\right) = \Pi_s F(v_1 + u_s). \tag{6.2}$$

Together with these notations (1.3) becomes equivalent to the following system posed for $x \in \mathbb{R}$

$$\begin{cases} \dfrac{\mathrm{d}v(x)}{\mathrm{d}x} = \mathcal{B}v(x) + G\left(v(x), u_s(x)\right), \\ -\left(\overline{\mathcal{D}_s + \mathcal{A}_s}\right) u_s(x) = H\left(v(x), u_s(x)\right). \end{cases} \tag{6.3}$$

Because of the above formulation and keeping in mind that at the next stage we shall incorporate parameters into the system to investigate Hopf bifurcation theorem, we will prove in the next section the existence of a global center manifold for (1.3). This will be achieved by proving a more general version (see Theorem 2.9) for a class of systems of the form (6.3).

6.2 Existence and smoothness of the center manifold

As mentioned above, in this section we will prove the existence of a global center manifold for a system of the form (6.3) before coming back to the specific case (1.3). To do so, let $(Z, \|.\|)$ be a Banach space and $\widehat{A} : D\left(\widehat{A}\right) \subset Z \to Z$ a linear operator satisfying Assumption 2.1 and 2.3. Next we set $Z_0 := \overline{D\left(\widehat{A}\right)}$ and we assume that

$$\omega_0\left(\widehat{A}_0\right) < 0,$$

wherein $\left(\widehat{A}_0, D\left(\widehat{A}_0\right)\right)$ denotes the part of $\widehat{A}$ in Z_0. Now let us consider $(Y, \|.\|)$ a finite dimensional Banach space and let $\mathcal{B} \in \mathcal{L}(Y)$ be a given bounded linear operator. Next we set

$$\sigma_s(\mathcal{B}) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\},\ \sigma_u(\mathcal{B}) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\} \text{ and } \sigma_c(\mathcal{B}) = \sigma(\mathcal{B}) \cap i\mathbb{R}.$$

We also denote by $\Pi_\alpha^{\mathcal{B}} \in \mathcal{L}(Y)$ the spectral projector on the spectral set $\sigma_\alpha(\mathcal{B})$ for $\alpha \in \{s, u, c\}$. Next we assume that

$$\sigma_c(\mathcal{B}) \neq \emptyset.$$

Let $G : Y \times Z_0 \to Y$ and $H : Y \times Z_0 \to Z$ be two Lipschitz continuous maps such that

$$G(0, 0) = 0 \text{ and } H(0, 0) = 0.$$

Then we consider the problem for $x \in \mathbb{R}$

$$\begin{cases} \dfrac{\mathrm{d}v(x)}{\mathrm{d}x} = \mathcal{B}v(x) + G\left(v(x), u(x)\right), \\ \dfrac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} - \gamma\dfrac{\mathrm{d}u(x)}{\mathrm{d}x} + \widehat{A}u(x) + H\left(v(x), u(x)\right) = 0. \end{cases} \tag{6.4}$$

In order to construct a global center manifold for the above problem, let us set for $\alpha \in \{s, c, u\}$

$$Y_\alpha = \Pi_\alpha^{\mathcal{B}} Y \text{ and } \mathcal{B}_\alpha = \mathcal{B}_{|Y_\alpha} \in \mathcal{L}\left(Y_\alpha\right).$$

We also set $\Pi_h^{\mathcal{B}} = \Pi_s^{\mathcal{B}} + \Pi_u^{\mathcal{B}}$ and $Y_h = \Pi_h^{\mathcal{B}} Y = \left(I - \Pi_c^{\mathcal{B}}\right) Y$. Before stating our result, let us fix $\beta > 0$ such that

$$0 < \beta < \min\left(-\omega_0\left(\mathcal{B}_s\right), \omega_0\left(\mathcal{B}_u\right), \eta_0\right),$$

wherein $\eta_0 > 0$ is the unique solution

$$\eta_0^2 + |\gamma|\eta_0 + \omega_0\left(\widehat{A}_0\right) = 0.$$

Let us now recall the definition of a center manifold for the above problem.

Definition 6.4 ($\eta-$center manifold) *Let $\eta \in (0, \beta)$ be given. The $\eta-$center manifold of (6.4) denoted by $\mathbb{V}_\eta$ is the set of all $(u_0, v_0) \in Z_0 \times Y$ such that there exist a pair of functions $(u, v) \in BC_{0,\eta}(\mathbb{R}, Z_0) \times BC_{0,\eta}(\mathbb{R}, Y)$ such that*

(i) $(u,v)(0)=(u_0,v_0)$,

(ii) (u,v) *is a mild solution of (6.4) in the following sense*

$$u=-\left(\overline{\mathcal{D}+\widehat{\mathcal{A}}}\right)^{-1}H\left(u(.),v(.)\right),$$
$$v(x+y)=e^{\mathcal{B}y}v(x)+\int_x^y e^{\mathcal{B}(y-s)}G\left(u(s),v(s)\right)\mathrm{d}s,\quad \forall x\in\mathbb{R},\ \forall y\geq 0.$$

In the sequel we will prove that under some suitable assumptions, $\mathbb{V}_\eta$ is a graph over Y_c. This is the aim of the next theorem. This result will then be used to system (1.3) to derive Theorem 2.9 as a specific case.

Theorem 6.5 (Existence of a global manifold) *Under the above assumptions, let $\eta\in(0,\beta)$ be given. There exists $\delta_0=\delta_0(\eta)>0$ such that for each $(H,G)\in\mathrm{Lip}(Y\times Z_0,Z)\times\mathrm{Lip}(Y\times Z_0,Y)$ such that*

$$\|H\|_{\mathrm{Lip}(Y\times Z_0,Z)}+\|G\|_{\mathrm{Lip}(Y\times Z_0,Y)}\leq\delta_0, \tag{6.5}$$

then there exists $\Psi_\eta\in\mathrm{Lip}\left(Y_c,Z_0\times(Y_s\oplus Y_u)\right)$ such that

$$\mathbb{V}_\eta=\left\{\begin{pmatrix}0\\ y_c\end{pmatrix}+\Psi_\eta(y_c),\ \ y_c\in Y_c\right\}.$$

Moreover the following statements are equivalent:

(i) $(u,v)\in BC_{0,\eta}(\mathbb{R},Z_0)\times BC_{0,\eta}(\mathbb{R},Y)$ *is a mild solution of (6.4).*

(ii) *Function $(u,v)\in C\left(\mathbb{R};Z_0\times Y\right)$ satisfies*

$$\begin{pmatrix}u(x)\\ \Pi_h^{\mathcal{B}}v(x)\end{pmatrix}=\Psi_\eta\left(\Pi_c^{\mathcal{B}}v(x)\right)\ \textit{for all } x\in\mathbb{R},$$

and the map $w=\Pi_c^{\mathcal{B}}v(.):\mathbb{R}\to Y_c$ is a solution of the following ordinary differential equation

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}=\mathcal{B}_c w(x)+\Pi_c^{\mathcal{B}}G\left(\begin{pmatrix}0\\ w(x)\end{pmatrix}+\Psi_\eta\left(w(x)\right)\right),\ \forall x\in\mathbb{R}. \tag{6.6}$$

Before proving this result, let us mention that Theorem 2.9 becomes a direct consequence of the above theorem. Indeed due Lemma 6.3 (see also (6.3)), Problem (1.3) re-writes as (6.4) with

$$Y=X_{cu}\times X_{cu},\ Z=X_s \text{ and } Z_0=X_{0s},$$

while $\left(\widehat{A},D\left(\widehat{A}\right)\right)=\left(A_s,D\left(A_s\right)\right)$,

$$\mathcal{B}=\begin{pmatrix}0 & I\\ -A_{cu} & \gamma I\end{pmatrix}\in\mathcal{L}(X_{cu}\times X_{cu}),$$

and wherein functions H and G are defined in (6.2). Note that $\omega_0\left(\widehat{A}_0\right) = \omega_0\left(A_{0s}\right) < 0$ since $\omega_{0,ess}\left(A_0\right) < 0$. Next recalling the notations in (2.4) and Assumption 2.7 one may observe that

$$\sigma_c\left(\mathcal{B}\right) = \sigma_{\mathcal{P}}(A) \text{ while } \Pi_c^{\mathcal{B}} = \begin{pmatrix} \Pi_{\mathcal{P}} & 0 \\ 0 & \Pi_{\mathcal{P}} \end{pmatrix},$$

so that $Y_c = X_{\mathcal{P}} \times X_{\mathcal{P}}$ and $Y_h = X_{\mathcal{Q}} \times X_{\mathcal{Q}}$. Due to this remark, Theorem 2.9 is a simple re-formulation of the more general result Theorem 6.5 in the above described context.

We are now in position to prove Theorem 6.5. This proof requires some preliminaries.

Let us first define for each $\eta \in [0, \beta]$ and each $f \in BC_{0,\eta}(\mathbb{R}, Y)$ the linear operator

$$\mathcal{K}(f)(t) = \int_0^t e^{\mathcal{B}(t-\tau)}\Pi_c^{\mathcal{B}} f(\tau)\mathrm{d}\tau - \int_t^{\infty} e^{\mathcal{B}(t-\tau)}\Pi_u^{\mathcal{B}} f(\tau)\mathrm{d}\tau + \int_{-\infty}^t e^{\mathcal{B}(t-\tau)}\Pi_s^{\mathcal{B}} f(\tau)\mathrm{d}\tau.$$

Then as proved in [77], the following estimates hold true.

Lemma 6.6 *For each $\eta \in [0, \beta)$, $\mathcal{K}$ is a bounded linear operator from $BC_{0,\eta}(\mathbb{R}, Y)$ into itself. Furthermore, there exists some constant $M > 0$ such that for each $\nu \in (-\beta, 0)$ there exists some constant $\widehat{C}_{s,\nu} > 0$ such that for all $\eta \in (0, -\nu)$:*

$$\|\mathcal{K}\|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},Y))} \leq \widehat{C}_{s,\nu} + \frac{M}{\beta - \eta} + \widetilde{C}_\eta,$$

wherein we have set

$$\widetilde{C}_\eta = \|\Pi_c^B\|_{\mathcal{L}(Y)} \max\left(\int_0^\infty \left\|e^{(\mathcal{B}_c - \eta)t}\right\| \mathrm{d}t, \int_0^\infty \left\|e^{-(\mathcal{B}_c + \eta)t}\right\| \mathrm{d}t\right).$$

Consider now the linear operator $\mathcal{K}_c$ defined by

$$\mathcal{K}_c(y_c)(t) = e^{t\mathcal{B}_c} y_c, \ \ t \in \mathbb{R}, \ y_c \in Y_c.$$

Then the following estimate holds true.

Lemma 6.7 *For each $\eta \in (0, \beta)$, $\mathcal{K}_c \in \mathcal{L}\left(Y_c, BC_{0\eta}(\mathbb{R}, Y_c)\right)$ and the following operator norm estimate holds true:*

$$\|\mathcal{K}_c\|_{\mathcal{L}(Y_c, BC_{0\eta}(\mathbb{R},Y_c))} \leq \max\left(\sup_{t\geq 0}\left\|e^{(\mathcal{B}_c - \eta)t}\right\|, \sup_{t \geq 0}\left\|e^{-(\mathcal{B}_c + \eta)t}\right\|\right).$$

We also refer to [77] for the proof of this estimate.

Using the above notations, we are able to state the fixed point problem that will be used to derive Theorem 6.5.

Lemma 6.8 *Let $\eta \in (0,\beta)$ be given. Then $(u_0,v_0) \in \mathbb{V}_\eta \subset Z_0 \times Y$ if and only if there exist $(u,v) \in BC_{0,\eta}(\mathbb{R},Z_0) \times BC_{0,\eta}(\mathbb{R},Y)$ such that*

$$
\begin{aligned}
&u(0) = u_0, \;\; v(0) = v_0, \\
&v = \mathcal{K}_c\left(\Pi_c^{\mathcal{B}} y\right) + \mathcal{K}G(u(.),v(.)), \\
&u = -\left(\overline{\mathcal{D}+\widehat{\mathcal{A}}}\right)^{-1} H(u(.),v(.)).
\end{aligned}
$$

The proof of this lemma is straightforward by using Lemma 4.9 in [54] as well as Definition 6.4.
Proof of Theorem 6.5. Let $\eta \in (0,\beta)$ be given. Let us recall that due to the resolvent estimate derived in Proposition 5.8 there exists some constant $C(\beta) > 0$ such that for each $\eta \in [0,\beta]$ one has

$$
\left\| -\left(\overline{\mathcal{D}+\widehat{A}}\right)^{-1} \right\|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},Z))} \leq C(\beta).
$$

For simplicity, in the sequel we will write $\mathcal{R} = -\left(\overline{\mathcal{D}+\widehat{A}}\right)^{-1}$. Consider now $\delta_0 > 0$ small enough arising in (6.5) such that

$$
\delta_0\left(\|\mathcal{K}\|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},Y))} + C(\beta)\right) < 1.
$$

Consider the map $T : BC_{0,\eta}(\mathbb{R},Z_0) \times B_{0,\eta}(\mathbb{R},Y) \to BC_{0,\eta}(\mathbb{R},Z_0) \times B_{0,\eta}(\mathbb{R},Y)$ defined by

$$
T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{R}H(u(.),v(.)) \\ \mathcal{K}G(u(.),v(.)) \end{pmatrix}. \tag{6.7}
$$

Then due to the choice of δ_0, the map $I - T$ is invertible and one has

$$
\left\|(I-T)^{-1}\right\|_{\text{Lip}} \leq \frac{1}{1-\delta_0\left(\|\mathcal{K}\|_{\mathcal{L}(BC_{0,\eta}(\mathbb{R},Y))} + C(\beta)\right)}.
$$

We define for each $y_c \in Y_c$ the pair of functions $(U(y_c),V(y_c)) \in BC_{0,\eta}(\mathbb{R},Z_0) \times B_{0,\eta}(\mathbb{R},Y)$ by

$$
\begin{pmatrix} U(y_c)(.) \\ V(y_c)(.) \end{pmatrix} = (I-T)^{-1}\begin{pmatrix} 0 \\ \mathcal{K}_c(y_c)(.) \end{pmatrix}.
$$

Next we set $\Psi_\eta : Y_c \to Z_0 \times Y_h$ defined by

$$
\Psi_\eta(y_c) = \begin{pmatrix} U(y_c)(0) \\ \Pi_h^{\mathcal{B}} V(y_c)(0) \end{pmatrix},
$$

so that the existence result follows.

It remains to show that (i) and (ii) are equivalent. To do so, let us now assume that $(u,v) \in BC_{0,\eta}(\mathbb{R},Z_0) \times BC_{0,\eta}(\mathbb{R},Y)$ is a mild solution of (6.4). Then from the definition of $\mathbb{V}_\eta$, one has $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in \mathbb{V}_\eta$ for all $t \in \mathbb{R}$. Hence

$\begin{pmatrix} u(t) \\ \Pi_h^{\mathcal{B}} v(t) \end{pmatrix} = \Psi_\eta\left(\Pi_c^{\mathcal{B}} v(t)\right)$ for all $t \in \mathbb{R}$. Then projecting the $v-$equation in (6.4) on Y_c implies that the function $w := \Pi_c^{\mathcal{B}} v(.)$ is solution of (6.6).

Assume now that $(u, v) \in C(\mathbb{R}, Z_0 \times Y)$ satisfies (ii). Then from the definition of $\mathbb{V}_\eta$, one has

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in \mathbb{V}_\eta \quad \forall t \in \mathbb{R}.$$

In particular $(u_0, v_0) := (u, v)(0) \in \mathbb{V}_\eta$ and therefore there exists $(U, V) \in BC_{0,\eta}(\mathbb{R}, Z_0) \times BC_{0,\eta}(\mathbb{R}, Y)$ a weak solution of (6.4) such that $(U, V)(0) = (u_0, v_0)$. As a consequence of the above part, function $W(t) = \Pi_c^{\mathcal{B}} U(t)$ satisfies (6.6) together with $W(0) = \Pi_c^{\mathcal{B}} v(0)$. The uniqueness result for ordinary differential equations with Lipschitz continuous nonlinearity implies that

$$\Pi_c^{\mathcal{B}} V(t) = \Pi_c^{\mathcal{B}} v(t), \quad \forall t \in \mathbb{R}.$$

As a consequence of this together with the first relation into (ii) implies that $(u, v)(t) \equiv (U, V)(t)$ and therefore we conclude that $(u, v) \in BC_{0,\eta}(\mathbb{R}, Z_0) \times BC_{0,\eta}(\mathbb{R}, Y)$ is a weak solution of (6.4). This completes the proof of the result. ■

We will now investigate Theorem 2.10. The proof relies on the fibre contraction theorem applied to the truncated nonlinearity. Fix the nonlinear functions $G : Y \times Z_0 \to Y$ and $H : Y \times Z_0 \to Z$ of the class C^k for some $k \geq 1$ such that

$$G(0, 0) = 0, \ H(0, 0) = 0 \text{ and } DG(0, 0) = 0, \ DH(0, 0) = 0.$$

In order to state the result, let us consider for each $r > 0$ the nonlinear functions $G_r : Y \times Z_0 \to Y$ and $H_r : Y \times Z_0 \to Z$ defined by

$$\begin{aligned} G_r(v, u) &= G(v, u)\chi_Y\left(r^{-1}v\right)\chi_Z\left(r^{-1}\|u\|_Z\right) \\ H_r(v, u) &= H(v, u)\chi_Y\left(r^{-1}v\right)\chi_Z\left(r^{-1}\|u\|_Z\right), \end{aligned}$$

wherein function $\chi_Y : Y \to \mathbb{R}^+$ is a C^∞ function such that $\chi_Y(y) = 1$ if $\|y\| \leq 1$ and $\chi_Y(y) = 0$ if $\|y\| \geq 2$. Recall that such a smooth map exists since Y is a finite dimensional space. Furthermore function $\chi_Z : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^∞ function such that $\chi_Z(z) = 1$ if $z \in [0, 1]$ and $\chi_Z(z) = 0$ for all $z \geq 2$. Now note that one has as $r \to 0^+$

$$\begin{aligned} &\sup_{(v,u) \in Y \times Z_0} \|G_r(v, u)\|_Y + \sup_{(v,u) \in Y \times Z_0} \|H_r(v, u)\|_Z = O\left(r^2\right), \\ &\|H_r\|_{\mathrm{Lip}(Y \times Z_0, Z)} + \|G_r\|_{\mathrm{Lip}(Y \times Z_0, Y)} = O(r), \\ &(G_r, H_r) \in C_b^k\left(Y \times B_{Z_0}(0, r), Y \times Z\right). \end{aligned}$$

Here the subscript b stands for bounded. Note now that because of the above estimate, Theorem 6.5 applies as soon as r is small enough. Then the following proposition holds true.

Proposition 6.9 *Under the above assumptions, let $\eta \in (0, \beta)$ be given small enough. Then there exists $r_0 = r_0(\eta)$ such that for each $r \in (0, r_0)$ the map Ψ constructed in Theorem 6.5 for (6.4) with (G, H) replaced by (G_r, H_r) satisfies*

$$\Psi \in C_b^k(Y_c; Z_0 \times Y_h), \ \Psi(0) = 0 \text{ and } D\Psi(0) = 0.$$

The proof of this result is an application of the fibre contraction theorem using the fixed point formulation associated to operator T in (6.7). We refer to Vanderbauwhede in [77], Vanderbauwhede and Iooss [81] and Magal and Ruan [54]. To see this we shall reformulate the fixed point problem as in [54] so that the methodology to prove such a smoothness property is similar. Consider the map $\Phi_{(H_r,G_r)} : BC_{0,\eta}(\mathbb{R}, Z_0) \times B_{0,\eta}(\mathbb{R}, Y) \to BC_{0,\eta}(\mathbb{R}, Z_0) \times B_{0,\eta}(\mathbb{R}, Y)$

$$\Phi_{(H_r,G_r)} \begin{pmatrix} u \\ v \end{pmatrix} (t) = \begin{pmatrix} H_r(u(t), v(t)) \\ G_r(u(t), v(t)) \end{pmatrix},$$

Consider also the linear operator $K_2 \in \mathcal{L}(BC_{0,\eta}(\mathbb{R}, Z_0) \times BC_{0,\eta}(\mathbb{R}, Y))$ defined by

$$K_2 = \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{K} \end{pmatrix},$$

so that operator T defined in (6.7) re-writes as $T = K_2 \Phi_{(H_r,G_r)}$. Consider for r small enough the nonlinear operator $\Gamma_0 = (I - T)^{-1}$ and the linear operator $L \in \mathcal{L}(BC_{0,\eta}(\mathbb{R}, Z_0) \times B_{0,\eta}(\mathbb{R}, Y), X \times Y)$ defined by

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \Pi_h^{\mathcal{B}} \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$$

Together with these notations, recall that the graph Ψ re-writes for all $y_c \in Y_c$ as

$$\Psi(y_c) = L\Gamma_0(K_1(y_c)),$$

wherein we have set $K_1(y_c) = \begin{pmatrix} 0 \\ \mathcal{K}(y_c) \end{pmatrix}$. Moreover Γ_0 satisfies

$$\Gamma_0(U) = U + K_2 \Phi_{(G_r,H_r)}(\Gamma_0(U)), \ \ \forall U \in \{0\} \times BC_{0,\eta}(\mathbb{R}, Y_c).$$

Hence Γ_0 is a fixed point for the following problem

$$\Gamma_0 = J + K_2 \Phi_{(G_r,H_r)}(\Gamma_0),$$

wherein J denotes the embedding from $\{0\} \times BC_{0,\eta}(\mathbb{R}, Y_c)$ into $BC_{0,\eta}(\mathbb{R}, Z_0) \times B_{0,\eta}(\mathbb{R}, Y)$. This fixed point formulation is similar to the one studied by Magal and Ruan [54] and, as explained above, the proof of Proposition 6.9 is similar.

Now let us notice that a solution (u, v) of (6.4) with (G, H) coincides together with a solution of (6.4) with (G_r, H_r) as soon as $(u, v)(x) \in \Omega := B_Y(0, r) \times B_{Z_0}(0, r)$ for all $x \in \mathbb{R}$. This remark directly implies the following local center manifold result for (6.4).

Theorem 6.10 *Let $G : Y \times Z_0 \to Y$ and $H : Y \times Z_0 \to Z$ be two given functions of the class C^k for some $k \geq 1$ such that*

$$G(0,0) = 0,\ H(0,0) = 0 \text{ and } DG(0,0) = 0,\ DH(0,0) = 0.$$

Next under the assumptions of Theorem 6.5 for the linear operator $\widehat{A}$ and $\mathcal{B}$, let $\eta \in (0, \beta)$ be given small enough. Then there exists $\Psi \in C^k(Y_c, Z_0 \times Y_h)$ with

$$\Psi(0) = 0 \text{ and } D\Psi(0) = 0,$$

and there exists Ω a bounded neighbourhood of 0 in $Z_0 \times Y$ satisfying the following properties:

(i) *If $v_c : \mathbb{R} \to Y_c$ is a solution of the reduced equation (6.6) then for some η small enough*

$$v_c \in BC_{0,\eta}(\mathbb{R}, Y_c).$$

Moreover if

$$\begin{pmatrix} 0 \\ v_c(x) \end{pmatrix} + \Psi(v_c(x))) \in \Omega, \forall x \in \mathbb{R},$$

then

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} 0 \\ v_c(x) \end{pmatrix} + \Psi\left(\Pi_c^{\mathcal{B}} v(t)\right) \text{ for all } t \in \mathbb{R},$$

is a classical solution of (6.4) on $\mathbb{R}$.

(ii) *If $(u, v) \in BC_{0,\eta}(\mathbb{R}, Y \times Z_0)$ is a weak solution of (6.4) such that*

$$(u, v)(x) \in \Omega, \forall x \in \mathbb{R},$$

then function $v_c = \Pi_c^{\mathcal{B}} v$ is a solution of the reduced equation (6.6).

Finally recalling that (1.3) re-writes in the form (6.4) (see the comments after Theorem 6.5), this also completes the proof of Theorem 2.10

7 Hopf bifurcation

This section is concerned with Hopf bifurcation. In the first subsection we state our general Hopf bifurcation theorem using the ODE reduction we have obtained in the previous section. Then we apply this general theorem to investigate the persistence of non-degenerate Hopf bifurcation for some singularly perturbed second order equation.

7.1 Proof of Theorem 2.11

This section is devoted to the study of Hopf bifurcation theorem. To that aim we consider the following problem depending on a real parameter $\mu \in \mathbb{R}$

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + F(\mu, u(x)) = 0, \ x \in \mathbb{R}. \tag{7.1}$$

In order to investigate the above problem, we first incorporate the parameter μ into the system by considering the following system of equation

$$\begin{cases} \dfrac{\mathrm{d}\mu(x)}{\mathrm{d}x} = 0, \\ \dfrac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - \gamma \dfrac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + F(\mu(x), u(x)) = 0. \end{cases} \tag{7.2}$$

Next projecting the second equation on X_s and X_{cu} and using the reduction introduced in the previous section, we obtain a similar form as the one proposed in (6.4). To be more precise, let us assume that Assumptions 2.1, 2.3 and 2.5 are satisfied, then the system (7.2) is equivalent to the following ones

$$\begin{cases} \dfrac{\mathrm{d}v(x)}{\mathrm{d}x} = \mathcal{B}v(x) + G(v(x), u_s(x)), \\ -(\mathcal{D}_s + \mathcal{A}_s) u_s = \Pi_s F(\mu, u_{cu} + u_s), \end{cases} \tag{7.3}$$

wherein we have set $\mathcal{B} \in \mathcal{L}(Y)$ with $Y = \mathbb{R} \times X_{cu} \times X_{cu}$ and

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -A_{cu} & \gamma I \end{pmatrix}, \quad v = \begin{pmatrix} \mu \\ u_{cu} \\ u'_{cu} \end{pmatrix},$$

and

$$G(v, u_s) = \begin{pmatrix} 0 \\ 0 \\ -\Pi_{cu} F(\mu, u_{cu} + u_s) \end{pmatrix} \text{ and } H(v, u_s) = \Pi_s F(\mu, u_{cu} + u_s).$$

Finally let us mention that Theorem 6.10 applies and provides a reduction technique that will be be used in the sequel to derive a Hopf bifurcation theorem. Before doing so, let us notice that the matrix operator $\mathcal{B}$ satisfies the following straightforward properties

Lemma 7.1 *Let Assumptions 2.1, 2.3 and 2.5 be satisfied. The linear operator $\mathcal{B}$ satisfies the following properties*

(i) *The spectrum of $\mathcal{B}$ is given by*

$$\sigma(\mathcal{B}) = \{0\} \cup \left\{ z \in \mathbb{C} : -z^2 + \gamma z \in \sigma(A_{cu}) \right\}.$$

(ii) *One has*

$$\Pi_c^{\mathcal{B}} = \begin{pmatrix} I & 0 & 0 \\ 0 & \Pi_{\mathcal{P}} & 0 \\ 0 & 0 & \Pi_{\mathcal{P}} \end{pmatrix} \in \mathcal{L}(Y) \text{ and } Y_c = \mathbb{R} \times X_{\mathcal{P}} \times X_{\mathcal{P}}.$$

We now focus on Hopf bifurcation theorem for (7.1). To do so we shall assume that the following set of assumptions are fulfilled.

Assumption 7.2 *Let $\eta > 0$ be given and $F \in C^k\left((-\eta,\eta) \times B_{X_0}(0,\eta)\right)$ for some $k \geq 4$. Assume that the following statements are satisfied:*

(i) *$F(\mu, 0) = 0$ for all $\mu \in (-\eta, \eta)$ and $\partial_u F(0,0) = 0$.*

(ii) *The linear operator A satisfies Assumptions 2.1, 2.3 and 2.5.*

(iii) *For each $\mu \in (-\eta,\eta)$, there exists a pair of conjugated simple eigenvalues of $\left(A + \partial_u F(\mu, 0)\right)_{cu}$, denoted by $\lambda^{(1)}(\mu)$ and $\overline{\lambda^{(1)}(\mu)}$, such that*

$$\lambda^{(1)}(0) = \omega_0^2 + i\gamma\omega_0 \text{ for some } \omega_0 > 0,$$

and

$$\sigma\left(A_0\right) \cap \mathcal{P} = \left\{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\right\} \text{ with } \mathcal{P} = \{\xi^2 + i\gamma\xi;\ \xi \in \mathbb{R}\}. \tag{7.4}$$

We furthermore assume that the map $\mu \mapsto \lambda^{(1)}(\mu)$ is continuously differentiable and satisfies

$$\operatorname{Re}\left[\frac{1}{\gamma - 2i\omega_0} \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu}\right] \neq 0. \tag{7.5}$$

Then the following Hopf bifurcation theorem holds true.

Theorem 7.3 (Hopf bifurcation theorem) *Let Assumption 7.2 be satisfied. Then, there exist a constant $\varepsilon^* > 0$ and three C^{k-1} maps, $\varepsilon \mapsto \mu(\varepsilon)$ from $(0,\varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \mapsto (x_\varepsilon, y_\varepsilon)$ from $(0, \varepsilon^*)$ into $X_0 \times X_0$, and $\varepsilon \to \gamma(\varepsilon)$ from $(0,\varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0,\varepsilon^*)$ there exists a $\gamma(\varepsilon)-$periodic function $u_\varepsilon \in C^k(\mathbb{R}, X_0)$, which is a weak solution of (7.1) with the parameter value $\mu = \mu(\varepsilon)$ and such that $u_\varepsilon(0) = x_\varepsilon$ and $u_\varepsilon'(0) = y_\varepsilon$.*

Remark 7.4 *Using Crandall and Rabinowitz's Hopf bifurcation theorem given in [18], the above theorem holds true when we only assume that F is the class C^2 and that condition (7.4) is replace by*

$$\sigma\left(A_0\right) \cap \mathcal{P}_{\xi_0} = \left\{\lambda(0), \overline{\lambda(0)}\right\} \text{ with } \mathcal{P}_{\xi_0} = \{\xi^2 + i\gamma\xi;\ \xi \in \xi_0\mathbb{N}\}. \tag{7.6}$$

The proof of the above result relies on the usual Hopf bifurcation theorem for ordinary differential equations after the reduction provided by Theorem 6.10. To see this, let us consider the matrix operator $\mathcal{B}(\mu)$ acting from $\mathbb{R} \times X_{cu} \times X_{cu}$ into itself and defined by

$$\mathcal{B}(\mu) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -\left[A + \partial_u F(\mu, 0)\right]_{cu} & \gamma I \end{pmatrix}.$$

Now note that Theorem 6.10 applies and allows us to reduce the problem to an ordinary differential system of equations involving $\mathcal{B}_c(\mu)$. To complete the proof of Theorem 7.3, we need to investigate the center spectrum of $\mathcal{B}(\mu)$ at $\mu = 0$ and check the transversality condition (see for instance [39] and [50]).

Note that when μ is small enough, the above matrix $\mathcal{B}(\mu)$ has the following properties:

(i) For each $\mu \in (-\eta, \eta)$, 0 is a simple eigenvalue.

(ii) Up to reduce η, for each $\mu \in (-\eta, \eta)$, there exists a map $\mu \mapsto \lambda^{(2)}(\mu)$ of the class C^1 such that $\lambda^{(2)}(\mu)$ and $\overline{\lambda^{(2)}(\mu)}$ are a pair of conjugated simple eigenvalues of $\mathcal{B}(\mu)$,

$$\lambda^{(2)}(0) = i\omega_0 \text{ and } \operatorname{Re} \frac{\mathrm{d}\lambda^{(2)}(0)}{\mathrm{d}\mu} \neq 0.$$

In order to justify the above claim, it is sufficient to prove (ii). To do so, let us notice that for each $\mu \in (-\eta, \eta)$

$$\lambda^{(2)}(\mu) \in \sigma\left(\mathcal{B}(\mu)\right) \Leftrightarrow -\left[\lambda^{(2)}(\mu)\right]^2 + \gamma\lambda^{(2)}(\mu) \in \sigma\left(\left[A + \partial_u F(\mu, 0)\right]_{cu}\right),$$

and

$$\lambda^{(2)}(\mu) \in i\mathbb{R} \Leftrightarrow -\left[\lambda^{(2)}(\mu)\right]^2 + \gamma\lambda^{(2)}(\mu) \in \mathcal{P}.$$

Finally let us mention that $\lambda^{(2)}(\mu)$ is a simple eigenvalue for $\mathcal{B}(\mu)$ if and only if $-\left[\lambda^{(2)}(\mu)\right]^2 + \gamma\lambda^{(2)}(\mu)$ is a simple eigenvalue of $\left[A + \partial_u F(\mu, 0)\right]_{cu}$.

On the other hand, let us notice that using the implicit function theorem, there exist $\eta_0 \in (0, \eta)$ and a map $\lambda \to \lambda^{(2)}(\mu)$ of the class C^1 defined from $(-\eta_0, \eta_0)$ such that

$$\begin{cases} \left[\lambda^{(2)}(\mu)\right]^2 - \gamma\lambda^{(2)}(\mu) + \lambda^{(1)}(\mu) = 0, \quad \forall \mu \in (-\eta_0, \eta_0), \\ \lambda^{(2)}(0) = i\omega_0. \end{cases}$$

Next note that

$$\frac{\mathrm{d}\lambda^{(2)}(0)}{\mathrm{d}\mu}\left[2i\omega_0 - \gamma\right] + \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} = 0,$$

so that (7.5) is equivalent to the usual transversality condition for $\lambda^{(2)}$ at $\mu = 0$, that reads as

$$\operatorname{Re} \frac{\mathrm{d}\lambda^{(2)}(0)}{\mathrm{d}\mu} \neq 0.$$

Finally Theorem 7.3 therefore follows.

7.2 Persistence of non-degenerate Hopf bifurcation

The aim of this section is to compare the non-degenerate Hopf bifurcation problem for a first order abstract equation of the form

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(\mu, u(t)), \ t \in \mathbb{R}, \tag{7.7}$$

together with the singularly perturbed second order equation

$$\varepsilon^2 \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) + F(\mu, u(t)) = 0. \tag{7.8}$$

Here $\varepsilon > 0$ is a small parameter while $\mu \in \mathbb{R}$ is a bifurcation parameter.

Remark 7.5 *Note that by setting $\gamma = \frac{1}{\varepsilon}$, then u is a solution of (7.8) if and only if the function $\tilde{u}(t) = u(\varepsilon t)$ is a solution of (1.3). Hence periodic solutions of (7.8) for small ε corresponds to fast wave train for (1.3), namely with γ very large.*

The aim of this section is to prove that non-degenerate Hopf bifurcation for (7.7) is persistent for (7.8) when ε is small enough. In order to make this statement more precise, we will assume that

Assumption 7.6 *We assume that operator A and function F satisfies Assumption 7.2 (i) and (ii). Next we assume that for each $\mu \in (-\eta, \eta)$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_u F(\mu, 0))_0$, denoted by $\lambda^{(1)}(\mu)$ and $\overline{\lambda^{(1)}(\mu)}$, such that*

$$\lambda^{(1)}(0) = i\omega_0 \textit{ for some } \omega_0 > 0,$$

and

$$\sigma(A_0) \cap i\mathbb{R} = \left\{ \lambda(0), \overline{\lambda(0)} \right\}. \tag{7.9}$$

We furthermore assume that the map $\mu \mapsto \lambda^{(1)}(\mu)$ is continuously differentiable and satisfies

$$\mathrm{Re}\, \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} \neq 0. \tag{7.10}$$

Recall that the above assumption implies that $\mu = 0$ is a Hopf bifurcation point for the first order equation (7.7) (we refer to [50] for such a result). Here we will prove the following result.

Theorem 7.7 *Let Assumption 7.6 be satisfied. Assume that*

$$\mathrm{Re}\, \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} > 0 \textit{ respectively} < 0. \tag{7.11}$$

Then there exist $0 < \widehat{\eta} \leq \eta$ and a map $\varepsilon \equiv \varepsilon(\mu)$ defined from $(0, \widehat{\eta})$ (resp. on $(-\widehat{\eta}, 0)$) into $(0, \infty)$ such that for each $\mu \in (0, \widehat{\eta})$ (respect for each $\mu \in (-\widehat{\eta}, 0)$, μ is a Hopf bifurcation point for (7.8) with $\varepsilon = \varepsilon(\mu)$.

Proof. In order to prove this result, we will assume, without loss of generality that (due to (7.10)) that

$$\mathrm{Re}\,\frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} > 0. \tag{7.12}$$

Then, up to reduce $\eta > 0$, one may assume that there exists $\delta > 0$ such that for each $\mu \in [0, \eta)$

$$\sigma\left[A_{cu} + \partial_u F(\mu, 0)\right]_{cu} \cap \{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z \leq \delta\} = \left\{\lambda^{(1)}(\mu), \overline{\lambda^{(1)}(\mu)}\right\}. \tag{7.13}$$

Now for each $\mu \in (0, \eta)$, there exists a unique $\varepsilon = \varepsilon(\mu)$ such that

$$\left\{\lambda^{(1)}(\mu), \overline{\lambda^{(1)}(\mu)}\right\} \subset \mathcal{P}_{\varepsilon(\mu)},$$

wherein $\mathcal{P}_\varepsilon$ denotes the parabola

$$\mathcal{P}_\varepsilon := \left\{\varepsilon^2\xi^2 - i\xi^2,\ \xi \in \mathbb{R}\right\}.$$

It is easy to check that for each $\mu \in (0, \eta)$, one has

$$\varepsilon(\mu) = \sqrt{\frac{\mathrm{Re}\,\lambda^{(1)}(\mu)}{\left(\mathrm{Im}\,\lambda^{(1)}(\mu)\right)^2}}. \tag{7.14}$$

Now, up to reduce η if necessary, there exists $M > 0$ such that for each $\mu \in [0, \eta)$

$$\sigma\left[A_{cu} + \partial_u F(\mu, 0)\right]_{cu} \cap \{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z \text{ and } |\mathrm{Im}\, z| \geq M\} = \emptyset.$$

As a consequence of this property and since $\varepsilon(\mu) \to 0$ when $\mu \to 0$ (see (7.14)), up to reduce η, one obtains that for each $\mu \in (0, \eta)$

$$\sigma\left[A_{cu} + \partial_u F(\mu, 0)\right]_{cu} \cap \mathcal{P}_{\varepsilon(\mu)} = \left\{\lambda^{(1)}(\mu), \overline{\lambda^{(1)}(\mu)}\right\}.$$

Now let $\mu^* \in (0, \eta)$ be given. We set $\varepsilon^* = \varepsilon(\mu^*)$. We will show that μ^* is a Hopf bifurcation point of (7.8) with $\varepsilon = \varepsilon^*$. To do so, it remains to check Assumption 7.2 (*iii*). More specifically, due to the above computations, it remains to check the transversality condition (2.10) at $\mu = \mu^*$. This conditions reads as

$$\mathrm{Re}\left[\frac{1}{1 - 2i\varepsilon^*\omega_0^*}\frac{\mathrm{d}\lambda^{(1)}(\mu^*)}{\mathrm{d}\mu}\right] \neq 0, \tag{7.15}$$

where $\omega_0^* > 0$ is given by

$$\omega_0^* = \mathrm{Im}\,\lambda^{(1)}(\mu^*).$$

To complete the proof, observe that due to (7.12), one has

$$\lim_{\mu \to 0^+} \mathrm{Re}\,\frac{\left(\lambda^{(1)}\right)'(\mu)}{1 - 2i\varepsilon(\mu)\mathrm{Im}\lambda^{(1)}(\mu)} = \mathrm{Re}\,\frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} > 0.$$

As a consequence, up to reduce μ^* if necessary, (7.15) holds true and Theorem 7.3 applies to complete the proof of the result. ■

From this result, let us notice that when (7.12) holds true, up to reduce η, the map $\mu \mapsto \varepsilon(\mu)^2$ is a bijection from $(0,\eta)$ onto $(0,\varepsilon^*)$ for some ε^*. As a consequence, one obtains that for each $\varepsilon \in (0,\varepsilon^*)$, there exists $\mu = \mu(\varepsilon) \in (0,\eta)$ such that $\mu = \mu(\varepsilon)$ is a Hopf bifurcation point of system (7.8) with ε. The same holds true when (7.12) is replaced by a non-positive condition. In the latter situation, the map $\mu \mapsto \varepsilon(\mu)$ is a bijection from $(-\eta,0)$ onto $(0,\varepsilon^*)$. Thus for each $\varepsilon \in (0,\varepsilon^*)$, there exists $\mu = \mu(\varepsilon) \in (-\eta,0)$ such that $\mu = \mu(\varepsilon)$ is a Hopf bifurcation point of (7.8). Due to the above remark, one obtains the following re-formulation of Theorem 7.7.

Corollary 7.8 *Let Assumption 7.6 be satisfied. Assume that*

$$\mathrm{Re}\,\frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} \neq 0.$$

Then there exists $\varepsilon^* > 0$ and $\widehat{\eta} > 0$ such that for each $\varepsilon \in (0,\varepsilon^*)$, there exists a unique $\mu = \mu(\varepsilon) \in (-\widehat{\eta},\widehat{\eta})$ such that $\mu = \mu(\varepsilon)$ is a Hopf bifurcation point for (7.8) with ε.

8 Application to the existence of wave trains for Problem (1.1)

In this section we come back to the age structured equation (1.1). We shall provide conditions that ensure the existence of periodic wave solutions emanating from Hopf bifurcation. Note that a wave train profile $U \equiv U(x,a)$ with speed $\gamma \in \mathbb{R}$ associated to this equation is a solution of the following problem

$$\begin{aligned} &\partial_x^2 U(x,a) - \gamma \partial_x U(x,a) - \partial_a U(x,a) - \mu U(x,a) = 0, \quad x \in \mathbb{R},\ a > 0, \\ &U(x,0) = \alpha f\left(\int_0^\infty \beta(a) U(x,a)\mathrm{d}a\right). \end{aligned} \tag{8.1}$$

Here $\mu > 0$ is a given and fixed parameter, $\alpha > 0$ is a bifurcation parameter while the nonlinear function f reads as the Ricker map described in Assumption 2.14. Furthermore assume that

$$\beta \in L^\infty_+(0,\infty) \text{ normalized by } \int_0^\infty \beta(a) e^{-\mu a}\mathrm{d}a = 1. \tag{8.2}$$

Under the above assumptions, let us first re-write (8.1) as a special case of (1.3). To do so, consider the Banach spaces

$$X = \mathbb{R} \times L^1(0,\infty) \text{ and } X_0 = \{0\} \times L^1(0,\infty),$$

as well as the non-densely defined linear operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \{0\} \times W^{1,1}(0,\infty) \text{ and } A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}.$$

Observe that $X_0 = \overline{D(A)} \neq X$. Next consider also the nonlinear map $G : X_0 \to X$ defined by

$$G\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} f\left(\int_0^\infty \beta(a)\varphi(a)\mathrm{d}a\right) \\ 0 \end{pmatrix}.$$

Next setting $u(x) = \begin{pmatrix} 0 \\ U(x,.) \end{pmatrix}$, system (8.1) re-writes as

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + \alpha G(u(x)) = 0, \ x \in \mathbb{R}. \tag{8.3}$$

Note that the linear operator $(A, D(A))$ is a Hille-Yosida operator so that Assumptions 2.1 and 2.3 are readily satisfied. Also note that $\omega_{0,ess}(A_0) \leq -\mu < 0$, so that Assumption 2.5 holds true. We refer to Magal and Ruan in [54] for more details.

Now let us observe that the stationary equation

$$A\overline{u} + \alpha G(\overline{u}) = 0, \ \ u \in D(A),$$

only has one solution $\overline{u} = 0$ when $\alpha \in (0,1]$ and two solutions when $\alpha > 1$ that are defined by

$$\overline{u} = 0 \text{ and } \overline{u}_\alpha = \begin{pmatrix} 0 \\ \overline{U}_\alpha \end{pmatrix}, \text{ with } \overline{U}_\alpha(a) \equiv \ln \alpha e^{-\mu a}, \ \ \forall a > 0.$$

In order to apply Theorem 7.7 or Corollary 7.8, we will first recall some known results for the first order differential equation

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = Au(x) + \alpha G(u(x)). \tag{8.4}$$

We refer to Chap. 5 in [54] for a detailed study of this equation. Consider for each $\alpha > 1$, the linear operator $B_\alpha : D(B_\alpha) \subset X \to X$ defined by

$$D(B_\alpha) = D(A), \ \ B_\alpha = A + \alpha DG(\overline{u}_\alpha).$$

As described in Assumption 2.14 we furthermore assume that

$$\beta(a) = \begin{cases} 0 \text{ if } a \in (0,\tau), \\ \delta(a-\tau)^n e^{-\zeta(a-\tau)} \text{ if } a \geq \tau, \end{cases} \qquad \text{with } \delta = \left(\int_\tau^\infty (a-\tau)^n e^{-\zeta(a-\tau)-\mu a}\mathrm{d}a\right)^{-1},$$

Then, the following results hold true.

Lemma 8.1 (Chap. 5 in [54]) *For each $\alpha > 1$ one has*

$$\omega_{0,ess}\left((B_\alpha)_0\right) \leq -\mu < 0.$$

There exist $\alpha^* > 1$ and $\theta^* > 0$ such that

(i) $\sigma(B_{\alpha^*}) \cap i\mathbb{R} = \{-i\theta^*, i\theta^*\}$,

(ii) $\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(B_{\alpha^*}) \setminus \{-i\theta^*, i\theta^*\}\} < 0$

(iii) *there exist a constant* $\eta > 0$ *and a continuously differentiable map* $\lambda : (\alpha^* - \eta, \alpha^* + \eta) \to \mathbb{C}$ *such that for each* $\alpha \in (\alpha^* - \eta, \alpha^* + \eta)$, $\lambda(\alpha)$ *and* $\overline{\lambda(\alpha)}$ *is a pair of simple eigenvalue of* B_α, $\lambda(\alpha^*) = i\theta^*$ *and*

$$\operatorname{Re}\frac{\mathrm{d}\lambda(\alpha^*)}{\mathrm{d}\alpha} > 0.$$

In order to apply Corollary 7.8, let us re-write System 8.3 for $\alpha > 1$ by setting $u = v + \overline{u}_\alpha$. This leads us to the following parametrized problem

$$\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}v(x)}{\mathrm{d}x} + B_\alpha v(x) + F(\alpha, v(x)) = 0,$$

where function $F : (1, \infty) \times X_0 \to X$ is defined by

$$F(\alpha, v) = \alpha\left[G(v + \overline{u}_\alpha) - G(\overline{u}_\alpha) - DG(\overline{u}_\alpha)v\right].$$

Then according to Lemma 8.1, Corollary 7.8 applies and leads us to the following result.

Theorem 8.2 *Let Assumption 2.14 be satisfied. Then there exist* $\gamma^* > 0$ *large enough and* $\eta^* > 0$ *such that for each* $\gamma \in (\gamma^*, \infty)$, *there exists* $\alpha = \alpha(\gamma) \in (\alpha^* - \eta^*, \alpha^* + \eta^*)$ *such that* $\alpha = \alpha(\gamma)$ *is a Hopf bifurcation point for (8.3) around* $\overline{u}_\alpha$.

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