Travelling wave solutions for an infection-age structured epidemic model with external supplies

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Abstract

The aim of this paper is to investigate the spatial invasion of some infectious disease. The contamination process is described by the age since infection. Compared with the classical Kermack and McKendrick's model, the vital dynamic is not omitted, and we allow some constant input flux into the population. This problem is rather natural in the context of epidemic problems and it has not been studied. Here we prove an existence and non-existence result for travelling wave solutions. We also describe the minimal wave speed. We are able to construct a suitable Lyapunov like functional decreasing along the travelling wave allowing to derive some qualitative properties, namely their convergence towards equilibrium points at $x = \pm\infty$.

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1. Introduction

The first significant work on epidemics modelling can be attributed to Ross [32] in 1916. We also refer to Ross and Hudson [33, 34]. Ross uses a system of ordinary differential equations to describe the transmission of diseases between susceptible and infected individuals.

In 1927, Kermack and McKendrick [21–23] extended Ross's ideas, by introducing the age since infection in some epidemic models. Nowadays, the age variable is widely used to describe either the age of individuals or the age since infection (which is the time since individuals got infected). We refer to Diekmann and Heesterbeek [6], Gurtin and MacCamy [17], Webb [41], Iannelli [20], Thieme [39], and the references cited therein.

First studied by Fisher in [16] and Kolmogorov, Petrovski and Piskunov in [24], the existence of travelling wave solutions in reaction–diffusion systems has attracted sustained attention of researchers. In the context of epidemic modelling, Bartlett [2] in 1956 predicted

the eventual establishment of an epidemic coming from an initially localized infected source with a constant speed. The existence and qualitative properties of travelling waves for various epidemic models based on reaction–diffusion systems have been extensively studied. We refer to the monographs of Murray [29], Rass and Radcliffe [31], a survey paper by Ruan [35], Ruan and Wu [36] and the references cited therein.

Most results on age-structured epidemic models focus on the existence, uniqueness and long time behaviour of the solutions; there are very few results on the existence of travelling waves in age-structured epidemic models [1, 7, 9, 11, 37]. The purpose of this paper is to establish the existence of travelling wave solutions in an infection-age structured model.

In order to describe the dynamic of the disease, we will decompose the population into the class S of susceptible, and the class I of infected. In the absence of infection within the population, the dynamic of the population is assumed to be described by the following model:

$$\partial_t S(t, x) = d_S \partial_x^2 S(t, x) + \Pi - \gamma S(t, x), \qquad \text{for } t > 0, \quad x \in \mathbb{R}, \qquad (1.1)$$

where the parameter $\Pi > 0$ represents the entering flux of susceptible individuals, while $\gamma > 0$ is the rate at which individuals die or leave the population. If we assume that the process of contamination occurs by mass action law, then the model we will consider in this work reads as

$$\begin{aligned}
\partial_t S &= d_S \partial_x^2 S + \Pi - \gamma S - S(t, x) \int_0^{+\infty} \beta(a) i(t, a, x) \, \mathrm{d}a, & x \in \mathbb{R}, \\
\partial_t i + \partial_a i &= d_I \partial_x^2 i - (\gamma + \mu(a)) i(t, a, x), & a > 0, \quad x \in \mathbb{R}, \qquad (1.2) \\
i(t, 0, x) &= S(t, x) \int_0^{+\infty} \beta(a) i(t, a, x) \, \mathrm{d}a, & x \in \mathbb{R},
\end{aligned}$$

where the function $\beta(a)$ denotes the infection age-specific contamination rate and the function $\mu(a)$ represents an additional mortality (or exit) rate due to the disease. Here the diffusion coefficients $d_S > 0$ and $d_I > 0$ describe the spatial motility of each class. Note that the diffusion coefficients may be different from susceptible to infected depending on the disease.

Throughout this work, we will make the following assumption.

Assumption 1.1. *We assume that d_S, d_I, Π and γ are positive constants, and the maps $a \to \beta(a)$ and $a \to \mu(a)$ are almost everywhere bounded, and, respectively, belong to $L^\infty(0, \infty)$ and $L^\infty_{\mathrm{loc}}([0, \infty))$.*

The corresponding spatially homogeneous system reads as

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \Pi - \gamma S(t) - S(t) \int_0^{+\infty} \beta(a) i(t, a) \, \mathrm{d}a, \\
\partial_t i + \partial_a i &= -(\gamma + \mu(a)) i(t, a), & a > 0, \qquad (1.3) \\
i(t, 0) &= S(t) \int_0^{+\infty} \beta(a) i(t, a) \, \mathrm{d}a,
\end{aligned}$$

and the basic reproductive number is given by

$$R_0 = \frac{\Pi}{\gamma} \int_0^{+\infty} \beta(a) l(a) \mathrm{e}^{-\gamma a} \, \mathrm{d}a \qquad \text{with } l(a) = \mathrm{e}^{-\int_0^a \mu(s) \, \mathrm{d}s}. \qquad (1.4)$$

System (1.3) has a unique disease-free equilibrium $(\overline{S}_F, \overline{i}_F) = (\frac{\Pi}{\gamma}, 0)$, and when $R_0 > 1$, system (1.3) has a unique endemic equilibrium $(\overline{S}_E, \overline{i}_E)$ defined by

$$\overline{S}_E := \frac{\overline{S}_F}{R_0}, \qquad \overline{i}_E(a) := \Pi \frac{R_0 - 1}{R_0} \mathrm{e}^{-\gamma a} l(a).$$

The asymptotic behaviour of system (1.3) has been studied in Magal *et al* [27]. The disease-free equilibrium $(\overline{S}_F, \overline{i}_F)$ is globally asymptotically stable whenever $R_0 \leqslant 1$, while $(\overline{S}_E, \overline{i}_E)$ is globally asymptotically stable whenever $R_0 > 1$.

Our goal in this paper is to study the existence and the non-existence of travelling wave solutions describing the spatial invasion of the infection within a population, where the infection is initially absent, namely travelling waves connecting the disease-free equilibrium $(\overline{S}_F, \overline{i}_F)$ to the endemic equilibrium $(\overline{S}_E, \overline{i}_E)$.

Definition 1.2 (Travelling waves). *We will say that the system (1.2) has a travelling wave solution if there exists a real number $c > 0$ and a pair of positive functions $(\widehat{S}, \widehat{i})$ (i.e. $S > 0$ on $\mathbb{R}$ and $i > 0$ on $[0, \infty) \times \mathbb{R}$) such that the maps*

$$\widehat{S} \in C_b^0(\mathbb{R}) \cap C^2(\mathbb{R}), \qquad \widehat{i} \in C_b^0([0, \infty) \times \mathbb{R}) \cap C^{1,2}((0, \infty) \times \mathbb{R}) \cap L^1(0, \infty; C_b^0(\mathbb{R})),$$

satisfy $\lim_{x \to -\infty} \widehat{S}(x) = \overline{S}_F$ and $\lim_{x \to -\infty} \widehat{i}(., x) = 0$ in $L^1(0, \infty; \mathbb{R})$, and such that the maps

$$S(t, x) = \widehat{S}(x + ct), \qquad i(t, a, x) = \widehat{i}(a, x + ct),$$

is a pair of entire solutions of the system (1.2). Here $C_b^0(\mathbb{R})$ denotes the space of bounded and continuous functions from $\mathbb{R}$ into itself.

Before stating our main result, let us define an important quantity for the following work. When $R_0 > 1$, define the minimal speed by

$$c^* := 2\sqrt{\alpha^*},$$

where $\alpha^* > 0$ is the unique solution of the equation

$$\frac{\Pi}{\gamma} \int_0^\infty \beta(a) l(a) e^{-(\alpha^* + \gamma)a} \, da = 1.$$

The main result of this paper is the following theorem.

Theorem 1.3. *Let assumption 1.1 be satisfied and assume that $d_S \geqslant d_I > 0$. The following results hold true:*

(i) If $R_0 \leqslant 1$ then system (1.2) has no travelling wave solution.
(ii) If $R_0 > 1$ and $c < c^*$ then system (1.2) has no travelling wave solution.
*(iii) If $R_0 > 1$, then for each $c > c^*$ system (1.2) has a travelling wave solution.*

Moreover, whenever such a travelling solution exists, it satisfies

$$\lim_{x \to \infty} \widehat{S}(x) = \overline{S}_E, \qquad \lim_{x \to \infty} \widehat{i}(a, x) = \overline{i}_E(a) \text{ in } L^1(0, \infty). \tag{1.5}$$

Remark 1.4. Let us stress that from the proof of the last point of the above theorem, the assumption $d_I \leqslant d_S$ is not required to prove the convergence property . The precise statement is the following one: let (S, i) be a bounded and travelling wave of (1.2) then it satisfies (1.5).

Remark 1.5. We also would like to comment the above theorem by stressing that no existence or non-existence of wave has been derived for the minimal wave speed c^* (in the case $R_0 > 1$). Let us mention that a usual limit procedure (looking at the convergence of a sequence a travelling wave with speed $\{c_n\}$ such that $c_n \searrow c^*$ as $n \to \infty$) is complicated because of possible non-monotone solutions (see figure 1). We expect that the minimal speed c^* is achieved but this question remains an open problem.

This above theorem extends some results obtained by Ducrot and Magal in [9] and Ducrot *et al* in [11] where the vital dynamics of the population were omitted, reducing the problem to some Kermack and McKendrick like models. Here a constant entering flux $\Pi > 0$ is considered in the model (see (1.1)). Due to this vital dynamics (external supplies) the mathematical analysis of the problem becomes much more difficult to handle. Indeed, assume that the infection spreads into the population, then new susceptible individuals enter the population spatially uniformly into the spatial domain (according to Π) and in particular, at some place where the outbreak of infection has passed. This may lead to a secondary outbreak of infection and thus may induce some spatio-temporal oscillations. The above theorem shows that (see
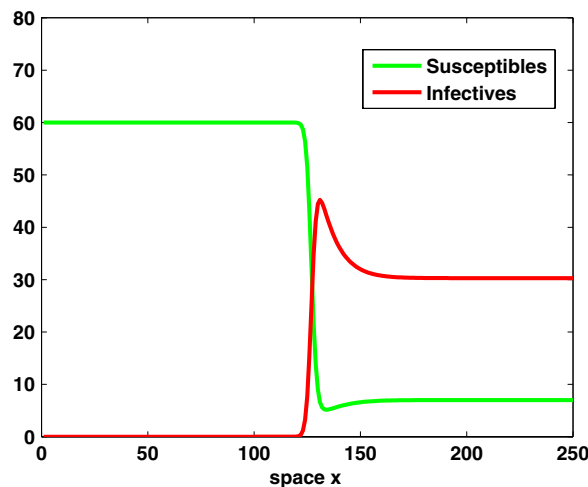
Figure 1. In this figure we plot a travelling wave for system (1.7) when $R_0 > 1$.

(This figure is in colour only in the electronic version)

also remark 1.4) if oscillations exist then they are damped, leading to a converging travelling wave. Let us mention that such a stabilization property does not hold for some non-local travelling wave problems (we refer to Berestycki *et al* [3], Ducrot [8], Fang and Zhao [14] and the references cited therein).

In this work, we shall first overcome the lack of monotonicity for system (1.2) by constructing suitable sub- and super-solution pairs. Next the boundedness of the solutions seems to be a difficult problem when the diffusion coefficients d_S and d_I are different. The derivation of some upper-bound for the infected component is solved with the restriction $d_S \geqslant d_I > 0$.

Using the boundedness result, the convergence towards the endemic equilibrium (1.5) is solved by building a suitable Lyapunov like functional. This kind of method was used by some authors in the context of diffusive predator–prey system, where the vital dynamics of prey may induce oscillatory converging wave to the coexistence state (see for instance Dunbar [12, 13] and Huang, Lu and Ruan [19]). Let us mention that in the latter results, the travelling wave problems generate a semiflow. This is no longer true for the problem under consideration in this work.

Finally, let us mention that the non-existence result (i) of solutions with the wave speed $c < c^*$ shows that c^* corresponds to the minimal speed of propagation of the disease. The proof of this result is fulfilled using a comparison argument for the scalar equation of the density of infected linearized at the disease-free equilibrium. This proof makes use of a maximum principle for age and space structured model that seems to be a new result and that is proved using the theory of integrated semigroups.

Theorem 1.3 has several consequences and corollaries. First, one may observe that it implies the existence of planar travelling wave solutions for the n-dimensional case in space. More precisely, if we consider

$$\partial_t S = d_S \Delta_x S + \Pi - \gamma S - S(t,x) \int_0^{+\infty} \beta(a) i(t,a,x)\,\mathrm{d}a, \qquad x \in \mathbb{R}^n,$$

$$\partial_t i + \partial_a i = d_I \Delta_x i - (\gamma + \mu(a))\, i(t,a,x), \qquad a > 0, \quad x \in \mathbb{R}^n,$$

$$i(t,0,x) = S(t,x) \int_0^{+\infty} \beta(a) i(t,a,x)\,\mathrm{d}a, \qquad x \in \mathbb{R}^n,$$

then the conclusions of theorem 1.3 hold for travelling wave solutions of the form

$$S(t, x) = \widehat{S}(x.e + ct), \qquad i(t, a, x) = \widehat{i}(a, x.e + ct),$$

when $e \in S^{n-1}$ (the unit sphere of $\mathbb{R}^n$).

Moreover, when $\mu(a) \equiv \widehat{\mu} \geqslant 0$ and $\beta(a) = \widehat{\beta}\chi_{[\tau,\infty)}(a)$, where $\tau \geqslant 0$ and $\chi_{[\tau,\infty)}$ is the characteristic function of the interval $[\tau, \infty)$, then by integrating the i-equation with respect to the age a in system (1.2), we obtain the following PDE with delay:

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= d_S \Delta S(t) + \Pi - \gamma S(t) - \widehat{\beta}\mathrm{e}^{-(\gamma+\widehat{\mu})\tau} S(t) T_{d_I\Delta}(\tau) I(t-\tau), \\
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= d_I \Delta I(t) - (\gamma+\widehat{\mu}) I(t) + \widehat{\beta}\mathrm{e}^{-(\gamma+\widehat{\mu})\tau} S(t) T_{d_I\Delta}(\tau) I(t-\tau),
\end{aligned} \qquad (1.6)$$

wherein $I(t, x) := \int_0^{+\infty} i(t, a, x)\,\mathrm{d}a$, and $\{T_{d\Delta}(t)\}_{t\geqslant 0}$ is the semigroup of bounded linear operators generated by $d\Delta$ on the space of $BUC(\mathbb{R}^n)$, the space of bounded and uniformly continuous functions from $\mathbb{R}$ into itself, which is explicitly given by

$$T_{d\Delta}(t)\varphi(x) := (4\pi\,\mathrm{d}t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x - y)\mathrm{e}^{-\frac{\|y\|^2}{4dt}}\,\mathrm{d}y.$$

In particular, the conclusions of theorem 1.3 hold for planar waves of system (1.6). Note that system (1.2) can be therefore viewed as a partial differential equation with distributed delay and we refer to Zou and Wu [42], Wang *et al* [40] and references therein for such equations.

Finally, let us note that when the delay $\tau = 0$ in (1.6), the original problem becomes

$$\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= d_S \Delta S(t) + \Pi - \gamma S(t) - \widehat{\beta} S(t) I(t), \\
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= d_I \Delta I(t) - (\gamma+\widehat{\mu}) I(t) + \widehat{\beta} S(t) I(t),
\end{aligned} \qquad (1.7)$$

As illustrated in figure 1, even in this simplified situation, system (1.7) gives rise to non-monotone travelling waves.

The rest of the paper is devoted to the proof of theorem 1.3. Section 2.1 is devoted to the existence of wave solutions while section 2.2 focuses on non-existence results and the minimal speed.

2. Proof of theorem 1.3

Note that using the change of variables

$$U(t, x) := \frac{\gamma}{\Pi} S\left(t, x\sqrt{d_I}\right), \qquad v(t, a, x) := \frac{\gamma}{\Pi} \frac{i\left(t, a, x\sqrt{d_I}\right)}{l(a)}.$$

and $d := \frac{d_S}{d_I}$, $\sigma(a) := \frac{\Pi}{\gamma}\beta(a)l(a)$, we obtain

$$\begin{aligned}
\partial_t U &= d\partial_x^2 U + \gamma - \gamma U - U(t, x) \int_0^{+\infty} \sigma(a)v(t, a, x)\,\mathrm{d}a, \qquad x \in \mathbb{R}, \\
\partial_t v + \partial_a v &= \partial_x^2 v - \gamma v(t, a, x), \qquad a > 0, \quad x \in \mathbb{R}, \\
v(t, 0, x) &= U(t, x) \int_0^{+\infty} \sigma(a)v(t, a, x)\,\mathrm{d}a, \qquad x \in \mathbb{R}.
\end{aligned}$$

So from here on, we make the following assumption.

Assumption 2.1. *We assume that $d_I = 1$, $\Pi = \gamma > 0$ and $\mu = 0$.*

In order to investigate a travelling wave solution of system (1.2) we need to consider the following system of elliptic equations for some $c > 0$:

$$d_S S''(x) - c S'(x) + \gamma - \gamma S(x) - S(x) \int_0^{+\infty} \beta(a) i(a, x) \, \mathrm{d}a = 0, \qquad x \in \mathbb{R},$$

$$\partial_a i(a, x) = \partial_x^2 i(a, x) - c \partial_x i(a, x) - \gamma i(a, x), \qquad (2.8)$$

$$i(0, x) = S(x) \int_0^{+\infty} \beta(a) i(a, x) \, \mathrm{d}a.$$

Together with assumption 2.1, we set the disease-free equilibrium and endemic equilibrium (when it exists)

$$S_F = 1, \quad i_F \equiv 0, \text{ and}$$
$$S_E = \frac{1}{R_0}, \quad i_E(a) = \gamma \left(1 - \frac{1}{R_0} \right) \mathrm{e}^{-\gamma a} \qquad \text{when } R_0 > 1. \qquad (2.9)$$

2.1. Existence result

In this subsection we prove assertion (iii) of theorem 1.3. The proof of this result consists of several steps. Let us first introduce some definitions that will be important in the following. Throughout this subsection, we fix $c > c^* = 2\sqrt{\alpha^*}$ and consider $\lambda > 0$ defined as the smallest root of the equation

$$\lambda^2 - c\lambda + \alpha^* = 0. \qquad (2.10)$$

Lemma 2.2. *The map $\bar{i}(a, x) = \mathrm{e}^{-(\alpha^* + \gamma)a} \mathrm{e}^{\lambda x}$ satisfies the equation:*

$$\partial_a i = \partial_x^2 i - c \partial_x i - \gamma i, \qquad a > 0, \quad x \in \mathbb{R},$$
$$i(0, x) = \int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a.$$

We observe that by construction $\int_0^{+\infty} \beta(a) \bar{i}(a, x) \, \mathrm{d}a = \mathrm{e}^{\lambda x}$, and have the following lemma.

Lemma 2.3. *For each $\delta^* > 0$ such that*

$$\delta^* < \lambda, \qquad c\delta^* - d_S \delta^{*2} + \gamma > 0,$$

and each $\beta^* > 0$ large enough, the map $\underline{S}(x) = 1 - \beta^* \mathrm{e}^{\delta^* x}$ satisfies the inequality:

$$d_S \underline{S}'' - c \underline{S}' + \gamma - \gamma \underline{S} - \mathrm{e}^{\lambda x} \underline{S} \geqslant 0.$$

Proof. Let $\delta^* > 0$ be sufficiently small and fixed. Then the map

$$x \to \frac{\mathrm{e}^{(\lambda - \delta^*)x}}{c\delta^* - d_S \delta^{*2} + \gamma + \mathrm{e}^{\lambda x}}$$

is bounded on the real line. Let $\beta^* > 0$ be large enough and greater than the maximum of the above map. Then we have

$$d_S \underline{S}'' - c \underline{S}' + \gamma - \gamma \underline{S} - \mathrm{e}^{\lambda x} \underline{S} = \beta^* \left(c\delta^* - d_S \delta^2 + \gamma + \mathrm{e}^{\lambda x} \right) \mathrm{e}^{\delta^* x} - \mathrm{e}^{\lambda x}.$$

From the definition of β^* we obtain that this last quantity is non-negative. This completes the proof of the lemma. ∎

Now we shall prove the following lemma.

Lemma 2.4. *Let $\kappa > 0$ be small enough. Then for each $k > 0$ sufficiently large, the map*

$$\underline{i}(a, x) = e^{\lambda x} e^{-(\gamma + \alpha^*)a} - k e^{(\lambda + \kappa)x} e^{-(\gamma + \alpha_\kappa)a},$$

where $\alpha_\kappa = c(\lambda + \kappa) - (\lambda + \kappa)^2 > 0$, satisfies

$$\partial_a \underline{i} = \partial_x^2 \underline{i} - c \partial_x \underline{i} - \gamma \underline{i}, \qquad a > 0, \quad x \in \mathbb{R},$$

$$\underline{i}(0, x) \leqslant \left(1 - \beta^* e^{\delta^* x}\right)^+ \int_0^\infty \beta(a) \underline{i}(a, x) \, da.$$

Proof. The differential inequality can be rewritten as

$$1 - k e^{\kappa x} \leqslant \left(1 - \beta^* e^{\delta^* x}\right)^+ \left(1 - R_\kappa e^{\kappa x}\right),$$

where we have set

$$R_\kappa := \int_0^\infty \beta(a) e^{-(\gamma + \alpha_\kappa)a} \, da.$$

Note that $\alpha_\kappa > \alpha^*$ when $\kappa > 0$ is small enough. Thus, $R_\kappa < 1$ when $\kappa > 0$ is small enough. ∎

The first main result of this section is the following proposition.

Proposition 2.5. *Let assumptions 1.1 and 2.1 be satisfied. Let $d_S > 0$ be given and fixed. Then there exists a pair of maps $(S, i) \in C(\mathbb{R}, \mathbb{R}^+) \times L_+^1(0, \infty; C(\mathbb{R}, \mathbb{R}))$ which are solutions of (2.8) such that*

$$0 < S(x) < 1, \quad 0 < i(a, x) \leqslant e^{\lambda x} e^{-(\alpha^* + \gamma)a}, \qquad a \geqslant 0, \quad x \in \mathbb{R}. \quad (2.11)$$

Proof. This proof is given in two steps. The first one considers a similar approximated problem on a bounded spatial domain $I_b = (-b, b)$ for any $b > 0$ large enough in order to have some compactness properties. The second step is a limit procedure as $b \to +\infty$.

First step. Let us introduce the maps

$$\overline{S}(x) \equiv 1, \quad \overline{i}(a, x) = e^{-(\alpha^* + \gamma)a} e^{\lambda x},$$
$$\underline{S}(x) = \max\left(0, 1 - \beta^* e^{\delta^* x}\right), \qquad \underline{i}(a, x) = \max\left(0, e^{\lambda x} e^{-(\gamma + \alpha^*)a} - k e^{(\lambda + \kappa)x} e^{-(\gamma + \alpha_\kappa)a}\right),$$

wherein β^* and δ^* are defined in lemma 2.3 while κ and α_κ are defined in lemma 2.4.

For any $b > 0$ define the closed convex set

$$\mathcal{C}_b = \left\{ i_0 \in C([-b, b]) : \ \underline{i}(x, 0) \leqslant i_0(x) \leqslant \overline{i}(x, 0) \right\}.$$

Then let us consider the operator $T : \mathcal{C}_b \to C([-b, b])$ given by

$$T(i_0) = S \int_0^\infty \beta(a) i(., a) \, da,$$

where i is the solution of the linear problem:

$$\partial_a i = \partial_x^2 i - c \partial_x i - \gamma i, \qquad a > 0, \quad x \in (-b, b),$$
$$i(0, x) = i_0(x), \qquad x \in (-b, b),$$
$$i(a, \pm b) = \underline{i}(a, \pm b), \qquad a > 0,$$

while S is a solution of the elliptic problem:

$$c S'(x) = d_S S''(x) + \gamma - \gamma S - S(x) \int_0^\infty \beta(a) i(a, x) \, da, \qquad x \in (-b, b),$$
$$S(\pm b) = \underline{S}(\pm b).$$

We claim that $T(\mathcal{C}_b) \subset \mathcal{C}_b$. This inclusion comes from suitable uses of comparison arguments. We refer to Ducrot and Magal [9] for similar arguments. The sketch of this proof is the following: when $i_0 \in \mathcal{C}_b$ is given, we derive that $i(a, x) \leqslant \bar{\mathbf{i}}(a, x)$ and from the S-equation that $S \leqslant \bar{\mathbf{S}}$ so that $T(i_0) \leqslant \bar{\mathbf{i}}(0, .)$. The same steps hold to derive the lower estimates. The details are left to the reader.

Moreover, the map T is completely continuous from $\mathcal{C}_b$ into $C([-b, b])$ and the Schauder fixed point theorem applies and provides the existence of a solution (S_b, i_b) of the problem

$$\partial_a i = \partial_x^2 i - c\partial_x i - \gamma i, \qquad a > 0, \quad x \in (-b, b)$$

$$cS'(x) = d_S S''(x) + \gamma - \gamma S - S(x) \int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a, \qquad x \in (-b, b)$$

$$i(0, x) = S(x) \int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a, \qquad x \in (-b, b)$$

$$i(a, \pm b) = \underline{\mathbf{i}}(a, \pm b), \qquad a > 0, \quad S(\pm b) = \underline{\mathbf{S}}(\pm b),$$

such that $\underline{\mathbf{S}} \leqslant S_b \leqslant \bar{\mathbf{S}}$ and $\underline{\mathbf{i}} \leqslant i_b \leqslant \bar{\mathbf{i}}$.

Second step. We consider a sequence $\{b_n\}_{n \geqslant 0}$ of positive numbers such that $b_n \to \infty$ when $n \to \infty$. Then due to elliptic and parabolic estimates, one can show, similarly as in [9], that the sequences of functions $\{S_{b_n}, i_{b_n}\}$ are relatively compact with respect to the topology of the uniform convergence on compact subsets. Using a standard limit procedure, this leads us to the existence of a solution (S, i) of system (2.8) such that

$$\underline{\mathbf{S}}(x) \leqslant S(x) \leqslant 1, \qquad \forall x \in \mathbb{R},$$
$$\underline{\mathbf{i}}(a, x) \leqslant i(a, x) \leqslant \bar{\mathbf{i}}(a, x), \qquad \forall (a, x) \in [0, \infty) \times \mathbb{R}.$$

Note now that from the strong maximum principle applied to the S-equation, one obtains that $S(x) < 1$ for all $x \in \mathbb{R}$. Since $S \int_0^\infty \beta(a) i(a, .) \, \mathrm{d}a \geqslant 0$, one obtains that $S = 0$ cannot be a minimal value for S so that $S(x) > 0$ for all $x \in \mathbb{R}$. From the strong comparison principle applied to the i-equation, we obtain $i(a, x) > 0$ for all $a > 0$ and $x \in \mathbb{R}$ and due to the boundary at $a = 0$, we also obtain that $i(0, x) > 0$ for all $x \in \mathbb{R}$. The result follows (see also [4, 9, 11] for similar treatments). ∎

In order to complete the proof of theorem 1.3, we shall derive some boundedness property when $d_S \geqslant 1$.

Lemma 2.6. *Let assumptions 1.1 and 2.1 be satisfied and assume that $d_S \geqslant 1$. Let (S, i) be a solution provided by proposition 2.5. Then we have*

$$i(a, x) \leqslant \|\beta\|_\infty \sqrt{d_S} \mathrm{e}^{-\gamma a}, \qquad \forall a \geqslant 0, \quad \forall x \in \mathbb{R}, \tag{2.12}$$

and

$$\frac{\gamma}{\gamma + \|\beta\|_\infty \sqrt{d_S}} \leqslant S(x) \leqslant 1, \qquad \forall x \in \mathbb{R}. \tag{2.13}$$

Proof. Let us set the maps

$$I(x) = \int_0^\infty i(a, x) \, \mathrm{d}a \quad \text{and} \quad g(x) = S(x) \int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a, \qquad x \in \mathbb{R}.$$

Integrating (2.8) over $a \in (0, \infty)$ we obtain

$$-d_S S'' + cS' + \gamma S = \gamma - g(x)$$
$$-I'' + cI' + \gamma I = g(x).$$

Thus, we obtain that

$$S(x) = 1 - f_{d_S}(x), \qquad I(x) = f_1(x),$$

where we have set

$$f_{d_S}(x) = \int_0^\infty e^{-\gamma t} T_{d_S \Delta}(t) g(x) \, dt = \int_0^\infty \frac{e^{-\gamma t}}{\sqrt{4\pi d_S t}} \int_{-\infty}^\infty g(x - y - ct) \exp\left(-\frac{y^2}{4d_S t}\right) dy \, dt.$$

Since $d_S \geqslant 1$ we obtain that

$$f_1(x) \leqslant \sqrt{d_S} f_{d_S}(x), \qquad \forall x \in \mathbb{R}.$$

On the other hand, we know that $0 \leqslant S(x) \leqslant 1$ for any $x \in \mathbb{R}$, thus for any $x \in \mathbb{R}$

$$I(x) = f_1(x) \leqslant \sqrt{d_S} f_{d_S}(x) = \sqrt{d_S} \left(1 - S(x)\right),$$

therefore $I(x) \leqslant \sqrt{d_S}$, for any $x \in \mathbb{R}$, and

$$i(0, x) \leqslant \|\beta\|_\infty \sqrt{d_S}, \qquad \forall x \in \mathbb{R}.$$

From the parabolic comparison principle, we obtain that

$$i(a, x) \leqslant \|\beta\|_\infty \sqrt{d_S} e^{-\gamma a}, \qquad \forall a \geqslant 0, \quad \forall x \in \mathbb{R}.$$

Moreover S satisfies the inequality

$$d_S S'' - cS' + \gamma - \left(\gamma + \|\beta\|_\infty \sqrt{d_S}\right) S(x) \leqslant 0,$$

the result follows from the maximum principle. ∎

In the following, we will use the following general result.

Lemma 2.7. *Let $v : \mathbb{R} \to \mathbb{R}$ be a positive, bounded and continuous map such that*

$$v(x) \geqslant v^- > 0 \qquad \forall x \in \mathbb{R},$$

for some constant $v^- > 0$. Let $\beta \in L^1_{+,\mathrm{loc}}([0, \infty))$ be some positive and locally integrable function such that there exists $a_\dagger > 0$ such that

$$\int_0^{a_\dagger} \beta(a) \, da > 0. \tag{2.14}$$

Let $u \equiv u(t, x)$ be a positive solution of the problem

$$\partial_t u = \partial_x^2 u - c\partial_x u - \gamma u, \qquad t > 0, \quad x \in \mathbb{R}$$
$$u(0, x) = v(x) \int_0^\infty \beta(t) u(x, t) \, dt.$$

Then there exists some constants $C > 0$ and $\theta > 0$ such that

$$u(t + \theta, x) \leqslant Cu(t, x), \qquad \forall x \in \mathbb{R}, \quad \forall t \geqslant 0. \tag{2.15}$$

Moreover, there exists some constant $M > 0$ such that

$$|\partial_x u(t, x)| \leqslant M\left(1 + \frac{1}{\sqrt{t}}\right) u(t, x), \qquad \forall t > 0, \quad \forall x \in \mathbb{R}.$$

Proof. First note that due to (2.14), there exist $t_1 > 0$ and $\varepsilon > 0$ such that

$$\int_{t_1}^{t_1+\varepsilon} \beta(a) \, da = \beta^- > 0.$$

Set $\theta = t_1/2$. First note that from the Harnack inequality (see [18, 25]), there exists some constant $M > 0$ such that

$$u(\theta, z) \leqslant M \min_{t_1 \leqslant s \leqslant t_1 + \varepsilon, \, |z-y| \leqslant 1} u(s, y).$$

Next recall that the map $K \equiv K(t, z)$ defined by

$$K(t, z) = \frac{e^{-\gamma t}}{\sqrt{4\pi t}} e^{-\frac{(z-ct)^2}{4t}}$$

is a Green function for the elliptic operator $\partial_{xx} - c\partial_x - \gamma$. Thus, we are able to rewrite u as a solution of the fixed point problem:

$$u(t, x) = \int_{\mathbb{R}} \int_0^\infty K(t, x - y)v(y)\beta(s)u(s, y) \, ds \, dy.$$

From this expression we have the following inequalities:

$$
\begin{aligned}
u(t, x) &= \frac{e^{-\gamma t}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y-ct)^2}{4t}} v(y) \int_0^\infty \beta(s)u(s, y) \, ds \, dy \\
&\geqslant v^- \frac{e^{-\gamma t}}{\sqrt{4\pi t}} \int_{|x-y| \leqslant 1} e^{-\frac{(x-y-ct)^2}{4t}} \int_{t_1}^{t_1+\varepsilon} \beta(s) \, ds \, dy \min_{t_1 \leqslant s \leqslant t_1 + \varepsilon, \, |x-y| \leqslant 1} u(s, y) \\
&\geqslant v^- \beta^- \frac{2e^{-\gamma t}}{\sqrt{4\pi t}} \int_0^1 e^{-\frac{(\xi-ct)^2}{4t}} \, d\xi \min_{t_1 \leqslant s \leqslant t_1 + \varepsilon, \, |x-y| \leqslant 1} u(s, y) \\
&\geqslant \left(\frac{v^- \beta^-}{M} \frac{2e^{-\gamma t}}{\sqrt{4\pi t}} \int_0^1 e^{-\frac{(\xi-ct)^2}{4t}} \, d\xi \right) u(\theta, x).
\end{aligned}
$$

Thus, since $u(0, x) = v(x) \int_0^\infty \beta(s)u(s, x) \, ds$, we obtain

$$u(0, x) \geqslant v^- \int_0^\infty \beta(t)u(t, x) \, dt \geqslant Cu(\theta, x),$$

where $C > 0$ is the positive constant defined by

$$C = \left(\frac{2v^- \beta^-}{M} \right) \int_0^\infty \beta(t) \frac{e^{-\gamma t}}{\sqrt{4\pi t}} \int_0^1 e^{-\frac{(\xi-ct)^2}{4t}} \, d\xi \, dt.$$

We have

$$u(0, x) \geqslant Cu(\theta, x), \qquad \forall x \in \mathbb{R},$$

and from the comparison principle we obtain

$$u(t, x) \geqslant Cu(t + \theta, x), \qquad \forall t \geqslant 0, \quad \forall x \in \mathbb{R},$$

which completes the proof of (2.15). From the interior parabolic regularity and Harnack inequality (see [18, 25, 26]), we have that there exists some constant $M > 0$ such that

$$|\partial_x u(t, x)| \leqslant M \left(1 + \frac{1}{\sqrt{t}} \right) u(t + \theta, x), \qquad \forall t > 0, \quad \forall x \in \mathbb{R}.$$

Therefore, due to (2.15) we obtain

$$|\partial_x u(t, x)| \leqslant CM \left(1 + \frac{1}{\sqrt{t}} \right) u(t, x), \qquad \forall t > 0, \quad \forall x \in \mathbb{R}.$$

This completes the proof of the result.   ∎

Lemma 2.8. *Let assumptions 1.1 and 2.1 be satisfied and assume that $d_S \geqslant 1$. Let (S, i) be a solution provided by proposition 2.5. Then there exists some constant $M > 0$ such that*

$$|\partial_x i(a, x)| \leqslant M \left(1 + \frac{1}{\sqrt{a}} \right) i(a, x), \qquad \forall a > 0, \qquad \forall x \in \mathbb{R}.$$

Proof. First note that due to lemma 2.6, there exists some constant $C > 0$ such that

$$S(x) \geqslant C, \qquad \forall x \in \mathbb{R}.$$

On the other hand, recall that i is a solution of the problem

$$\partial_a i = \partial_x^2 i - c \partial_x i - \gamma i, \qquad a > 0, \quad x \in \mathbb{R}$$
$$i(0, x) = S(x) \int_0^\infty \beta(a) i(a, x) \, da.$$

Thus, lemma 2.7 applies with $u = i$ and $v(x) \equiv S(x)$ and the proof is complete. ∎

In order to complete the proof of theorem 1.3-(iii), it remains to prove the convergence of the solutions as $x \to +\infty$. This result is a consequence of some suitable Lyapunov functional.

Lyapunov function. We complete the proof of theorem 1.3-(iii). Recalling (2.9), we set

$$g(x) = x - 1 - \ln x, \quad \alpha(a) = \int_a^\infty \beta(s) i_E(s) \, ds, \qquad a \geqslant 0. \tag{2.16}$$

Next consider the set

$$\mathcal{C} = \left\{ \begin{array}{l} (\varphi, \psi) \in C^1(\mathbb{R}, (0, \infty)) \times C^{1,0}(\mathbb{R} \times [0, \infty); (0, \infty)) : \\ \varphi > 0, \quad \psi > 0, \\ \exists M > 0, \quad \dfrac{\psi(a, x)}{i_E(a)} \leqslant M, \quad \left| \dfrac{\partial_x \psi(a, x)}{\psi(a, x)} \right| \leqslant M \left(1 + \dfrac{1}{\sqrt{a}} \right), \\ \alpha(.) g \left(\dfrac{\psi(., x)}{i_E(.)} \right) \in L^1(0, \infty), \quad \forall x \in \mathbb{R} \end{array} \right\},$$

and for each $(\varphi, \psi) \in \mathcal{C}$ consider the functional $V(\varphi, \psi) : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} V(\varphi, \psi)(x) = {} & c W(\varphi, \psi)(x) + d_S \varphi'(x) \left(\frac{1}{\varphi(x)} - \frac{1}{S_E} \right) \\ & + \int_0^\infty \alpha(a) \partial_x \psi(a, x) \left(\frac{1}{\psi(a, x)} - \frac{1}{i_E(a)} \right) da, \end{aligned} \tag{2.17}$$

where

$$W(\varphi, \psi)(x) = g \left(\frac{\varphi(x)}{S_E} \right) + \int_0^\infty \alpha(a) g \left(\frac{\psi(a, x)}{i_E(a)} \right) da.$$

Then we shall show the following result:

Lemma 2.9 (Lyapunov properties). *Let assumptions 1.1 and 2.1 be satisfied and assume that $d_S \geqslant 1$. Let (S, i) be a positive solution of system (2.8) such that there exists some constant $M > 1$,*

$$\frac{1}{M} \leqslant S(x) \leqslant 1, \qquad \forall x \in \mathbb{R},$$
$$i(a, x) \leqslant M i_E(a), \qquad \forall a \geqslant 0, \quad \forall x \in \mathbb{R},$$
$$|\partial_x i(a, x)| \leqslant M \left(1 + \frac{1}{\sqrt{a}} \right) i(a, x) \qquad \forall a \geqslant 0, \quad \forall x \in \mathbb{R}.$$

Then there exists some constant $m > 0$ (only depending on M) such that

$$-m \leqslant V(S, i)(x) \leqslant m + cW(S, i)(x) < \infty \qquad \forall x \in \mathbb{R}, \tag{2.18}$$

and the map $x \to V(S, i)(x)$ is non-increasing. Moreover, if $x \to V(S, i)(x)$ is constant then

$$S \equiv S_E, \qquad i \equiv i_E, \qquad S' \equiv 0 \qquad \text{and} \qquad i_x \equiv 0.$$

The proof of this lemma is postponed. We shall use this lemma to prove the final step in the proof of theorem 1.3-(iii).

Proof of theorem 1.3-(iii). First recall that due to lemma 2.6 there exists some constant $\widetilde{M}$ such that

$$i(a, x) \leqslant \min\left(e^{\lambda x} e^{-\alpha^* a}, \widetilde{M}\right) e^{-\gamma a}.$$

Thus, there exists some constant $M > 1$ such that

$$\frac{i(a, x)}{i_E(a)} \leqslant M. \tag{2.19}$$

Moreover, due to (2.13) and lemma 2.8, taking M sufficiently large, we have

$$\frac{1}{M} \leqslant S(x) \leqslant 1, \qquad \forall x \in \mathbb{R}$$
$$|\partial_x i(a, x)| \leqslant M\left(1 + \frac{1}{\sqrt{a}}\right) i(a, x) \qquad \forall a \geqslant 0, \quad \forall x \in \mathbb{R}. \tag{2.20}$$

Next consider an increasing sequence $\{x_n\}_{n \geqslant 0}$ of positive real numbers such that $x_n \to +\infty$ when $n \to +\infty$ as well as the sequences of functions

$$S_n(x) = S(x + x_n), \qquad i_n(a, x) = i(a, x + x_n).$$

Due to parabolic and elliptic estimates, up to a subsequence, one may assume that the sequences $\{S_n\}$ and $\{i_n\}$ convergence towards some functions $\widetilde{S}$ and $\widetilde{i}$ for the topology of $C^1_{\text{loc}}(\mathbb{R}) \times C^1_{\text{loc}}(\mathbb{R} \times (0, \infty))$. Moreover, since the map $x \to V(S, i)(x)$ is decreasing we obtain that for each $n \geqslant 0$

$$V(S_n, i_n)(x) = V(S, i)(x + x_n) \leqslant V(S, i)(x), \qquad \forall x \in \mathbb{R}.$$

Since it is bounded from above, there exists $l \in \mathbb{R}$ such that

$$\lim_{n \to \infty} V(S_n, i_n)(x) = l, \qquad \forall x \in \mathbb{R}.$$

On the other hand, due to (2.19) and (2.20) functions $(\widetilde{S}, \widetilde{i})$ satisfy

$$\frac{\widetilde{i}(a, x)}{i_E(a)} \leqslant M,$$
$$\frac{1}{M} \leqslant \widetilde{S}(x) \leqslant 1, \qquad \forall x \in \mathbb{R},$$
$$|\partial_x \widetilde{i}(a, x)| \leqslant M\left(1 + \frac{1}{\sqrt{a}}\right) \widetilde{i}(a, x) \qquad \forall a \geqslant 0, \quad \forall x \in \mathbb{R}.$$

Once again due to (2.19) and (2.20) together with Lebesgue convergence theorem we have

$$\lim_{n \to \infty} V(S_n, i_n)(x) = V(\widetilde{S}, \widetilde{i})(x), \qquad \forall x \in \mathbb{R}.$$

Thus,

$$V(\widetilde{S}, \widetilde{i})(x) \equiv l.$$

To conclude, it is sufficient to note that $(\widetilde{S}, \widetilde{i})$ is a solution of system (2.8). The last part of lemma 2.9 applies and provides that

$$\widetilde{S} \equiv S_E, \qquad \widetilde{i} \equiv i_E, \qquad \widetilde{S}' \equiv 0 \qquad \text{and} \qquad \partial_x \widetilde{i} \equiv 0.$$

This shows that

$$\lim_{x \to \infty} S(x) = S_E, \quad \lim_{x \to \infty} i(a, x) = i_E(a) \text{ in } C^1_{\text{loc}}(0, \infty).$$

It remains to show that the last convergence holds in $L^1(0, \infty)$. Indeed, due to the above convergence and Lebesgue convergence theorem we have

$$\lim_{x \to \infty} S(x) \int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a = i_E(0).$$

Thus, $\lim_{x \to \infty} i(0, x) = i_E(0)$. On the other hand, we have

$$i(a, x) - i_E(a) = \int_{\mathbb{R}} K(a, y)(i(0, x - y) - i_E(0)) \, \mathrm{d}y,$$

where

$$K(t, z) = \frac{e^{-\gamma t}}{\sqrt{4\pi t}} e^{-\frac{(z - ct)^2}{4t}}.$$

Since for any $y \in \mathbb{R}$ we have $\lim_{x \to \infty} i(0, x - y) - i_E(0) = 0$ and $i(0, z) \leqslant M i_E(0)$, we obtain from Lebesgue convergence theorem that the limit for i holds in $L^1(0, \infty)$. This completes the proof of the lemma. ∎

It remains to prove lemma 2.9.

Proof of lemma 2.9. Since $i(a, x) \leqslant M i_E(a)$ we obtain

$$\int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a \leqslant M \int_0^\infty \beta(a) i_E(a) \, \mathrm{d}a.$$

Since S is bounded, we obtain that S is also bounded in $W^{2, \infty}(\mathbb{R})$. Therefore we obtain

$$\left| d_S \frac{S'(x)}{S(x)} \left(1 - \frac{S(x)}{S_E} \right) + \int_0^\infty \alpha(a) \frac{\partial_x i(a, x)}{i(a, x)} \left(1 - \frac{i(a, x)}{i_E(a)} \right) \, \mathrm{d}a \right|$$
$$\leqslant d_S M \|S'\|_\infty \left(1 + \frac{1}{S_E} \right) + \int_0^\infty \alpha(a) M \left(1 + \frac{1}{\sqrt{a}} \right) \, \mathrm{d}a \, (1 + M).$$

Due to the definition of the function g we have $0 \leqslant W(S, i)(x)$ for all $x \in \mathbb{R}$. Next we claim that

$$W(S, i)(x) < \infty \qquad \text{for all } x \in \mathbb{R}. \tag{2.21}$$

To do so it is sufficient to show that

$$\alpha(.) g \left(\frac{i(., x)}{i_E(.)} \right) \in L^1(0, \infty), \qquad \forall x \in \mathbb{R}. \tag{2.22}$$

To check this, let $x_0 > 0$ be given. Since $i(0, x) > 0$ for all $x \in \mathbb{R}$, there exists $\varepsilon > 0$ such that

$$i(0, x) \geqslant \varepsilon, \qquad \forall x \in [-x_0, x_0].$$

We consider $\lambda_D \in \mathbb{R}$ the first eigenvalue and u an associated eigenvector of the problem

$$u''(x) - c u'(x) - \gamma u(x) = \lambda_D u(x), \qquad x \in (-x_0, x_0), \qquad u(\pm x_0) = 0.$$

Moreover, we select u such that $0 < u(x) \leqslant \varepsilon$ on $(-x_0, x_0)$ and set

$$w(a, x) = e^{\lambda_D a} u(x), \qquad a \geqslant 0, \quad x \in [-x_0, x_0].$$

Due to the comparison principle we have $w(a, x) \leqslant i(a, x)$ for all $a \geqslant 0$ and $x \in [-x_0, x_0]$. As a consequence we obtain that for each $x \in [-\frac{x_0}{2}, \frac{x_0}{2}]$,

$$\left| \ln \left(\frac{i(a, x)}{i_E(a)} \right) \right| \leqslant \max \left(|\ln M|, \left| \ln \frac{u(x)}{i_E(0)} \right| + (|\lambda_D| + \gamma)a \right).$$

As a consequence, we obtain that for each $x_0 > 0$, there exist two constants $M_{x_0} > 0$ and $m_{x_0} > 0$ such that

$$\left| \ln \left(\frac{i(a, x)}{i_E(a)} \right) \right| \leqslant M_{x_0} + m_{x_0}a, \qquad \forall a \geqslant 0, \quad \forall x \in [-x_0, x_0].$$

Recalling definition (2.16) we obtain that the map $a \to (1 + a)\alpha(a)$ belongs to $L^1(0, \infty)$ and (2.21) follows. Then since $c > 0$, (2.18) holds.

Let us now show that the map $x \to V(S, i)(x)$ is decreasing.

$$\frac{\mathrm{d}V(S, i)(x)}{\mathrm{d}x} = (d_S S'' - cS') \left(\frac{1}{S(x)} - \frac{1}{S_E} \right) + \int_0^\infty \alpha(a) \left(\frac{1}{i(a, x)} - \frac{1}{i_E(a)} \right) (\partial_x^2 i - c\partial_x i)$$

$$- d_S \frac{S'^2}{S^2} - \int_0^\infty \alpha(a) \frac{(\partial_x i)^2}{i} \, \mathrm{d}a$$

$$= \left(\frac{1}{S(x)} - \frac{1}{S_E} \right) \left(\gamma(S - S_E) + S_E \int_0^\infty \beta(a) i_E(a) \, \mathrm{d}a - S \int_0^\infty \beta(a) i(a, x) \, \mathrm{d}a \right)$$

$$+ \int_0^\infty \alpha(a) \left(\frac{1}{i(a, x)} - \frac{1}{i_E(a)} \right) (\partial_a i + \gamma i)$$

$$- d_S \frac{S'^2}{S^2} - \int_0^\infty \alpha(a) \frac{(\partial_x i)^2}{i} \, \mathrm{d}a$$

$$= -\gamma \frac{(S - S_E)^2}{SS_E} + \int_0^\infty \beta(a) i_E(a) \left(1 - \frac{iS}{i_E S_E} - \frac{S_E}{S} + \frac{i}{i_E} \right) \mathrm{d}a$$

$$+ \int_0^\infty \alpha(a) \partial_a \left(g \left(\frac{i}{i_E} \right) \right) \mathrm{d}a - d_S \frac{S'^2}{S^2} - \int_0^\infty \alpha(a) \frac{(\partial_x i)^2}{i} \, \mathrm{d}a$$

$$= -\gamma \frac{(S - S_E)^2}{SS_E} + \int_0^\infty \beta(a) i_E(a) \left(1 - \frac{iS}{i_E S_E} - \frac{S_E}{S} + \frac{i}{i_E} \right) \mathrm{d}a$$

$$+ \alpha(0) g \left(\frac{i(0, x)}{i_E(0)} \right) - \int_0^\infty \alpha'(a) g \left(\frac{i}{i_E} \right) \mathrm{d}a - d_S \frac{S'^2}{S^2} - \int_0^\infty \alpha(a) \frac{(\partial_x i)^2}{i} \, \mathrm{d}a$$

Thus, we obtain

$$\frac{\mathrm{d}V(S, i)(x)}{\mathrm{d}x} = -\gamma \frac{(S - S_E)^2}{SS_E} + \int_0^\infty \beta(a) i_E(a) \left(1 - \frac{iS}{i_E S_E} - \frac{S_E}{S} + \frac{i}{i_E} \right) \mathrm{d}a$$

$$+ \int_0^\infty \beta(a) i_E(a) \left(\frac{i(0, x)}{i_E(0)} - \frac{i(a, x)}{i_E(a)} - \ln \frac{i(0, x)}{i_E(0)} + \ln \frac{i(a, x)}{i_E(a)} \right) \mathrm{d}a$$

$$- d_S \frac{S'^2}{S^2} - \int_0^\infty \alpha(a) \frac{(\partial_x i)^2}{i} \, \mathrm{d}a$$

and

$$\frac{\mathrm{d}V(S, i)(x)}{\mathrm{d}x} = -\gamma \frac{(S - S_E)^2}{SS_E} - d_S \frac{S'^2}{S^2} - \int_0^\infty \alpha(a) \frac{(\partial_x i)^2}{i} \, \mathrm{d}a + K(x),$$

wherein

$$K(x) = \int_0^\infty \beta(a) i_E(a) \left(1 - \frac{iS}{i_E S_E} - \frac{S_E}{S} + \frac{i(0, x)}{i_E(0)} - \ln \frac{i(0, x)}{i_E(0)} + \ln \frac{i(a, x)}{i_E(a)} \right) \mathrm{d}a.$$

Note now that

$$\int_0^\infty \beta(a) i_E(a) \left(\frac{i(0,x)}{i_E(0)} - \frac{iS}{i_E S_E} \right) da = 0.$$

Indeed, we have

$$\int_0^\infty \beta(a) i_E(a) \left(\frac{i(0,x)}{i_E(0)} - \frac{iS}{i_E S_E} \right) da$$
$$= \frac{i(0,x)}{i_E(0)} \int_0^\infty \beta(a) i_E(a) \, da - \frac{S(x)}{S_E} \int_0^\infty \beta(a) i(a,x) \, da$$
$$= \frac{i(0,x)}{S_E} - \frac{i(0,x)}{S_E} = 0.$$

We deduce that

$$\int_0^\infty \beta(a) i_E(a) \left(1 - \frac{i(a,x) S(x) i_E(0)}{i_E(a) S_E i(0,x)} \right) da = 0. \tag{2.23}$$

Thus, K simplifies to

$$K(x) = \int_0^\infty \beta(a) i_E(a) \left(1 - \frac{S_E}{S} - \ln \frac{i(0,x)}{i_E(0)} + \ln \frac{i(a,x)}{i_E(a)} \right) da. \tag{2.24}$$

Adding (2.23) and (2.24) and adding and subtracting $\ln S(x)/S_E$ we obtain

$$K(x) = \int_0^\infty \beta(a) i_E(a) C(a,x) \, da,$$

wherein

$$C(a,x) = 2 - \frac{i(a,x) S(x) i_E(0)}{i_E(a) S_E i(0,x)} - \frac{S_E}{S} - \ln \frac{i(0,x)}{i_E(0)} + \ln \frac{i(a,x)}{i_E(a)} + \ln \frac{S(x)}{S_E} - \ln \frac{S(x)}{S_E}$$
$$= - \left\{ g\left(\frac{S_E}{S(x)} \right) + g\left(\frac{i(a,x) S(x) i_E(0)}{i_E(a) S_E i(0,x)} \right) \right\}.$$

Since $C(a,x) < 0$ for all $a \geqslant 0$ and $x \in \mathbb{R}$, this completes the proof of lemma 2.9. ∎

2.2. Non-existence results

In this subsection, we focus on assertions (i)–(ii) of theorem 1.3.

2.2.1. Non-existence when $R_0 < 1$.
Assume that $R_0 < 1$, and there is a pair of maps (S, i) of non-negative solutions of (2.8) such that

$$0 \leqslant S(x) \leqslant 1,$$
$$\partial_a i = \partial_x^2 i - c \partial_x i - \gamma i, \qquad a > 0, \quad x \in \mathbb{R},$$
$$i(0,x) = S(x) \int_0^\infty \beta(a) i(a,x) \, da, \qquad x \in \mathbb{R}.$$

Since $i(0,.)$ is assumed to be bounded, using the comparison principle we obtain for any $a \geqslant 0$ that

$$\sup_{x \in \mathbb{R}} i(a,x) \leqslant e^{-\gamma a} \|i(0,.)\|_\infty.$$

Due to the boundary condition we obtain

$$\|i(0,.)\|_\infty \leqslant \int_0^\infty \beta(a) e^{-\gamma a} \, da \|i(0,.)\|_\infty = R_0 \|i(0,.)\|_\infty,$$

which implies that $i \equiv 0$.

2.2.2. Non-existence for $R_0 = 1$.

Lemma 2.10. *Assume that $R_0 = 1$. Let $c > 0$ be given and consider $S \in C^2(\mathbb{R})$ and $i \in C^{1,2}([0, \infty) \times \mathbb{R})$ two positive bounded functions such that*

$$d_S S''(x) - c S'(x) + \gamma - \gamma S(x) - S(x) \int_0^{+\infty} \beta(a) i(a, x) \, da = 0, \qquad x \in \mathbb{R},$$

$$\partial_a i(a, x) = \partial_x^2 i(a, x) - c \partial_x i(a, x) - \gamma i(a, x),$$

$$i(0, x) = S(x) \int_0^{+\infty} \beta(a) i(a, x) \, da.$$

Then $S(x) \equiv 1$ and $i(a, x) \equiv 0$.

Proof. Let us first note that S cannot have a local maximum with value larger than one, so that $0 \leqslant S(x) \leqslant 1$ for all $x \in \mathbb{R}$. Now consider a sequence $\{x_n\}_{n \geqslant 0} \subset \mathbb{R}$ such that

$$\lim_{n \to \infty} i(0, x_n) = M := \sup_{x \in \mathbb{R}} i(0, x).$$

Note that in order to prove the lemma, it is sufficient to show that $M = 0$. To do so, let us argue by contradiction by assuming that $M > 0$. Next consider the sequence of maps $\{S_n(x) = S(x + x_n)\}_{n \geqslant 0}$ and $\{i_n(a, x) = i(a, x + x_n)\}_{n \geqslant 0}$. Since S and i are smooth functions, one may assume, possibly along a subsequence, that $\{S_n\}_{n \geqslant 0}$ and $\{i_n\}_{n \geqslant 0}$ converge as $n \to \infty$ locally uniformly towards to some limit functions $\widehat{S}$ and $\widehat{i}$ satisfying

$$d_S \widehat{S}''(x) - c \widehat{S}'(x) + \gamma - \gamma \widehat{S}(x) - \widehat{S}(x) \int_0^{+\infty} \beta(a) \widehat{i}(a, x) \, da = 0, \qquad x \in \mathbb{R},$$

$$\partial_a \widehat{i}(a, x) = \partial_x^2 \widehat{i}(a, x) - c \partial_x \widehat{i}(a, x) - \gamma \widehat{i}(a, x),$$

$$\widehat{i}(0, x) = \widehat{S}(x) \int_0^{+\infty} \beta(a) \widehat{i}(a, x) \, da,$$

$$\widehat{i}(0, 0) = M, \quad \widehat{i}(0, x) \leqslant M \quad \text{and} \quad 0 \leqslant \widehat{S}(x) \leqslant 1 \qquad \forall x \in \mathbb{R}.$$

We infer from the comparison principle that $\widehat{i}(a, x) \leqslant M e^{-\gamma a}$ so that plugging this into the boundary condition leads us to

$$\widehat{i}(0, x) \leqslant \widehat{S}(x) M \int_0^{\infty} \beta(a) e^{-\gamma a} \, da, \qquad \forall x \in \mathbb{R}.$$

Since $R_0 = \int_0^{\infty} \beta(a) e^{-\gamma a} \, da = 1$, then with $x = 0$, one obtains that

$$M \leqslant \widehat{S}(0) M.$$

Since $M > 0$ and $\widehat{S}(x) \leqslant 1$, this leads us to $\widehat{S}(0) = 1$, so that $\widehat{S}(x) \equiv 1$ and therefore due to the $\widehat{S}$-equation, we obtain that $\int_0^{\infty} \beta(a) \widehat{i}(a, x) \, da = 0$ for all $x \in \mathbb{R}$ and therefore $\widehat{i} \equiv 0$, a contradiction together with $\widehat{i}(0, 0) = M > 0$. This completes the proof of the result. ∎

2.2.3. Non-existence when $R_0 > 1$ and $0 < c < c^*$.

The aim of this section is to prove that when $R_0 > 1$ then there is no wave solution with a wave speed $c < c^*$. Assume that there exists a bounded positive solution (S, i) of (2.8) such that

$$\lim_{x \to -\infty} \begin{pmatrix} S(x) \\ i(a, x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{in } \mathbb{R} \times L^1(0, \infty).$$

For each $\delta \in [0, 1)$ we define $\alpha_\delta^* > 0$ such that

$$1 = (1 - \delta) \int_0^{\infty} \beta(a) e^{-(\alpha_\delta^* + \gamma) a} \, da.$$

Note that the map $\delta \to \alpha_\delta^*$ is non-increasing. Then fix $\delta \in (0, 1)$ such that $c < 2\sqrt{\alpha_\delta^*}$. Fix $M_\delta > 0$ such that $1 - \delta \leqslant S(x) \leqslant 1$, $\forall x < -M_\delta$. Let us fix $c_0 \in (0, c)$ such that

$$\theta := 2\sqrt{\frac{4\alpha_\delta^* - c_0^2}{4\alpha_\delta^* - c^2}} \in \mathbb{Q}, \tag{2.25}$$

and consider the maps $w_{\widehat{c}}(a, x) = e^{\frac{\widehat{c}x}{2}} \sin(\Gamma_{\widehat{c}}x) e^{-(\alpha_\delta^* + \gamma)a}$, wherein

$$\Gamma_{\widehat{c}} = \frac{\sqrt{4\alpha_\delta^* - \widehat{c}^2}}{2}. \tag{2.26}$$

Then, the maps w_c and w_{c_0} satisfy the equations

$$\partial_a w_{\widehat{c}} = \partial_x^2 w_{\widehat{c}} - \widehat{c} \partial_x w_{\widehat{c}} - \gamma w_{\widehat{c}},$$
$$w_{\widehat{c}}(0, x) = (1 - \delta) \int_0^\infty \beta(a) w_{\widehat{c}}(a, x) \, da.$$

Then, we shall prove the following result.

Lemma 2.11. *Let $k_0 \in \mathbb{N}$ be given. Then there exist $p \in \mathbb{N}$ and $j \in \mathbb{N}$ such that*

$$(2j + (2k_0 + 1)) \frac{\Gamma_{c_0}}{\Gamma_c} \geqslant 2p + 1 \geqslant 2p \geqslant (2j + 2k_0) \frac{\Gamma_{c_0}}{\Gamma_c}.$$

Proof. Note now that since $c_0 < c$ then $\Gamma_{c_0} > \Gamma_c$. Next fix $\delta_0 > 0$ such that $\delta_0 < \frac{\Gamma_{c_0}}{\Gamma_c} - 1$. Then $\theta = 2\frac{\Gamma_{c_0}}{\Gamma_c} \in \mathbb{Q}$, we find $j \in \mathbb{N}$ such that $(j + k_0) \frac{\Gamma_{c_0}}{\Gamma_c} \in \mathbb{N}$. Fix $p := (j + k_0) \frac{\Gamma_{c_0}}{\Gamma_c}$, we have

$$\frac{2j\Gamma_{c_0}}{\Gamma_c} + \frac{2k_0\Gamma_{c_0}}{\Gamma_c} \leqslant 2p \leqslant \frac{2j\Gamma_{c_0}}{\Gamma_c} + \frac{2k_0\Gamma_{c_0}}{\Gamma_c} + \delta_0,$$

thus

$$2p + 1 \leqslant \frac{2j\Gamma_{c_0}}{\Gamma_c} + \frac{2k_0\Gamma_{c_0}}{\Gamma_c} + \delta_0 + 1 \leqslant \frac{2j\Gamma_{c_0}}{\Gamma_c} + \frac{(2k_0 + 1)\Gamma_{c_0}}{\Gamma_c},$$

and the result follows. ∎

Now let $k_0 \in \mathbb{N}$ be such that

$$-\frac{2k_0\pi}{\Gamma_c} \leqslant -M_\delta.$$

Let j and p be given by lemma 2.11 and set

$$y_1 = -\frac{2j\pi}{\Gamma_c} - \frac{(2k_0 + 1)\pi}{\Gamma_c}, \qquad y_2 = -\frac{2j\pi}{\Gamma_c} - \frac{2k_0\pi}{\Gamma_c}$$
$$x_1 = -\frac{(2p + 1)\pi}{\Gamma_{c_0}}, \qquad x_2 = -\frac{2p\pi}{\Gamma_{c_0}}.$$

From the definitions of p and j we have $y_1 \leqslant x_1 \leqslant x_2 \leqslant y_2 \leqslant -M_\delta$. Since the map $x \to \sin \Gamma_{c_0} x$ is strictly positive on (x_1, x_2) and vanishes at $x = x_1$ and $x = x_2$ while the map $x \to \sin \Gamma_c x$ is strictly positive on $[x_1, x_2]$, there exists $\varepsilon > 0$ such that

$$\varepsilon w_{c_0}(a, x) \leqslant w_c(a, x), \qquad \forall a > 0, \quad \forall x \in [x_1, x_2] \tag{2.27}$$

and

$$w_{c_0}(a, x_i) = 0, \qquad a \geqslant 0, \quad i = 1, 2.$$

By the assumption on the existence of function i, the map $x \to i(0, x)$ is strictly positive. Thus, there exists $\widehat{\varepsilon} > 0$ such that

$$\widehat{\varepsilon} w_c(0, x) \leqslant i(0, x), \qquad \forall x \in [y_1, y_2].$$

Then due to the definition of y_1, y_2 we have $w_c(0, y_i) = 0$ for $i = 1, 2$ and the comparison principle applies and implies that

$$\widehat{\varepsilon} w_c(a, x) \leqslant i(a, x), \qquad \forall a \geqslant 0, \quad \forall x \in [y_1, y_2].$$

We infer from (2.27) that

$$\varepsilon \widehat{\varepsilon} w_{c_0}(a, x) \leqslant i(a, x), \qquad \forall a \geqslant 0, \quad \forall x \in [x_1, x_2].$$

Consider the map $\widetilde{i}(t, a, x) = i(a, x + (c_0 - c)t)$ that satisfies the equation for $t > 0$, $a > 0$ and $x \in (x_1, x_2)$.

$$\partial_t \widetilde{i} + \partial_a \widetilde{w} = \partial_x^2 \widetilde{i} - c_0 \partial_x \widetilde{i} - \gamma \widetilde{i},$$
$$\widetilde{i}(t, 0, x) = S(x + (c_0 - c)t) \int_0^\infty \beta(a) \widetilde{i}(t, a, x) \, da.$$

Note that since $(c_0 - c) < 0$, we have $x + (c_0 - c)t \leqslant -M_\delta$ for any $t \geqslant 0$ and $x \in [x_1, x_2]$, so that $S(x + (c_0 - c)t) \geqslant 1 - \delta$. We conclude that $\widehat{i}$ satisfies the following problem:

$$\partial_t \widetilde{i} + \partial_a \widetilde{w} = \partial_x^2 \widetilde{i} - c_0 \partial_x \widetilde{i} - \gamma \widetilde{i},$$
$$\widetilde{i}(t, 0, x) \geqslant (1 - \delta) \int_0^\infty \beta(a) \widetilde{i}(t, a, x) \, da.$$

Next we introduce the map

$$u(t, a, x) = i(a, x + (c_0 - c)t) - \varepsilon \widehat{\varepsilon} w_{c_0}(a, x),$$

that satisfies for $t > 0$, $a > 0$ and $x \in (x_1, x_2)$ the problem

$$\begin{aligned} &\partial_t u + \partial_a u = \partial_x^2 u - c_0 \partial_x u - \gamma u, \\ &u(t, 0, x) \geqslant (1 - \delta) \int_0^\infty \beta(a) u(t, a, x) \, da, \\ &u(0, a, x) \geqslant 0 \qquad a > 0, \quad x \in (x_1, x_2), \\ &u(t, a, x_i) \geqslant 0. \end{aligned} \qquad (2.28)$$

We claim that

$$u(t, a, x) \geqslant 0, \text{ for any } t \geqslant 0, a \geqslant 0 \text{ and } x \in [x_1, x_2]. \qquad (2.29)$$

Before proving this claim we first complete the proof of theorem 1.3. Note that (2.29) yields to

$$\varepsilon \widehat{\varepsilon} w_{c_0}(a, \xi) \leqslant i(a, \xi + (c_0 - c)t), \qquad t \geqslant 0, \quad a > 0, \quad x_1 \leqslant \xi \leqslant x_2.$$

Since $c_0 - c < 0$ we obtain a contradiction by letting $t \to \infty$ because $i(a, x) \to 0$ when $x \to -\infty$.

It remains to apply the following lemma to prove (2.29).

Lemma 2.12 (Maximum principle). *Let $u : [0, \infty) \times [0, \infty) \times [x_1, x_2] \to \mathbb{R}$ be a bounded and smooth map satisfying (2.28). Then we have*

$$u(t, a, x) \geqslant 0, \qquad \forall (t, a, x) \in [0, \infty) \times [0, \infty) \times [x_1, x_2].$$

Proof. The proof of this result is based on an integrated semigroup argument (see [10, 28, 38] for more details). In order to simplify the notation and without loss of generality we shall assume that $x_1 = 0$ and $x_2 = 1$. Next we consider the Banach spaces

$$\begin{aligned} Y &= \mathbb{R} \times \mathbb{R} \times L^1(0, 1; \mathbb{R}), & Y_+ &= \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+(0, 1; \mathbb{R}), \\ Y_0 &= \{0\} \times \{0\} \times L^1(0, 1; \mathbb{R}), & Y_{0+} &= Y_+ \cap Y_0, \end{aligned}$$

and the linear operator $B : D(B) \subset Y \to Y$ defined by

$$D(B) = \{0\} \times \{0\} \times W^{2,1}(0, 1; \mathbb{R}),$$

$$B \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi(1) \\ \varphi'' - c_0 \varphi' \end{pmatrix}.$$

The linear operator $B : D(Y) \subset Y \to Y$ is a Hille–Yosida and $(0, \infty) \subset \rho(B)$. Moreover, using classical elliptic maximum principle, we deduce that B is resolvent positive, or more precisely

$$(\lambda - B)^{-1} Y_+ \subset Y_{0+}, \qquad \forall \lambda > 0.$$

As a consequence, the linear operator $B_0 : D(B_0) \subset Y_0 \to Y_0$, the part of B in Y_0 defined by

$$D(B_0) = \{x \in D(B) : Bx \in Y_0\}, \qquad B_0 x = Bx \quad \forall x \in D(B_0),$$

is the infinitesimal generator of a positive C_0-semigroup on Y_0 denoted by $\{T_{B_0}(t)\}_{t \geqslant 0}$. Furthermore, the linear operator B is the infinitesimal generator of a non-degenerate integrated semigroup on Y denoted by $\{S_B(t)\}_{t \geqslant 0}$.

We consider the Banach spaces and the positive cones

$$X = Y_0 \times L^1(0, \infty; Y), \qquad X_+ = Y_+ \times L^1(0, \infty; Y_+),$$
$$X_0 = \{0\} \times L^1(0, \infty; Y_0), \qquad X_{0+} = X_0 \cap X_+.$$

Let us introduce the family of linear bounded operators $\{R_\lambda : X \to X_0\}_{\lambda > 0}$ defined by

$$R_\lambda \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \Leftrightarrow \varphi(a) = e^{-\lambda a} T_{B_0 - \gamma}(a)\alpha + \left(S_{B-\gamma-\lambda} \diamond \psi(.)\right)(a),$$

wherein we have set

$$\left(S_{B-\gamma-\lambda} \diamond \psi(.)\right)(a) = \frac{\mathrm{d}}{\mathrm{d}a} \int_0^a S_{B-\gamma-\lambda}(a - s)\psi(s)\,\mathrm{d}s,$$

and wherein $\{T_{B_0 - \gamma}(t)\}_{t \geqslant 0}$ denotes the C_0-semigroup on Y_0 generated by $B_0 - \gamma$ while $\{S_{B-\gamma-\lambda}(t)\}_{t \geqslant 0}$ denotes the non-degenerate integrated on Y generated by $B - (\gamma + \lambda)$.

Next we observe that $\{R_\lambda\}_{\lambda > 0}$ is a pseudo resolvent on X, that is

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu, \qquad \forall \lambda, \quad \mu > 0.$$

We have

$$R_\lambda x = 0, \qquad x \in X \implies x \in X_0$$

and

$$\lim_{\lambda \to \infty} \lambda R_\lambda x = x, \qquad \forall x \in X_0.$$

Therefore, using the results of Pazy [30, pp 36–7], we conclude that there exists a unique closed operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ such that

$$\overline{D(\mathcal{A})} = X_0, \quad R_\lambda = (\lambda - \mathcal{A})^{-1}, \qquad \forall \lambda > 0.$$

Moreover, we can check that it is a Hille–Yosida operator and B is resolvent positive, we obtain that $R_\lambda X_+ \subset X_{0+}$ for each $\lambda > 0$. Consider the bounded linear operator $\mathcal{B} : X_0 \to X$ defined by

$$\mathcal{B} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \int_0^\infty \beta(a)\varphi(a)\,\mathrm{d}a \\ 0 \end{pmatrix}.$$

Note that this operator is a positive operator, that is $\mathcal{B}X_{0+} \subset X_+$. Therefore, $\mathcal{A} + \mathcal{B} : D(\mathcal{A}) \subset X \to X$ is a Hille–Yosida operator with a positive resolvent operator. The result follows. ∎

Acknowledgments

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