Asymptotic behaviour of a non-local diffusive logistic equation

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Abstract

The aim of the manuscript is to investigate the long time behaviour of a logistic type equation modelling the motion of cells. The equation we consider takes into account birth and death processes using a simple logistic effect as well as a non-local motion of cells using non-local Darcy's law with regular kernel. Using the periodic framework we first investigate the well posedness of the problem before deriving some information about its long time behaviour. The lack of asymptotic compactness of the system is overcome by making use of Young measure theory. This allows us to conclude that the semiflow converges for the Young measure topology.

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1 Introduction

This work is concerned with the study of the integro-differential equation

$$\begin{cases} \partial_t u(t,x) + \text{div}\left(u(t,x)\mathbf{J}\left[u(t,.)\right](x)\right) = f\left(u(t,x)\right), \ t>0, \ x \in \mathbb{R}^N \\ \mathbf{J}\left[u(t,.)\right](x) = -\nabla\left(K \circ u(t,.)\right)(x), \\ u(t,x+2\pi k) \equiv u(t,x), \ \ \forall t>0, \ x\in\mathbb{R}^N, \ k\in\mathbb{Z}^N. \end{cases} \tag{1.1}$$

Here $\circ$ denotes the convolution operator on the $N-$dimensional torus $\mathbb{T}^N = \mathbb{R}^N/2\pi\mathbb{Z}^N$ defined for each $\mathbb{T}^N-$periodic and measurable functions f and g by

$$\left(f\circ g\right)(x) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N} f(x-y)g(y)dy,$$

whenever the above expression makes sense for almost every $x\in\mathbb{T}^N$.
Equation (1.1) is supplemented together with some initial datum

$$u(0,x) = u_0(x) \text{ with } u_0 \text{ non-negative and } \mathbb{T}^N-\text{periodic function.} \tag{1.2}$$

Problem (1.1) arises when looking at spatially periodic solutions of the integro-differential equation

$$\begin{cases} \partial_t u(t,x) + \nabla.\big(u(t,x)\mathbf{v}(t,x)\big) = f\left(u(t,x)\right),\ t>0,\ x\in\mathbb{R}^N \\ \mathbf{v}(t,x) = -\nabla\left(\int_{\mathbb{R}^N} \rho(x-y)u(t,y)dy\right), \end{cases} \tag{1.3}$$

associated to $\mathbb{T}^N-$periodic initial datum and fast decaying kernel function ρ. Indeed, in such a context if $u\equiv u(t,x)$ is spatially $\mathbb{T}^N-$periodic solution of (1.3) then it becomes a solution of (1.1) with the $\mathbb{T}^N-$periodic kernel K defined by

$$K(x) = \sum_{k\in\mathbb{Z}^N} \rho\left(x+2\pi k\right). \tag{1.4}$$

The fast decay of ρ is obviously used to ensure the convergence of the above formula.

Problem of form (1.1) or (1.3) describes the spatio-temporal interactions of cells or individuals, those density at time $t>0$ and located at $x\in\mathbb{R}^N$ is denoted $u(t,x)$. The non-local flux $\mathbf{J}$, namely the convolution operator takes into account the non-local spatial interactions between individuals while the vital dynamics of the population is modelled by the non-linear function f. Throughout this work function $f:\mathbb{R}^+\to\mathbb{R}$ will be assumed to be of logistic type (see Assumption 1.1 for more precise assumptions), with a prototypical shape $f(u)=u(1-u)$.

The modelling of the motion of cells has a long history. If one omits, for the moment, the vital dynamics of the population, that is $f(u)\equiv 0$, the non-local flux operator $\mathbf{J}$ can be derived from particles interaction using the so-called *mean-field* or *Vlasov-limit*. This form allows to take into account long range interaction between cells. Let us now briefly described such a mean-field approximation. Let us consider n particles in $\mathbb{R}^N$ interacting through a potential V_n. Then the spatial location of each particle is denoted by $X_k^n(t)$. Then law of motion reads as

$$\frac{dX_k^n(t)}{dt} = -\frac{1}{n}\sum_{m=1}^n \nabla\rho_n\left(X_k^n(t)-X_m^n(t)\right),\quad k=1,..,n, \tag{1.5}$$

where the potential is assumed to satisfy some scaling property:

$$\rho_n(x) = n^\beta \rho\left(n^{\frac{\beta}{N}}x\right), \tag{1.6}$$

for some $\beta\in[0,1]$ and where ρ is fixed given potential. Using the above particle interaction modelling, the corresponding macroscopic law of motion is obtained by investigating the convergence $n\to\infty$ of the empirical measure $\mu_n(t)$ defined by

$$\mu_n(t) = \frac{1}{n}\sum_{k=1}^n \delta_{X_k^n(t)}.$$

Using this framework, the scaling with $\beta=0$ (resp. $\beta\in(0,1)$, resp. $\beta=1$) corresponds to the so-called Vlasov-limit (resp. meso-scale limit, resp. hydro-dynamics limit) (see [27, 25]). Roughly speaking in one-space dimension, that is $N = 1$, n^{-1} corresponds to the characteristic length between particles while $n^{-\beta}$ corresponds to the characteristic length of interaction induced by the potential ρ_n. The Vlasov framework, namely $\beta = 0$, corresponds to order 1 length of interaction, that is long range interactions.

All these re-scaled limits have been rigorously or formally investigated in the literature for deterministic system (1.5) but also for noisy perturbation of such systems. Let us mention the works of Oelschläger [26] where the author derived the flux formulation $\mathbf{J}$ from the noisy system (1.5) with (1.6) and $\beta = 0$. In this work the author also consider the case $\beta \in (0, 1)$ and derived Darcy's law for moderately interacting potential (fast decay). In [27] the same author investigated the limit behaviour $n \to \infty$ of (1.5) (without noise perturbation) for the case $\beta \in (0, 1)$ as well as the one-dimensional hydrodynamic limit. We also refer to Morale et al [25], Bodnar and Velazquez [12] for more modelling details and formal derivations. Let us finally refer to Capasso and Morale [15] for for rigorous derivation of macroscopic equation coupling several scales and stochastic effects.

Note that the non-local law of motion described by equation (1.3) posed on $\mathbb{R}^N$ with $f(u) \equiv 0$ arises in many applicative fields and has attracted the attention of many researchers. The equation takes the form

$$\partial_t u(t,x) = \nabla.\big(u(t,x)\nabla\left[\rho * u(t,.)\right](x)\big). \tag{1.7}$$

We refer the reader to the introduction of the recent paper of Bertozzi et al [5] for the list of applications for different fields. This class of system has been recently studied in [5, 13] and [29] with additional heterogeneous transport term (see also the references therein). Further results in [2, 3, 4, 5, 14, 17] have been obtained for equation (1.7) with a linear or non linear diffusive perturbation.

In the context of population dynamics, nonlocal equation similar to (1.1) with diffusive perturbation has been previously considered in [24]. We also refer to Leverentz et al [22] and the references therein for swarming models and numerical experiments. In the context of cell-cell adhesion, several works based on the article of Armstrong et al [1] consider some non regular kernel K (see also [6, 7] for more results on this topic). Let us also mention that System (1.1) or (1.3) is also closely related to the so-called hyperbolic Keller-Segel equation. Indeed the hyperbolic Keller-Segel equation with linear sensitivity and a logistic perturbation takes the following form (see [28] for more results on this subject)

$$\begin{cases} \partial_t u(t,x) = \nabla.\left[\chi u(t,x)\nabla v(t,x)\right] + f(u), \text{ for } t > 0, \text{ and } x \in \mathbb{R}^N, \\ \left(1 - \varepsilon^2\Delta\right) v(t,x) = u(t,x) \\ u(0,.) = u_0 \in L^\infty_+\left(\mathbb{R}^N\right). \end{cases} \tag{1.8}$$

Here $\chi \in \mathbb{R}^*$ denotes the sensitivity parameter. It is negative when the substance with concentration is denoted by v is attractive while $\chi > 0$ if it has a chemo-repulsive effect. Next recall that that

$$\left(I - \varepsilon^2\Delta\right)^{-1}(\varphi)(x) = \int_0^\infty e^{-l} T_{\varepsilon^2\Delta}(l)(\varphi)(x)dl$$

and

$$T_{\varepsilon^2\Delta}(t)\,(\varphi)\,(x) = \frac{1}{(4\pi\varepsilon^2 t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \varphi\,(x-y)\, e^{-\frac{\|y\|^2}{4\pi\varepsilon^2 t}} dy.$$

It follows that

$$v - \varepsilon^2 \Delta v = u \Leftrightarrow v(t,x) = \int_{\mathbb{R}^N} \rho_N(x-y) u(t,y) dy$$

where

$$\rho_N(x) = \int_0^\infty \frac{e^{-l}}{(4\pi\varepsilon^2 l)^{\frac{N}{2}}} e^{-\frac{\|x\|^2}{4\pi\varepsilon^2 l}} dl.$$

In particular for $N=1$ we have

$$\rho_1(x) := \frac{\varepsilon^{-1}}{2} e^{-\varepsilon^{-1}|x|}.$$

Hence the hyperbolic Keller-Segel equation re-writes as (1.1) with the kernel $\rho \equiv \chi\rho_N$. For this class of systems the kernel K defined in (1.4) has a singularity only at $x=0$. Such a situation will not enter the framework of this work.

Coming back to (1.1) the assumptions of this work read as follows

Assumption 1.1 *Function $f : \mathbb{R}^+ \to \mathbb{R}$ is concave and takes the form of*

$$f(u) = uh(u), \forall u \geq 0,$$

for some function $h : \mathbb{R}^+ \to \mathbb{R}$ of the class C^1 that satisfies $h(1) = 0$ and the following sign changed condition

$$h(u) > 0, \;\; \forall u \in [0,1) \text{ and } h(u) < 0, \;\; \forall u > 1, \text{ and } \lim_{u \to \infty} h(u) = -\infty. \tag{1.9}$$

Note that such an assumption holds true for the prototypical function $h(u) \equiv 1-u$. Our second assumption is related to the properties of the kernel operator K.

Assumption 1.2 *Function $K : \mathbb{R}^N \to \mathbb{R}$ satisfies the following properties:*

(i) K is a $\mathbb{T}^N-$periodic function of the class C^3 on $\mathbb{R}^N$.

(ii) The Fourier's coefficients of function K on $\mathbb{T}^N$ denoted by $\{c_n[K]\}_{n\in\mathbb{Z}^N}$ satisfy $c_n[K] > 0, \;\; \forall n \in \mathbb{Z}^N \backslash \{0\}$. Here the Fourier coefficients are defined by

$$c_n[K] = |\mathbb{T}^N|^{-1} \int_{\mathbb{T}^N} e^{-inx} K(x) dx, \;\; \forall n \in \mathbb{Z}^N.$$

Note that condition (ii) in the above assumption is related to a repulsive property of the kernel, so that (1.1) exhibits a diffusion like effect. This will be used in a forthcoming modelling work on cell motion [20] (see also [21] for more results on this subject).

If we come back to the formulation (1.3) with kernel ρ, let us notice that Assumption 1.2 (ii) can be re-written in terms of Fourier transform of ρ using the relationship (1.4) as:

$$c_n[K] = |\mathbb{T}^N|^{-1}\widehat{\rho}\left(\frac{n}{2\pi}\right) > 0, \ \ \forall n \in \mathbb{Z}^N \setminus \{0\},$$

where $\widehat{\rho}$ denotes the Fourier transform of ρ defined by

$$\widehat{\rho}\,(\xi) := \int_{\mathbb{R}^N} \rho(x)e^{-2\pi i x\cdot\xi}dx \text{ for } \xi \in \mathbb{R}^{\mathbb{N}}.$$

Due to this remark, if $\rho : \mathbb{R}^N \to \mathbb{R}$ satisfies $\rho \in W^{3,1}\left(\mathbb{R}^N\right) \cap C^3\left(\mathbb{R}^N\right)$; for each multi-index α of length $|\alpha| \leq 3$ the series

$$x \mapsto \sum_{k\in\mathbb{Z}^N} D^\alpha \rho(x + 2\pi k),$$

is uniformly converging on $[-\pi,\pi]^N$ and $\widehat{\rho}(\xi) > 0$ for all $\xi \in \mathbb{R}^N$ then function K defined by (1.4) satisfies Assumption 1.2.

Example 1.3 *Assuming that $N = 1$, it is readily checked that the kernel*

$$\rho(x) = e^{-x^2} \ \ \textit{(respectively } \rho(x) = \frac{2}{1+4\pi^2x^2}\textit{)}$$

satisfies all the aforementioned assumptions and its Fourier transform is

$$\widehat{\rho}\,(\xi) = \sqrt{\pi}e^{-(\pi\xi)^2} \ \ \textit{(respectively } \widehat{\rho}\,(\xi) = e^{-|\xi|}\textit{).}$$

Of course the kernel corresponding to the one dimensional hyperbolic Keller-Segel equation $\rho\,(x) = e^{-|x|}$ also has a positive Fourier transform. But as mentioned above, due the singularity of ρ at 0, we will not consider this class of problems.

Example 1.4 *As mentioned above the regularity assumption for ρ is not satisfied for the 1d hyperbolic Keller-Segel equation. However the assumptions considered in this work allow us to deal with the following modified version of the hyperbolic Keller-Segel equation with chemo-repulsive effect*

$$\begin{cases} \partial_t u(t,x) = \partial_x\left(u(t,x)\partial_x v(t,x)\right) + f\left(u(t,x)\right), \ \textit{for } t \geq 0, \ \textit{and } x \in \mathbb{R}, \\ \left(1-\varepsilon^2\partial_x^2\right)^k v(t,x) = u(t,x), \\ u(0,.) = u_0 \in L^\infty_+\left(\mathbb{R}\right), \end{cases} \tag{1.10}$$

with $k \geq 2$. Indeed, in that case one has $v = \left(\rho_1^{*k} * u\right)$ with $\rho_1(x) := \frac{\varepsilon^{-1}}{2}e^{-\varepsilon^{-1}|x|}$. For $k \geq 2$ we have $\rho_1^{*k} \in W^{2k-1,1}\left(\mathbb{R}\right) \cap C^{2k-2}\left(\mathbb{R}\right)$. Moreover the Fourier transform is positive since $\widehat{\rho_1^{*k}}\,(\xi) = \widehat{\rho}_1\,(\xi)^k > 0$. Hence we deduce that Assumption 1.1 is satisfied for $k \geq 3$.

Assumption 1.2 is a relatively strong assumption on the interaction kernel. This type of condition has been used by Chayes and Panferov [17, Corollary 2.7] to prove a convergence results for similar system with a linear diffusive perturbation and without logistic term. Bernoff and Topaz [2] discusses the positivity condition on the Fourier transform of the interaction kernel for this class of model problems (without the logistic term). As far as we know logistic effect together with non local diffusive motion (namely (1.1)) has not been considered in the literature.

2 Preliminary

The aim of this preliminary is to derive the existence and uniqueness of solution for (1.1)-(1.2) and to provide estimates of the solutions that will be used in the sequel to derive their asymptotic behaviours.

Before going further let us introduce some notations that will be used in what follows.

For each $k \in \mathbb{N}$, let us denote by $C_\sharp^k\left(\mathbb{R}^N\right)$ the Banach space of functions of the class C^k from $\mathbb{R}^N$ into $\mathbb{R}$ and $[0,2\pi]^N$-periodic endowed with the usual sup-norm

$$\|\varphi\|_{k,\infty} = \sum_{p=0}^{k} \sup_{x\in\mathbb{R}^N} |D^p\varphi(x)| .$$

For each $p \in [1,+\infty]$, let us denote by $L_\sharp^p\left(\mathbb{R}^N\right)$ the space of measurable and $[0,2\pi]^N-$periodic functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that

$$\|\varphi\|_{L_\sharp^p(\mathbb{R}^N)} := \|\varphi\|_{L^p\left((-\pi,\pi)^N\right)} < +\infty.$$

Then $L_\sharp^p\left(\mathbb{R}^N\right)$ endowed with the norm $\|\varphi\|_{L_\sharp^p(\mathbb{R}^N)}$ is a Banach space. We also introduce its positive cone $L_{\sharp,+}^p\left(\mathbb{R}^N\right)$ consisting of function in $L_\sharp^p\left(\mathbb{R}^N\right)$ almost everywhere positive.

Finally if (X,d) is a metric space, then $\text{Lip}(X)$ denotes the Banach space of bounded and Lipschitz continuous functions from X into $\mathbb{R}$ endowed with the usual Lipschitz norm

$$\|f\|_{\text{Lip}} = \sup_{x\in X}|f(x)| + \sup_{(x,y)\in X^2,\ x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}, \quad \forall f \in \text{Lip}(X). \tag{2.11}$$

2.1 Solution integrated along the characteristics

In this section we investigate the existence of solution of (1.1) with initial data $u_0 \in L_\sharp^\infty\left(\mathbb{R}^N\right)$. To do so let us first investigate the characteristic curves of the problem.

Lemma 2.1 *Let Assumption 1.2 (i) be satisfied. Let* $u \in C\left([0,\tau], L^1_\sharp\left(\mathbb{R}^N\right)\right)$*. Then the map* $\mathbf{v}(t,x) = (\nabla K \circ u(t,.))(x)$ *belongs to* $C\left([0,\tau], C^1_\sharp\left(\mathbb{R}^N\right)\right)^N$*, and satisfies the following estimates*

$$\|\mathbf{v}(t,.)\|_{0,\infty} \leq \|\nabla K\|_{0,\infty} \|u(t,.)\|_{L^1_\sharp(\mathbb{R}^N)},$$
$$\|\operatorname{div}\mathbf{v}(t,.)\|_{0,\infty} \leq \|\Delta K\|_{0,\infty} \|u(t,.)\|_{L^1_\sharp(\mathbb{R}^N)}. .$$

The proof of this lemma is omitted. Next the following lemma will allow us to define the characteristic curves:

Lemma 2.2 *Let Assumption 1.2 (i) be satisfied. Let* $u \in C\left([0,\tau], L^1_\sharp\left(\mathbb{R}^N\right)\right)$ *be given. Then by setting for* $\mathbf{v}(t,x) = (\nabla K \circ u(t,.))(x)$ *the following non-autonomous system posed for each* $s \in [0,\tau]$ *and each* $x \in \mathbb{R}^N$*:*

$$\begin{cases} \partial_t \Pi_{\mathbf{v}}(t,s;x) = -\mathbf{v}(t, \Pi_{\mathbf{v}}(t,s;x)), \text{ for each } t \in [0,\tau], \\ \Pi_{\mathbf{v}}(s,s;x) = x, \end{cases} \tag{2.12}$$

generates a unique non-autonomous continuous flow $\{\Pi_{\mathbf{v}}(t,s)\}_{t,s\in[0,\tau]}$*, that is to say that*

$$\Pi_{\mathbf{v}}(t,r;x)\Pi_{\mathbf{v}}(r,s;x) = \Pi_{\mathbf{v}}(t,s;x), \forall t,s,r \in [0,\tau], \text{ and } \Pi_{\mathbf{v}}(s,s;.) = I$$

and the map $(t,s,x) \to \Pi_{\mathbf{v}}(t,s;x)$ *is continuous. Moreover for each* $t,s \in [0,\tau]$*, we have*

$$\Pi_{\mathbf{v}}(t,s;x+2\pi k) = \Pi_{\mathbf{v}}(t,s;x) + 2\pi k, \forall x \in \mathbb{R}^{\mathbb{N}}, \, k \in \mathbb{Z}^N,$$

the map $x \to \Pi_{\mathbf{v}}(t,s;x)$ *is continuously differentiable and one has:*

$$\det(\partial_x \Pi_{\mathbf{v}}(t,s;x)) = \exp\left(-\int_s^t \operatorname{div}\mathbf{v}\left(l, \Pi_{\mathbf{v}}(l,s;x)\right) dl\right).$$

Proof. This result follows by using classical arguments on ordinary differential equations and the estimations obtained in Lemma 2.1. ∎

In order to precise the notion of solution we will use in this work, assume first that $u \in C^1([0,\tau]\times\mathbb{R},\mathbb{R}) \cap C\left([0,\tau], C^0_{\sharp,+}(\mathbb{R}^N)\right)$ is a classical solution of (1.1)-(1.2). By setting for $0 \leq s \leq t \leq \tau$:

$$V_{\mathbf{v}}(t,s;x) = \exp\left(-\int_s^t \operatorname{div}\mathbf{v}\left(l, \Pi_{\mathbf{v}}(l,s;x)\right) dl\right), \tag{2.13}$$

then one has

$$\begin{aligned}\frac{d}{dt}\Big(u(t,\Pi_{\mathbf{v}}(t,0)x)V_{\mathbf{v}}(t,0;x)\Big) = \Big[&\partial_t u(t,\Pi_{\mathbf{v}}(t,0;x)) - \mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x)\nabla u(t,\Pi_{\mathbf{v}}(t,0;x)) \\ &- u(t,\Pi_{\mathbf{v}}(t,0;x)\operatorname{div}\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x))\Big]V_{\mathbf{v}}(t,0;x).\end{aligned}$$

Hence a classical solution of (1.1)-(1.2) satisfies

$$\frac{d}{dt}\Big(u(t,\Pi_{\mathbf{v}}(t,0;x))V_{\mathbf{v}}(t,0;x)\Big)=h\Big(u\left(t,\Pi_{\mathbf{v}}(t,0;x)\right)\Big)u(t,\Pi_{\mathbf{v}}(t,0;x))V_{\mathbf{v}}(t,0;x).$$

It follows by using (2.13) that

$$u(t,\Pi_{\mathbf{v}}(t,0;x))=\exp\left(\int_0^t h\big(u(l,\Pi_{\mathbf{v}}(l,0;x))+\text{div }\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right)u_0\left(x\right), \tag{2.14}$$

or equivalently

$$u(t,x)=\exp\left(\int_0^t h\big(u(l,\Pi_{\mathbf{v}}(l,t;x))\big)dl+\text{div }\mathbf{v}(l,\Pi_{\mathbf{v}}(l,t;x))dl\right)u_0\left(\Pi_{\mathbf{v}}(0,t;x)\right). \tag{2.15}$$

Setting

$$\mathbf{v}(t,x):=\int_{\mathbb{T}^N}\nabla K(x-z)u(t,z)dz. \tag{2.16}$$

By using the change of variable $\widehat{z}=\Pi_{\mathbf{v}}(0,t;z)$ we obtain

$$\mathbf{v}(t,x):=\int_{\mathbb{T}^N}\nabla K(x-\Pi_{\mathbf{v}}(t,0;\widehat{z}))u(t,\Pi_{\mathbf{v}}(t,0;\widehat{z}))\exp\left(-\int_0^t \text{div }\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;\widehat{z}))dl\right)d\widehat{z}$$

Combining the above formula together with (2.14) we obtain

$$\mathbf{v}(t,x)=\int_{\mathbb{T}^N}\nabla K(x-\Pi_{\mathbf{v}}(t,0;\widehat{z}))e^{\int_0^t h\left(u(l,\Pi_{\mathbf{v}}(l,0;\widehat{z}))\right)dl}u_0\left(\widehat{z}\right)d\widehat{z}. \tag{2.17}$$

The above computations leads us to the following definition of solution

Definition 2.3 (Solution integrated along the characteristics) *Let $u_0\in L^{\infty}_{\sharp,+}\left(\mathbb{R}^N\right)$ be given. Let $\tau>0$ be given. A function $u\in C\left(\left[0,\tau\right],L^1_{\sharp,+}\left(\mathbb{R}^N\right)\right)\cap L^{\infty}\left((0,\tau),L^{\infty}_{\sharp,+}\left(\mathbb{R}^N\right)\right)$ is said to be a solution integrated along the characteristics of (1.1)-(1.2), if u satisfies (2.15) with $\mathbf{v}$ defined in (2.16).*

Then our existence result reads as:

Theorem 2.4 *Let Assumptions 1.1-1.2 (i) be satisfied. For each $u_0\in L^{\infty}_{\sharp,+}\left(\mathbb{R}^N\right)$, System (1.1)-(1.2) has a unique solution integrated along the characteristics $t\to U(t)u_0$ in $C\left(\left[0,+\infty\right),L^1_{\sharp,+}\left(\mathbb{R}^N\right)\right)\cap L^{\infty}_{loc}\left([0,\infty),L^{\infty}_{\sharp,+}\left(\mathbb{R}^N\right)\right)$. Moreover $\left\{U(t)\right\}_{t\geq 0}$ is a continuous semiflow on $L^{\infty}_{\sharp,+}\left(\mathbb{R}^N\right)$, that is to say that*

- *(i) $U(t)U(s)=U(t+s),\forall t,s\geq 0$ and $U(0)=I$;*
- *(ii) The map $(t,u_0)\to U(t)u_0$ maps every bounded set of $[0,+\infty)\times L^{\infty}_{\sharp,+}\left(\mathbb{R}\right)$ into a bounded set of $L^{\infty}_{\sharp,+}\left(\mathbb{R}\right)$;*

(iii) ***(Continuity)*** *If* $\{t_n\}_{n\in\mathbb{N}} (\subset [0,+\infty)) \to t < +\infty$ *and* $\{u_0^n\}_{n\in\mathbb{N}}$ *is bounded sequence in* $L^\infty_{\sharp,+}\left(\mathbb{R}^N\right)$ *such that* $\|u_0^n - u_0\|_{L^1_\sharp(\mathbb{R}^N)} \to 0$ *as* $n \to +\infty$*, then*

$$\|U(t_n)u_0^n - U(t)u_0\|_{L^1_\sharp(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty.$$

The semiflow U *also satisfies the two following properties*

$$U(t)u_0 \geq 0, \forall u_0 \geq 0, \forall t \geq 0, \tag{2.18}$$

$$\|U(t)u_0\|_{L^1_\sharp(\mathbb{R}^N)} \leq e^{Mt}\|u_0\|_{L^1_\sharp(\mathbb{R}^N)}, \forall t \geq 0 \text{ with } M = \sup_{u\geq 0} h(u). \tag{2.19}$$

Furthermore, if in addition $u_0 \in W^{1,1}_\sharp\left(\mathbb{R}^N\right)$ *then* $U(.)u_0 \in C^1\left([0,+\infty), L^1_\sharp\left(\mathbb{R}^N\right)\right)$*, and if in addition* $u_0 \in C^1_\sharp\left(\mathbb{R}^N\right)$ *then* $u(t,x) = U(t)(u_0)(x)$ *belongs to* $C^1\left([0,+\infty)\times\mathbb{R}^N, \mathbb{R}\right)$ *and* $u(t,x)$ *is a classical solution of system* (1.1)-(1.2)*.*

In order to prove the above lemma we set

$$w(t,x) := u(t, \Pi_{\mathbf{v}}(t,0;x)). \tag{2.20}$$

Next (2.14) yields to

$$w(t,x) = \exp\left(\int_0^t h\big(w(l,x)\big) + \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right)u_0(x). \tag{2.21}$$

Hence one obtains

$$w \in C\left([0,\tau], L^\infty_{\sharp,+}\left(\mathbb{R}^N\right)\right).$$

On the other hand, note that (2.17) provides

$$\mathbf{v}(t,x) := \int_{\mathbb{T}^N} \nabla K(x - \Pi_{\mathbf{v}}(t,0;\widehat{z}))e^{\int_0^t h\left(w(l,\widehat{z})\right)dl}u_0(\widehat{z})\,d\widehat{z}. \tag{2.22}$$

Then (2.21)-(2.22) leads us to the following fixed point problem:

$$\begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} \in C\left([0,\tau], L^\infty_{\sharp,+}\left(\mathbb{R}^N\right)\right) \times C\left([0,\tau], C^1_\sharp\left(\mathbb{R}^N\right)^N\right) \text{ and } \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} = \mathcal{T}\begin{pmatrix} w \\ \mathbf{v} \end{pmatrix},$$

wherein the operator $\mathcal{T}$ is defined by $\mathcal{T}\begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} w' \\ \mathbf{v}' \end{pmatrix}$ and

$$\begin{aligned} w'(t,x) &= \exp\left(\int_0^t h\big(w(l,x)\big) + \operatorname{div}\mathbf{v}'(l,\Pi_{\mathbf{v}}(l,0;x))\,dl\right)u_0(x), \\ \mathbf{v}'(t,x) &= \int_{\mathbb{T}^N} \nabla K(x - \Pi_{\mathbf{v}}(t,0;\widehat{z}))e^{\int_0^t h\left(w(l,\widehat{z})\right)dl}u_0(\widehat{z})\,d\widehat{z}. \end{aligned} \tag{2.23}$$

We will now sketch the proof of Theorem 2.4 by showing that the contraction mapping theorem applies for $\mathcal{T}$ as soon as $\tau > 0$ is small enough. This will ensure the existence and uniqueness of a local solution. To do so we fix $\tau > 0$,

a valued that will chosen latter on and we consider the Banach space X defined by
$$X = C\left([0,\tau], L^\infty_\sharp\left(\mathbb{R}^N\right)\right) \times C\left([0,\tau], C^1_\sharp(\mathbb{R}^N)^N\right),$$
endowed with the usual product norm:
$$\left\| \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} \right\|_X = \|w\|_{C\left([0,\tau], L^\infty_\sharp(\mathbb{R}^N)\right)} + \|\mathbf{v}\|_{C\left([0,\tau], C^1_\sharp(\mathbb{R}^N)^N\right)}.$$
We also introduce the closed subset $X^+ \subset X$ defined by:
$$X^+ = C\left([0,\tau], L^\infty_{\sharp,+}\left(\mathbb{R}^N\right)\right) \times C\left([0,\tau], C^1_\sharp(\mathbb{R}^N)^N\right).$$
Now we consider the operator $\mathcal{T} : X^+ \to X$ defined by $\mathcal{T}\begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} w' \\ \mathbf{v}' \end{pmatrix}$, with $(w', \mathbf{v})$ defined in (2.23). Note that due to (2.15) one has
$$\mathcal{T}(X^+) \subset X^+. \tag{2.24}$$
Next let us set
$$U_0(t) \equiv \begin{pmatrix} u_0 \\ \mathbf{v_0} \end{pmatrix} \in X^+ \text{ with } \mathbf{v_0} = \nabla K \circ u_0.$$
For each, $U \in X$ and $\kappa > 0$ we denote by $\overline{B}_X(U,\kappa)$ the closed ball in X of center U and radius κ.
Now let $\kappa > 0$ be given. We claim that there exists $\widehat{\tau} > 0$ such that for each $\tau \in (0, \widehat{\tau})$:
$$\mathcal{T}\left(X^+ \cap \overline{B}_X(U_0,\kappa)\right) \subset X^+ \cap \overline{B}_X(U_0,\kappa). \tag{2.25}$$
To prove this claim, let $\begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} \in X^+ \cap \overline{B}_X(U_0,\kappa)$ be given. Note that
$$\left\| \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix} \right\|_X \leq \|U_0\|_X + \kappa =: M(\kappa).$$
Then recalling the definition of $(w', \mathbf{v}')$ in (2.23) and setting $H(\kappa) = \sup_{s \in [0, M(\kappa)]} h(s)$ one has:
$$\sup_{t \in [0,\tau]} \|\text{div } \mathbf{v}'(t,.)\|_{L^\infty_\sharp} \leq \|\Delta K\|_{0,\infty} e^{\tau H(\kappa)} \|U_0\|_X =: K_1(\tau), \tag{2.26}$$
and
$$\sup_{t \in [0,\tau]} \|w'(t,.) - u_0(.)\|_{L^\infty_\sharp} \leq \tau \|U_0\|_X \left(H(\kappa) + K_1(\tau)\right) e^{\tau(H(\kappa) + K_1(\tau))}.$$
Moreover one has:
$$\sup_{t \in [0,\tau]} \|\mathbf{v}'(t,.) - \mathbf{v_0}\|_{\left(C^1_\sharp\right)^N} \leq 2\tau \|U_0\|_X \|K\|_{3,\infty} e^{\tau H(\kappa)} |\mathbb{T}^N| \left[M(\kappa) + H(\kappa)\right].$$

Recalling (2.24), the above computation completes the proof of (2.25).

We now claim that there exists $\tau^* \in (0, \widehat{\tau})$ such that for each $\tau \in (0, \widehat{\tau})$ there exists $L(\tau) \in (0,1)$ such that for each $\begin{pmatrix} w_1 \\ \mathbf{v_1} \end{pmatrix}$, $\begin{pmatrix} w_2 \\ \mathbf{v_2} \end{pmatrix} \in X^+ \cap \overline{B}_X (U_0, \kappa)$:

$$\left\| \mathcal{T} \begin{pmatrix} w_1 \\ \mathbf{v_1} \end{pmatrix} - \mathcal{T} \begin{pmatrix} w_2 \\ \mathbf{v_2} \end{pmatrix} \right\|_X \leq L(\tau) \left\| \begin{pmatrix} w_1 \\ \mathbf{v_1} \end{pmatrix} - \begin{pmatrix} w_2 \\ \mathbf{v_2} \end{pmatrix} \right\|_X . \tag{2.27}$$

To prove this claim let $\tau \in (0, \widehat{\tau})$ be given. Let $\begin{pmatrix} w_1 \\ \mathbf{v_1} \end{pmatrix}$, $\begin{pmatrix} w_2 \\ \mathbf{v_2} \end{pmatrix} \in X^+ \cap \overline{B}_X (U_0, \kappa)$ be given and let us set $(w_1', \mathbf{v_1'})$ and $(w_2', \mathbf{v_2'})$ be defined as in (2.23) with $(w, \mathbf{v})$ replaced respectively by $(w_1, \mathbf{v_1})$ and $(w_2, \mathbf{v_2})$. Now before proving (2.27), let us first observe that due to (2.12) and Gronwall inequality one has for all $t \in [0, \tau]$ and $x \in \mathbb{T}^N$:

$$\|\Pi_{\mathbf{v_1}}(t, 0, x) - \Pi_{\mathbf{v_2}}(t, 0, x)\| \leq \tau \|\mathbf{v_1} - \mathbf{v_2}\|_{C([0,\tau], C^1_\sharp)} e^{M(\kappa)\tau} \tag{2.28}$$

From the above estimate we obtain:

$$\begin{aligned} \|\mathbf{v_1'} - \mathbf{v_2'}\| \leq \; & \|K\|_{3,\infty} \tau \|\mathbf{v_1} - \mathbf{v_2}\|_{C([0,\tau], C^1_\sharp)} e^{M(\kappa)\tau} e^{\tau H(\kappa)} |\mathbb{T}^N| \|u_0\|_\infty \\ & + \|K\|_{2,\infty} e^{\tau H(\kappa)} \tau H'(\kappa) \|w_1 - w_2\|_{C([0,\tau], L^\infty_\sharp)} |\mathbb{T}^N| \|u_0\|_\infty . \end{aligned} \tag{2.29}$$

Here we have set

$$H'(\kappa) = \sup_{s \in [0, M(\kappa)]} |h'(s)|.$$

Recalling (2.26) one gets for each $t \in [0, \tau]$:

$$\begin{aligned} \|w_1'(t,.) - w_2'(t,.)\|_{L^\infty_\sharp} \leq \tau \|U_0\|_X e^{\tau H(\kappa) + \tau K_1(\tau)} \Big\{ & H'(\kappa) \|w_1 - w_2\|_{C([0,\tau], L^\infty_\sharp)} \\ & + \sup_{l \in [0,\tau]} \|\text{div } \mathbf{v_1'} \left(l, \Pi_{\mathbf{v_1}}(l, 0; .)\right) - \text{div } \mathbf{v_2'} \left(l, \Pi_{\mathbf{v_2}}(l, 0; .)\right) \|_{L^\infty_\sharp} dl \Big\} . \end{aligned} \tag{2.30}$$

To complete the proof of (2.27) let us notice that for each $l \in [0, \tau]$ one has:

$$\begin{aligned} \|\text{div } \mathbf{v_1'} & \left(l, \Pi_{\mathbf{v_1}}(l, 0; .)\right) - \text{div } \mathbf{v_2'} \left(l, \Pi_{\mathbf{v_2}}(l, 0; .)\right) \|_{L^\infty_\sharp} \\ & \leq 2 \|U_0\|_X e^{\tau H(\kappa)} \|K\|_{3,\infty} \sup_{x \in \mathbb{T}^N} \|\Pi_{\mathbf{v_1}}(l, 0, x) - \Pi_{\mathbf{v_2}}(l, 0, x)\| \\ & + \|U_0\|_X \|K\|_{2,\infty} e^{\tau H(\kappa)} H'(\kappa) \tau \|w_1 - w_2\|_{C([0,\tau], L^\infty_\sharp)} . \end{aligned}$$

Hence (2.27) follows from (2.29)-(2.30) making use of (2.28).

Finally one concludes from (2.25) and (2.27) that for τ small enough, the contraction mapping theorem applies to operator $\mathcal{T}$. Hence the operator $\mathcal{T}$ has a unique fixed point in $X^+ \cap \overline{B}_X (U_0, \kappa)$. Recalling (2.20), this ensures the existence and uniqueness of the local solution integrated along the characteristic of (1.1) as well as (2.18).

Let us mention that the smoothness of the kernel K (namely of the class C^3) is a key ingredient to derive the existence and the uniqueness of the solution. We refer to [6, 7] for other methods whenever K is less smooth (and without logistic non-linearity). We also refer [23] for other existence and uniqueness results for the 2D Euler equation in fluid mechanics.

It remains to show that the semiflow is globally defined. Recalling the definition of M in (2.19) one deduces from (2.15) that

$$u(t,x) \leq \exp(Mt)\, V_{\mathbf{v}}\left(t,0;\Pi_{\mathbf{v}}(0,t;x)^{-1}\right) u_0\left(\Pi_{\mathbf{v}}(0,t;x)\right), \tag{2.31}$$

that completes the proof (2.19). Recalling Lemma 2.1 and $u_0 \in L^\infty_\sharp\left(\mathbb{R}^N\right)$, one obtains that there exists some constant $\widehat{M} > 0$ such that:

$$\|\mathbf{v}(t,.)\|_{0,\infty} + \|\mathrm{div}\ \mathbf{v}(t,.)\|_{0,\infty} \leq \widehat{M} \exp(Mt). \tag{2.32}$$

Recalling that

$$V_{\mathbf{v}}\left(t,0;\Pi_{\mathbf{v}}(0,t;x)^{-1}\right) = \exp\left(\int_0^t \mathrm{div}\ \mathbf{v}(l,\Pi_v(l,0)x)dl\right),$$

one infers from (2.31) and (2.32) that there exists some constant $\widetilde{M} > 0$ such that

$$\|u(t,.)\|_{0,\infty} \leq \exp(Mt)\exp\left(\frac{\widehat{M}}{M}\exp(Mt)\right)\|u_0\|_{0,\infty},\ \forall t \geq 0.$$

The result follows.

Remark 2.5 (Conservation law) *The above computations leads us to the following conservation law: for each Borel set $A \subset \mathbb{T}^N$ and each $0 \leq s \leq t$:*

$$\int_{\Pi_{\mathbf{v}}(t,s;.)A} u(t,x)dx = \int_A \exp\left[\int_s^t h\left(u\left(l,\Pi_{\mathbf{v}}(l,s;z)\right)\right)\right] u(s,z)dz.$$

3 A priori estimates and energy functional

When dealing with (1.3) for $f \equiv 0$ and with initial datum in $L^1(\mathbb{R}^N)$ then it is well known (see for instance [14, 29] that the functional $\mathcal{E}[u(t,.)] := \int_{\mathbb{R}^N} \left(\rho * u(t,.)\right)(x)u(t,x)dx$ is decreasing along the trajectories. In the context of periodic initial datum, one can also check that the functional $\mathcal{E}_\sharp$ defined by

$$\mathcal{E}_\sharp[u(t,.)] := \int_{\mathbb{T}^N} \left(K \circ u(t,.)\right)(x)u(t,x)dx,$$

is also decreasing along the trajectories of (1.1). However here we shall make use of an other energy functional. Let $u \equiv u(t,x)$ be a classical solution of (1.1)-(1.2). Then let us consider the functional

$$E[u(t,.)] = \frac{1}{|T^N|}\int_{\mathbb{T}^N} G\left(u(t,x)\right)dx, \tag{3.33}$$

wherein function $G:[0,\infty)\to[0,\infty)$ is defined by

$$G(u)=u\ln(u)-u+1. \tag{3.34}$$

Then the following lemma holds true:

Lemma 3.1 *Let Assumptions 1.1-1.2 (i) be satisfied. Let $u_0\in L^\infty_{\sharp,+}(\mathbb{R}^N)$ be given. Let $u\equiv u(t,x)$ be the solution of (1.1)-(1.2). Then one has for each $(t,\tau)\in(0,\infty)^2$ with $t-\tau\geq 0$:*

$$\begin{aligned} E[u(t+\tau,.)]-E[u(t-\tau,.)] =& -\int_{t-\tau}^{t+\tau}\sum_{n\in\mathbb{Z}^N}\|n\|^2c_n(K)\left|c_n\left(u(s,.)\right)\right|^2ds \\ & -\frac{1}{|\mathbb{T}^N|}\int_{t-\tau}^{t+\tau}\int_{\mathbb{T}^N}u(s,x)\left|\ln\left(u(s,x)\right)\right|\left|h\left(u(s,x)\right)\right|dxds. \end{aligned} \tag{3.35}$$

Proof. Let u be a classical solution of (1.1). Then one has

$$\frac{d}{dt}E[u(t,.)]=-\frac{1}{|T^N|}\int_{\mathbb{T}^N}u\left|\ln(u)\right|\left|h(u)\right|dx+\frac{1}{|T^N|}\int_{\mathbb{T}^N}u\left(\Delta K\circ u\right)dx.$$

On the other hand, let us notice that for each $\varphi\in L^2_\sharp\left(\mathbb{R}^N\right)$ one has

$$\frac{1}{|T^N|}\int_{\mathbb{T}^N}\varphi\left(\Delta K\circ\varphi\right)dx=\sum_{k\in\mathbb{Z}^N}\overline{c_k\left[\varphi\right]}c_k\left[\Delta K\circ\varphi\right]=-\sum_{k\in\mathbb{Z}^N}\|k\|^2c_k[K]\left|c_k\left[\varphi\right]\right|^2.$$

Hence (3.35) holds true for classical solutions.

Let $u_0\in L^\infty_\sharp(\mathbb{R}^N)$ be given and let u denotes the corresponding solution of (1.1). Let us consider a sequence $\{u_0^n\}_{n\geq 0}\subset C^1_\sharp(\mathbb{R}^N)$ such that $\|u_0^n-u_0\|_{L^1}\to 0$ as $n\to\infty$. If we denote by u^n the classical solution of (1.1) with initial data u_0^n then Theorem 2.4 (iii) ensures that $u^n\to u$ in $C_{loc}\left([0,\infty);L^1_\sharp\left(\mathbb{R}^N\right)\right)$. Since u^n satisfies (3.35), u also satisfies (3.35) using the above convergence as well as Lebesgue convergence theorem. The result follows. ∎

Before using this functional, let us first derive some estimates on the solution of (1.1)-(1.2) provided by Theorem 2.4. Our first estimate reads as:

Lemma 3.2 *Let Assumptions 1.1-1.2 (i) be satisfied. Let $u_0\in L^\infty_{\sharp,+}(\mathbb{R}^N)$ be given. Then one has*

$$\|U(t)u_0\|_{L^1_\sharp(\mathbb{R}^N)}\leq\max\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)},|\mathbb{T}^N|\right),\forall t>0.$$

Moreover we have the following dissipativity property

$$\limsup_{t\to+\infty}\|U(t)u_0\|_{L^1_\sharp(\mathbb{R}^N)}\leq|\mathbb{T}^N|.$$

The above lemma shows the semiflow U is bounded dissipative in $L^1_\sharp\left(\mathbb{R}^N\right)$.
Proof. To prove this estimate we will first deal with classical solution. To do so let us assume that $u_0 \in C^1_\sharp\left(\mathbb{R}^N\right)$ with $u_0 \geq 0$. Let us denote by $u \equiv u(t,x)$ the corresponding classical solution of (1.1) (see Theorem 2.4). Then integrating (1.1) over $\mathbb{T}^N$ yields

$$\frac{d}{dt}\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} u(t,x)dx = \frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} f\left(u(t,x)\right)dx \leq f\left(\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} u(t,x)dx\right).$$

Note that the above inequality holds true using Jensen inequality (recall that f is assumed to be concave see Assumption 1.1). The results follows using usual ordinary differential arguments with Assumption 1.1. The case of solutions integrated along the characteristics (see Definition 2.3 is handled using a regularisation sequence of initial data. ■

Our next estimate reads as follows.

Lemma 3.3 *Let Assumptions 1.1-1.2 be satisfied. The function*

$$\mathbf{v}(t,x) := \left(\nabla K \circ u(t,.)\right)(x) \text{ satisfies } \mathbf{v} \in L^\infty\left((0,\infty), W^{1,\infty}_\sharp(\mathbb{R}^N)\right)^N,$$

and one has for each $t \geq 0$,

$$\|\operatorname{div}\mathbf{v}(t,.)\|_{L^\infty_\sharp(\mathbb{R}^N)} \leq \|\Delta K\|_\infty \max\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)}, |\mathbb{T}^N|\right),$$

and

$$\limsup_{t\to+\infty}\|\operatorname{div}\mathbf{v}(t,.)\|_\infty \leq \|\Delta K\|_\infty|\mathbb{T}^N|.$$

The proof of the above lemma is straightforward using Lemma 3.2. It will allow us to derive the following uniform bound.

Lemma 3.4 *Let Assumptions 1.1-1.2 be satisfied. Then for each initial datum $u_0 \in L^\infty_\sharp(\mathbb{R}^N)$, one has for each $t > 0$ and almost every $x \in \mathbb{R}^N$*

$$u(t,x) \leq \max\left(\|u_0\|_\infty, M_0\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)}\right)\right),$$

where $M_0 = M_0(z)$ is defined using (1.9) by

$$M_0(z) = \inf\left\{t > 1 :\ C(z) + h(y) \leq 0,\ \forall y \geq t\right\} \text{ and } C(z) := \|\Delta K\|_{0,\infty}\max\left(z, |\mathbb{T}^N|\right).$$

Moreover we have the following dissipativity property

$$\limsup_{t\to+\infty}\|u(t,.)\|_{0,\infty} \leq M_0\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)}\right).$$

Proof. Here again we only prove the above estimates for classical solution. Usual limiting procedure allows to extend this estimate for solutions integrated along the characteristics according to Definition 2.3. Let $u_0 \in C^1_\sharp(\mathbb{R}^N)$ with $u_0 \geq 0$

be given. Let $u\equiv u(t,x)$ be the corresponding classical solution of (1.1). Then one has for each $t>0$ and $x\in\mathbb{T}^N$:

$$\partial_t u=\mathbf{v}\nabla u+u\left(\mathrm{div}\mathbf{v}+h(u)\right).$$

On the other hand, recalling the notations of Lemma 2.2 one gets for each $t>0$ and $x\in\mathbb{T}^N$:

$$\frac{du\left(t,\Pi_{\mathbf{v}}(t,0;x)\right)}{dt}=\partial_t u(t,\Pi_{\mathbf{v}}(t,0;x))-\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x))\nabla u(t,\Pi_v(t,0;x)).$$

Hence by setting $X(t):=u\left(t,\Pi_{\mathbf{v}}(t,0;x)\right)$ one obtains

$$\frac{dX(t)}{dt}=X(t)\left[\mathrm{div}\mathbf{v}(t,\Pi_v(t,0)x)+h\left(X(t)\right)\right].$$

Due to Lemma 3.3 we obtain

$$\frac{dX(t)}{dt}\leq X(t)\left[C\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)}\right)+h(X(t))\right]\text{ and }X(0)=u_0(x).$$

Therefore this yields to

$$u(t,\Pi_v(t,0;x))=X(t)\leq\max\left(\|u_0\|_\infty,M_0\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)},\right)\right),$$

and since the map $x\to\Pi_v(t,0;x)$ is invertible, the proof is completed whenever u_0 is smooth enough. ∎

Let us prove an identity that will be used in the sequel.

Lemma 3.5 *Let Assumptions 1.1-1.2 be satisfied. Let $u_0\in L^\infty_{\sharp,+}(\mathbb{R}^N)$ be given and let us denote by $u\equiv u(t,x)$ the solution of (1.1)-(1.2). For each $F\in C^1\left(\mathbb{T}^N\times\mathbb{R}\right)$ the map $t\mapsto\int_{\mathbb{T}^N}F\left(x,u\right)dx$ is of the class C^1 and one has*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^N}F\left(x,u(t,x)\right)dx=&\int_{\mathbb{T}^N}\left[u(t,x)F_u(x,u(t,x))-F(x,u(t,x))\right]\mathrm{div}\;\mathbf{v}(t,x)\;dx\\&-\int_{\mathbb{T}^N}\nabla_x F(x,u(t,x))\mathbf{v}(t,x)dx\\&+\int_{\mathbb{T}^N}F_u(x,u(t,x))u(t,x)h(u(t,x))dx,\end{aligned}$$

with $\mathbf{v}(t,x)\equiv\left(\nabla K\circ u(t,.)\right)(x)$.

Proof. Let u be a classical solution of (1.1). Let $F\in C^2\left(\mathbb{T}^N\times\mathbb{R}\right)$ be given. Then multiplying (1.1) by $F_u(x,u)$ and integrating over $\mathbb{T}^N$ leads us to

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^N}F(x,u)dx&=\int_{\mathbb{T}^N}F_u(x,u)\mathrm{div}\left(u\nabla K\circ u\right)dx+\int_{\mathbb{T}^N}F_u(x,u)uh(u)dx\\&=-\int_{\mathbb{T}^N}u\nabla\left[F_u(x,u)\right]\mathbf{v}dx+\int_{\mathbb{T}^N}F_u(x,u)uh(u)dx.\end{aligned}$$

Next, one has:

$$u\nabla\left[F_u(x,u)\right]=\nabla\left[uF_u(x,u)-F(x,u)\right]+\nabla_x F(x,u),$$

that yields to

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^N}F(x,u)dx=&\int_{\mathbb{T}^N}\left[uF_u(x,u)-F(x,u)\right]\mathrm{div}\,\mathbf{v}dx\\&-\int_{\mathbb{T}^N}\nabla_x F(x,u)\mathbf{v}dx+\int_{\mathbb{T}^N}F_u(x,u)uh(u)dx.\end{aligned}$$

This completes the proof of the identity for classical solutions and C^2 test function. The case of C^1 test function is obtained using limiting argument and the case of solution integrated along the characteristics can handled by using a classical regularisation procedure. ■

4 Asymptotic behaviour

In this section we investigate the long time behaviour of (1.1)-(1.2). In order to overcome the possible lack of asymptotic smoothness of the trajectories we will the Young measure framework. First notice that due to Lemma 3.5 (and Lemmas 3.3 and 3.4), for each $n\in\mathbb{Z}^N$ the map $t\to c_n\left[u(t,.)\right]$ is of the class C^1 bounded up to its first derivative (by choosing the test function $F(x,u)=e^{inx}u$). Next due to the a priori estimate (3.35) one obtains that for each solution

$$\lim_{t\to+\infty}c_n\left[u(t,.)\right]=0 \text{ for each } n\in\mathbb{Z}^N\setminus\{0\}\,.$$

By using the convergence of the Fourier coefficients $c_n\left[u(t,.)\right]$ it is clear that for a given sequence $\{t_k\}_{k\geq 0}\to+\infty$ one can find a subsequence $t_{k_p}\to+\infty$ and a constant $c\in\mathbb{R}$ such that

$$u(t_{k_p},.)\rightharpoonup c \text{ in } L^2\left(\mathbb{T}^N\right).$$

The limit solution may depend on the sequence $\{t_k\}_{k\geq 0}$.

Remark 4.1 *From the above remark one concludes that*

$$\lim_{t\to+\infty}\|\mathbf{v}(t,.)\|_{1,\infty}=0. \tag{4.36}$$

Indeed if $\{t_k\}_{k\geq 0}$ is a given sequence tending to ∞ then from the regularity of the kernel (of the class C^3) the sequence $\{\mathbf{v}(t_k,.)\}_{k\geq 0}$ is bounded $C^2_\sharp(\mathbb{R}^N)^N$ and thus relatively compact in $C^1_\sharp(\mathbb{R}^N)^N$. Let $\tilde{\mathbf{v}}$ be a limit point of the above sequence in $C^1_\sharp(\mathbb{R}^N)^N$. This means that there exists a subsequence $\{t_{k_p}\}_{p\geq 0}$ such that

$$\lim_{p\to\infty}\mathbf{v}(t_{n_p},.)=\tilde{\mathbf{v}} \textit{ in } C^1_\sharp(\mathbb{R}^N)^N.$$

From the above remark, possibly along a subsequence, one may assume that there exists some constant $c \in \mathbb{R}$ such that $u\left(t_{k_p}, .\right) \rightharpoonup c$ weakly in $L^2\left(\mathbb{T}^N\right)$. Hence one gets that for each $x \in \mathbb{T}^N$:

$$\int_{\mathbb{T}^N} \nabla K(x-y) u\left(t_{k_p}, y\right) dy \to c \int_{\mathbb{T}^N} \nabla K(x-y) dy = 0.$$

This justifies (4.36).

Therefore the goal of this section is to obtain more detailed convergence results on the original distribution $u(t,.)$. One of the main goal of this section will be to prove the weak convergence of $u(t,.)$ to a unique constant distribution as t goes to $+\infty$. Actually, we will prove a stronger convergence result by showing that there exists a unique constant $E_\infty := \lim_{t\to+\infty} E[u(t,.)]$ such that

$$f\left(u(t,.)\right) \rightharpoonup \left[E_\infty f(0) + \left(1 - E_\infty\right) f(1)\right] \text{ in } L^2\left(\mathbb{T}^N\right) \text{ as } t \to +\infty$$

for each continuous function $f : \mathbb{R} \to \mathbb{R}$. We will use this type of convergence result to derive a strong convergence result for some specific initial data.

In order to state our result, let us recall some definitions related to Young measures theory.

Definition 4.2 (Young measure) *Let (X, d) be a separable metric space and let $\mathcal{P}(X)$ be the set of probability measures on (X, d). Let Ω be a given set endowed with a $\sigma-$algebra $\mathcal{A}$. A map $\nu : \Omega \to \mathcal{P}(X)$ is said to be* ***a Young measure*** *if for each Borel set $B \in \mathcal{B}(X)$ the function $x \to \nu_x(B)$ is measurable from $(\Omega, \mathcal{A})$ into $[0,1]$. The set of all Young measure from $(\Omega, \mathcal{A})$ into X is denoted by $Y(\Omega, \mathcal{A}; X)$.*

Definition 4.3 (Narrow convergence topology) *Let (X, d) be a separable metric space and let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. The set $Y(\Omega, \mathcal{A}; X)$ is endowed with* ***the narrow convergence topology****; and this topological space is denoted by $(Y(\Omega, \mathcal{A}; X); \mathcal{T})$; which is defined as the weakest topology on $Y(\Omega, \mathcal{A}; X)$ such that all the functionals*

$$Y(\Omega, \mathcal{A}; X) \ni \nu \to \int_A \left(\int_X \eta(\omega) \nu_x(d\omega) \right) \mu(dx) \in \mathbb{R},$$

for $A \in \mathcal{A}$ and $\eta \in C_b(X; \mathbb{R})$ are continuous. Here $C_b(X; \mathbb{R})$ denotes the space of continuous and bounded maps from X into $\mathbb{R}$.

Remark 4.4 *Using the above notations, note that a sequence $\{\nu^n\}_{n\in\mathbb{N}} \subset Y(\Omega, \mathcal{A}; X)$ narrow converges to $\nu \in Y(\Omega, \mathcal{A}; X)$ if and only if for each continuous function $\eta \in C_b(X; \mathbb{R})$ one has*

$$\lim_{n\to\infty} \int_X \eta(\omega) \nu_x^n(d\omega) = \int_X \eta(\omega) \nu_x(d\omega),$$

wherein the above convergence holds for the weak$-*$ *topology of* $L^\infty(\Omega,\mathcal{A};\mathbb{R})$ *with respect to the parameter* $x\in\Omega$*. This also re-writes as for each* $\eta\in C_b(X;\mathbb{R})$ *and each* $\varphi\in L^1(\Omega,\mathcal{A};\mathbb{R})$,

$$\lim_{n\to\infty}\int_\Omega\varphi(x)\left[\int_X\eta(\omega)\nu_x^n(d\omega)\right]\mu(dx)=\int_\Omega\varphi(x)\left[\int_X\eta(\omega)\nu_x(d\omega)\right]\mu(dx)$$

In the sequel we will denote

$$Y_\sharp(\mathbb{R}^N,X):=Y\left(\mathbb{T}^N,\mathcal{B}(\mathbb{T}^N);X\right),$$

and for each $\Omega\in\mathcal{B}(\mathbb{R})$:

$$Y_\sharp(\Omega\times\mathbb{R}^N,X)=Y\left(\Omega\times\mathbb{T}^N,\mathcal{B}(\Omega)\otimes\mathcal{B}(\mathbb{T}^N);X\right).$$

The above spaces will always be endowed with the narrow convergence topology. For this reason, in the sequel we will not explicitly write down the topology $\mathcal{T}$.

Definition 4.5 (Local narrow convergence topology) *Let* (X,d) *be a separable metric space and let* $(\Omega,\mathcal{A},\mu)$ *a finite measure space. The set* $Y(\mathbb{R}\times\Omega,\mathcal{B}(\mathbb{R})\otimes\mathcal{A};X)$ *is endowed with* ***the local narrow convergence topology*** *denoted by* $\mathcal{T}_{loc}$ *which is defined as the weakest topology on* $Y(\mathbb{R}\times\Omega,\mathcal{B}(\mathbb{R})\otimes\mathcal{A};X)$ *such that all the functionals*

$$Y(\Omega,\mathcal{A};X)\ni\nu\to\int_{I\times A}\left(\int_X\eta(\omega)\nu_.(d\omega)\right)(dt\otimes\mu)\in\mathbb{R},$$

for each $I\subset\mathbb{R}$ *bounded interval,* $A\in\mathcal{A}$ *and* $\eta\in C_b(X;\mathbb{R})$ *are continuous.*

In the following we will use the above notion with $(\Omega,\mathcal{A})=\left(\mathbb{T}^N,\mathcal{B}\left(\mathbb{T}^N\right)\right)$ and $X=[0,\gamma]$ some real interval. Moreover we shall denote by $Y_{\sharp,loc}\left(\mathbb{R}\times\mathbb{R}^N;[0,\gamma]\right)$ the topological space $Y_\sharp\left(\mathbb{R}\times\mathbb{R}^N;[0,\gamma]\right)$ endowed with the local narrow convergence topology $\mathcal{T}_{loc}$.

Remark 4.6 *Let us also recall that since* $\left(\mathbb{T}^N,\mathcal{B}(\mathbb{T}^N)\right)$ *is a countably generated* $\sigma-$*algebra then the topology* $Y_\sharp(\mathbb{R}^N,X)$ *is metrizable (see for instance the monograph of Castaing et al [16]).*

Using these definition, the main result of this section reads as follows:

Theorem 4.7 *Let Assumptions 1.1-1.2 be satisfied. Let* $u_0\in L^\infty_{\sharp,+}(\mathbb{R}^N)$ *be given and let* $u\equiv u(t,x)$ *be the solution of* (1.1)-(1.2) *provided by Theorem 2.4. Recalling definition* (3.33) *let* $E_\infty\geq 0$ *be the quantity defined by*

$$E_\infty:=\lim_{t\to\infty}E[u(t,.)].$$

Let us denote by $\gamma=\max\left(\|u_0\|_\infty,M_0\left(\|u_0\|_{L^1_\sharp(\mathbb{R}^N)}\right)\right)$ *(see Lemma 3.4 for the notations). Then for each* $t\geq 0$ *the map* $x\mapsto\delta_{u(t,x)}$ *belongs to* $Y_\sharp\left(\mathbb{R}^N,[0,\gamma]\right)$*;* $E_\infty\in[0,1]$ *and*

$$\lim_{t\to\infty}\delta_{u(t,.)}=E_\infty\delta_0+(1-E_\infty)\delta_1$$

wherein the above convergence holds for the narrow convergence topology of $Y_\sharp\left(\mathbb{R}^N,[0,\gamma]\right)$.

Before dealing with the proof of the above theorem, let us state and prove several corollaries of this result. The first one is related to the velocity field $\mathbf{v}$. In particular this provides an other of the convergence result stated in Remark 4.1.

Corollary 4.8 *Under the same assumptions and notations as in Theorem 4.7, the velocity field* $\mathbf{v}(t,x) \equiv \nabla K \circ u(t,.)(x)$ *satisfies*

$$\lim_{t\to\infty} \mathbf{v}(t,x) = 0 \text{ in } C^1_\sharp(\mathbb{R}^N). \tag{4.37}$$

We furthermore have the following point dissipativity estimate

$$\limsup_{t\to\infty} \|u(t,.)\|_\infty \leq 1.$$

Proof. Note that the convergence (4.37) directly follows from Theorem 4.7. Indeed recall that one has:

$$\mathbf{v}(t,x) = |\mathbb{T}^N|^{-1} \int_{\mathbb{T}^N} \nabla K(x-y) \left[\int_{[0,\gamma]} \omega \; \delta_{u(t,y)}(d\omega) \right] dy.$$

Hence one obtains using Theorem 4.7 that for each $x \in \mathbb{T}^N$:

$$\lim_{t\to\infty} \mathbf{v}(t,x) = \frac{1-E_\infty}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \nabla K(x-y) dy = 0.$$

In addition, since $K \in C^3_\sharp(\mathbb{R}^N)$, $\{\mathbf{v}(t,.)\}_{t>0}$ is uniformly bounded in C^2 that leads us to $\mathbf{v}(t,.) \to 0$ as $t \to \infty$ for the topology of $C^1_\sharp(\mathbb{R}^N)$ and completes the proof of (4.37). Hence recalling the notations introduced in the proof of Lemma 3.4 one has:

$$\frac{dX(t)}{dt} = X(t) \left[\text{div } \mathbf{v}(t, \Pi_v(t,0)x) + h\left(X(t)\right) \right].$$

Thus since $\|\text{div } \mathbf{v}(t,.)\|_{0,\infty} \to 0$ as $t \to \infty$, one obtains that for each $T > 0$ and $x \in \mathbb{T}^N$

$$\limsup_{t\to\infty} u(t,x) \leq \Psi(T) := \inf \left\{ z > 1 : \; \sup_{t\geq T} \|\text{div } \mathbf{v}(t,.)\|_{0,\infty} + h\left(y\right) \leq 0, \; \forall y \geq z \right\}.$$

Finally note that because of Assumption 1.1, one has $\Psi(T) \to 1$ as $T \to \infty$ and the result follows. ∎

Our second corollary allows us to characterize the asymptotic compactness of the trajectories.

Corollary 4.9 *Under the same assumptions and notations as in Theorem 4.7, let* $\{t_n\}_{n\geq 0}$ *be a sequence tending to* ∞ *as* $n \to \infty$. *Then the sequence* $\{u(t_n,.)\} \subset L^\infty_\sharp(\mathbb{R}^N)$ *is relatively compact in* $L^1_\sharp(\mathbb{R}^N)$ *if and only if* $E_\infty \in \{0,1\}$. *We furthermore have*

$$E_\infty = 0 \text{ (resp. } 1) \Leftrightarrow \lim_{t\to\infty} u(t,x) = 1 \text{ (resp. } 0) \text{ in } L^1_\sharp(\mathbb{R}^N).$$

This corollary is a direct consequence of Young measure properties (see Corollary 3.1.5 in [16]).

Our next corollary described some situations ensuring that $E_\infty = 0$.

Corollary 4.10 *Under the same assumptions and notations as in Theorem 4.7, let us assume that there exists $\delta > 0$ such that $u_0(x) \geq \delta$ a.e. $x \in \mathbb{R}^N$. Then one has $E_\infty = 0$ and $\lim_{t\to\infty} u(t,x) = 1$ in $L^1_\sharp(\mathbb{R}^N)$.*

Proof. Recalling the notations introduced in the proof of Lemma 3.4 one has:

$$\frac{dX(t)}{dt} = X(t)\left[\mathrm{div}\, \mathbf{v}(t, \Pi_v(t,0)x) + h\left(X(t)\right)\right].$$

From this one may observe that $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{T}^N$. Since $\|\mathrm{div}\, \mathbf{v}(t,.)\|_{0,\infty} \to 0$ as $t \to \infty$, there exists $T > 0$ large enough such that

$$\|\mathrm{div}\, \mathbf{v}(t,.)\|_{0,\infty} \leq \varepsilon, \ \ \forall t \geq T,$$

where $\varepsilon > 0$ is a given parameter such that $h(0) - \varepsilon > 0$. Now using the above integration along characteristic, one obtains that for each $t \geq T$ and each $x \in \mathbb{T}^N$:

$$u(t,x) \geq \phi(t),$$

wherein $\phi(t)$ is the solution of the ordinary differential equation

$$\phi'(t) = \phi(t)\left[h\left(\phi(t)\right) - \varepsilon\right], \ t \geq T \text{ and } \phi(T) = \min_{x\in\mathbb{T}^N} u(T,x).$$

Since $h(0) - \varepsilon > 0$, there exists $\eta \in (0,1)$ and $T > 0$ such that

$$u(t,x) \geq \eta, \ \ \forall t \geq T, \ x \in \mathbb{T}^N. \tag{4.38}$$

Let $\widetilde{f} : [0,\gamma] \to \mathbb{R}$ be a given continuous functions such that

$$\widetilde{f}(s) = 0 \text{ if } s \geq \eta \text{ and } \widetilde{f}(0) = 1.$$

Then according to Theorem 4.7 one has

$$\lim_{t\to\infty} \int_{\mathbb{T}^N} \widetilde{f}(u(t,x))dx = E_\infty.$$

On the other hand, because of (4.38) and the definition of $\widetilde{f}$ one has $\widetilde{f}(u(t,.)) = 0$ for each $t \geq T$ that implies that $E_\infty = 1$. Hence $\delta_{u(t,.)} \to \delta_1$ as $t \to \infty$ and the L^1-convergence follows from standard Young measures properties (see [16, 30]).
■

It remains to prove Theorem 4.7. This proof requires several lemmas. Before going into the proof of Theorem 4.7, let us first recall further definitions and notations. Recalling the definition of γ in Theorem 4.7 let us consider the probability space $\mathcal{P}\left(\mathbb{T}^N \times [0,\gamma]\right)$ and let us recall that the usual weak$*-$topology on

$\mathcal{P}\left(\mathbb{T}^N \times [0,\gamma]\right)$ is metrizable using the so-called bounded dual Lipschitz metric defined by

$$\Theta(\mu,\nu) = \sup\left\{\left|\int_{\mathbb{T}^N\times[0,\gamma]} fd(\mu-\nu)\right|,\ f\in \mathrm{Lip}\ \left(\mathbb{T}^N\times[0,\gamma]\right),\ \|f\|_{\mathrm{Lip}}\le 1\right\}.$$

Recall that the Lipschitz norm is defined in equation (2.11). Therefore the above distance does not correspond to the 1-Wasserstein distance. We refer to Theorem 18 of Dudley [19] and to the textbook of Billingsley [10] for the equivalence between the weak $*-$topology on $\mathcal{P}\left(\mathbb{T}^N \times [0,\gamma]\right)$ and the topology induced by $\Theta(.,.)$. In the sequel the probability space $\mathcal{P}\left(\mathbb{T}^N \times [0,\gamma]\right)$ is always endowed with the metric topology induced by Θ without further precision.
Let $\{t_n\}_{n\ge 0}$ be a given increasing sequence tending to ∞ as $n\to\infty$. Using the above definition, one will show the following result:

Lemma 4.11 *Under the assumptions of Theorem 4.7, let $T>0$ be given. Then the sequence of maps $\mu^n \in C^0\left([-T,T];\mathcal{P}\left(\mathbb{T}^N\times[0,\gamma]\right)\right)$ defined by*

$$\mu^n_t = |\mathbb{T}^N|^{-1}dx\otimes\delta_{u(t+t_n,x)},$$

is relatively compact in $C^0\left([-T,T];\mathcal{P}\left(\mathbb{T}^N\times[0,\gamma]\right)\right)$. Note that the definition of μ^n_t means that for each continuous function $f\in C\left(\mathbb{T}^N\times[0,\gamma];\mathbb{R}\right)$:

$$\int_{\mathbb{T}^N\times[0,\gamma]} f(x,y)d\mu^n_t(x,y) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N} f\left(x,u(t+t_n,x)\right)dx. \tag{4.39}$$

Proof. This result is a consequence of the identity provided in Lemma 3.5. Let us first show that there exists some constant $M>0$ such that for each $(t,s)\in[0,\infty)^2$ then

$$\Theta\left(|\mathbb{T}^N|^{-1}dx\otimes\delta_{u(t,x)}, |\mathbb{T}^N|^{-1}dx\otimes\delta_{u(s,x)}\right)\le M|t-s|. \tag{4.40}$$

Let $F\equiv F(x,u)\in C^1\left(\mathbb{T}^N\times\mathbb{R}\right)$ be given. Let $(t,s)\in[0,\infty)^2$ be given such that $s\le t$. Then Lemma 3.5 yields to

$$\int_{\mathbb{T}^N} F(x,u(t,x))dx - \int_{\mathbb{T}^N} F(x,u(s,x))dx = \int_s^t \Gamma_F(l)dl, \tag{4.41}$$

wherein we have set

$$\begin{aligned}\Gamma_F(t) := &\int_{\mathbb{T}^N}\left[u(t,x)F_u(x,u(t,x)) - F(x,u(t,x))\right]\mathrm{div}\ \mathbf{v}(t,x)\ dx\\ &-\int_{\mathbb{T}^N}\nabla_x F(x,u(t,x))\mathbf{v}(t,x)dx\\ &+\int_{\mathbb{T}^N}F_u(x,u(t,x))u(t,x)h(u(t,x))dx.\end{aligned}$$

On the other hand, since $0 \leq u(t,x) \leq \gamma$ and using the estimates for $\mathbf{v}$ in Lemma 3.3, one directly obtains that there exist some constant $M = M(\gamma)$ such that for all $F \in C^1\left(\mathbb{T}^N \times \mathbb{R}\right)$ and each $l \in [0,\infty)$:

$$|\Gamma_F(l)| \leq M\|F\|_{\mathrm{Lip}\,(\mathbb{T}^N\times[0,\gamma];\mathbb{R})}. \tag{4.42}$$

Finally (4.40) follows from (4.41) and (4.42).

To complete the proof of the lemma, it is sufficient to notice that Prohorov's compactness theorem (see [10]) implies that $\mathcal{P}\left(\mathbb{T}^N \times [0,\gamma]\right)$ is a compact metric space and the result follows from Arzela-Ascoli theorem. ∎

Now using well known results about Young measures (see for instance [16, 30]), since u is uniformly bounded, the following compactness result holds true:

Lemma 4.12 *The sequence $\left\{\delta_{u(t_n,.)}\right\}_{n\geq 0}$ is relatively compact for the narrow convergence topology of $Y_\sharp\left(\mathbb{R}^N,[0,\gamma]\right)$.*

Using the two above results, namely Lemma 4.11 and Lemma 4.12, one obtains that up to a subsequence, one may assume that

$$\lim_{n\to\infty} \mu_t^n = \mu_t, \text{ and } \lim_{n\to\infty} \delta_{u(t_n,x)} = \nu_x \tag{4.43}$$

where the limits hold respectively for the topology of $C^0_{loc}\left(\mathbb{R};\mathcal{P}\left(\mathbb{T}^N\times[0,\gamma]\right)\right)$ and for the narrow convergence topology of $Y_\sharp\left(\mathbb{R}^N,[0,\gamma]\right)$. Here we would like to recall that the limits μ_t and ν_x depends on the chosen and fixed sequence $\{t_n\}_{n\geq 0}$.

Next due to (4.39) one has for each continuous function $f \in C\left(\mathbb{T}^N\times[0,\gamma];\mathbb{R}\right)$ and each $n \geq 0$:

$$\int_{\mathbb{T}^N\times[0,\gamma]} f(x,y)d\mu_0^n(x,y) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N}\int_{[0,\gamma]} f\left(x,y\right)d\delta_{u(t_n,x)}(y)\,dx.$$

Passing to the limit $n\to\infty$ yields to

$$\int_{\mathbb{T}^N\times[0,\gamma]} f(x,y)d\mu_0(x,y) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N}\int_{[0,\gamma]} f\left(x,y\right)d\nu_x(y)\,dx,$$

for each continuous function $f \in C\left(\mathbb{T}^N\times[0,\gamma];\mathbb{R}\right)$. This re-writes as:

$$\mu_0 = |\mathbb{T}^N|^{-1}dx\otimes\nu_x \tag{4.44}$$

Next the following lemma is also a direct consequence of Young measures properties.

Lemma 4.13 *The sequence $\left\{\delta_{u(t+t_n,x)}\right\}_{n\geq 0}$ is relatively compact for the local narrow convergence topology of $Y_{\sharp,loc}\left(\mathbb{R}\times\mathbb{R}^N;[0,\gamma]\right)$.*

Using the above result, up to a subsequence, one may assume that there exists a Young measure $\nu \equiv \nu_{t,x} \in Y_\sharp\left(\mathbb{R}\times\mathbb{R}^N;[0,\gamma]\right)$ such that

$$\lim_{n\to\infty} \delta_{u(t_n+t,x)} = \nu_{t,x} \text{ for the topology of } Y_{\sharp,loc}\left(\mathbb{R}\times\mathbb{R}^N;[0,\gamma]\right). \tag{4.45}$$

As a consequence one has

$$\mu_t = |\mathbb{T}^N|^{-1} dx \otimes \nu_{t,x} \text{ a.e. } (t,x) \in \mathbb{R}\times\mathbb{T}^N. \tag{4.46}$$

Here let us also recall that $\nu_{t,x}$ depends on the chosen and fixed sequence $\{t_n\}_{n\geq 0}$.
The aim of the next lemmas is to identify the family of measures $\nu_{t,x}$. Our next result describes first property of $\nu_{t,x}$:

Lemma 4.14 *There exists a measurable map $a : \mathbb{R}\times\mathbb{T}^N \to \mathbb{R}$ such that $0 \leq a(t,x) \leq 1$ a.e. $(t,x) \in \mathbb{R}\times\mathbb{T}^N$ and*

$$\nu_{t,x} = [1-a(t,x)]\,\delta_0 + a(t,x)\delta_1, \;\; a.e.\ (t,x) \in \mathbb{R}\times\mathbb{T}^N.$$

Proof. The proof of this lemma relies on the energy functional E (see (3.33)). Let $T > 0$ be given. Then (3.35) implies that

$$\lim_{n\to\infty} \iint_{[-T,T]\times\mathbb{T}^N} \widehat{G}\left(u(t_n+t,x)\right) dtdx = 0, \tag{4.47}$$

wherein $\widehat{G} : [0,\infty) \to [0,\infty)$ denotes the continuous function defined as $\widehat{G}(s) = s\left|\ln(s)\right|\left|h(s)\right|$. Recalling the definition of μ_t^n in Lemma 4.11, one obtains that for all $n \geq 0$:

$$\frac{1}{|\mathbb{T}^N|}\iint_{[-T,T]\times\mathbb{T}^N} \widehat{G}\left(u(t_n+t,x)\right) dtdx = \int_{-T}^{T}\left[\int_{\mathbb{T}^N\times[0,\gamma]} \widehat{G}(y)\; d\mu_t^n(x,y)\right].$$

Combining (4.43), (4.46) and (4.47) one obtains that

$$\int_{-T}^{T}\int_{\mathbb{T}^N}\left[\int_{[0,\gamma]} \widehat{G}\left(y\right) d\nu_{t,x}(y)\right] dxdt = 0, \;\; \forall T > 0.$$

Since the map $u \mapsto \widehat{G}(u)$ is positive and only vanishes at $u = 0$ and $u = 1$ one gets that

$$\operatorname{supp} \nu_{t,x} \subset \{0\}\cup\{1\}, \;\; a.e.\ (t,x) \in \mathbb{R}\times\mathbb{T}^N.$$

The above characterisation of the support allows us to re-write (see [18])

$$\nu_{t,x} = \nu_{t,x}\left(\{0\}\right)\delta_0 + \nu_{t,x}\left(\{1\}\right)\delta_1, \; a.e.\ (t,x) \in \mathbb{R}\times\mathbb{T}^N.$$

Finally set $a(t,x) \equiv \nu_{t,x}\left(\{1\}\right)$. Recalling that $(t,x) \mapsto \nu_{t,x}$ is measurable with value in $\mathcal{P}\left([0,\gamma]\right)$, function a is measurable and the result follows. ∎

Our next result reads as

Lemma 4.15 *There exists a measurable map $b:\mathbb{R}\to\mathbb{R}$ such that function $a\equiv a(t,x)$ provided by Lemma 4.14 satisfies $a(t,x)\equiv b(t)$ a.e. $(t,x)\in\mathbb{R}\times\mathbb{T}^N$.*

Proof. Here again the proof of this lemma relies on (3.35). Recalling Assumption 1.2 (iv) let us notice that (3.35) yields that for each $\tau>0$:

$$\lim_{n\to\infty}\int_{-\tau}^{\tau}\sum_{k\in\mathbb{Z}^N\setminus\{0\}}\|k\|^2c_k[K]\left|c_k[u(t+t_n,.)]\right|^2dt=0.$$

This more particularly implies that for each given $k\in\mathbb{Z}^N\setminus\{0\}$ and each $\tau>0$ one has

$$\lim_{n\to\infty}\int_{-\tau}^{\tau}\left|c_k[u(t+t_n,.)]\right|^2dt=0.$$

Now Hölder inequality yields for each $\varphi\in L^2_{loc}(\mathbb{R})$, for each $\tau>0$ and for each $k\in\mathbb{Z}^N\setminus\{0\}$ that

$$\lim_{n\to\infty}\int_{-\tau}^{\tau}\varphi(t)c_k[u(t+t_n,.)]dt=0.$$

Using the definition of $\nu_{t,x}$ in (4.45) and its description in Lemma 4.14 one obtains that for each $\tau>0$ and for each $k\in\mathbb{Z}^N\setminus\{0\}$:

$$\lim_{n\to\infty}\int_{-\tau}^{\tau}\varphi(t)c_k[u_n(t,.)]dt=\int_{-\tau}^{\tau}\varphi(t)c_k[a(t,.)]dt=0.$$

This implies that $c_k[a(t,.)]=0$, $\forall k\in\mathbb{Z}^N\setminus\{0\}$ *a.e.* $t\in\mathbb{R}$ and the result follows. ∎

In order to complete the proof of Theorem 4.7, we will prove the following lemma

Lemma 4.16 *The following hold true: $E_\infty\in[0,1]$ and $b(t)\equiv 1-E_\infty$ a.e. $t\in\mathbb{R}$.*

Proof. The proof of this result is based on the identity derived in Lemma 3.5 with $F(x,s)\equiv s$ applied with $u(t+t_n,x)$ that reads as for each $n\geq 0$ and each $t>-t_n$

$$\partial_t\int_{\mathbb{T}^N}u(t+t_n,x)dx=\int_{\mathbb{T}^N}u(t+t_n,x)h(u(t+t_n,x))dx.$$

Let $\varphi\in C^1_c(\mathbb{R})$ be given. Multiplying the above equation by φ for each $n\geq 0$ large enough such that $\operatorname{supp}\varphi\subset[-t_n,\infty)$ and integrating over $\mathbb{R}$ leads to

$$-\int_{\mathbb{R}\times\mathbb{T}^N}\varphi'(t)u(t+t_n,x)dtdx=\int_{\mathbb{R}}\varphi(t)\int_{\mathbb{T}^N}\left[\int_{[0,\gamma]}f(z)\,d\delta_{u(t+t_n,x)}(z)\right]dxdt.$$

Letting $n\to\infty$ and using Lemma 4.15 leads us to for all $\varphi\in C^1_c(\mathbb{R})$

$$\int_{\mathbb{R}}\varphi'(t)b(t)dt=\int_{\mathbb{R}}\varphi(t)\int_{\mathbb{T}^N}\left[\int_{[0,\gamma]}f(z)\left((1-b(t))\,\delta_0+b(t)\delta_1\right)(dz)\right]dxdt=0.$$

This implies that there exists some constant $b^* \in [0,1]$ such that $b(t) \equiv b^*$.

Finally recalling that $E[u(t+t_n,.)] \to E_\infty$ as $n \to \infty$ for each $t \in \mathbb{R}$ directly yields $E_\infty = 1 - b^*$, and the result follows. ∎

To complete the proof of Theorem 4.7 note that we infer from the above lemmas that $E_\infty \in [0,1]$ while $\nu_{t,x} = E_\infty \delta_0 + (1-E_\infty)\delta_1$ *a.e.* $(t,x) \in \mathbb{R} \times \mathbb{T}^N$. Recalling that function $t \to \mu_t$ defined in (4.43) is continuous from $\mathbb{R}$ into $\mathcal{P}(\mathbb{T}^N \times [0,\gamma])$ and that (4.46) implies that for all $t \in \mathbb{R}$:

$$\mu_t \equiv |\mathbb{T}^N|^{-1} dx \otimes (E_\infty \delta_0 + (1-E_\infty)\delta_1).$$

Finally recalling (4.44) one obtains that

$$\nu_x \equiv E_\infty \delta_0 + (1-E_\infty)\delta_1 \ a.e.\ x \in \mathbb{T}^N.$$

Since the sequence $\{t_n\}_{n \geq 0}$ denotes any sequence increasing tending tending to infinity, recalling the definition of $x \mapsto \nu_x$ in (4.43), Lemma 4.12 and Remark 4.6 this completes the proof of Theorem 4.7.

5 Concluding remarks

In this work we have given a detailed description of the asymptotic behaviour of system (1.1)-(1.2) (see Theorem 4.7). The situation $E_\infty \in (0,1)$ remains a pathological behaviour (see Corollary 4.9). However we conjecture that as soon as $u_0 \not\equiv 0$ then the corresponding solution $u \equiv u(t,x)$ strongly converges to 1 (namely $E_\infty = 0$). This means that the Lebesgue measure of the set $Z_t = \{x \in \mathbb{T}^N : \ u(t,x) = 0\}$ tends to zero as t goes to ∞. However we are not able to prove this result for general initial data (see Corollary 4.10).

Let us finally notice that even if such a pathological behaviour ($E_\infty \in (0,1)$) occurs then it is "unstable" with respect to diffusive perturbation. Indeed if we consider the following equation on $\mathbb{T}^N$

$$\partial_t u(t,x) = \Delta u(t,x) + \operatorname{div}\left(u(t,x)\left(K \circ u(t,.)\right)(x)\right) + f\left(u(t,x)\right), \tag{5.48}$$

supplemented together with some initial data $u_0 \in L^\infty_{\sharp,+}\left(\mathbb{R}^N\right)$. Then it is easy to see that under the regularity Assumption 1.2 (i) for the kernel K that the solution goes to 1 uniformly as $t \to \infty$ as soon as $u_0 \not\equiv 0$. To see this, let us first notice that using strong maximum principle one has $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{T}^N$. Next uniform bound of the solution follows from similar arguments as in Lemma 3.2 coupled with maximum principle. Because of the uniform bound of the solution, the convection term is also uniformly bounded and usual parabolic regularity applies and ensures that the solution is asymptotically relatively compact in $C\left(\mathbb{T}^N\right)$. Moreover the functional $t \to E[u(t,.)]$ defined in (3.33) is decreasing along the trajectories and satisfies a similar property as in (3.35), that reads as:

$$\begin{aligned}\frac{d}{dt}E[u(t,.)] = &- \frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N}\left[\frac{|\nabla u|^2}{u} + u|\ln u||h(u)|\right]dx \\ &+ \frac{1}{|\mathbb{T}^N|^2}\int_{\mathbb{T}^N \times \mathbb{T}^N} u(x)\Delta K(x-y)u(y)dxdy.\end{aligned}$$

Recalling that Assumption 1.2 (*ii*) ensures that the last term in the above equality is negative, one directly obtains the convergence to 1 for the solution.

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