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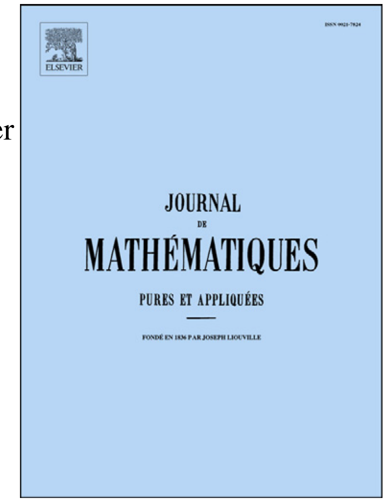

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Convergence to generalized transition waves for some Holling-Tanner prey-predator reaction-diffusion system

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Abstract

This manuscript is devoted to the study of some spreading properties of Holling-Tanner prey-predator reaction-diffusion system. We more particularly focus on the invasion of introduced predator in some environment which is initially well-populated of prey. We first prove that, for the arbitrary dimensional problem, the system has a spreading speed property. We derive more precise information for the one-dimensional system for which the long time behaviour is studied and it is proved that the solution converges (in some sense) towards a generalized transition wave with some determined global mean speed of propagation.

Résumé

L'objet de ce travail est d'étudier l'invasion d'un prédateur introduit dans un environment spatial homogonène riche en proie. Le modèle mathématique que nous considérons est un système de réaction-diffusion posé dans tout l'espace avec des interactions proies-prédateurs de type Holling-Tanner. Dans un premier temps, sans faire d'hypothèse sur la dimension de l'espace, nous caractérisons la zone d'expansion de la population de prédateurs. Une étude plus approfondie est ensuite menée dans le cadre monodimensionnel en espace. Dans ce cas, nous montrons que le comportement asymptotique des solutions est décrit par des ondes de transition généralisées dont la vitesse moyenne est explicitée en fonction des paramètres du modèle.

Keywords: Generalized transition waves, long time behaviour, prey-predator system

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1. Introduction

We consider in this work the following Holling-Tanner predator-prey reaction diffusion system

$$\begin{aligned} &\partial_t u - d\Delta u = u(1-u) - \Pi(u)v, \\ &\partial_t v - \Delta v = rv\left(1 - \frac{v}{u}\right), \end{aligned} \tag{1.1}$$

posed for $t > 0$ and $x \in \mathbb{R}^N$ and supplemented together with some initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \tag{1.2}$$

whose properties will be described below.

Here $d > 0$ describes the (normalized) diffusivity of prey, $\Pi \equiv \Pi(u)$ denotes the functional response to predation while $r > 0$ denotes the growth rate of predator.

The underlying ordinary differential system of (1.1) with

$$\Pi(u) = \frac{mu}{A+u},$$

for some given constants $m > 0$ and $A > 0$ was suggested by Tanner [15]. (See also Murray [11] and Renshaw [14]). Since the last decades, (1.1) as well as the underlying ordinary differential system have attracted the attention of many researchers. One can for instance refer to May [9], Murray [11] and Hsu and Huang [12] for the study of the underlying ordinary differential equations and Du and Hsu [7] for the study the reaction-diffusion system with $\Pi(u) = \alpha u$ on a bounded domain. Typical examples of functions Π are given by Holling type functional response, that reads as

$$\Pi(u) = \frac{eu^n}{h+u^n}, \tag{1.3}$$

for some constant $e > 0$ and $h > 0$ and some power $n \geq 1$. Let us also mention the case where $\Pi(u) \equiv \alpha u$ some constant given constant $\alpha > 0$, that corresponds to the classical Lotka-Volterra functional response. Here we will not assume a specific form for function Π. We shall assume that function $\Pi : [0,\infty) \to [0,\infty)$ satisfies

$$\Pi(u) = u\pi(u), \quad \forall u \in [0,\infty), \tag{1.4}$$

wherein $\pi : [0,\infty) \to [0,\infty)$ is of the class C^1 such that $\pi(u) > 0$ for all $u \in (0,1]$. Furthermore, we assume that

Assumption 1.1. *There exists $\delta \in (0,1)$ such that*

$$1 - \delta = \pi(\delta).$$

This assumption is a simple condition to ensure a positive and uniform lower bound for the prey population. All the proofs of this work are crucially based on this lower bound property. Here we do not focus on finding more general conditions on function π to satisfy such a property. Here we mainly focus on the predator invasion phenomenon. Note that when $n \geq 2$, function Π described in (1.3) satisfies the above assumption.

In this work we shall focus on the response of system (1.1) to a localized introduction of predator. To do so, one shall consider that the prey population is initially uniformly well distributed (at its carrying capacity $u_0 \equiv 1$ for simplicity and see Assumption 1.2 below for more precise assumption) while predator population is initially introduced in some localized location, namely v_0 is compactly supported. Using such a framework we are interesting in deriving some information about the invasion of predator in such an environment. Before stating our first result, let us precise the assumptions we shall use on the initial data. We assume that the initial data u_0 and v_0 satisfy the following conditions

Assumption 1.2. *Functions u_0 and v_0 arising in (1.2) satisfy*

(i) They are both nonnegative and continuous functions,

(ii) Function u_0 satisfies $\delta \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}^N$,

(iii) Function v_0 is nonzero and compactly supported and $0 \leq v_0(x) \leq 1$ for all $x \in \mathbb{R}^N$.

The first main result of this work is concerned with the spreading speed property for (1.1) that reads as

Theorem 1.3 (Spreading speed)**.** *Let Assumptions 1.1-1.2 be satisfied. Set $c^* = 2\sqrt{r}$. Let (u, v) be the solution of (1.1). Then the following holds true:*

$$\lim_{t\to\infty} v(t, x + ect) = 0, \quad \lim_{t\to\infty} u(t, x + ect) = 1, \tag{1.5}$$

for all unit vector e and for all $c > c^*$ locally uniformly with respect to $x \in \mathbb{R}^N$. For each $c \in (0, c^*)$, one has

$$\liminf_{t\to\infty} \inf_{\|x\|\le ct} v(t, x) \ge \delta, \quad \limsup_{t\to\infty} \sup_{\|x\|\le ct} u(t, x) < 1, \tag{1.6}$$

wherein $\|.\|$ denotes the Euclidian norm in $\mathbb{R}^N$.

The second main result of this work is concerned together with the large time behaviour of the one-space dimensional system (1.1). In order to derive and state the convergence result, let us notice that due to Assumption 1.1 and $\pi(1) > 0$, there exists $\gamma \in (\delta, 1)$ such that

$$\frac{1-\gamma}{\pi(\gamma)} = \delta. \tag{1.7}$$

Using this definition, we will assume the following additional condition.

Assumption 1.4. *We assume that the map $h : (0, 1] \to \mathbb{R}$ defined by*

$$h(u) = \frac{1-u}{\pi(u)}, \quad \forall u \in (0, 1],$$

satisfies

(i) $h(u) < \delta$ for each $u \in (\gamma, 1]$,

(ii) $h(u) < 1$ for all $u \in (\delta, 1]$.

Then the following result holds true:

Theorem 1.5. *Let us assume that $N = 1$. Let Assumptions 1.1, 1.2 and 1.4 be satisfied. Let (u, v) be the solution of (1.1). Consider the quantity $m(t)$ defined at least for large time by*

$$m(t) = \sup\left\{x \ge 0 : \quad v(t, x) = \frac{\delta}{2}\right\}.$$

There exist $\widehat{H} > 0$ *and two non-increasing maps* $\mathbf{V}^{\pm} : \mathbb{R} \to (0,1)$ *such that*

$$\lim_{x\to\infty} \mathbf{V}^{\pm}(x) = 0, \quad \lim_{x\to-\infty} \mathbf{V}^{+}(x) = 1, \quad \lim_{x\to-\infty} \mathbf{V}^{-}(x) = \delta,$$

and two non-decreasing maps $\mathbf{U}^{\pm} : \mathbb{R} \to (0,1)$ *such that*

$$\lim_{x\to\infty} \mathbf{U}^{\pm}(x) = 1, \quad \lim_{x\to-\infty} \mathbf{U}^{+}(x) = \gamma, \quad \lim_{x\to-\infty} \mathbf{U}^{-}(x) = \delta,$$

such that for each $\{t_k\}_{k\geq 0}$ *tending to* $+\infty$ *when* $k \to \infty$, *there exists a subsequence still denoted by* $\{t_k\}_{k\geq 0}$ *and some constant* $H \in \left[-\widehat{H}, \widehat{H}\right]$ *such that the sequences* $\{u_k(t,x) = u(t+t_k, x+m(t_k))\}_{k\geq 0}$ *and* $\{v_k(t,x) = v(t+t_k, x+m(t_k))\}_{k\geq 0}$ *converge locally uniformly with respect to* $(t,x) \in \mathbb{R}^2$ *towards two functions* $\overline{u} \equiv \overline{u}(t,x)$ *and* $\overline{v} \equiv \overline{v}(t,x)$, *an entire solution of (1.1) and such that for all* $(t,x) \in \mathbb{R}^2$

$$\begin{aligned} \overline{v}(0,0) &= \frac{\delta}{2}, \\ \mathbf{V}^{-}(x - c^*t + H) \leq &\overline{v}(t,x) \leq \mathbf{V}^{+}(x - c^*t + H), \\ \mathbf{U}^{-}(x - c^*t + H) \leq &\overline{u}(t,x) \leq \mathbf{U}^{+}(x - c^*t + H). \end{aligned}$$

Remark 1.6. *From the proof of this result, one also obtains the following asymptotic behaviour for the quantity* $m(t)$:

$$m(t) = c^*t - \frac{3}{2c^*}\ln t + o(\ln t) \text{ as } t \to \infty.$$

The entire solutions constructed in the above result look like generalized transition waves. We refer to Berestycki and Hamel [5, 4] for more information on such a notion. However within the general framework of Assumption 1.4, we are not able to prove that the entire solutions constructed in Theorem 1.5 are generalized transition waves. To prove such a result, we will use the following additional assumption

Assumption 1.7. *We assume that the map* $h : (0,1] \to [0,\infty)$ *defined in Assumption 1.4 is non-increasing.*

Under the above additional assumption, we will prove that the entire solutions constructed in Theorem 1.5 are generalized transition waves of (1.1). Before stating this result, let us recall some definitions taken from Berestycki and Hamel [5, 4]. We refer to this article for more general definitions (see also Mellet et al in[10] for some convergence results to generalized transition wave for some scalar equation in general heterogeneous environment and combustion type nonlinearity). First note that under Assumption 1.7, the equation

$$h(a) = a,$$

has a unique solution denoted by $a^* \in (0,1)$.

Definition 1.8 (Generalized transition waves)**.** *Assume that* $N = 1$. *An entire classical solution* $(\overline{u}, \overline{v})$ *of (1.1) is said to be* **an almost planar generalized transition wave** *of (1.1) connecting the stationary states* $(1,0)$ *and* (a^*, a^*) *if there exists* $\{\xi_t\}_{t\in\mathbb{R}} \subset \mathbb{R}$ *such that for each* $\varepsilon > 0$ *there exists* $M > 0$ *such that*

$$\forall t \in \mathbb{R},\ \ \forall x \in (\xi_t, \infty),\ \ x \geq \xi_t + M \ \ \Rightarrow \ \ \left\| \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}(t,x) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq \varepsilon,$$

and

$$\forall t \in \mathbb{R},\ \ \forall x \in (-\infty, \xi_t),\ \ x \leq \xi_t - M \ \ \Rightarrow \ \ \left\| \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}(t,x) - \begin{pmatrix} a^* \\ a^* \end{pmatrix} \right\| \leq \varepsilon,$$

This generalized transition wave is said to have **a global mean speed of propagation** $c \in \mathbb{R}$ *if*

$$\frac{|\xi_t - \xi_s|}{|t-s|} \to c \ \text{as} \ |t-s| \to \infty.$$

Our main last result is the following:

Theorem 1.9. *Let Assumptions 1.1, 1.2, 1.4 and 1.7 be satisfied. Then each entire solution* $(\overline{u}, \overline{v})$ *of (1.1) constructed in Theorem 1.5 is an almost planar generalized transition invasion wave of predators with global mean speed of propagation* c^* *and connecting the two stationary states*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \text{and} \ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a^* \\ a^* \end{pmatrix},$$

wherein $a^* \in (0,1)$ *is the unique solution of the equation*

$$h(a^*) = a^*.$$

Furthermore, recalling Definition 1.8, one can choose $\xi_t = c^* t$ *for all* $t \in \mathbb{R}$.

Remark 1.10. *According to Definition 1.4 given by Berestycki and Hamel [4], in the context of Theorem 1.9, the stationary coexistence state* $\begin{pmatrix} a^* \\ a^* \end{pmatrix}$ *invades the predator free stationary state* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Let us notice that in the context of spatially periodic scalar equations, Berestycki and Hamel recently proved in [4] that under some assumptions, generalized transition waves such that $\xi_t = c^* t$ correspond to planar travelling wavefronts. In the context of Theorem 1.9, one expects that the almost planar transition waves correspond to travelling wave solutions of (1.1), that is entire solution of the form:

$$(\overline{u}, \overline{v})(t, x) \equiv \left(\overline{U}, \overline{V}\right)(x - c^* t).$$

However this question remains an open problem.

The organization of this work is the following: Section 2 is devoted to recall some well known results on the asymptotic behaviour of the Fisher-KPP equation. Section 3 is devoted to the study of spreading speed property. Section 4 focuses on the one-dimensional system (1.1) and provides some information on the asymptotic shape of the solutions. Finally Section 5 is devoted to the proof of Theorem 1.9.

2. Preliminaries on the Fisher-KPP equation

This aim of this section is to recall some important and well known properties of the Fisher-KPP equation that will play a crucial role in the sequel. Consider a map $f : [0, 1] \to \mathbb{R}$ of the class C^1. Assume that the map $uf(u)$ satisfies the Fisher-KPP assumptions, namely

(i) the map f is non-increasing on $[0, 1]$ and

(ii) $f(1) = 0$.

Next consider the following so-called Fisher-KPP parabolic equation

$$\partial_t u - \Delta u = uf(u), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{2.8}$$

supplemented together with some initial data $u(0, x) \equiv u_0(x)$ where $u_0 \in C\left(\mathbb{R}^N, [0, 1]\right)$ is some given compactly supported function. Then the following two important properties will be used in the sequel:

Theorem 2.1 (Spreading speed)**.** *Set $c^* = 2\sqrt{f(0)}$ and let u be the solution of (2.8) with initial data u_0. Then for each $c > c^*$ and each $e \in S^{N-1}$ one has*

$$\lim_{t\to\infty} u(t, x + cte) = 0, \text{ locally uniformly in } x \in \mathbb{R}^N,$$

and for each $c \in (0, c^*)$, one has

$$\lim_{t\to\infty} \inf_{\|x\|\le ct} u(t,x) = 1.$$

One refers for instance to Aronson and Weinberger [2] for the proof of such a result (see also [1], [6] and the references cited therein for more results on the heterogeneous equation).

In the one dimensional situation, namely $N = 1$, one obtains more precise information about the long time behaviour of the solution. Consider for each time large enough the quantity

$$m(t) = \sup\left\{x \in \mathbb{R} : \; u(t,x) = \frac{1}{2}\right\}.$$

Note that the later quantity is well defined due to Theorem 2.1. Then the following result holds true

Theorem 2.2. *Assume that $N = 1$. Let $u \equiv u(t,x)$ be a solution of (2.8). Then*

$$\lim_{t\to\infty} u(t, x + m(t)) = U^*(x),$$

uniformly with respect to x in each semi-infinite interval of the form $[-K, \infty)$ and $K > 0$. Here U^* denotes the unique travelling front with speed c^* of (2.8) with $U^*(0) = \frac{1}{2}$, namely U^* satisfies

$$\begin{cases} (U^*)''(x) + c^*(U^*)'(x) + U^*(x) f(U^*(x)) = 0, \; x \in \mathbb{R}, \\ U^*(\infty) = 0, \;\; U^*(-\infty) = 1, \;\; U^*(0) = \frac{1}{2}. \end{cases} \tag{2.9}$$

The proof of this result can found, for instance, in the article of Uchiyama [16] (see also Bramson [3] and Lau [13] for more results on the asymptotic behaviour of the Fisher-KPP equation). We also refer to Ducrot et al [8], for recent results on the long behaviour for quite general one-dimensional scalar equation with spatial periodicity and Heaviside like initial data .

In the sequel of this work, we will need some notations. For each $\alpha > 0$ and $N \geq 1$, we set $\mathcal{D}_\alpha^N$ the set defined as follows

$$\mathcal{D}_\alpha^N := \left\{ g : \mathbb{R}^N \to [0,\alpha] : \; g \not\equiv 0 \text{ positive, continuous and compactly supported} \right\}.$$

Next for each $\alpha > 0$ and each $g \in \mathcal{D}_\alpha^N$, function $U(t,x;\alpha;g)$ denotes the solution of

$$\begin{cases} \partial_t U - \Delta U = rU\left(1 - \frac{U}{\alpha}\right), \\ U(0,.) = g(.). \end{cases} \tag{2.10}$$

Next, for each $\alpha > 0$, each $g \in \mathcal{D}_\alpha^1$ and each $\eta \in (0,\alpha)$ we set

$$m(t;\alpha;g;\eta) = \sup\left\{x \geq 0 : \;\; U(t,x;\alpha;g) = \eta\right\}, \tag{2.11}$$

which is at least defined for t large enough due to the spreading speed property.

Let us also recall Lemma 8.5 given by Uchiyama [16]

Lemma 2.3. *For each g and f in $\mathcal{D}_1^1$ the quantity $m\left(t;1;g;\frac{1}{2}\right) - m\left(t;1;f;\frac{1}{2}\right)$ is bounded for large time t.*

Using Theorem 2.2 one derives the following layer property:

Lemma 2.4. *Let $g \in \mathcal{D}_1^1$ be given. For each $\varepsilon \in \left(0,\frac{1}{2}\right)$, there exists $h_\varepsilon > 0$ and $T_\varepsilon > 0$ such that for each $t > T_\varepsilon$:*

$$\{x \geq 0 : \;\; U(t,x;1;g) \in [\varepsilon, 1-\varepsilon]\} \subset m\left(t;1;g;\frac{1}{2}\right) + [-h_\varepsilon, h_\varepsilon].$$

3. Spreading speed

The aim of this section is to prove Theorem 1.3. The proof of this result relies on deriving suitable asymptotic estimates.

Lemma 3.1. *Let Assumptions 1.1 and 1.2 be satisfied. Let (u,v) be the solution of (1.1). Then we have for all $(t,x) \in [0,\infty) \times \mathbb{R}^N$*

$$\delta \leq u(t,x) \leq 1, \;\; 0 < v(t,x) \leq 1.$$

Proof. The proof of this result relies on the comparison principle. The proof is straightforward due to Assumptions 1.1 and 1.2. ☐

Lemma 3.2. *Let Assumptions 1.1 and 1.2 be satisfied. Then the following holds true*

$$\liminf_{t\to\infty} v(t,x) \geq \delta,$$

locally uniformly with respect to $x \in \mathbb{R}^N$.

Proof. Since $u \geq \delta$ then v satisfies

$$\partial_t v - \Delta v = rv\left(1 - \frac{v}{u}\right) \geq rv\left(1 - \frac{v}{\delta}\right).$$

Thus

$$\partial_t v - \Delta v - rv\left(1 - \frac{v}{\delta}\right) \geq 0.$$

Recalling Definition (2.10), since $\delta \in (0,1]$, one obtains, due to the comparison principle, that

$$v(t,x) \geq U(t,x;\delta;\delta v_0), \quad \forall t \geq 0, \ \forall x \in \mathbb{R}^N,$$

and the result follows (see Aronson et Weinberger [2]). □

Lemma 3.3. *Let Assumptions 1.1 and 1.2 be satisfied. Let* (u,v) *be a solution of (1.1). Then for each* $e \in S^{N-1}$ *(the unit sphere of* $\mathbb{R}^N$*) and each* $c > c^*$ *one has*

$$\lim_{t\to\infty} v(t, x + cte) = 0, \ \textit{locally uniformly with respect to } x \in \mathbb{R}^N,$$

and for each $c \in (0, c^*)$*, one has*

$$\liminf_{t\to\infty} \inf_{\|x\|\leq ct} v(t,x) \geq \delta.$$

Proof. From Lemma 3.1 and the comparison principle, one obtains that

$$U(t,x;\delta;\delta v_0) \leq v(t,x) \leq U(t,x;1;v_0), \quad \forall t \geq 0, \ x \in \mathbb{R}^N.$$

One the other hand, let us notice that

$$U(t,x;\delta;\delta v_0) \equiv \delta U(t,x;1;v_0),$$

so that we obtain that

$$\delta U(t,x;1;v_0) \leq v(t,x) \leq U(t,x;1;v_0), \quad \forall t \geq 0, \ x \in \mathbb{R}^N. \tag{3.12}$$

Using Theorem 2.1, the result of Lemma 3.3 follows. □

We are now able to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. As done by Berestycki et al in [6] (see Theorem 1.3 and Lemma 2.2), to complete the proof of Theorem 1.3 (1.6), it is sufficient to show that for each $c \in (0, c^*)$, there exists $\kappa \in (0,1)$ such that for all $e \in S^{N-1}$

$$\limsup_{t\to\infty} u(t, ect) \le \kappa.$$

To do so, let $c \in (0, c^*)$ be given and let us argue by contradiction by assuming that there exists a sequence $\{t_n \to \infty\}$ and a sequence $\{e_n\} \subset S^{N-1}$ such that

$$\lim_{n\to\infty} u(t_n, e_n c t_n) = 1.$$

Set $\{x_n := e_n c t_n\}_{n\ge 0}$ and consider the following sequence of map for $n \ge 0$

$$u_n(t,x) = u\left(t + t_n, x + x_n\right), \quad v_n(t,x) = v\left(t + t_n, x + x_n\right).$$

Up to a subsequence, one may assume that u_n and v_n converges locally uniformly to U and V, entire solutions of the following system of equations

$$\begin{cases} \partial_t U - d\Delta U = U(1-U) - U\pi(U)V, \\ \partial_t V - \Delta V = rV\left(1 - \frac{V}{U}\right), \\ 0 \le V \le 1, \;\; \delta \le U \le 1. \end{cases}$$

Note that due to Lemma 3.3, one has

$$\lim_{n\to\infty} v_n(0,0) \ge \delta,$$

while

$$\lim_{n\to\infty} u_n(0,0) = 1.$$

Thus one gets that

$$U(0,0) = 1 \text{ and } V(0,0) \ge \delta.$$

The strong comparison principle implies that $U(t,x) \equiv 1$ and from the $U-$equation, $V \equiv 0$, a contradiction. Thus we have obtained that for each $c \in (0, c^*)$, there exists $\kappa = \kappa(c) \in (0,1)$ such that

$$\limsup_{t\to\infty} u(t, ect) \le \kappa, \quad \forall e \in S^{N-1}.$$

This completes the proof of Theorem 1.3 (1.6).

Finally it remains to prove (1.5) for the $u-$component. To do so, let $c > c^*$ be given. Let $e \in S^{N-1}$ be given. Let $\{t_k\}_{k\geq 0}$ be a given sequence tending to infinity as $k \to \infty$. Consider the sequence of map defined for $k \geq 0$ by

$$u_k(t,x) = u\left(t + t_k, x + cet_k\right), \quad v_k(t,x) = v\left(t + t_k, x + cet_k\right).$$

Due to property (1.5) for the $v-$component (see Lemma 3.3), one obtains that

$$\lim_{k\to\infty} v_k(0,x) = 0 \text{ locally uniformly.}$$

Due to parabolic estimates, up to a subsequence, one may assume that $\{u_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 0}$ converges locally uniformly towards some functions U and V, an entire solution of the following system of equations

$$\begin{cases} \partial_t U - d\Delta U = U(1-U) - U\pi(U)V, \\ \partial_t V - \Delta V = rV\left(1 - \frac{V}{U}\right), \\ 0 \leq V \leq 1, \ \ V(0,x) = 0, \\ \delta \leq U \leq 1. \end{cases}$$

From the strong comparison principle, one gets that $V(t,x) \equiv 0$ and therefore U becomes an entire solution of the Fisher-KPP equation

$$\begin{aligned} &\partial_t U - d\Delta U = U(1-U), \\ &\delta \leq U \leq 1. \end{aligned}$$

It remains to prove that such an entire solution U satisfies $U(t,x) \equiv 1$. The proof of this claim follows from the next lemma:

Lemma 3.4. *Let $d > 0$ be given. Let $\eta \in (0,1)$ be given. Let U be a given entire super-solution of the equation*

$$\partial_t U - d\Delta U - U(1-U) \geq 0, \ \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

with $\eta \leq U(t,x) \leq 1$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Then $U(.,.) \equiv 1$.

This completes the proof of Theorem 1.3. □

It remains to prove Lemma 3.4.

Proof of Lemma 3.4. Let us now show that $U(t,x) \equiv 1$. To do so, consider the function $\bar{u} \equiv \bar{u}(t)$ defined by

$$\begin{cases} \frac{d\bar{u}(t)}{dt} = \bar{u}(t)\left(1 - \bar{u}(t)\right), & t > 0, \\ \bar{u}(0) = \eta. \end{cases}$$

Then from the comparison principle, for each $s \in \mathbb{R}$ and $t \in \mathbb{R}$ one has

$$\bar{u}(t-s) \leq U(t,x), \quad \forall x \in \mathbb{R}, \quad t \geq s.$$

Since $\bar{u}(t) \to 1$ when $t \to \infty$, one obtains that for each $t \in \mathbb{R}$:

$$\lim_{s \to -\infty} \bar{u}(t-s) \leq U(t,x) \leq 1,$$

uniformly with respect to $x \in \mathbb{R}$. This implies that $U(t,x) \equiv 1$ and completes the proof of the result. □

4. The one-dimensional system $N = 1$

The aim of this section is to derive more precise information on the large time behaviour of the one dimensional system (1.1), namely with $N = 1$.

The next lemma shows that the location of the invasion front of predator is strongly related to the one of the Fisher-KPP equation.

Lemma 4.1. *Let Assumptions 1.1 and 1.2 be satisfied. Let (u,v) be the solution of (1.1). Due to Theorem 1.3 one can consider at least for large time t the quantity $m(t)$ defined by*

$$m(t) = \sup\left\{x \geq 0 : \quad v(t,x) = \frac{\delta}{2}\right\}.$$

Then recalling Definition (2.11), the quantity $m(t) - m\left(t; 1; v_0; \frac{1}{2}\right)$ is bounded for large time.

Remark 4.2. *Note that the above result proves Remark 1.6 by using the asymptotic of $m\left(t; 1; v_0; \frac{1}{2}\right)$ provided by Uchiyama in [16].*

Proof. The proof of this result relies on Lemma 2.4. Indeed due to (3.12), one gets that for t large enough,

$$m(t) \in \left\{x \geq 0 : \quad U(t,x;1;v_0) \in \left[\frac{\delta}{2}, \frac{1}{2}\right]\right\}.$$

Therefore due to Lemma 2.4, there exists $h > 0$ and $T > 0$ such that for all $t \geq T$:

$$m(t) - m\left(t; 1; v_0; \frac{1}{2}\right) \in [-h, h],$$

and the result follows. □

In order to prove Theorem 1.5, we will need to investigate the large time behaviour of the following families of parabolic problems for $s \in \mathbb{R}$ and $h \in \mathbb{R}$

$$\begin{cases} \partial_t U - d\partial_{xx} U = U(1-U) - \delta U \pi(U) U^* \left(x - c^* t + h\right), \quad t > s, \\ U(s, s, x; h) \equiv 1 \end{cases} \tag{4.13}$$

whose solution will be denoted by $U \equiv U(t, s, x; h)$ for $t \geq s$ and wherein U^* is the Fisher front defined in (2.9). We will also need some information on the large time behaviour of the following scalar equation

$$\begin{cases} \partial_t V - d\partial_{xx} V = V(1-V) - V \pi(V) U^* \left(x - c^* t + h\right), \quad t > s, \\ V(s, s, x; h) \equiv \delta, \end{cases} \tag{4.14}$$

whose solution will be denoted by $V \equiv V(t, s, x; h)$ for $t \geq s$.

Concerning the above problems, namely (4.13) and (4.14), one will derive the following result:

Theorem 4.3 (Behaviour of (4.13))**.** *Let Assumptions 1.1 and 1.4 (i) be satisfied. There exists a map $\overline{U} : \mathbb{R} \to \mathbb{R}$ such that*

(i) $\overline{U}$ is increasing and

$$\lim_{x \to \infty} \overline{U}(x) = 1, \quad \lim_{x \to -\infty} \overline{U}(x) = \gamma,$$

(ii) $\overline{U}$ satisfies for all $x \in \mathbb{R}$

$$d\overline{U}''(x) + c^* \overline{U}'(x) + \overline{U}(x)\left(1 - \overline{U}(x)\right) - \delta \overline{U} \pi\left(\overline{U}(x)\right) U^*(x) = 0,$$

(iii) the map $\overline{U}$ satisfies for each $t \in \mathbb{R}$ and each $h \in \mathbb{R}$:

$$\lim_{s \to -\infty} \sup_{x \in \mathbb{R}} |U(t, s, x; h) - \overline{U}(x + h - c^* t)| = 0.$$

Then we will show a similar result as the one described in Theorem 4.3 for (4.14), that reads

Theorem 4.4 (Behaviour of (4.14)). *Let Assumptions 1.1 and 1.4 (ii) be satisfied. There exists a map* $\underline{U}:\mathbb{R}\to\mathbb{R}$ *such that*

(i) $\underline{U}$ *is increasing and*

$$\lim_{x\to\infty}\underline{U}(x)=1,\quad \lim_{x\to-\infty}\underline{U}(x)=\delta,$$

(ii) $\underline{U}$ *satisfies for each* $x\in\mathbb{R}$*:*

$$d\underline{U}''(x)+c^*\underline{U}'(x)+\underline{U}(x)\left(1-\underline{U}(x)\right)-\underline{U}\pi\left(\underline{U}(x)\right)U^*(x)=0,$$

(iii) the map $\underline{U}$ *satisfies for each* $t\in\mathbb{R}$ *and each* $h\in\mathbb{R}$*:*

$$\lim_{s\to-\infty}\sup_{x\in\mathbb{R}}\left|V(t,s,x;h)-\underline{U}(x+h-c^*t)\right|=0.$$

The proofs of these results are postponed. We will first complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Due to Lemma 4.1, define $\widehat{H}>0$ such that

$$m(t)-m\left(t;1;v_0;\frac{1}{2}\right)\in\left[-\widehat{H},\widehat{H}\right],\text{ for all } t \text{ large enough.}$$

Let $\{t_k\}_{k\geq 0}$ be a given sequence tending to ∞ as $k\to\infty$. Recalling the definition of $m(t)$ in Lemma 4.1, we consider the sequences of maps $\{u_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 0}$ defined by

$$\left(u_k,v_k\right)(t,x)=(u,v)\left(t+t_k,x+m\left(t_k\right)\right).$$

Due to parabolic estimates, one may assume possibly along a subsequence, still denoted by $\{t_k\}_{k\geq 0}$ that $\{(u_k,v_k)\}_{k\geq 0}$ converges towards some function $(\overline{u},\overline{v})$ locally uniformly with respect to $(t,x)\in\mathbb{R}^2$. Moreover $(\overline{u},\overline{v})$ becomes an entire solution of the following problem

$$\begin{aligned}&\partial_t\overline{u}-d\partial_{xx}\overline{u}=\overline{u}\left(1-\overline{u}\right)-\overline{u}\pi\left(\overline{u}\right)\overline{v},\\&\partial_t\overline{v}-\partial_{xx}\overline{v}=r\overline{v}\left(1-\frac{\overline{v}}{\overline{u}}\right),\\&\delta\leq\overline{u}(t,x)\leq 1,\quad 0\leq\overline{v}(t,x)\leq 1.\end{aligned}\tag{4.15}$$

Due to the definition of $m(t)$ in Lemma 4.1 one has that

$$\overline{v}(0,0) = \frac{\delta}{2}.$$

If we come back to (3.12), one obtains that for each $k \geq 0$

$$\delta U\left(t+t_k, x+m\left(t_k\right)\right) \leq v_k(t,x) \leq U\left(t+t_k, x+m\left(t_k\right)\right), \quad \forall t \geq -t_k,\ x \in \mathbb{R},$$

wherein we have set for short $U(t,x) = U(t,x;1;v_0)$. Next due to Lemma 4.1, the sequence $\{H_k\}$ defined by

$$H_k := m(t_k) - m\left(t_k; 1; v_0; \frac{1}{2}\right) \in \left[-\widehat{H}, \widehat{H}\right],$$

is bounded. Up to a subsequence one may assume that it converges to some value $H \in \left[-\widehat{H}, \widehat{H}\right]$. Therefore, using Theorem 2.2, we obtain that for each $(t,x) \in \mathbb{R}^2$

$$\begin{aligned} \lim_{k\to\infty} U\left(t+t_k, x+m\left(t_k\right)\right) &= \lim_{k\to\infty} U\left(t+t_k, x+m\left(t_k; 1; v_0; \frac{1}{2}\right) + H_k\right) \\ &= U^*\left(x - c^*t + H\right). \end{aligned}$$

As a conclusion we derive that for each $(t,x) \in \mathbb{R}^2$:

$$\delta U^*\left(x-c^*t+H\right) \leq \overline{v}(t,x) \leq U^*\left(x-c^*t+H\right).$$

Plugging this last estimate into the $u-$equation in (4.15), one obtains that for each $(t,x) \in \mathbb{R}^2$:

$$\begin{aligned} &\partial_t \overline{u} - d\partial_{xx}\overline{u} \geq \overline{u}\left(1-\overline{u}\right) - \overline{u}\pi\left(\overline{u}\right)U^*\left(x-c^*t+H\right), \\ &\overline{u}(t,x) \geq \delta, \end{aligned} \tag{4.16}$$

and for each $(t,x) \in \mathbb{R}^2$:

$$\begin{aligned} &\partial_t \overline{u} - d\partial_{xx}\overline{u} \leq \overline{u}\left(1-\overline{u}\right) - \delta\overline{u}\pi\left(\overline{u}\right)U^*\left(x-c^*t+H\right), \\ &\overline{u}(t,x) \leq 1, \end{aligned} \tag{4.17}$$

As a consequence of these differential inequalities and using the comparison principle, we obtain that for each $(t,x) \in \mathbb{R}^2$ and each $s \leq t$ that

$$V(t,s,x;H) \leq \overline{u}(t,x) \leq U(t,s,x;H).$$

Letting $s \to -\infty$ allows us to obtain that

$$\underline{U}\left(x - c^*t + H\right) \leq \overline{u}(t,x) \leq \overline{U}\left(x - c^*t + H\right), \ \ \forall (t,x) \in \mathbb{R}^2.$$

This completes the proof of Theorem 1.5. □

It remains to prove Theorem 4.3 and 4.4. The proof of these results are similar to each other. We will only prove Theorem 4.3 and we focus on (4.13). Notice that the study of such a problem may be reduced to the study of the latter equation with $h = 0$ and $s = 0$. Indeed from the uniqueness of the solution, one may observe that for each $s \in \mathbb{R}$, each $h \in \mathbb{R}$, each $t \geq s$ and $x \in \mathbb{R}$, we have

$$U(t,s,x;h) \equiv U\left(t - s, 0, x - c^*s + h; 0\right).$$

It is therefore sufficient to study the problem

$$\begin{cases} \partial_t U - d\partial_{xx} U = U(1-U) - u\pi(U)U^*\left(x - c^*t\right), \ \ t > 0, \ x \in \mathbb{R}, \\ U(0,x) \equiv 1, \end{cases} \tag{4.18}$$

wherein U^* is defined in (2.9). We shall prove the following result:

Theorem 4.5. *Let Assumptions 1.1 and 1.4 (i) be satisfied. Let $U \equiv U(t,x)$ be the solution of (4.18). There exists a map $\overline{U} : \mathbb{R} \to \mathbb{R}$ such that*

(i) $\overline{U}$ is increasing and

$$\lim_{x\to\infty} \overline{U}(x) = 1, \quad \lim_{x\to-\infty} \overline{U}(x) = \gamma,$$

(ii) $\overline{U}$ satisfies for all $x \in \mathbb{R}$

$$d\overline{U}''(x) + c^*\overline{U}'(x) + \overline{U}(x)\left(1 - \overline{U}(x)\right) - \delta\overline{U}\pi\left(\overline{U}(x)\right)U^*(x) = 0,$$

(iii) the map $\overline{U}$ satisfies for each $t \in \mathbb{R}$ and each $h \in \mathbb{R}$:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |U(t,x) - \overline{U}(x - c^*t)| = 0.$$

It is clear that due to the uniform converge explained in (*iii*), Theorem 4.3 holds true. Thus it is sufficient to prove Theorem 4.5.

Proof of Theorem 4.5. Denote by $U \equiv U(t,x)$ the solution of (4.18). Then from the comparison principle we have

$$\gamma \leq U(t,x) \leq 1, \;\; \forall t \geq 0, \; x \in \mathbb{R}.$$

Next consider the map $w \equiv w(t,x) :\equiv U(t, x+c^*t)$ that satisfies the following problem

$$\begin{aligned} &\partial_t w = d\partial_{xx} w + c^* \partial_x w + w(1-w) - \delta w \pi(w) U^*(x), \\ &w(0,x) \equiv 1. \end{aligned} \tag{4.19}$$

Then we infer from the comparison principle and $U^*(x) > 0$ and $(U^*)'(x) < 0$ for all $x \in \mathbb{R}$ that w is decreasing with respect to t and increasing with respect to $x \in \mathbb{R}$. Since $w(t,x) \in [\gamma, 1]$ we obtain that there exists $\overline{U} : \mathbb{R} \to (\gamma, 1)$ an increasing map such that

(i) $\overline{U}$ satisfies for all $x \in \mathbb{R}$:

$$d\overline{U}''(x) + c\overline{U}'(x) + \overline{U}(x)\left(1 - \overline{U}(x)\right) - \delta \overline{U}(x) \pi\left(\overline{U}(x)\right) U^*(x) = 0,$$

(ii) $\lim\limits_{t \to \infty} w(t,x) = \overline{U}(x)$ locally uniformly with respect to $x \in \mathbb{R}$.

Furthermore, since $\overline{U}$ is increasing, one obtains that

$$\lim_{x \to -\infty} \overline{U}(x) = \gamma \text{ and } \lim_{x \to \infty} \overline{U}(x) = 1.$$

We now aim to prove that the above convergence in (ii) is uniform with respect to $x \in \mathbb{R}$. To prove that, we will argue by contradiction by assuming that the convergence of w to $\overline{U}$ is not uniform with respect to $x \in \mathbb{R}$. Thus there exists $\varepsilon > 0$, a sequence $\{t_n\}_{n \geq 0}$ and $\{x_n\}_{n \geq 0}$ such that

$$\begin{aligned} &t_n \to \infty, \;\; |x_n| \to \infty, \text{ when } n \to \infty \\ &|w(t_n, x_n) - \overline{U}(x_n)| \geq \varepsilon, \;\; \forall n \geq 0. \end{aligned}$$

Next consider the sequence of map $\{w_n\}_{n \geq 0}$ defined by

$$w_n(t,x) := w(t + t_n, x + x_n).$$

We now split the argument into two parts:
Let us first assume that up to a subsequence that $x_n \to +\infty$ when $n \to \infty$.

Due to parabolic estimates, possibly along a subsequence, one may assume that the sequence $\{w_n\}$ converges to some W^* locally uniformly with respect to $(t,x) \in \mathbb{R}^2$. Moreover since $U^*(x) \to 0$ and $\overline{U}(x) \to 1$ when $x \to \infty$, one obtains that

$$W^*(0,0) = \lim_{n\to\infty} w_n(0,0) \leq 1-\varepsilon,$$
$$\partial_t W^* = d\partial_{xx}W^* - c\partial_x W^* + W^*\left(1-W^*\right), \quad (t,x) \in \mathbb{R}^2,$$
$$\gamma \leq W^*(t,x) \leq 1, \quad \forall (t,x) \in \mathbb{R}^2.$$

Using Lemma 3.4, one obtains that $W \equiv 1$, a contradiction with $W^*(0,0) < 1$.

Assume now that, up to a subsequence that $x_n \to -\infty$ when $n \to \infty$. Due to parabolic estimates, possibly along a subsequence, one may assume that the sequence $\{w_n\}$ converges to some W^* locally uniformly with respect to $(t,x) \in \mathbb{R}^2$. Moreover since $\overline{U}(x) \to \gamma$ and $U^*(x) \to 1$ when $x \to -\infty$, one obtains that

$$W^*(0,0) = \lim_{n\to\infty} w_n(0,0) > \gamma,$$
$$\partial_t W^* = d\partial_{xx}W^* - c\partial_x W^* + W^*\left(1-W^*-\delta\pi\left(W^*\right)\right), \quad (t,x) \in \mathbb{R}^2,$$
$$\gamma \leq W^*(t,x) \leq 1, \quad \forall (t,x) \in \mathbb{R}^2.$$

We claim that $W^* \equiv \gamma$. Let $w(t,s)$ with $t \geq s$ be the solution of

$$\partial_t w(t,s) = w(t,s)F\left(w(t,s)\right), \quad w(s,s) = 1,$$

wherein we have set

$$F(w) = 1 - w - \delta\pi(w).$$

Then from the comparison principle, one has for each $t \geq s$.

$$W^*(t,x) \leq w(t,s), \quad \forall x \in \mathbb{R}.$$

Note that $w(t,s) = w(t-s,0)$ for all $t \geq s$. One the other hand, since $F(w) < 0$ for each $w \in (\gamma, 1]$ (see (i) in Assumption 1.4) then $t \mapsto w(t,0)$ is decreasing and since $F(\gamma) = 0$, one obtains

$$\lim_{t\to\infty} w(t,0) = \gamma.$$

As a consequence for each $t \in \mathbb{R}$ and each $x \in \mathbb{R}$, one gets

$$W^*(t,x) \leq \lim_{s\to-\infty} w(t-s,0) = \gamma.$$

Recalling that $W^* \geq \gamma$ leads us to $W^* \equiv \gamma$ and the claim is proved. Note that the former property leads to a contradiction together with $W^*(0,0) > \gamma$. and this completes the proof of Theorem 4.5. □

5. Proof of Theorem 1.9

The aim of this section is to complete the proof of Theorem 1.9. This step relies on the following proposition:

Proposition 5.1. *Let Assumptions 1.1, 1.2, 1.4 and 1.7 be satisfied. Let $(u,v) = (u,v)(t,x)$ be an entire solution of (1.1) such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}$*

$$\delta \leq u(t,x) \leq \gamma \text{ and } \delta \leq v(t,x) \leq 1,$$

wherein δ is defined in Assumption 1.1 while γ is defined in (1.7). Then (u,v) satisfies:

$$(u,v)(t,x) \equiv (a^*, a^*).$$

Before proving Proposition 5.1, let us complete the proof of Theorem 1.9.

Proof of Theorem 1.9. Let $(\overline{u}, \overline{v})$ be an entire solution of (1.1) constructed in Theorem 1.5. Set $\xi_t = c^* t$ for all $t \in \mathbb{R}$ and let $H \in \mathbb{R}$ be such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}$:

$$\begin{cases} \mathbf{V}^-(x - c^*t + H) \leq \overline{v}(t,x) \leq \mathbf{V}^+(x - c^*t + H), \\ \mathbf{U}^-(x - c^*t + H) \leq \overline{u}(t,x) \leq \mathbf{U}^+(x - c^*t + H), \end{cases} \tag{5.20}$$

and wherein functions $\mathbf{U}^{\pm}$ and $\mathbf{V}^{\pm}$ are provided by Theorem 1.5. Recalling Definition 1.8, let us prove that $(\overline{u}, \overline{v})$ is an almost planar generalized transition wave of (1.1) connecting the stationary states $(1,0)$ and (a^*, a^*).

Let $\varepsilon > 0$ be given. Since $(\mathbf{U}^{\pm}, \mathbf{V}^{\pm})(x) \to (1,0)$ when $x \to \infty$, there exists $M > 0$ such that for all $x \geq M - H$,

$$0 \leq 1 - \mathbf{U}^{\pm}(x) \leq \varepsilon \text{ and } 0 \leq \mathbf{V}^{\pm}(x) \leq \varepsilon.$$

Hence due to (5.20) for each $t \in \mathbb{R}$ and each $x \geq c^*t + M$ one has

$$\left\| \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} (t,x) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq \varepsilon.$$

Let us now prove that for each $\varepsilon > 0$ there exists $M > 0$ such that for each $(t,x) \in \mathbb{R} \times \mathbb{R}$:

$$x - \xi_t \leq -M \quad \Rightarrow \quad \left\| \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} (t,x) - \begin{pmatrix} a^* \\ a^* \end{pmatrix} \right\| \leq \varepsilon.$$

In order to prove this result let us argue by contradiction by assuming that there exists $\varepsilon > 0$, a sequence $\{(t_n, x_n)\}_{n\geq 0}$ such that for each $n \geq 0$

$$x_n - c^* t_n \leq -n \quad \text{and} \quad \left\| \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} (t_n,x_n) - \begin{pmatrix} a^* \\ a^* \end{pmatrix} \right\| > \varepsilon. \tag{5.21}$$

Next consider the sequence entire solution $(u_n, v_n)(t,x) = (\overline{u}, \overline{v})(t + t_n, x + x_n)$. Due to parabolic estimates, possibly along a subsequence, one may assume that $(u_n, v_n) \to (u_\infty, v_\infty)$ locally uniformly for $(t,x) \in \mathbb{R}^2$ where (u_∞, v_∞) is an entire solution of (1.1). Furthermore the second inequality in (5.21) ensures that

$$\left\| \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} (0,0) - \begin{pmatrix} a^* \\ a^* \end{pmatrix} \right\| > \varepsilon. \tag{5.22}$$

On the other hand (5.20) yields for each $n \geq 0$, for each $(t,x) \in \mathbb{R}^2$:

$$\begin{cases} \mathbf{V}^-(x_n - c^* t_n + x - c^* t + H) \leq v_n(t,x) \leq \mathbf{V}^+(x_n - c^* t_n + x - c^* t + H), \\ \mathbf{U}^-(x_n - c^* t_n + x - c^* t + H) \leq u_n(t,x) \leq \mathbf{U}^+(x_n - c^* t_n + x - c^* t + H). \end{cases}$$

Recalling that $x_n - c^* t_n \to -\infty$ as $n \to \infty$, we infer from the asymptotic behaviour close to $x = -\infty$ of $\mathbf{U}^\pm$ and $V^\pm$ that for all $(t,x) \in \mathbb{R}^2$:

$$\delta \leq u_\infty(t,x) \leq \gamma \text{ and } \delta \leq v_\infty(t,x) \leq 1.$$

Finally Proposition 5.1 applies and provides a contradiction together with (5.22). This completes the proof of Theorem 1.9. □

It remains to prove Proposition 5.1. To do so, recalling Assumption 1.7, one introduces the sequence $\{\gamma_n\}_{n\geq 0}$ defined by

$$\begin{cases} \gamma_0 = 1, \\ h(\gamma_{n+1}) = \gamma_n, \quad n \geq 0. \end{cases} \tag{5.23}$$

Due to the definition of δ in Assumption 1.1 and of γ in (1.7), one has $\gamma_1 = \delta$ and $\gamma_2 = \gamma$. Then the following lemma holds true:

Lemma 5.2. *Using the above definition, the two sequences* $\{\gamma_{2n}\}_{n\geq 0}$ *and* $\{\gamma_{2n+1}\}_{n\geq 0}$ *are adjacent. They converge to* $a^* \in (0,1)$*, the unique solution of* $h(a^*) = a^*$ *and satisfy for each* $n \geq 0$*:*

$$\gamma_1 < \gamma_3 < \ldots < \gamma_{2n+1} < \ldots < a^* < .. < \gamma_{2n} < .. < \gamma_2 < \gamma_0.$$

The proof of this lemma is straightforward.

Using the definition of $\{\gamma_n\}_{n\geq 0}$ we are now able to complete the proof of Proposition 5.1.

Proof of Proposition 5.1. The proof of this result is based on deriving the following "sandwich" estimates for all $n \geq 0$ and $(t,x) \in \mathbb{R}^2$:

$$\gamma_{2n+1} \leq u(t,x) \leq \gamma_{2n+2} \text{ and } \gamma_{2n+1} \leq v(t,x) \leq \gamma_{2n}. \tag{5.24}$$

Note that because of Lemma 5.2, (5.24) completes the proof the Proposition 5.1. As a consequence we only need to prove (5.24).
Let us first notice that this inequality holds true for $n = 0$ because of the assumptions of the proposition. Let us now argue by induction on n. Let us assume that (5.24) holds true for some $n \geq 0$ and let us prove that (5.24) holds true for $n+1$.

Firstly since $u \leq \gamma_{2n+2}$ then $v \equiv v(t,x)$ satisfies:

$$\partial_t v - \partial_{xx} v - rv\left(1 - \frac{v}{\gamma_{2n+2}}\right) \leq 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Since v is bounded one concludes from the comparison principle that $v(t,x) \leq \gamma_{2n+2}$ for all $(t,x) \in \mathbb{R}^2$. Next $u \equiv u(t,x)$ satisfies

$$\partial_t u - d\partial_{xx} u - u(1-u) + \gamma_{2n+2} u\pi(u) \geq 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Thus using (5.24) for n and the comparison principle, one obtains that for each $t \in \mathbb{R}$, $s \leq t$ and $x \in \mathbb{R}$:

$$\underline{u}(t-s) \leq u(t,x), \tag{5.25}$$

where $\underline{u}$ is the solution of the ordinary differential equation:

$$\begin{cases} \partial_t \underline{u} = \underline{u}\left(1 - \underline{u}\right) - \gamma_{2n+2} \underline{u} \pi\left(\underline{u}\right) \text{ for } t \geq 0, \\ \underline{u}(0) = \gamma_{2n+1}. \end{cases}$$

Now let us notice that the map $t \mapsto \underline{u}(t)$ is increasing. Indeed one has for

$$\begin{aligned}(1-\gamma_{2n+1})-\gamma_{2n+2}\pi\left(\gamma_{2n+1}\right) &= \pi\left(\gamma_{2n+1}\right)\left[h\left(\gamma_{2n+1}\right)-\gamma_{2n+2}\right] \\ &= \pi\left(\gamma_{2n+1}\right)\left[\gamma_{2n}-\gamma_{2n+2}\right] > 0,\end{aligned}$$

where the last inequality arises because of the monotonic property of $\{\gamma_n\}_{n\geq 0}$ stated in Lemma 5.2. Hence $\underline{u}$ converges as $t \to \infty$ to γ_{2n+3}. Therefore using (5.25) and letting $s \to -\infty$ yields to $\gamma_{2n+3} \leq u(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. Using the same arguments as before and $\gamma_{2n+3} \leq u(t,x)$, one concludes that $v(t,x) \geq \gamma_{2n+3}$ and this latter inequality is used to complete $u(t,x) \leq \gamma_{2n+4}$. Thus (5.24) holds true for $n+1$ and this completes the proof of Proposition 5.1. □

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