On the large time behaviour of the multi-dimensional Fisher-KPP equation with compactly supported initial data

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Abstract

This paper is concerned with the study of the asymptotic behaviour of a multi-dimensional Fisher-KPP equation posed in an asymptotically homogeneous medium and supplemented together with a compactly supported initial datum. We derive precise estimates for the location of the front before proving the convergence of the solutions towards travelling front. In particular we show that the location of the front drastically depends on the rate at which the medium becomes homogeneous at infinity. Fast rate of convergence only changes the location by some constant while lower rate of convergence induces further logarithmic delay.

Key words: Fisher-KPP equation, travelling waves, long time behaviour.

1 Introduction

In this work we consider the following initial value problem of Fisher-KPP type

$$\begin{cases} u_t - \Delta u = G\left(x, u\right), \ t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \ \ \forall x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where the initial datum u_0 is assumed to satisfy

$$0 \leq u_0(x) \leq 1 \text{ is non-trivial, continuous and compactly supported.} \tag{2}$$

The nonlinearity $G = G(x,u)$ is assumed to be asymptotically homogeneous in space in the sense that, the following convergence holds true, locally uniformly,

$$G(x,u) \to F(u) \text{ as } \|x\| \to \infty,$$

and where the limit nonlinearity $F(u)$ is of Fisher-KPP type on the interval $[0,1]$.

The goal of this paper is to understand the relationship between the long time behaviour of the solution of (1) and the one of the homogeneous Fisher-KPP equation, that reads

$$u_t - \Delta u = F(u),\ t > 0,\ x \in \mathbb{R}^N. \tag{3}$$

Here function $F : [0,1] \to \mathbb{R}$ is of the class C^1 and satisfies $F(0) = F(1) = 0$, $F(u) > 0$ for all $u \in (0,1)$, together with the so-called KPP assumption

$$F(u) \leq F'(0)u,\ \ \forall u \in [0,1].$$

This equation has a long history and was introduced in particular in the pioneer works of Fisher [15] and Kolmogorov, Petrovskii and Piskunov [23] to model some problems in population dynamics. One of the main property of (3) is that there exists a minimal speed $c^* = 2\sqrt{F'(0)}$ such for all $c \geq c^*$, (3) admits travelling wave solutions with speed c, that is entire solutions of the form

$$u(t,x) = U\left(x \cdot e - ct\right),$$

where $e \in \mathbb{S}^{N-1}$ is a given direction while the profile U is non-increasing and satisfies $U(-\infty) = 1$ and $U(\infty) = 0$.
The role of the critical travelling front, that is the travelling wave solution with the minimal wave speed c^*, is of particular importance when looking at the dynamical properties of the Cauchy problem (3) supplemented together with an initial datum $u(0,x) = u_0(x)$ satisfying (2). In that context, Aronson and Weinberger proved in the late 70's that the solution $u = u(t,x)$ enjoys the so-called asymptotic speed of spread property:

$$\begin{aligned} &\lim_{t\to\infty} \sup_{\|x\|\geq ct} u(t,x) = 0,\ \ \forall c > c^*, \\ &\lim_{t\to\infty} \sup_{\|x\|\leq ct} |1 - u(t,x)| = 0,\ \ \forall c \in [0,c^*)\,. \end{aligned} \tag{4}$$

Here $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^N$. The above spreading speed result shows that the solution exhibits the propagation of a transition zone between the two equilibrium points. Let us mention that Kolmogorov et al [23] already gave a proof for the one-dimensional attractivity of the critical wave with respect to Heaviside initial datum. The boundedness of the transition zone as well as the convergence to the critical travelling front in a moving frame $x = m(t)$ attached to the level line $\frac{1}{2}$ has been proved by Uchiyama in [33] for the one-dimensional equation (3) and associated to more general initial data. In the same spirit, one can also refer to the work of Lau [25] where convergence to the critical wave front has been studied (see also [24, 32, 35, 36] and the references therein). Let us also mention [17, 10, 27] where convergence to critical travelling wave and propagating terrace have been investigated using intersection number arguments and for one-dimensional periodic equations. More refined information has been obtained by Bramson in [7, 8] using probabilistic methods. He proved the following asymptotic expansion for $m(t)$, the location of the level line $u = \frac{1}{2}$:

$$m(t) = c^* t - \frac{3}{c^*} \ln t - x_0 - x_1 t^{-1/2} + O\left(t^{-1}\right).$$

Here $x_0 \in \mathbb{R}$ and $x_1 > 0$ are two constants. Recently Hamel, Nolen, Roquejoffre and Ryzhik [18, 19] reconsidered this problem for the one-dimensional Fisher-KPP equation in a homogeneous and periodic medium respectively. Using partial differential equation methods, the authors proved that the front is located at $x = c^* t - \frac{3}{c^*} \ln t + O(1)$. We also refer to Ebert et al [11] for a formal derivation of the location of pulled front.

The multi-dimensional Fisher-KPP equation (3) posed on the whole space and equipped with a compactly supported initial datum has been studied by Gärtner in [16] also using probabilistic arguments. In that context the front has proved to be located at $\|x\| = c^* t - \frac{N+2}{c^*} \ln t + O(1)$. As a corollary of this work (see Corollary 1.7), we shall recover this result. Let us also mention the work of Uchiyama [33] where the author derived the asymptotic behaviour of the solution of some monostable and non-KPP problem and where convergence to pushed front is proved for a class of radially symmetric initial data. We finally refer to Mallordy and Roquejoffre [26] and the references cited therein for the asymptotic behaviour of the solutions of KPP equation in cylinders.

The goal of this work is to understand how is localized the transition zone of the solution of (1) between the two equilibrium points of function $F(u)$ for large time. We shall more precisely show that the location of the transition zone may depend on the rate at which the nonlinearity $G(x, u)$ approaches its homogeneous limit $F(u)$ as $\|x\| \to \infty$. Roughly speaking when this rate is integrable, meaning that the convergence to the homogeneous medium is sufficiently fast, then the level lines of the solution behave similarly to the ones of the homogeneous case. To be more precise, the transition zone of the solution remains at a uniformly bounded distance of the one of the solution of the homogeneous equation (3). When the approach of the homogeneous medium is not sufficiently fast, non-integrable rate of convergence, we obtain that the transition zone of the solution is perturbed with an additional logarithm term. Let us mention that some results in this direction has been investigated by Ducrot and Giletti in [9] for a one-dimensional and periodic equation with compactly supported perturbation as a forcing term and zero initial datum. Let us also emphasize that Nolen, Roquejoffre, Ryzhik and Zlatoš in [29] considered a one-dimensional Fisher-KPP equation in an asymptotically homogeneous medium and proved that under spectral conditions on the linearized elliptic operator at $u = 0$, transition front does not exist while bump-like solutions do exist. However we show in this work that the solution of (1) approaches the critical travelling wave for large time.

Before stating our main results, let us first precisely state the assumptions we shall make use throughout this work:

Assumption 1.1 *We assume that:*

(i) The function $G = G(x,u) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is assumed to be continuous and C^1 with respect to u and there exists $M \geq 1$ such that

$$G(x,0) = 0,\ \forall x \in \mathbb{R}^N \text{ and } G(x,u) \leq 0,\ \ \forall x \in \mathbb{R}^N,\ \forall u \geq M.$$

(ii) There exists a decreasing function $\Gamma : [0,\infty) \to [0,\infty)$ such that

$$\lim_{r\to\infty} \Gamma(r) = 0,$$

and a function $F : \mathbb{R}^+ \to \mathbb{R}$ of the class C^2 such that

a) $F(0) = F(1) = 0$, $F(u) > 0$ for all $u \in (0,1)$ and $F(u) < 0$ for all $u > 1$.

b) Function $u \mapsto \frac{F(u)}{u}$ is decreasing on $(0,1)$.

such that the following estimate holds true:

$$|G(x,u) - F(u)| \leq \Gamma(\|x\|)\, u,\ \ \forall x \in \mathbb{R}^N,\ u \in [0,M].$$

Note that the first assumption ensures that the solution $u = u(t,x)$ of (1) and (2) satisfies $0 < u(t,x) \leq M$ for all $t > 0$ and $x \in \mathbb{R}^N$. The second assumption means that the nonlinear term becomes asymptotically homogeneous with the rate of convergence $\Gamma(\|x\|)$ and the limit nonlinearity $F(u)$ satisfies the strong KPP assumption. Note since $\Gamma(r) \to 0$ as $r \to \infty$ and $F(u) < 0$ for $u > 1$ then for each positive function $\varepsilon \equiv \varepsilon(t)$ tending to ∞ as $t \to \infty$ one has

$$u(t,x) \leq 1 + o(t) \text{ as } t \to \infty \text{ uniformly for } \|x\| \geq \varepsilon(t). \tag{5}$$

As mentioned above we aim at understanding the location of the transition layer for the solution of (1)-(2).
Our first results deal with the case where the rate of convergence Γ decreases to 0 sufficiently fast. This assumption will be formalized as the condition $\Gamma \in L^1(0,\infty)$. In this context, our first result reads as

Theorem 1.2 *In addition to Assumption 1.1, let us furthermore assume that $\Gamma \in L^1(0,\infty)$ Let us consider the shift function $\xi(t) := c^*t - \frac{N+2}{c^*}\ln t$ with $c^* = 2\sqrt{F'(0)}$. Then the solution $u \equiv u(t,x)$ of (1)-(2) satisfies that for each $\varepsilon > 0$ small enough, for each $c \in (0,c^*)$, there exists $h_\varepsilon > 0$ large enough such that for all large enough time:*

$$1 - \varepsilon \leq \inf_{ct \leq \|x\| \leq \xi(t) - h_\varepsilon} u(t,x) \leq 1 + \varepsilon,$$

and

$$\sup_{\|x\| \geq \xi(t) + h_\varepsilon} u(t,x) \leq \varepsilon.$$

The above result state that when $\Gamma \in L^1$ then the transition zone is uniformly bounded and located at the radius $r = \xi(t) + O(1)$ for all large time.
Before stating our next result, related to the asymptotic behaviour of the solution, let us introduce for each direction $e \in \mathbb{S}^{N-1}$, the unit sphere in $\mathbb{R}^N$, the quantity $m(t;e)$ defined for all large enough time by

$$m(t;e) = \sup\left\{ r > 0 : \ u(t, re) = \frac{1}{2} \right\}. \tag{6}$$

Note that due to Theorem 1.2 one has

$$m(t;e) = \xi(t) + O(1) \text{ as } t \to \infty, \text{ uniformly with respect to } e \in \mathbb{S}^{N-1}.$$

In order to state our next convergence result, we introduce for each $K > 0$, each speed $c \geq 0$ and each direction $e \in \mathbb{S}^{N-1}$ the time dependent cylinder $\Omega^t(K, c, e)$ defined by

$$\Omega^t(K,c,e) = \left\{ x \in \mathbb{R}^N : \ x \cdot e \geq ct \text{ and } \|x - (x \cdot e)\, e\| \leq K \right\}.$$

Let us also introduce the function U, the unique critical travelling wave associated to (3) normalized by $U(0) = \frac{1}{2}$, that is the unique solution of the problem

$$\begin{cases} U''(z) + c^* U'(z) + F(U(z)) = 0, \\ U'(z) < 0, \ \forall z \in \mathbb{R}, \\ U(-\infty) = 1, \ U(\infty) = 0 \text{ and } U(0) = \frac{1}{2}. \end{cases} \tag{7}$$

Using the above notations our next result reads as:

Theorem 1.3 *Let Assumption* (1.1) *be satisfied and assume furthermore that* $\Gamma \in L^1(0,\infty)$ *and* $F'(1) < 0$. *Let* $u \equiv u(t,x)$ *be the solution of* (1)-(2). *Then for each* $c \in (0, c^*)$, *for each* $K > 0$ *the following convergence holds true:*

$$\lim_{t\to\infty} \sup_{\substack{x \in \Omega^t(K,c,e) \\ e \in \mathbb{S}^{N-1}}} \left| u(t,x) - U(x \cdot e - m(t;e)) \right| = 0,$$

wherein the function U *is defined in* (7).

As a direct corollary of the above result, and more precisely using the uniformity with respect to the direction, one derives the following convergence property:

Corollary 1.4 *Under the assumptions of Theorem 1.3, the solution* u *of* (1)-(2) *satisfies for each* $c \in (0, c^*)$:

$$\lim_{t\to\infty} \sup_{\|x\| \geq ct} \left| u(t,x) - U\left(\|x\| - m\left(t; \frac{x}{\|x\|}\right)\right) \right| = 0.$$

Here the function U *is defined in* (7).

In order to study the influence of the rate of convergence $\Gamma = \Gamma(\|x\|)$ of the nonlinear term $G(x,u)$ toward the homogeneous medium $F(u)$ and to indicate that our results are somehow sharp, we consider the following example of asymptotically homogeneous Fisher-KPP equation

$$\begin{cases} u_t - \Delta u = G(x,u) \text{ with } G(x,u) = F(u) - \frac{\lambda u}{1+\|x\|}, \\ u(0,x) = u_0(x). \end{cases} \tag{8}$$

Here the initial datum u_0 satisfies (2) while $\lambda \in \mathbb{R}$ is some given constant. Instead of assuming that function F satisfies the strong KPP assumption (see Assumption 1.1 $(ii-b)$) we shall assume that it only satisfies the usual KPP assumption, that reads as

Assumption 1.5 *We assume that function $F : \mathbb{R}^+ \to \mathbb{R}$ is of the class C^2 and satisfies:*

$$\begin{aligned} & F(0) = F(1) = 0, \\ & F(u) > 0,\ \forall u \in (0,1) \text{ and } F(u) < 0,\ \forall u > 1, \\ & F(u) \le F'(0)u,\ \forall u \ge 0 \text{ and } F'(1) < 0. \end{aligned}$$

In this example one has $G(x,u) \to F(u)$ as $\|x\| \to \infty$ locally uniformly and the convergence rate is given by $\Gamma(r) = \frac{\lambda}{1+r} \notin L^1(0,\infty)$ as soon as $\lambda \neq 0$. Our next result is concerned with the location of the transition zone for (8). We shall show that it depends on λ through a logarithmic perturbation of the location of homogeneous Fisher-KPP case (3) (corresponding to the case $\lambda = 0$).
The choice of the perturbation term Γ is in some sense a limit case. It is motivated by the fact that $\int_0^r \Gamma(s)ds \sim \lambda \ln r$ for large radius so that when looking at $r = c^*t$ for large time then $\int_0^{c^*t} \Gamma(s)ds$ is of the same order as the expected logarithmic shift. More general non-integrable perturbation terms can also be considered, such as $\Gamma(r) \sim \frac{1}{(1+r)^\alpha}$ for some $\alpha \in (0,1)$. In such a case, the expected additional shift is of order $kt^{1-\alpha}$ for some constant $k \in \mathbb{R}$. Since such a shift is rather large as time goes to infinity, several arguments used in this work need to be modified and such a problem remains open for the moment.

In order to discuss the large time behaviour of (8), as above, we introduce the directional propagating radius associated to the level line $u = \frac{1}{2}$ defined as in (6) for the solution of (8). Then the precise result we shall show in this work reads as:

Theorem 1.6 *Let Assumption 1.5 be satisfied. Let $\lambda \ge 0$ be given. Let $u \equiv u(t,x)$ be the solution of (8). Then one has:*

$$m(t;e) = c^*t - \left(\frac{2\lambda}{(c^*)^2} + \frac{N+2}{c^*}\right) \ln t + O(1),$$

as time is large enough and uniformly with respect to the direction $e \in \mathbb{S}^{N-1}$. Moreover for each $c \in (0,c^*)$, for each $K > 0$ the following convergence holds *true:*

$$\lim_{t\to\infty} \sup_{\substack{x\in\Omega^t(K,c,e)\\ e\in\mathbb{S}^{N-1}}} |u(t,x) - U(x\cdot e - m(t;e))| = 0,$$

wherein the function U is defined in (7)*.*

Let us also mention that the above result applies in particular in the homogeneous setting, namely $\lambda = 0$. As already mentioned above, in the homogeneous case, the localization of the transition zone (see Theorem 1.2) has been derived by Gärtner using probabilistic methods. Here this localisation result as well as the convergence result hold for the homogeneous Fisher-KPP equation. Keeping this in mind as well as (4) one obtains the following corollary for the homogeneous Fisher-KPP equation:

Corollary 1.7 *Under Assumption 1.5, let $u = u(t,x)$ denotes the solution of* (3) *equipped with a initial datum u_0 satisfying* (2)*. Then, for each $\alpha > 0$, the following convergence holds true:*

$$\lim_{t\to\infty} \sup_{\|x\|\geq\alpha} \left| u(t,x) - U\left(\|x\| - m_{KPP}\left(t; \frac{x}{\|x\|}\right)\right)\right| = 0,$$

wherein U is defined in (7)*. Furthermore the following asymptotic holds true uniformly with respect to the direction:*

$$m_{KPP}(t;e) = c^* t - \frac{N+2}{c^*}\ln t + O(1).$$

The proof of the above results are based on some comparison of the solution together with the linear equation with Dirichlet boundary condition

$$\begin{aligned} &v_t - \Delta v = F'(0)v, \;\; t>0, \; \|x\| \geq X(t), \\ &v(t,x) = 0, \; t>0 \text{ and } \|x\| = X(t), \end{aligned}$$

where $X(t)$ is a suitable moving frame. This approach has been recently developed by Hamel et al in [18] to study Fisher-KPP equation in the one-dimensional homogeneous setting and in [19] in one-dimensional periodic medium. Note that such a comparison was also observed and used by Gärtner [16] in a probabilistic context. Explained in a different way, the formal computations provided by Ebert et al [11] are also based on such an idea. We also mention the recent work of Nolen et al [28] where the authors studied the influence of time varying diffusivity of the location of the front using some of these ideas.

This manuscript is organized as follows: Section 2 is devoted to the derivation of suitable estimates for a class of linear problems in some moving frame located at $X(t) = c^* t - \delta \ln t$. These estimates are then used in Section 3 to control the solution of (1) around the transition zone and complete the proof of Theorem 1.2. Theorem 1.3 is proved in Section 4 while Section 5 is devoted to the proof of Theorem 1.6.

2 Preliminary

This preliminary section is devoted to the derivation of suitable lower and upper estimates for the solution of a linear diffusion equation posed in some particular moving domain. Let $c > 0$, $\widehat{r} > 0$, $\delta \geq 0$ and $\lambda \in \mathbb{R}$ be given. Consider $T > 0$ large enough such that the map

$$\xi(t) := ct - \delta \ln \frac{t+T}{T}, \tag{9}$$

is non-decreasing from $[0, \infty)$ into itself. Note that such a condition can be re-written as $\frac{\delta}{cT} < 1$.

The aim of this section is to derive accurate lower and upper estimate for the following linear problem:

$$\begin{cases} v_t = v_{rr} + \frac{N-1}{r} v_r + \left(\frac{c^2}{4} + \frac{\lambda}{1+r} \right) v, \ t > 0, \ r > \xi(t) + \widehat{r}, \\ v(t, \xi(t) + \widehat{r}) = 0, \ t > 0 \end{cases} \tag{10}$$

supplemented together with some initial datum v_0, a non-negative, non-trivial and compactly supported function.

Our first estimate is concerned with the following lower bound:

Lemma 2.1 *[Lower estimate] Let $r_0 > \widehat{r}$ be given. Let $v_0 : [\widehat{r}, \infty) \to \mathbb{R}^+$ be a non-negative, non-trivial and compactly supported initial datum. Let $v \equiv v(t, r)$ be the solution of* (10) *associated to the initial datum v_0. Then there exist $\varrho > 0$, $t_\varrho > 0$ large enough, $\gamma > 0$ and $\beta > 0$ such that for all $t > t_\varrho$ and $r \in \left[r_0 + \xi(t), \xi(t) + \varrho\sqrt{t}\right]$:*

$$v(t, r) \geq \gamma t^{\frac{\delta c}{2} + \frac{\lambda}{c} - \frac{N+2}{2}} (r - r_0 - \xi(t)) \, e^{-\frac{c}{2}(r - \xi(t))} \left[1 - \beta t^{-\frac{1}{2}} \right].$$

In particular when $\delta = 0$ (see (9)) and $\lambda = 0$, one derives the following corollary:

Corollary 2.2 *Assume that $\delta = 0$ and $\lambda = 0$. Let $r_0 > \widehat{r}$ be given. Under the same condition as in Lemma 2.1, there exist $\varrho > 0$, $t_\varrho > 0$ large enough, $\gamma > 0$ and $\beta > 0$ such that for all $t > t_\varrho$ and $r \in \left[r_0 + ct, ct + \varrho\sqrt{t}\right]$:*

$$v(t, r) \geq \gamma t^{-\frac{N+2}{2}} (r - r_0 - ct) \, e^{-\frac{c}{2}(r - ct)} \left[1 - \beta t^{-\frac{1}{2}} \right].$$

We now state some upper estimate that will be used in the sequel.

Lemma 2.3 *[Upper estimate] The following upper estimates holds true:*

(i) There exists $\widehat{r}_0 > 0$ large enough depending on N, λ and c such that for each initial datum $v_0 : [\widehat{r}, \infty) \to \mathbb{R}^+$, a non-negative, non-trivial and compactly supported initial function, for each $\widehat{r} > \widehat{r}_0$, for each $\varrho > 0$ there exist $\bar{t} > 0$ large enough and some constants $\alpha, \beta > 0$ such that for each $t > \bar{t}$ and $r \in \left[\widehat{r} + \xi(t), \xi(t) + \varrho\sqrt{t}\right]$:

$$v(t, r) \leq \alpha t^{\frac{\delta c}{2} + \frac{\lambda}{c} - \frac{N+2}{2}} (r - \widehat{r} - \xi(t)) \, e^{-\frac{c}{2}(r - \xi(t))} \left[1 + \beta t^{-\frac{1}{4}} \right].$$

(ii) If $\delta = 0$ and if in addition v_0 is smooth enough (at least C^2) then for each $\widehat{r} > 0$ large enough (depending only on N and c), there exists some constant $\gamma > 0$ such that for all $t \geq 0$ and $r \geq ct + \widehat{r}$ one has:

$$v(t,r) \leq \gamma (t+1)^{\frac{\lambda}{c} - \frac{N+2}{2}}.$$

The proofs of the above results are based on self similar change of variables. Before proving these results, we shall first recall some functional framework and basic properties of the linear differential operator $\mathcal{L}\varphi = \varphi'' + \frac{y}{2}\varphi' + \varphi$ that will be used in the sequel.

2.1 Functional framework

Let us introduce the weight function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho(y) = \exp\left(\frac{y^2}{4}\right),$$

as well as the weighted spaces

$$H := L^2_\rho = \left\{\varphi \in L^2(0,\infty;\mathbb{R}) : \sqrt{\rho}\varphi \in L^2(0,\infty;\mathbb{R})\right\},$$

endowed with the usual norm denoted by $\|.\|_{2,\rho}$ and defined by

$$\|\varphi\|_{2,\rho} = \|\sqrt{\rho}\varphi\|_{L^2(0,\infty;\mathbb{R})}, \quad \forall \varphi \in H.$$

Note that it is a Hilbert space endowed with the usual inner product

$$\langle u, v\rangle_\rho = \int_0^\infty \rho(y)u(y)v(y)dy, \quad \forall (u,v) \in H^2.$$

We also introduce for each integer $m \geq 1$ the weighted Sobolev space

$$H^m_\rho = \left\{u \in H^m(0,\infty;\mathbb{R}) : u^{(k)} \in L^2_\rho(0,\infty), \quad \forall k = 0,..,m\right\}.$$

Next let us consider the linear operator $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ defined by

$$D(\mathcal{L}) = H^2_\rho \cap H^1_0(0,\infty;\mathbb{R}) \text{ and } \mathcal{L}\varphi = \rho^{-1}\left(\rho\varphi'\right)' + \varphi = \varphi'' + \frac{y}{2}\varphi' + \varphi.$$

Then the following lemma holds true:

Lemma 2.4 *The linear operator $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ satisfies the following properties:*

(a) It generates a strongly continuous analytic, compact and positive semigroup on H.

(b) *The operator* $-\mathcal{L}$ *is a self adjoint operator with the null space generated by the simple eigenvector* $\widehat{e_0} \in D(\mathcal{L})$ *defined by*

$$\widehat{e_0}(y) = \frac{1}{\|e_0\|_{2,\rho}} e_0(y),\ y \geq 0,\ \textit{with } e_0(y) \equiv y e^{-\frac{y^2}{4}}, \tag{11}$$

that is $\widehat{e_0} = \left(2\sqrt{\pi}\right)^{-1/2} e_0$.
The quadratic form associated to $-\mathcal{L}$ *denoted by* $\mathcal{Q} : H_0^1(0,\infty) \cap H_\rho^1 \to \mathbb{R}^+$ *and defined by*

$$\begin{aligned} \mathcal{Q}(\varphi) &= \int_0^\infty \rho(y) \left[|\varphi'(y)|^2 - \varphi^2(y) \right] dy, \\ &= \int_0^\infty \left[\left| \left(\rho^{1/2} \varphi \right)' (y) \right|^2 + \left(\frac{y^2}{16} - \frac{3}{4} \right) \rho(y) \varphi^2(y) \right] dy, \end{aligned} \tag{12}$$

satisfies

$$\mathcal{Q}(\varphi) \geq \|\varphi\|_{2,\rho}^2,\ \ \forall \varphi \in \langle \widehat{e_0} \rangle^\perp .$$

(c) *For each* $\varphi \in H_\rho^1 \cap H_0^1(0,\infty)$ *one has*

$$\left| \langle \varphi, \varphi' \rangle_\rho \right| = \frac{1}{4} \int_0^\infty \rho(y) y \varphi(y)^2 dy \leq \mathcal{Q}(\varphi) + \|\varphi\|_{2,\rho}^2.$$

(d) *The linear operator* $\mathcal{L}_s$ *defined as the part of* $\mathcal{L}$ *in* $H_s := \langle e_0 \rangle^\perp$, *that is*

$$\begin{cases} D\left(\mathcal{L}_s\right) = \left\{ \varphi \in D\left(\mathcal{L}\right)\ \ \mathcal{L}\varphi \in H_s \right\}, \\ \mathcal{L}_s \varphi = \mathcal{L}\varphi,\ \ \forall \varphi \in D\left(\mathcal{L}_s\right), \end{cases}$$

enjoys the maximal parabolic regularity, that is that for each $p \in (1,\infty)$ *there exists some constant* $M_p > 0$ *such that for each* $f \in L^p(0,\infty; H_s)$:

$$\left\| \int_0^t e^{(t-l)\mathcal{L}_s} f(l) dl \right\|_{W^{1,p}(0,\infty;H_s) \cap L^p(0,\infty;D(\mathcal{L}_s))} \leq M_p \|f\|_{L^p(0,\infty;H_s)},\ \forall t \geq 0.$$

(e) *The following estimates hold true for each* $\varphi \in D\left(\mathcal{L}\right)$:

$$\begin{aligned} &\int_0^\infty \rho(y) y^2 \varphi^2(y) dy \leq 16 \left\langle \left(I - \mathcal{L}\right) \varphi, \varphi \right\rangle_\rho, \\ &\|\varphi'(.)\|_{2,\rho} \leq \left\langle \left(I - \mathcal{L}\right) \varphi, \varphi \right\rangle_\rho, \end{aligned} \tag{13}$$

and setting $\psi = \rho^{1/2} \varphi$

$$\|\psi'\|_{L^2(0,\infty)} \leq \|\varphi'\|_{2,\rho}\ \textit{and}\ \|\psi''\|_{L^2(0,\infty)}^2 \leq \|\left(I - \mathcal{L}\right)\varphi\|_{2,\rho}^2 + \frac{1}{8}\|\varphi\|_{2,\rho}^2. \tag{14}$$

Remark 2.5 *Note that, due to Gagliardo-Nirenberg inequality,* (13) *and* (14)*, there exists some constant* $C > 0$ *such that for each* $\varphi \in D(\mathcal{L}_s)$ *one has*

$$\|\rho^{1/2}\varphi\|_\infty + \left\| \left(\rho^{1/2}\varphi\right)' \right\|_\infty \leq C\|(-\mathcal{L}_s)\varphi\|_{2,\rho}^{3/4}\|\varphi\|_{2,\rho}^{1/4}.$$

As a consequence one obtains using usual results on fractional powers (see Henry [20] and Pazy [30]) that for each $\beta \in \left(\frac{3}{4}, 1\right)$ *there exists some constant* $C_\beta > 0$ *such that*

$$\left\| \left(\rho^{1/2}\varphi\right)' \right\|_\infty \leq C_\beta \left\| (-\mathcal{L}_s)^\beta \varphi \right\|_{2,\rho}, \quad \forall \varphi \in D\left((-\mathcal{L}_s)^\beta\right). \tag{15}$$

Proof. The proof of the above lemma $(a) - (c)$ is classical. We refer for instance to Kavian [22] and Escobedo and Kavian [12] where the authors studied a similar operator on the whole space and to Hamel et al [18] for details for this operator on the half line with Dirichlet boundary condition.

The proof of (d) follows from the usual maximal parabolic regularity. We refer to Amann [1], Hieber and Prüss [21], Prüss [31] and the references therein for results on maximal parabolic regularity.

It remains to prove (e). Note that the two estimates in (13) follow from the definition as well as the alternated formulation of $\mathcal{Q}$ in (12). Next (14) follows from the weighted Sobolev estimates proved by Escobedo and Kavian in [12] and Kavian in [22]. We would like to mention that such estimates have been proved for the operator on the whole space. In the sequel we mimic this proof in order to check that similar estimates hold for the operator $\mathcal{L}$ on the half line with Dirichlet boundary condition. Let $\varphi \in D(\mathcal{L})$ be given. Set $\psi = \rho^{1/2}\varphi$ and note that one has $\psi' - \frac{y}{4}\psi = \rho^{1/2}\varphi'$. Hence one gets:

$$\int_0^\infty \rho|\varphi'|^2 dy = \int_0^\infty \left[|\psi'|^2 + \frac{y^2}{16}\psi^2\right] dy - \int_0^\infty \frac{y}{2}\psi\psi' dy.$$

However one has

$$\int_0^\infty \frac{y}{2}\psi\psi' dy = -\frac{1}{4}\int_0^\infty \psi^2 dy.$$

Hence one gets

$$\int_0^\infty \rho|\varphi'|^2 dy = \int_0^\infty \left[|\psi'|^2 + \frac{1}{4}\psi^2 + \frac{y^2}{16}\psi^2\right] dy.$$

This completes the first estimate in (14). Next observe that ψ'' satisfies the equation

$$\psi'' = \rho^{1/2}(\mathcal{L} - I)\varphi + \left(\frac{1}{4} + \frac{y^2}{16}\right)\psi.$$

Multiplying this equation by ψ'' and integrating over $(0, \infty)$ yields

$$\int_0^\infty |\psi''|^2 = \int_0^\infty \psi''\rho^{1/2}(\mathcal{L} - I)\varphi - \int_0^\infty \left(\frac{1}{4}|\psi'|^2 + \frac{y^2}{16}|\psi'|^2\right) + \frac{1}{16}\int_0^\infty \psi^2$$

Hence using Young inequality, the second estimate in (14) follows. ■

Remark 2.6 *Note that $\sigma(\mathcal{L}_s) \leq -1$ (here σ denotes the spectral bound) so that using Theorem 6.13 p.74 in Pazy [30], for each $\delta > 0$, $\alpha \in [0,1]$ there exists some constant $M_\alpha(\delta) > 0$ such that*

$$\left\| (-\mathcal{L}_s)^\alpha e^{t\mathcal{L}_s} \right\|_{\mathcal{L}(H_s)} \leq M_\alpha(\delta) t^{-\alpha} e^{-(1-\delta)t}, \ \ \forall t > 0.$$

In the sequel, for each $\alpha \in (0,1]$ we shall denote $H_s^\alpha := D\left((-\mathcal{L}_s)^\alpha\right)$ that is a Banach space endowed with the usual graph norm defined by

$$\|\varphi\|_{H_s^\alpha} := \|(-\mathcal{L}_s)^\alpha \varphi\|, \ \ \forall \varphi \in H_s^\alpha.$$

2.2 Proof of Lemma 2.1

In this subsection we will prove Lemma 2.1. In order to prove this result, we follow some argument developed in [19] and we define the new time variable τ by $c\tau = \xi(t)$ and consider the non-decreasing map $h : [0,\infty) \to [0,\infty)$ defined by

$$t = h(\tau).$$

Note that one has $h'(\tau) = \frac{c}{\xi'(t)}$ so that

$$1 \leq h'(\tau) \leq \left(1 - \frac{\delta}{cT}\right)^{-1}, \ \forall \tau \geq 0.$$

Moreover one has

$$\frac{1}{h'(\tau)} = 1 - \omega(\tau) \text{ with } \omega(\tau) = \frac{\delta}{c\left(h(\tau) + T\right)}. \tag{16}$$

Note that for some constants $K^\pm > 0$ one has:

$$\delta K^-(\tau + T)^{-1} \leq \omega(\tau) \leq \delta K^+(\tau + T)^{-1}, \ \ \forall \tau \geq 0. \tag{17}$$

Using this new time variable, namely τ, the function $\tilde{v}(\tau, r) := v(t, r)$ satisfies the following parabolic problem:

$$\begin{cases} \frac{1}{h'(\tau)} \tilde{v}_\tau = \tilde{v}_{rr} + \frac{N-1}{r} \tilde{v}_r + \left(\frac{c^2}{4} + \frac{\lambda}{1+r}\right) \tilde{v}, \ \tau > 0, \ r > c\tau + \widehat{r}, \\ \tilde{v}\left(\tau, c\tau + \widehat{r}\right) = 0, \ \tau > 0, \\ \tilde{v}(0, r) = v_0(r), \ r > \widehat{r}. \end{cases} \tag{18}$$

Next we set $\tilde{v}(\tau, r) = e^{-\frac{c}{2}(r - c\tau)} \alpha(\tau) w(\tau, r)$ wherein the function α is defined as a solution of the equation

$$\frac{\alpha'(\tau)}{\alpha(\tau)} = \frac{c^2}{2}(h'(\tau) - 1) \text{ with } \alpha(0) = 1.$$

Note that simple computation ensures that

$$\alpha(\tau) = \tau^{\frac{\delta c}{2}} \left(1 + O\left(\tau^{-\frac{1}{2}}\right)\right) \text{ as } \tau \to \infty.$$

Furthermore the function w satisfies the following parabolic equation

$$\begin{cases} \frac{1}{h'(\tau)} w_\tau = w_{rr} + \left[\frac{N-1}{r} - c\right] w_r + \left(\frac{\lambda}{1+r} - \frac{c}{2}\frac{N-1}{r}\right) w, \ r > c\tau + \widehat{r}, \\ w(\tau, \widehat{r} + c\tau) = 0, \ \tau > 0, \\ w(0,r) = w_0(r), \ r > \widehat{r}, \end{cases}$$

where w_0 is a non-trivial, non-negative and compactly supported function. Note that one has $w(\tau, r) > 0$ for all $\tau > 0$ and $r > c\tau + \widehat{r}$.

Now we fix $r_0 > \widehat{r}$ and we consider the map $\tilde{w}(\tau, r) = w(\tau, r_0 + r + c\tau)$ so that $\tilde{w}(\tau, 0) > 0$ for all $\tau > 0$ and it satisfies for all $\tau > 0$ and $r > 0$:

$$[1-\omega(\tau)]\, \tilde{w}_\tau = \tilde{w}_{rr} + \left[\frac{N-1}{r+r_0+c\tau} + c\omega(\tau)\right] \tilde{w}_r + \left[\frac{\lambda}{1+r+r_0+c\tau} - \frac{c}{2}\frac{N-1}{r+r_0+c\tau}\right] \tilde{w}.$$

This problem is supplemented together with the conditions

$$\begin{cases} \tilde{w}(\tau, 0) = w(\tau, r_0 + c\tau) > 0, \ \forall \tau > 0, \\ \tilde{w}(0, r) = w(0, r_0 + r), \ \forall r > 0. \end{cases} \tag{19}$$

Now in order to prove Lemma 2.1 we shall make use of self-similar variables. Let us introduce $t_1 > 0$ such that $r_0 = ct_1$ and let us consider the new variables

$$y = r(\tau + t_1)^{-1/2} \text{ and } s = \ln \frac{\tau + t_1}{t_1}, \tag{20}$$

as well as the map $\hat{w}(s, y) = \tilde{w}(\tau, r)$. Then the function $\hat{w}(s, y)$ satisfies for all $s > 0$ and $y > 0$ the following problem:

$$\begin{aligned} [1-\omega(\tau)] \left[\hat{w}_s - \frac{y}{2}\hat{w}_y\right] =& \hat{w}_{yy} + (\tau + t_1)^{\frac{1}{2}} \left[\frac{(N-1)}{r+r_0+c\tau} + c\omega(\tau)\right] \hat{w}_y \\ &+ \left[\frac{\lambda(\tau+t_1)}{1+r+r_0+c\tau} - \frac{c}{2}\frac{(N-1)(\tau+t_1)}{r+r_0+c\tau}\right] \hat{w}. \end{aligned}$$

Next, note that one has

$$\frac{c}{2}\frac{(N-1)(\tau+t_1)}{r+r_0+c\tau} = \frac{N-1}{2} - a(\tau, r) \text{ with } a(\tau, r) = \frac{N-1}{2}\frac{r}{r+r_0+c\tau},$$

while

$$\frac{\lambda(\tau+t_1)}{1+r+r_0+c\tau} = \frac{\lambda}{c} - \tilde{a}(\tau, r) \text{ with } \tilde{a}(\tau, r) := \frac{\lambda}{c}\frac{1+r}{1+r+r_0+c\tau}.$$

Hence the function $V = V(s, y)$ defined by $\hat{w}(s, y) = e^{\left(\frac{\lambda}{c} - \frac{N+1}{2}\right)s} V(s, y)$ satisfies the following equation:

$$[1-\omega(\tau)] \left[V_s - \frac{y}{2}V_y\right] = V_{yy} + V + b(\tau, r)V_y + c(\tau, r)V,$$

wherein we have set

$$b(\tau,r)=(\tau+t_1)^{1/2}\left[\frac{(N-1)}{r+r_0+c\tau}+c\omega(\tau)\right],$$
$$c(\tau,r)=a(\tau,r)-\tilde{a}(\tau,r)+\left[\frac{\lambda}{c}-\frac{(N+1)}{2}\right]\omega(\tau).$$

Recalling (19) function V satisfies:

$$V(s,0)>0 \text{ and } V(0,y)=w_0\left(r_0+t^{1/2}y\right).$$

Now note that one has

$$a(\tau,r)\geq 0,\ \forall\tau>0,\ r>0.$$

Hence, since $V(s,y)\geq 0$ and $0\leq \max\left[0,\left(\frac{(N+1)}{2}-\frac{\lambda}{c}\right)\right]\omega(\tau)\leq K(\tau+t_1)^{-1},\ \forall\tau\geq 0$ for some given constant $K>0$ (see (17)), function $V\equiv V(s,y)$ satisfies for all $s>0$ and $y>0$, the following differential inequality:

$$L[V](s,y):=(1-\omega(\tau))\left[V_s-\frac{y}{2}V_y\right]-V_{yy}-V-b(\tau,r)V_y+\left[\tilde{a}(\tau,r)+\frac{K}{\tau+t_1}\right]V\geq 0.$$

We are now looking for a sub-solution of the form

$$\underline{V}(s,y)=e_0(y)+\beta e^{-s/2}\left[12e_0(y)-e_1(y)\right].$$

Here e_0 is defined in (11) and e_1 is defined by

$$e_1(y)=y^3e^{-\frac{y^2}{4}},\ \forall y\geq 0. \tag{21}$$

Parameter $\beta>0$ will be chosen large enough using the following computational lemma.

Lemma 2.7 *There exist $\beta>0$, $\varrho>0$ and $s_0>0$ such that the function $\underline{V}\equiv\underline{V}(s,y)$ satisfies*

$$L\left[\underline{V}\right](s,y)\leq 0,\ \ \forall s\geq s_0,\ \forall y\in\left[0,\varrho e^{\frac{s}{2}}\right].$$

Before proving this computational lemma, let us complete the proof of Lemma 2.1. To complete the proof of this lemma, let us notice that one has

$$e^{\frac{\varrho^2e^s}{4}}\underline{V}\left(s,\varrho e^{\frac{s}{2}}\right)=\varrho\left[e^{\frac{s}{2}}+12\beta-\beta\varrho^2e^s\right]\to-\infty \text{ as } s\to\infty.$$

Since $\underline{V}(s,0)=0$ for all $s>0$, the parabolic comparison principle applies and ensures that there exists some constant $\gamma>0$ small enough and $s_1\geq s_0$ large enough such that

$$\gamma\underline{V}(s,y)\leq V(s,y),\ \ \forall s\geq s_1\ \forall y\in\left[0,\varrho e^{s/2}\right].$$

This completes the proof of Lemma 2.1 recalling (20) and coming back to the original function.

Now it remains to complete the proof of Lemma 2.7.

Proof of Lemma 2.7. To prove this lemma we aim at finding $\beta > 0$ large enough, $s_0 > 0$ large enough and $\varrho > 0$ such that $L[\underline{V}] \leq 0$ on the set $Q_{(s_0,\varrho)}$ defined by

$$Q_{(s_0,\varrho)} = \left\{(s,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \ s \geq s_0, \ y \leq \varrho e^{s/2}\right\}.$$

To do so, note that one has:

$$\begin{aligned} L[\underline{V}](s,y) =& \frac{-\beta}{2}(1-\omega(\tau))\, e^{-\frac{s}{2}}\left[12e_0(y) - e_1(y)\right] \\ &+ \omega(\tau)\frac{y}{2}\left[e_0'(y) + \beta e^{-\frac{s}{2}}\left(12e_0'(y) - e_1'(y)\right)\right] \\ &+ \beta e^{-\frac{s}{2}}\mathcal{L}e_1 - b(\tau,r)\left[e_0' + \beta e^{-\frac{s}{2}}\left(12e_0' - e_1'\right)\right] \\ &+ \left[\tilde{a}(\tau,r) + \frac{K}{\tau+t_1}\right]\left[e_0 + \beta e^{-\frac{s}{2}}\left(12e_0 - e_1\right)\right]. \end{aligned}$$

Recalling definition (11) and (21), it is easy to check that

$$\mathcal{L}e_1 = 6e_0 - e_1. \tag{22}$$

Using this formula one obtains that

$$\begin{aligned} e^{\frac{s}{2}}e^{\frac{y^2}{4}}L[\underline{V}](s,y) =& -\beta\frac{y^3}{2} - t_1^{\frac{1}{2}}\left[\frac{(N-1)e^s}{t_1^{\frac{1}{2}}ye^{\frac{s}{2}} + ct_1e^s} + ce^s\omega(\tau)\right]\left[1 - \frac{y^2}{2}\right] \\ &+ \frac{\omega(\tau)}{2}\beta\left[12y - y^3\right] \\ &+ e^{\frac{s}{2}}\omega(\tau)\frac{y}{2}\left[1 - \frac{y^2}{2} + \beta e^{-\frac{s}{2}}\left(12 - 9y^2 + \frac{y^4}{2}\right)\right] \\ &- t_1^{\frac{1}{2}}e^s\left[\frac{(N-1)}{t_1^{\frac{1}{2}}ye^{\frac{s}{2}} + ct_1e^s} + c\omega(\tau)\right]\left[\beta e^{-\frac{s}{2}}\left(12 - 9y^2 + \frac{y^4}{2}\right)\right] \\ &+ e^{\frac{s}{2}}\left[\frac{\lambda}{c}\frac{1 + t_1^{\frac{1}{2}}e^{\frac{s}{2}}y}{1 + t_1^{\frac{1}{2}}e^{\frac{s}{2}}y + r_0 + ct_1e^s} + \frac{K}{\tau+t_1}\right]\left[y + \beta e^{-\frac{s}{2}}\left(12y - y^3\right)\right]. \end{aligned}$$

Next, recalling (17) and (20), there exist some constants $\omega^{\pm} > 0$ such that

$$\delta\omega^- e^{-s} \leq \omega(\tau) \leq \omega^+ e^{-s}, \ \ \forall s \geq 0.$$

Hence there exists some constant $C > 0$ independent of s, y and β such that

$$\begin{aligned} e^{\frac{s}{2}}e^{\frac{y^2}{4}}L[\underline{V}](s,y) \leq& -\beta\frac{y^3}{2} - t_1^{\frac{1}{2}}\left[\frac{(N-1)e^s}{t_1^{\frac{1}{2}}ye^{\frac{s}{2}} + ct_1e^s} + c\delta\omega^-\right] + Cy^2 \\ &+ (1+\beta)Ce^{-\frac{s}{2}}y + \beta Ce^{-s}y^5 + (1+\beta)Ce^{-\frac{s}{2}}y^2. \end{aligned}$$

Now let us choose $\beta > 0$ large enough such that

$$-\beta\frac{y^3}{2} - t_1^{\frac{1}{2}}\left[\frac{(N-1)e^s}{t_1^{\frac{1}{2}}ye^{\frac{s}{2}} + ct_1e^s} + c\delta\omega^-\right] + Cy^2 \le -\frac{\beta}{4}y^3 - \beta^{-1},\ \ \forall y \ge 0,\ s \ge 0.$$

Note that when $\delta > 0$ the existence of such $\beta > 0$ is obvious since $\delta\omega^- > 0$. When $\delta = 0$ then the inequality becomes

$$-\beta\frac{y^3}{2} - t_1^{\frac{1}{2}}\frac{(N-1)e^s}{t_1^{\frac{1}{2}}ye^{\frac{s}{2}} + ct_1e^s} + Cy^2 \le -\frac{\beta}{4}y^3 - \beta^{-1}.$$

Then the second term allows to find such a $\beta > 0$ large enough so that the inequality holds true uniformly for $s \ge 0$ and $y \in [0,1]$. For $y \ge 1$ in order to satisfy this inequality it is sufficient to have

$$-\beta\frac{y^3}{2} + Cy^2 \le -\frac{\beta}{4}y^3 - \beta^{-1},\ \forall y \ge 1,$$

that holds true for $\beta > 0$ large enough.

Next with such a choice of β, one obtains:

$$e^{\frac{s}{2}}e^{\frac{y^2}{4}}L\left[\underline{V}\right](s,y) \le -\beta\frac{y^3}{4} - \beta^{-1} + (1+\beta)Ce^{-\frac{s}{2}}(y+y^2) + \beta Ce^{-s}y^5.$$

Now note that there exists $\varrho > 0$ such that for all $s \ge 0$:

$$-\beta\frac{y^3}{8} + \beta Ce^{-s}y^5 \le 0,\ \forall y \in \left[0, \varrho e^{\frac{s}{2}}\right].$$

In addition, there exists $s_0 > 0$ large enough such that for all $y \ge 0$ and $s \ge s_0$:

$$(1+\beta)Ce^{-\frac{s}{2}}(y+y^2) \le \beta\frac{y^3}{8} + \beta^{-1}.$$

As a consequence of the above computations we obtain that there exist $\beta > 0$ large enough, $\varrho > 0$ and $s_0 > 0$ such that the function $\underline{V}(s,y)$ satisfies

$$L\left[\underline{V}\right](s,y) \le 0,\ \ \forall s \ge s_0,\ y \in \left[0, \varrho e^{\frac{s}{2}}\right].$$

This completes the proof of Lemma 2.7. ∎

2.3 Proof of Lemma 2.3

The aim of this section is to provide an upper estimate for the solution of (10) as stated in Lemma 2.3. Let $\varepsilon \in \left(0, \frac{1}{6}\right)$ be given. Let us fix $t_1 = c^{-1}\widehat{r}$ where $\widehat{r}$ is fixed large enough such that

$$\left[\frac{N-1}{2} + \frac{|\lambda|}{c}\right]\frac{4}{ct_1^{1/2}} < 1, \tag{23}$$

and

$$\frac{(N-1)}{2c^2t_1}+\frac{8}{ct_1^{1/2}}\left[\frac{N-1}{2}+\frac{|\lambda|}{c}\right]<\frac{\varepsilon}{2} \text{ and } \frac{4}{ct_1^{1/2}}\left[\frac{N-1}{2}+\frac{|\lambda|}{c}\right]<\frac{\varepsilon}{2}. \tag{24}$$

Similarly to the proof of the lower estimate, let us define the new time variable τ by $c\tau=\xi(t)$ and consider the non-decreasing map $h:[0,\infty)\to[0,\infty)$ defined by $t=h(\tau)$. Then the function $v(\tau,r):=v(t,r)$ satisfies:

$$\begin{cases}\frac{1}{h'(\tau)}v_\tau=v_{rr}+\frac{N-1}{r}v_r+\left[\frac{c^2}{4}+\frac{\lambda}{1+r}\right]v,\ t>0,\ r>c\tau+\widehat{r}\\ v\left(\tau,c\tau+\widehat{r}\right)=0,\ \tau>0\\ v(0,r)=v_0(r)\ r>\widehat{r}.\end{cases} \tag{25}$$

Next we introduce the new function $w=w(\tau,r)$ defined by $v(\tau,r)=e^{-\frac{c}{2}(r-c\tau)}\alpha(\tau)w(\tau,r)$ and wherein function $\alpha>0$ satisfies

$$\frac{\alpha'(\tau)}{\alpha(\tau)}=\frac{c^2}{2}(h'(\tau)-1),\ \alpha(0)=1.$$

Similarly as above the function α satisfies

$$\alpha(\tau)=\tau^{\frac{\delta c}{2}}\left(1+O\left(\tau^{-\frac{1}{2}}\right)\right) \text{ as } \tau\to\infty.$$

Hence the function w satisfies the following equation

$$\begin{cases}\frac{1}{h'(\tau)}w_\tau=w_{rr}+\left[\frac{N-1}{r}-c\right]w_r+\left[\frac{\lambda}{1+r}-\frac{c}{2}\frac{N-1}{r}\right]w,\ r>c\tau+\widehat{r},\\ w(\tau,\widehat{r}+c\tau)=0,\ \tau>0,\\ w(0,r)=w_0(r),\ r>\widehat{r}.\end{cases}$$

We now introduce $\tilde{w}(\tau,r)=w(\tau,\widehat{r}+r+c\tau)$. Recalling (16), the function $\tilde{w}$ satisfies for all $\tau>0$ and $r>0$ the following parabolic equation

$$(1-\omega(\tau))\,\tilde{w}_\tau=\tilde{w}_{rr}+\left[\frac{N-1}{r+\widehat{r}+c\tau}+c\omega(\tau)\right]\tilde{w}_r+\left[\frac{\lambda}{1+r+\widehat{r}+c\tau}-\frac{c}{2}\frac{N-1}{r+\widehat{r}+c\tau}\right]\tilde{w},$$

supplemented together with

$$\tilde{w}(\tau,0)=0 \text{ and } \tilde{w}(0,r)=w(0,\widehat{r}+r).$$

Now in order to prove Lemma 2.3 we shall make use of self-similar variables by considering the new variables

$$y=r(\tau+t_1)^{-1/2} \text{ and } s=\ln\frac{\tau+t_1}{t_1}, \tag{26}$$

and the map $V\equiv V(s,y)$ defined by $\tilde{w}(\tau,r)=e^{\left(\frac{\lambda}{c}-\frac{N+1}{2}\right)s}V(s,y)$. Here recall that $t_1=c^{-1}\widehat{r}$. Then the function $V(s,y)$ satisfies the following equation

$$(1-\widehat{\omega}(s))V_s-\mathcal{L}V=\left[\widehat{\omega}(s)\frac{y}{2}+B(s,y)\right]V_y+C(s,y)V. \tag{27}$$

Here, recalling (26), we have set $\widehat{\omega}(s) = \omega(\tau)$,

$$B(s,y) = \left[\frac{(N-1)}{y + ct_1^{1/2}e^{s/2}} + c\widehat{\omega}(s)\right],$$

and

$$C(s,y) = \frac{N-1}{2}\frac{y}{y + ct_1^{1/2}e^{s/2}} - \frac{\lambda}{c}\frac{1 + t_1^{1/2}ye^{-s/2}}{1 + t_1^{1/2}ye^{-s/2} + ct_1e^s} + \left[\frac{\lambda}{c} - \frac{(N+1)}{2}\right]\widehat{\omega}(s).$$

Now in order to prove Lemma 2.3 we shall study (27) by using the functional framework introduced in Subsection 2.1. We first derive a uniform $L^2_\rho-$estimate for the function V. More particularly we claim that

Claim 2.8 *There exists some constant $K > 0$ such that*

$$\|V(s,.)\|_{2,\rho} \leq K, \ \ \forall s \geq 0.$$

Throughout this proof and in the sequel of the proof of Lemma 2.3, we shall use K to denote any constant depending on the parameters of the system and on function v_0 (or V_0), that may change from line to line.
To prove this claim, we take the inner product of (27) with V, that yields

$$\begin{aligned}(1-\widehat{\omega}(s))\frac{d}{2ds}\|V\|^2_{2,\rho} + \mathcal{Q}\left[V(s,.)\right] = &\int_0^\infty \rho(y)\left[-\frac{\widehat{\omega}(s)}{2} + \frac{(N-1)}{(y + ct_1^{1/2}e^{s/2})^2}\right]\left(\frac{V^2}{2}\right)dy \\ &- \int_0^\infty \rho(y)\frac{y}{2}\left[\widehat{\omega}(s)\frac{y}{2} + B(s,y)\right]\left(\frac{V^2}{2}\right)dy \\ &+ \int_0^\infty \rho(y)C(s,y)V^2dy.\end{aligned}$$

This leads us to the following identity

$$\begin{aligned}(1-\widehat{\omega}(s))\frac{d}{2ds}\|V(s,.)\|^2_{2,\rho} + \mathcal{Q}\left[V(s,.)\right] \leq &Ke^{-s/2}\|V(s,.)\|^2_{2,\rho} \\ &+ \left[\frac{N-1}{2} + \frac{|\lambda|}{c}\right]\frac{e^{-s/2}}{ct_1^{1/2}}\int_0^\infty \rho(y)yV^2(s,y)dy.\end{aligned}$$

Next Lemma 2.4 (c) yields that for some constant K one has

$$(1-\widehat{\omega}(s))\frac{d}{2ds}\|V(s,.)\|^2_{2,\rho} + \left[1 - \left(\frac{N-1}{2} + \frac{|\lambda|}{c}\right)\frac{4e^{-s/2}}{ct_1^{1/2}}\right]\mathcal{Q}\left[V(s,.)\right] \leq Ke^{-s/2}\|V(s,.)\|^2_{2,\rho}.$$

Recalling the choice of t_1 in (23), one obtains that

$$\|V(s,.)\|^2_{2,\rho} \leq \|V(0,.)\|^2_{2,\rho}\exp\left(K\int_0^\infty \frac{e^{-t/2}}{1-\widehat{\omega}(t)}dt\right), \ \ \forall s \geq 0. \tag{28}$$

This completes the proof of Claim 2.8.
Using this uniform $L^2_\rho-$bound we shall complete the proof of Lemma 2.3 (i). To do so, we decompose function V as

$$V(s,.) = p(s)\widehat{e_0} + \tilde{V}(s,.) \text{ with } \tilde{V}(s,.) \in \langle \widehat{e_0} \rangle^{\perp},\ \forall s \geq 0. \tag{29}$$

Note that due to (28), one knows that there exists some constant $K >$ such that

$$|p(s)| \leq K \text{ for all } s \geq 0. \tag{30}$$

On the other hand function $\tilde{V}$ satisfies:

$$\begin{aligned}(1-\widehat{\omega}(s))\left[p'(s)\widehat{e_0} + \tilde{V}_s\right] - \mathcal{L}\tilde{V} = &\left[\widehat{\omega}(s)\frac{y}{2} + B(s,y)\right]\left[p(s)\widehat{e_0}' + \tilde{V}_y\right] \\ &+ C(s,y)\left[p(s)\widehat{e_0} + \tilde{V}\right].\end{aligned} \tag{31}$$

Taking the inner product of the above equation with $\tilde{V}$ yields

$$\begin{aligned}(1-\widehat{\omega}(s))\frac{d}{2ds}\|\tilde{V}\|^2_{2,\rho} + \mathcal{Q}\left[\tilde{V}(s,.)\right] = &\int_0^\infty \rho(y)\left[\widehat{\omega}(s)\frac{y}{2} + B(s,y)\right]\left[p(s)\widehat{e_0}'\tilde{V} + \tilde{V}\tilde{V}_y\right]dy \\ &+ \int_0^\infty \rho(y)C(s,y)\left[p(s)\widehat{e_0}\tilde{V} + |\tilde{V}|^2\right]dy.\end{aligned} \tag{32}$$

Note that due to (28), Hölder inequality and integration by parts, one obtains

$$\begin{aligned}&\int_0^\infty \rho(y)\left[\widehat{\omega}(s)\frac{y}{2} + B(s,y)\right]\left[p(s)\widehat{e_0}'\tilde{V} + \tilde{V}\tilde{V}_y\right]dy \\ =& -\int_0^\infty \rho(y)\left[\widehat{\omega}(s)\frac{y}{2} + B(s,y)\right]p(s)\widehat{e_0}'\tilde{V}dy \\ &- \int_0^\infty \rho(y)\frac{y}{4}\left[\widehat{\omega}(s)\frac{y}{2} + B(s,y)\right]\frac{|\tilde{V}|^2}{2}dy \\ &- \int_0^\infty \rho(y)\left[\widehat{\omega}(s)\frac{1}{2} - \frac{(N-1)}{\left(y + ct_1^{1/2}e^{s/2}\right)^2}\right]\frac{|\tilde{V}|^2}{2}dy \\ \leq& K\left[\widehat{\omega}(s)\|\cdot\widehat{e_0}'(.)\|_{2,\rho} + e^{-s/2}\|\widehat{e_0}'\|_{2,\rho}\right]\|\tilde{V}\|_{2,\rho} + \frac{(N-1)e^{-s}}{2c^2t_1}\|\tilde{V}\|^2_{2,\rho}.\end{aligned}$$

One the other hand, Hölder inequality yields

$$\begin{aligned}\int_0^\infty \rho(y)C(s,y)\left[p(s)\widehat{e_0}\tilde{V} + |\tilde{V}|^2\right]dy \leq& K\left[\widehat{\omega}(s)\|\widehat{e_0}\|_{2,\rho} + e^{-s/2}\|\cdot\widehat{e_0}(.)\|_{2,\rho}\right]\|\tilde{V}\|_{2,\rho} \\ &+ \left[\frac{N-1}{2} + \frac{|\lambda|}{c}\right]\frac{e^{-s/2}}{ct_1^{1/2}}\int_0^\infty \rho(y)y|\tilde{V}(s,y)|^2dy.\end{aligned}$$

In addition, we infer from Lemma 2.4 (c) that

$$\int_0^\infty \rho(y)C(s,y)\left[p(s)\widehat{e_0}\tilde{V}+|\tilde{V}|^2\right]dy \leq K\left[\widehat{\omega}(s)\|\widehat{e_0}\|_{2,\rho}+e^{-s/2}\|\cdot\widehat{e_0}(.)\|_{2,\rho}\right]\|\tilde{V}\|_{2,\rho}$$
$$+\left[\frac{N-1}{2}+\frac{|\lambda|}{c}\right]\frac{e^{-s/2}}{ct_1^{1/2}}\left[4\mathcal{Q}\left[\tilde{V}(s,.)\right]+8\|\tilde{V}(s,.)\|_{2,\rho}^2\right].$$

As a consequence, by setting

$$M(t_1)=\frac{(N-1)}{2c^2t_1}+\frac{8}{ct_1^{1/2}}\left[\frac{N-1}{2}+\frac{|\lambda|}{c}\right] \text{ and } N(t_1)=\frac{4}{ct_1^{1/2}}\left[\frac{N-1}{2}+\frac{|\lambda|}{c}\right],$$

we infer from (32) and the two above estimates that

$$\begin{aligned}&(1-\widehat{\omega}(s))\frac{d}{2ds}\|\tilde{V}\|_{2,\rho}^2+\mathcal{Q}\left[\tilde{V}(s,.)\right]\\&\leq Ke^{-s}\|\tilde{V}(s,.)\|_{2,\rho}+M(t_1)e^{-s/2}\|\tilde{V}\|_{2,\rho}^2+N(t_1)e^{-s/2}\mathcal{Q}\left[\tilde{V}(s,.)\right].\end{aligned} \tag{33}$$

Recalling the choice of t_1 in (24) and that $\tilde{V}(s,.)\in\langle\widehat{e_0}\rangle^\perp$ for all $s\geq 0$, one obtains using Lemma 2.4 (b), and Young inequality that

$$(1-\widehat{\omega}(s))\frac{d}{2ds}\|\tilde{V}\|_{2,\rho}^2+\left(1-\frac{3}{2}\varepsilon e^{-s/2}\right)\|\tilde{V}(s,.)\|_{2,\rho}^2\leq Ke^{-\frac{s}{2}}.$$

One deduces from the above inequality that

$$\|\tilde{V}(s,.)\|_{2,\rho}\leq Ke^{-\frac{s}{4}} \text{ for all } s \text{ large enough.} \tag{34}$$

Now applying parabolic estimates to (31) yields that for each $\rho>0$, there exists some constant $K_\rho>0$ such that

$$|\tilde{V}_y(s,y)|\leq K_\rho e^{-\frac{s}{4}} \text{ for all } s \text{ large enough and } y\in[0,\rho].$$

Therefore since $\tilde{V}(s,0)=0$ then

$$|\tilde{V}(s,y)|\leq K_\rho y e^{-\frac{s}{4}} \text{ for all } s \text{ large enough and } y\in[0,\rho].$$

Coming back to the original function $v\equiv v(t,r)$, this completes the proof of Lemma 2.3 (i).

We now turn to the proof of (ii) with $\delta=0$. Let us first notice that from the above estimate (34) and integrating (33) over $s\in[0,\infty)$ ensures that

$$\int_0^\infty \mathcal{Q}\left[\tilde{V}(s,.)\right]ds\leq K.$$

Recalling the definition of $\mathcal{Q}$ in Lemma 2.4 implies that

$$\int_0^\infty\left[\|\tilde{V}_y(s,.)\|_{2,\rho}^2+\|\cdot\tilde{V}(s,\cdot)\|_{2,\rho}^2\right]ds\leq K. \tag{35}$$

Now let us notice that when $\delta = 0$ then $\widehat{\omega}(s) \equiv 0$ and the equation for V re-writes as

$$V_s - \mathcal{L}V = \widehat{B}(s,y)V_y + \widehat{C}(s,y)V, \tag{36}$$

with

$$\widehat{B}(s,y) = \frac{(N-1)}{y + ct_1^{1/2}e^{s/2}}$$
$$\widehat{C}(s,y) = \frac{N-1}{2}\frac{y}{y+ct_1^{1/2}e^{s/2}} - \frac{\lambda}{c}\frac{1+t_1^{1/2}ye^{-s/2}}{1+t_1^{1/2}ye^{-s/2}+ct_1e^s}.$$

Introducing the projector $\Pi_s \in \mathcal{L}(H)$ on $H_s = \widehat{e_0}^{\perp}$ defined by

$$\Pi_s\varphi = \varphi - \langle \varphi, \widehat{e_0}\rangle_\rho \widehat{e_0},$$

and using the constant variation formula, one obtains that $\tilde{V}(s,.) = \Pi_s V(s,.)$ satisfies for all $s \geq s_0 \geq 0$

$$\tilde{V}(s,.) = e^{(s-s_0)\mathcal{L}_s}\tilde{V}(s_0,.) + \int_{s_0}^{s} e^{(s-s')\mathcal{L}_s}\Pi_s\left[\widehat{B}(s',.)V_y(s',.) + \widehat{C}(s',.)V(s',.)\right]ds'. \tag{37}$$

Here $\left\{e^{t\mathcal{L}_s}\right\}_{t\geq 0} \subset \mathcal{L}(H_s)$ denotes the analytic semigroup generated by the linear operator $\mathcal{L}_s$.
Now we investigate some first properties of the function f defined by

$$f(s) := \widehat{B}(s,.)V_y(s,.) + \widehat{C}(s,.)V(s,.).$$

Recalling (29), note that one has

$$f(s) = f_1(s) + f_2(s),$$

wherein we have set

$$f_1(s) = p(s)\left[\widehat{B}(s,.)\widehat{e_0}' + \widehat{C}(s,.)\widehat{e_0}\right],$$
$$f_2(s) = \widehat{B}(s,.)\tilde{V}_y(s,.) + \widehat{C}(s,.)\tilde{V}(s,.).$$

Hence due to (30) one obtains

$$\|f_1(s)\|_{2,\rho} \leq Ke^{-s/2},\ \forall s \geq 0,$$

while

$$\|f_2(s)\|_{2,\rho} \leq Ke^{-s/2}\left[\|\tilde{V}_y(s,.)\|_{2,\rho} + \|y\tilde{V}(s,.)\|_{2,\rho} + e^{-s/2}\right].$$

Next due to (35), note that $s \mapsto e^{s/2}f_2(s) \in L^2(0,\infty;H)$ As a consequence, one may apply maximal regularity stated in Lemma 2.4 (d) to (37) to conclude (recalling that $\tilde{V}(0,.) \in D(\mathcal{L}_s)$ because the initial datum is assumed to smooth enough) that

$$\tilde{V} \in L^2(0,\infty;D(\mathcal{L}_s)) \cap W^{1,2}(0,\infty;H_s).$$

Hence, usual interpolation yields $\tilde{V} \in L^4\left(0,\infty; H_s^{\frac{1}{2}}\right)$. As a consequence, using (13) one gets that the map $s \mapsto e^{s/2} f_2(s) \in L^4(0,\infty; H)$. Next we set $g(s) = e^{s/2}\|f_2(s)\|$ with $g \in L^4(0,\infty;\mathbb{R}^+)$. Now let $\alpha \in \left(0,\frac{3}{4}\right)$ be a given fractional power. Then using (37) with $s_0 = 0$ and the properties of fractional powers recalled in Lemma 2.4 and Remark 2.6, one obtains that for some $\delta \in \left(0,\frac{1}{2}\right)$:

$$\|\tilde{V}(s,.)\|_{H_s^\alpha} \leq Ke^{-(1-\delta)s} + K\int_0^s \frac{e^{-(1-\delta)(s-s')}}{(s-s')^\alpha} e^{-s'/2}\left[1+g(s')\right]ds'. \tag{38}$$

Using Young inequality one gets

$$\begin{aligned}\|\tilde{V}(s,.)\|_{H_s^\alpha} \leq & Ke^{-(1-\delta)s} + Ke^{-s/2}\int_0^s \frac{e^{-(1-\delta+\frac{1}{2})l}}{l^\alpha}dl \\ & + K\left[\int_0^s \frac{e^{-\frac{4}{3}(1-\delta)(s-s')}}{(s-s')^{\frac{4\alpha}{3}}} e^{-2s'/3}ds'\right]^{3/4}\|g\|_{L^4}.\end{aligned}$$

Hence we derive that for each power $\alpha \in \left(0,\frac{3}{4}\right)$ there exists some constant $K_\alpha > 0$ such that

$$\|\tilde{V}(s,.)\|_{H_s^\alpha} \leq K_\alpha e^{-s/2},\ \forall s \geq 0.$$

Due to the above estimates and (13) one obtains with $\alpha = \frac{1}{2}$ that

$$\|f_2(s)\|_{2,\rho} \leq Ke^{-s/2},\ \ \forall s \geq 0.$$

As a consequence one knows that $g \in L^\infty(0,\infty;\mathbb{R})$ and by using (38) with $g \in L^\infty(0,\infty;\mathbb{R}^+)$, one obtains that for each $\alpha \in (0,1)$ there exists some constant $K_\alpha > 0$ such that

$$\|\tilde{V}(s,.)\|_{H_s^\alpha} \leq K_\alpha e^{-s/2},\ \forall s \geq 0.$$

Finally with any $\alpha \in \left(\frac{3}{4},1\right)$ and estimate (15), one concludes that

$$\|\rho^{1/2}\tilde{V}(s,.)\|_{W^{1,\infty}} \leq Ke^{-s/2},\ \forall s \geq 0.$$

As a consequence, one has obtained that for each $s \geq 0$ and $y \geq 0$:

$$V(s,y) \leq K\widehat{e_0}(y) + Ke^{-s/2}y\rho^{-1/2}(y) \leq Ky.$$

This completes the proof of Lemma 2.3 (ii) by coming back to the original variables and function.

3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Throughout this section the condition $\Gamma \in L^1(0,\infty)$ is assumed.
We shall first study the asymptotic speed spread of the solution of (1). Then we shall derive some lower and upper estimates for the solution in a moving frame located at the radius $r = \xi(t) := c^*t - \frac{N+2}{c^*}\ln t$, to finally conclude to the proof of Theorem 1.2.

3.1 Asymptotic speed of spread

In this section we shall investigate the asymptotic speed of spread for an asymptotically homogeneous Fisher-KPP equation. Let $\gamma : [0, \infty) \to \mathbb{R}$ be a function such that $\gamma(r) \to 0$ as $r \to \infty$. Consider the problem

$$\begin{cases} u_t - \Delta u = F(u) + \gamma(\|x\|)u, \ t > 0, \ x \in \mathbb{R}^N, \\ u(0, x) = u_0(x). \end{cases}$$

Here function F satisfies that the usual KPP assumption (see Assumption 1.5). Then our inner spreading result reads as:

Lemma 3.1 *Recalling that the initial datum u_0 satisfies* (2)*, the solution $u = u(t, x)$ of the above problem satisfies for each $0 < c' < c'' < c^*$:*

$$\lim_{t\to\infty} \sup_{c't \le \|x\| \le c''t} |1 - u(t, x)| = 0.$$

The proof of this result relies on standard arguments. We refer for instance to [3] or [6]. We however give a sketch of the proof for the sake of completeness.
Proof. Let $0 < c' < c'' < c^*$ be given. Let $c \in [c', c'']$ be given. Let $R > 0$ be given and consider the eigenvalue problem on the open ball $B_R \subset \mathbb{R}^N$ of center 0 and radius R

$$\begin{cases} -\Delta\varphi_R = \mu_R\varphi_R \text{ in } B_R, \\ \varphi_R = 0 \text{ on } \partial B_R \text{ and } \varphi_R > 0 \text{ on } B_R. \end{cases}$$

Recall first that $\mu_R > 0$ and $\mu_R \to 0$ as $R \to \infty$. We normalize φ_R by $\varphi_R(x) \le \varphi_R(0) = 1$ and we extend it outside the ball by 0. Let $e \in \mathbb{S}^{N-1}$ be a given direction. Consider for some given $\eta > 0$ the map

$$\underline{u}(t, x) = \eta e^{-\frac{c}{2}(x\cdot e - ct)}\varphi_R(x - cte)$$

Next we compute on the set $x - cte \in B_R$

$$L[\underline{u}] := \underline{u}_t - \Delta\underline{u} - F(\underline{u}) - \gamma(\|x\|)\underline{u} = \underline{u}\left[\frac{c^2}{4} - \frac{F(\underline{u})}{\underline{u}} + \mu_R + \gamma(\|x\|)\right].$$

On the other hand one has $\underline{u} \le \eta e^{\frac{cR}{2}}$ and there exists $\kappa > 0$ such that

$$F'(0) \ge \frac{F(u)}{u} \ge F'(0) - \kappa u, \ \ \forall u.$$

Hence one obtains

$$L[\underline{u}] \le \underline{u}\left[\frac{c^2}{4} - F'(0) + \kappa\eta e^{cR/2} + \mu_R + \sup_{r\in[ct-R,ct+R]} |\gamma(r)|\right].$$

Since $c'' < c^* = 2\sqrt{F'(0)}$, $\mu_R \to 0$ as $R \to \infty$ and $\gamma(r) \to 0$ as $r \to \infty$, we can adjust all the parameters in order to obtain that $L[\underline{u}] \le 0$ for all t large enough.

Still decreasing η if necessary, one obtains from the comparison principle that there exists $t_0 > 0$ large enough, $R > 0$ large enough, $\eta > 0$ small enough such that for all $t \geq t_0$, all $x \in \mathbb{R}^N$, all $e \in \mathbb{S}^{N-1}$ and all $c \in [c', c'']$:

$$u(t,x) \geq \eta e^{-\frac{c}{2}(x\cdot e - ct)}\varphi_R(x - cte).$$

Hence for all t large enough, this yields the following lower estimate:

$$\inf_{c't \leq \|x\| \leq c''t} u(t,x) \geq \eta\varphi_R(0).$$

The result follows by passing to the limit into the equation for u and recalling that estimate (5) holds true in this context. ■

We now derive an upper estimate for the spreading speed of (1)

Lemma 3.2 *Let $u \equiv u(t,x)$ be the solution of* (1)*. It satisfies for each $\sigma > 0$:*

$$\lim_{t\to\infty} \sup_{\|x\| \geq c^*t + \sigma\sqrt{t}} u(t,x) = 0.$$

Proof. Note that due to Assumption 1.1 (ii) the function u satisfies on the set $\|x\| \geq c^* t$:

$$L[u](t,x) := u_t - \Delta u - u\left[F'(0) + \Gamma(c^*t)\right] \leq 0.$$

Let $e \in \mathbb{S}^{N-1}$ be given. Consider, for some constant $K > 0$, the map

$$\overline{u}(t,x) = K\beta(t)e^{-\frac{c^*}{2}(x\cdot e - c^*t)} \text{ with } \beta(t) = e^{\int_0^t \Gamma(c^*s)ds}.$$

Then one has:

$$e^{\frac{c^*}{2}(x\cdot e - c^*t)}L[\overline{u}](t,x) = K\left(\beta'(t) - \beta(t)\Gamma(c^*t)\right) = 0.$$

Hence if we choose $K > 0$ sufficient large so that $\overline{u}(0,x) \geq u_0(x)$ for all $x \in \mathbb{R}^N$, one gets from the comparison principle that for all $t \geq 0$ and $\|x\| \geq c^*t$:

$$u(t,x) \leq Ke^{\int_0^t \Gamma(c^*s)ds}e^{-\frac{c^*}{2}(x\cdot e - c^*t)}, \;\; \forall e \in \mathbb{S}^{N-1}.$$

This implies that for all t and $\|x\| \geq c^*t$:

$$u(t,x) \leq Ke^{\int_0^t \Gamma(c^*s)ds}e^{-\frac{c^*}{2}(\|x\| - c^*t)}.$$

The result follows. ■

3.2 Lower estimate

In this section we derive a lower estimate of the solution $u \equiv u(t,x)$ of (1). To do so, let $\eta \in (0,1)$ be given. Consider a function $\tilde{f}_\eta : [0,1] \to \mathbb{R}$ of the class C^1 such that

$$\begin{cases} \tilde{f}_\eta(u) \leq u^{-1}F(u) \; \forall u \in (0,1] \text{ and } \tilde{f}_\eta(0) = F'(0), \\ 0 < \tilde{f}_\eta(u) \leq \tilde{f}_\eta(0), \; \forall u \in [0,\eta[\text{ and } \tilde{f}_\eta\,(\eta) = 0. \end{cases} \tag{39}$$

Note that the nonlinear function $u \mapsto u\tilde{f}_\eta(u)$ satisfies the usual KPP assumptions. Then according to Aronson and Weinberger's results [3], there exists a unique (up to translation) travelling wave with minimal speed $c^*_\eta := 2\sqrt{\tilde{f}_\eta(0)} = c^*$ associated to the modified non-linearity $u \mapsto u\tilde{f}_\eta(u)$. In the sequel we denote by $\widetilde{U}_\eta$ this travelling wave, that is the unique non-increasing solution of the problem

$$\begin{cases} \widetilde{U}''_\eta(z) + c^*\widetilde{U}'_\eta(z) + \widetilde{U}_\eta(z)\tilde{f}_\eta\left(\widetilde{U}_\eta(z)\right) = 0, \ z \in \mathbb{R}, \\ \widetilde{U}_\eta(-\infty) = \eta, \ \widetilde{U}_\eta(\infty) = 0 \text{ and } \widetilde{U}_\eta(0) = \frac{\eta}{2}. \end{cases} \tag{40}$$

Using these notations, our first estimate reads as

Lemma 3.3 *Let $u \equiv u(t,x)$ be the solution of (1). There exists $\varrho > 0$ such that for each $\eta \in (0,1)$ and each $c \in (0,c^*)$, there exist $h_\eta > 0$ large enough and $t_\eta > 0$ large enough such that for all $t \geq t_\eta$ and $\|x\| \in \left[ct, c^*t + \varrho\sqrt{t}\right]$:*

$$u(t,x) \geq \alpha_c \widetilde{U}_\eta\left(\|x\| - c^*t + \frac{N+2}{c^*}\ln t + h_\eta\right).$$

Here we have set $\alpha_c = \exp\left(-\frac{1}{c}\int_0^\infty \Gamma(s)ds\right)$.

The proof of this lower estimate is based on the following claim coupled together with the construction of a suitable sub-solution involving function $\widetilde{U}_\eta$. First we claim that:

Claim 3.4 *There exists $\varrho > 0$ such that function $u \equiv u(t,x)$, the solution of (1), satisfies*

$$\liminf_{t\to\infty} t^{\frac{N+1}{2}} e^{\frac{\varrho c^* \sqrt{t}}{2}} \inf_{\|x\| = c^*t + \varrho\sqrt{t}} u(t,x) > 0.$$

Before proving this claim let us first complete the proof of Lemma 3.3.

Proof of Lemma 3.3. As mentioned above, the proof of this result relies on the construction of a suitable sub-solution involving $\widetilde{U}_\eta$. Since $\eta \in (0,1)$ is fixed, we omit the dependence with respect to η during this proof. We also fix $c \in (0,c^*)$. Let us first recall that there exists some constant $\alpha > 0$ such that the following asymptotic holds true:

$$\lim_{z\to\infty} \frac{e^{\frac{c^*z}{2}}}{z}\widetilde{U}(z+h) = \alpha e^{-\frac{c^*h}{2}}, \ \ \forall h \in \mathbb{R}. \tag{41}$$

Let us fix $c_1 > c$ such that

$$\frac{N-1}{N+2}c^* < c_1 < c^*. \tag{42}$$

Next for each $h \in \mathbb{R}$ define the function $\underline{w}^h$ by

$$\underline{w}^h(t,x) := \beta(t)\widetilde{U}\left(\|x\| - c^*t + \frac{(N+2)}{c^*}\ln(t) + h\right),$$

wherein we have set

$$\beta(t) = e^{-\int_0^t \Gamma(cs)ds}.$$

Here recall that $\widetilde{U}$ is defined as the solution of (40). Note that due to Assumption 1.1 (ii) the function u satisfies on the set $Q := \{(t,x) : \ \|x\| \geq ct\}$ the following inequality:

$$L[u](t,x) = u_t - \Delta u - F(u) + u\Gamma(ct) \geq 0.$$

Now let $h > 0$ be given. Then one has:

$$\begin{aligned} L\left[\underline{w}^h\right](t,x) =& \beta'(t)\widetilde{U}(Z) + \left(-c^* + \frac{(N+2)}{c^*t}\right)\beta(t)\widetilde{U}'(Z) - \beta(t)\widetilde{U}''(Z) \\ &- \frac{N-1}{\|x\|}\beta(t)\widetilde{U}'(Z) + \beta(t)\Gamma(ct)\widetilde{U}(Z) - F\left(\beta(t)\widetilde{U}(Z)\right), \end{aligned}$$

wherein we have set $Z = \|x\| - c^*t + \frac{(N+2)}{c^*}\ln(t) + h$. Next because of (40) and since $\beta'(t) + \Gamma(ct)\beta(t) = 0$ and $\tilde{f}(u) \leq u^{-1}F(u)$, one obtains that

$$\beta(t)^{-1}L\left[\underline{w}^h\right](t,x) \leq \left(\frac{(N+2)}{c^*t} - \frac{N-1}{\|x\|}\right)\widetilde{U}'(Z) + \widetilde{U}\left[\frac{F(\widetilde{U})}{\widetilde{U}} - \frac{F(\beta(t)\widetilde{U})}{\beta(t)\widetilde{U}}\right].$$

Since $u \mapsto u^{-1}F(u)$ is decreasing and $\beta(t) \leq 1$ one obtains that

$$\beta(t)^{-1}L\left[\underline{w}^h\right](t,x) \leq \left(\frac{(N+2)}{c^*t} - \frac{N-1}{\|x\|}\right)\widetilde{U}'(Z).$$

Recalling the definition of c_1 in (42), for each $\|x\| \geq c_1 t$ and all t large enough one has:

$$\frac{N+2}{c^*t} - \frac{N-1}{\|x\|} \geq \frac{N+2}{c^*t} - \frac{N-1}{c_1 t} \geq 0.$$

As a consequence, since $\widetilde{U}' < 0$, we infer that

$$L\left[\underline{w}^h\right](t,x) \leq 0 \leq L[u](t,x),$$

for all t large enough and $\|x\| \geq c_1 t$.

Now due to the asymptotic speed of spread recalled in Lemma 3.1 and recalling that $0 < c < c_1 < c^*$, one knows that

$$\lim_{t\to\infty} \sup_{ct\leq\|x\|\leq c_1 t} |1 - u(t,x)| = 0. \tag{43}$$

Hence since $\widetilde{U}$ is decreasing, $\underline{w}^h \leq \eta < 1$ and there exists $t_1 > 0$ large enough such that for all $h \in \mathbb{R}$

$$\sup_{\|x\|=c_1 t} \underline{w}^h(t,x) \leq \inf_{ct\leq\|x\|\leq c_1 t} u(t,x), \ \ \forall t \geq t_1.$$

Now recalling (41) as well as Claim 3.4, there exists $h_0 > 0$ large enough such that

$$\liminf_{t\to\infty} \frac{\inf_{\|x\|=ct+\varrho\sqrt{t}} u(t,x)}{\sup_{\|x\|=c^*t+\varrho\sqrt{t}} \underline{w}^h(t,x)} > 1,$$

uniformly with respect to $h > h_0$ (indeed function $\widetilde{U}$ is non-increasing so is function $\underline{w}^h(t,x)$ with respect to h). This means there exists $\widehat{t} > t_1$ large enough such that for all $t \geq \widehat{t}$ and $x \in \mathbb{R}^N$ such that $\|x\| = c^*t + \varrho\sqrt{t}$ and all $h > h_0$:

$$u(t,x) \geq \underline{w}^h(t,x).$$

Finally, since $\widetilde{U}(z) \to 0$ as $z \to \infty$, let us choose h large enough such that

$$u\left(\widehat{t},x\right) \geq \underline{w}^h\left(\widehat{t},x\right) \text{ for all } c_1\widehat{t} \leq \|x\| \leq c^*\widehat{t} + \varrho\sqrt{\widehat{t}}.$$

Then the comparison principle applies and ensures that there exists $\widehat{t} > 0$ large enough such that for each $t \geq \widehat{t}$ and each $x \in \mathbb{R}^N$ with $c_1 t \leq \|x\| \leq c^*t + \varrho\sqrt{t}$:

$$u(t,x) \geq \beta(t)\widetilde{U}\left(\|x\| - \xi(t) + h_0\right).$$

Here let us recall that $\xi(t) = c^*t - \frac{N+2}{c^*}\ln t$. Once again let us recall that $\widetilde{U} \leq \eta$. Hence because of (43) the lower estimate stated in Lemma 3.3 follows, possibly by increasing $\widehat{t}$ if necessary. ■

To conclude this lower estimate it remains to prove Claim 3.4. This proof follows some ideas of [19].

Proof of Claim 3.4. To prove this claim we shall construct a suitable radially symmetric sub-solution for u. Let $r_0 > \widehat{r}_0$ be given large enough wherein $\widehat{r}_0$ is provided by Lemma 2.3 with $c = c^*$. We shall construct such a sub-solution on the set $\|x\| \geq c^*t + r_0$ and $t > 0$. To do so let us notice that there exists some constant $\kappa > 0$ such that

$$\frac{F(u)}{u} \geq F'(0) - \kappa u, \quad \forall u \in [0,M].$$

Hence because of Assumption 1.1 (ii) the function u satisfies on the set $\|x\| \geq c^*t + r_0$:

$$u_t - \Delta u - u\left[F'(0) - \kappa u - \Gamma(c^*t + r_0)\right] \geq 0.$$

Consider the map $\widehat{u}(t,x) = u(t+1,x)$. Then it satisfies for all $t \geq 0$ and $\|x\| \geq c^*t + \tilde{r}_0$ with $\tilde{r}_0 = r_0 + c^*$:

$$L[\widehat{u}](t,x) := u_t - \Delta u - u\left[F'(0) - \kappa u - \tilde{\Gamma}(t)\right] \geq 0.$$

Here we have set $\tilde{\Gamma}(t) = \Gamma(c^*(t+1) + r_0)$. Let us chose a non-trivial radially symmetric function $v_0 \geq 0$ smooth and compactly supported such that

$$\widehat{u}(0,x) = u(1,x) \geq v_0(\|x\|), \ \forall \|x\| \geq r_0.$$

Then we look for a sub-solution of the parabolic operator L of the form

$$\underline{w}(t,x)=\alpha(t)v(t,r),\ r=\|x\|,$$

where v is a solution of the linear problem

$$\begin{cases} v_t=v_{rr}+\frac{N-1}{r}v_r+\frac{(c^*)^2}{4}v,\ r>c^*t+\tilde{r}_0,\\ v\left(t,c^*t+\tilde{r}_0\right)=0,\\ v(0,r)=v_0(r),\ r>\tilde{r}_0. \end{cases}$$

Let us first notice that due to Lemma 2.3 (ii) there exists some constant $\gamma>0$ such that

$$v(t,r)\leq \frac{\gamma}{(t+1)^{1+\frac{N}{2}}},\ \ \forall r>c^*t+\tilde{r}_0.$$

Next, recalling that $F'(0)=\frac{(c^*)^2}{4}$, one has

$$\begin{aligned} L\left[\underline{w}\right](t,x)=&\alpha'(t)v+\alpha\left[v_t-v_{rr}-\frac{N-1}{r}v_r-F'(0)v\right]+\kappa\alpha(t)^2v^2+\tilde{\Gamma}(t)v\alpha(t)\\ &\leq v\left[\alpha'(t)+\tilde{\Gamma}(t)\alpha(t)+\kappa\gamma\alpha(t)^2(1+t)^{-1-\frac{N}{2}}\right]. \end{aligned}$$

Hence let us fix $\alpha_0\in(0,1)$ and consider the map

$$\alpha(t)=e^{-\int_0^t\tilde{\Gamma}(s)ds}\pi(t)\text{ with }\pi(t)=\frac{\alpha_0}{1+2\alpha_0\frac{\kappa\gamma}{N}\left(1-(1+t)^{-N/2}\right)}.$$

Note that one has $\pi'(t)+\kappa\gamma\pi(t)^2(1+t)^{-1-\frac{N}{2}}=0$. Then this yields

$$\alpha'(t)+\tilde{\Gamma}(t)\alpha(t)+\kappa\gamma e^{\int_0^t\tilde{\Gamma}(s)ds}\alpha(t)^2(1+t)^{-1-\frac{N}{2}}=0.$$

Finally since $\tilde{\Gamma}\geq 0$ one concludes that

$$\alpha'(t)+\tilde{\Gamma}(t)\alpha(t)+\kappa\gamma\alpha(t)^2(1+t)^{-1-\frac{N}{2}}\leq 0.$$

On the other hand note that

$$\frac{\alpha_0e^{-\int_0^\infty\tilde{\Gamma}(s)ds}}{1+2\alpha_0\frac{\kappa\gamma}{N}}\leq\alpha(t)\leq\alpha_0\leq 1,\ \forall t\geq 0.$$

Together with such a function, one obtains that $L\left[\underline{w}\right](t,x)\leq 0$ for all $t\geq 0$, $\|x\|\geq c^*t+\tilde{r}_0$. Since $\underline{w}(t,x)=0$ for all $\|x\|=c^*t+\tilde{r}_0$ and $t\geq 0$ and $\underline{w}(0,x)\leq v_0(\|x\|)\leq u(1,x)$ for all $\|x\|\geq\tilde{r}_0$, the comparison principle applies and provides that

$$\underline{w}(t,x)\leq u(t+1,x),\ \ \forall t\geq 0,\ \|x\|\geq c^*t+\tilde{r}_0.$$

Hence

$$\frac{\alpha_0e^{-\int_0^\infty\tilde{\Gamma}(s)ds}}{1+2\alpha_0\frac{\kappa\gamma}{N}}v(t,\|x\|)\leq u(t+1,x),\ \ \forall t\geq 0,\ \|x\|\geq c^*t+\tilde{r}_0.$$

Then using Lemma 2.1 with $\delta=0$, there exists $\varrho>0$ such that

$$\liminf_{t\to\infty}t^{\frac{N+1}{2}}e^{\frac{\varrho c^*\sqrt{t}}{2}}v\left(t,c^*t+\varrho\sqrt{t}\right)>0.$$

Finally coupling the two above estimates completes the proof of Claim 3.4. ■

3.3 Upper estimates at the leading edge

The aim of this section is to prove an accurate upper estimate of the function u at the leading edge of the propagation. We shall more precisely prove the following result:

Lemma 3.5 *Let $u \equiv u(t,x)$ be the solution of (1). Set $\xi(t) = c^* t - \frac{N+2}{c^*} \ln \frac{t+T}{T}$ with $T > 0$ is fixed so that $\xi'(t) > 0$ for all $t \geq 0$. Then there exist some constant $\alpha > 0$, $\beta > 0$, $\tilde{r} > 0$ and a time $\tilde{t} > 0$ large enough such that for all $t \geq \tilde{t}$ and all $\|x\| \in \left[\tilde{r} + \xi(t), \xi(t) + \sqrt{t}\right]$*

$$u(t,x) \leq \alpha \left(\|x\| - \xi(t)\right) e^{-\frac{c}{2}(\|x\| - \xi(t))} \left[1 + \beta t^{-\frac{1}{4}}\right].$$

Proof. Let $\widehat{r} > \widehat{r}_0$ be given. Here $\widehat{r}_0$ is the value provided by Lemma 2.3 with $c = c^*$. Recalling that u_0 is compactly supported and $u_0 \leq 1$, consider $A > 0$ large enough such that

$$u_0(x) \leq v_0(\|x\|) := \chi_{[\widehat{r}, \widehat{r}+A]}(\|x\|), \quad \forall \|x\| \geq \widehat{r}. \tag{44}$$

We have denoted by χ_I, the characteristic function of the interval $I \subset \mathbb{R}$.
Next consider the linear problem

$$\begin{cases} v_t = v_{rr} + \frac{N-1}{r} v_r + \frac{(c^*)^2}{4} v, \ t > 0, \ r > \xi(t) + \widehat{r}, \\ v(t, \xi(t) + \widehat{r}) = 0, \ t > 0 \\ v(0,r) = v_0(r), \ r > \widehat{r}. \end{cases} \tag{45}$$

Note that due to Lemma 2.1 and 2.3 one has:

$$0 < \inf_{t \geq 0} v\left(t, \xi(t) + \widehat{r} + \frac{A}{2}\right) \leq \sup_{t \geq 0} v\left(t, \xi(t) + \widehat{r} + \frac{A}{2}\right) < \infty.$$

Recalling the definition of M in Assumption 1.1 (i), consider $B \geq 1$ large enough such that

$$M \leq Bv\left(t, \xi(t) + \widehat{r} + \frac{A}{2}\right), \quad \forall t \geq 0.$$

Next because of Assumption 1.1 (ii) and recalling that $c^* = 2\sqrt{F'(0)}$, the function u satisfies on the set $t \geq 0$ and $\|x\| \geq \xi(t) + \widehat{r} + \frac{A}{2}$ the following differential inequality:

$$L[u](t,x) := u_t - \Delta u - \left[\frac{(c^*)^2}{4} + \tilde{\Gamma}(t)\right] u \leq 0.$$

Here we have set $\tilde{\Gamma}(t) = \Gamma\left(\xi(t) + \widehat{r} + \frac{A}{2}\right)$. Next let us consider the function defined for $t \geq 0$ and $\|x\| \geq \xi(t) + \widehat{r}$ by

$$\overline{u}(t,x) = e^{\int_0^t \tilde{\Gamma}(s)ds} Bv\left(t, \|x\|\right).$$

Hence for all $t \geq 0$ and $r \geq \xi(t) + \widehat{r} + \frac{A}{2}$ one gets

$$L[u](t,x) \leq L[\overline{u}](t,x).$$

Moreover one has

$$\sup_{\|x\|=\xi(t)+\widehat{r}+\frac{A}{2}} u(t,x) \leq M \leq e^{\int_0^t \tilde{\Gamma}(s)ds} Bv\left(t, \xi(t) + \widehat{r} + \frac{A}{2}\right), \quad \forall t \geq 0,$$

and due to the choice of A in (44) and since $B \geq 1$, one has:

$$u_0(x) \leq Bv_0(\|x\|), \quad \forall \|x\| \geq \widehat{r} + \frac{A}{2}.$$

Thus the comparison principle applies and ensures that

$$u(t,x) \leq e^{\int_0^t \tilde{\Gamma}(s)ds} Bv(t, \|x\|), \quad \forall t \geq 0, \ \forall \|x\| \geq \xi(t) + \widehat{r} + \frac{A}{2}.$$

The upper estimate stated in Lemma 3.5 follows from the upper estimate provided by Lemma 2.3 (i) remembering that $\tilde{\Gamma} \in L^1(0,\infty)$. ■

3.4 Proof of Theorem 1.2

In this section we shall complete the proof of Theorem 1.2. For that purpose let $c \in (0, c^*)$ be given and fixed. Consider $\eta \in (0,1)$. Then applying Lemma 3.3 with this value of η one obtains that there exists $t_\eta > 0$ large enough and $k_\eta > 0$ small enough such that

$$u(t,x) \geq k_\eta, \quad \forall t \geq t_\eta, \forall \|x\| \in \left[\frac{c}{2}t, c^*t - \frac{N+2}{c^*}\ln t\right]. \tag{46}$$

From this estimate we shall derive that for each $\varepsilon > 0$ small enough there exist $t_\varepsilon > 0$ and $h_\varepsilon > 0$ large enough such that

$$1 + \varepsilon \geq u(t,x) \geq 1 - \varepsilon, \quad \forall t \geq t_\varepsilon, \forall \|x\| \in \left[ct, c^*t - \frac{N+2}{c^*}\ln t - h_\varepsilon\right]. \tag{47}$$

To prove (47) let us first observe that estimate (5) applies and ensures that the above upper bound is satisfied.
Next to prove the lower estimate we argue by contradiction by assuming that there exists $\varepsilon_0 \in (0,1)$ and three sequences $\{t_n\}_{n\geq 0}$ tending to infinity; $\{h_n\}_{n\geq 0}$ tending to infinity and $x_n \in \mathbb{R}^N$ such that

$$(c^* - c)\,t_n - \frac{N+2}{c^*}\ln t_n - h_n \to \infty,$$

such that $\|x_n\| \in \left[ct_n, c^*t_n - \frac{N+2}{c^*}\ln t_n - h_n\right]$ and

$$u(t_n, x_n) \leq \varepsilon_0, \quad \forall n \geq 0.$$

Next consider the sequence $u^n(t,x) := u(t+t_n, x+x_n)$ that converges, possibly up to a subsequence to $u^\infty \equiv u^\infty(t,x)$ such that, due to (46),

$$\begin{aligned}
&u_\infty(t,x) \geq k_\eta,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\\
&u_t^\infty - \Delta u^\infty = F\left(u^\infty\right),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N,\\
&u^\infty(0,0) \leq \varepsilon_0 < 1.
\end{aligned}$$

The two first conditions imply that $u^\infty(t,x) \equiv 1$ that contradicts the third normalisation condition. Thus (47) holds true.

Now it remains to prove the estimate for the leading edge. According to Lemma 3.5, one concludes that for each $\varepsilon > 0$ small enough, there exist some constant $h_\varepsilon > 0$ large enough and a time $t_\varepsilon > 0$ large enough such that

$$\sup_{\|x\|\in\left[\xi(t)+h_\varepsilon,\xi(t)+\sqrt{t}\right]} u(t,x) \leq \varepsilon,\ \ \forall t \geq t_\varepsilon.$$

Hence to complete the proof of Theorem 1.2, it is sufficient to show that

$$\lim_{t\to\infty} \sup_{\|x\|\geq\xi(t)+\sqrt{t}} u(t,x) = 0.$$

Since $\xi(t) + \sqrt{t} >> c^*t + \frac{1}{2}\sqrt{t}$, the above convergence follows from Lemma 3.2. This completes the proof of Theorem 1.2.

Remark 3.6 *From this proof and especially from Lemma 3.3 and Lemma 3.5, one has derived an important estimate of the leading edge of the solution that will be used to derive our asymptotic result in the next section. To be more precise, if $x \mapsto u_0(x) \in [0,1]$ is a non-trivial and compactly supported initial datum then the associated solution $u(t,x)$ of (1) satisfies the following estimate at the leading edge: there exists $\tilde{t} > 0$ large enough and some constants $0 < \alpha < \beta$ and $\tilde{r} > 0$ large enough such that for all $t \geq \tilde{t}$ and all $x \in \mathbb{R}^N$ such that $\|x\| \in \left[\xi(t)+\tilde{r}, \xi(t)+\sqrt{t}\right]$*

$$\alpha\left(\|x\|-\xi(t)\right)e^{-\frac{c^*}{2}(\|x\|-\xi(t))} \leq u\left(t,x\right) \leq \beta\left(\|x\|-\xi(t)\right)e^{-\frac{c^*}{2}(\|x\|-\xi(t))},$$

wherein $\xi(t) = c^*t - \frac{N+2}{c^*}\ln t$.

4 Proof of Theorem 1.3

The aim of this section is to prove Theorem 1.3. As recalled in the introduction, because the asymptotic spreading speed (see Lemma 3.1 and Lemma 3.2), one can defined for each direction $e \in \mathbb{S}^{N-1}$ the following directional spreading radius

$$m(t;e) = \sup\left\{r > 0;\ u(t,re) = \frac{1}{2}\right\}.$$

Next according to Theorem 1.2 the following asymptotic holds true:

$$m(t;e) = \xi(t) + a(t,e),$$

where $a(t;e)$ is a bounded function uniformly with respect to time large enough and to the direction $e \in \mathbb{S}^{N-1}$. Here recall that we have defined $\xi(t) = c^*t - \frac{N+2}{c^*}\ln t$.

Now in view of Theorem 1.2, in order to prove Theorem 1.3, it is sufficient to prove the following lemma:

Lemma 4.1 *Let $u \equiv u(t,x)$ be the solution of (1)-(2). Let $e \in \mathbb{S}^{N-1}$ be a given direction. For each $K > 0$ and $h > 0$ consider the bounded cylinder $\Theta_{K,h}(e)$ defined by*

$$\Theta_{K,h}(e) = \left\{x \in \mathbb{R}^N : \ -h \le x \cdot e \le h \text{ and } \|\tilde{x}\| \le K \text{ with } \tilde{x} := x - (x \cdot e)\, e\right\}.$$

Then, for each $K > 0$ and $h > 0$, the following convergence holds true

$$\lim_{t\to\infty} \sup_{\substack{x\in\Theta_{K,h}(e)\\ e\in\mathbb{S}^{N-1}}} |u\,(t, x + \xi(t)e + a(t;e)e) - U\,(x \cdot e)| = 0.$$

Proof. In order to prove this result, let us argue by contradiction by assuming that Lemma 4.1 does not holds true. This means that there exist $\varepsilon > 0$, a sequence $\{t_n\}_{n\ge0}$, a sequence $\{e_n\}_{n\ge0} \subset \mathbb{S}^{N-1}$ and $x_n \in \Theta_{K,h}(e_n)$ such that $t_n \to \infty$ and

$$|u\,(t_n, x_n + \xi(t_n)e_n + a_n e_n) - U\,(x_n \cdot e_n)| \ge \varepsilon, \ \forall n \ge 0. \tag{48}$$

Here we have set $a_n = a(t_n, e_n)$. Up to sub-sequence one may assume that

$$\lim_{n\to\infty} e_n = e_\infty \in \mathbb{S}^{N-1} \text{ and } \lim_{n\to\infty} x_n = x_\infty \in \Theta_{K,h}\,(e_\infty)\,.$$

Let us now consider the sequence of maps

$$u_n(t,x) = u\,(t + t_n, x + \xi(t_n)e_n + a_n e_n)\,.$$

Then due to parabolic regularity, possibly along a subsequence, one may assume that $u_n(t,x) \to u_\infty(t,x)$ locally uniformly in $\mathbb{R} \times \mathbb{R}^N$ where u_∞ is an entire solution of (3) such that

$$0 \le u_\infty(t,x) \le 1 \text{ and } u_\infty(0,0) = \frac{1}{2},$$

while (48) ensures that

$$|u_\infty\,(0, x_\infty) - U\,(x_\infty \cdot e_\infty)| \ge \varepsilon. \tag{49}$$

Now we claim that

Claim 4.2 *Function u_∞ satisfies*

$$\lim_{A\to-\infty} \inf_{\substack{x\in\mathbb{R}^N\\ x.e_\infty - c^*t \le -A}} u_\infty(t,x) = 1. \tag{50}$$

Proof. To prove the above claim, note that for each $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ one has for all n large enough:

$$\begin{aligned}
&\|x + (\xi(t_n) + a_n)\, e_n\| - \xi(t + t_n) \\
&= \|x - c^* t e_n + \left(\xi(t + t_n) + O\left(t_n^{-1}\right) + a_n\right) e_n\| - \xi(t + t_n) \\
&= x \cdot e_n - c^* t + a_n + O\left(t_n^{-1}\right).
\end{aligned}$$

Let $\varepsilon > 0$ be given. Recalling that the sequence $\{a_n\}_{n \geq 0}$ is bounded, let $A > 0$ be given large enough such that

$$-\frac{A}{2} + \limsup_{n \to \infty} a_n < -2h_\varepsilon.$$

Here $h_\varepsilon > 0$ is defined in Theorem 1.2. Let $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ be given such that $x.e_\infty \leq c^* t - A$. Then because of the above expansion, one obtains that for all n large enough:

$$\|x + (\xi(t_n) + a_n)\, e_n\| \leq \xi(t + t_n) - h_\varepsilon.$$

Hence the inner part of Theorem 1.2 applies and ensures that

$$u\left(t + t_n, x + \xi(t_n) e_n + a_n e_n\right) \geq 1 - \varepsilon.$$

Letting $n \to \infty$ provides $u_\infty(t, x) \geq 1 - \varepsilon$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ such that $x \cdot e_\infty - c^* t \leq -A$. Hence Claim 4.2 holds true. ■

Now we claim that

Claim 4.3 *There exist some constants $\alpha < \beta$ and $\widehat{r} > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $x \cdot e_\infty \geq \widehat{r}$:*

$$\alpha x \cdot e_\infty e^{-\frac{c^*}{2} x \cdot e_\infty} \leq u_\infty(t, c^* t e_\infty + x) \leq \beta x \cdot e_\infty e^{-\frac{c^*}{2} x \cdot e_\infty}. \tag{51}$$

Proof. Let $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ be given. Then note that one has

$$\begin{aligned}
&\|x + c^* t e_n + \xi(t_n) e_n + a_n e_n\| - \xi(t + t_n) \\
&= \sqrt{\left(x \cdot e_n + c^* t + \xi(t_n) + a_n\right)^2 + \|x - (x \cdot e_n)\, e_n\|^2} - \xi(t + t_n) \\
&= x \cdot e_n + a_n + O\left(t_n^{-1}\right).
\end{aligned}$$

Recalling the definition of $\tilde{r} > 0$ in Remark 3.6, for each $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ such that $x \cdot e_\infty > \widehat{r} := \tilde{r} - \liminf_{n \to \infty} a_n$ one has for all n large enough

$$\|x + c^* t e_n + \xi(t_n) e_n + a_n e_n\| \in \left[\xi(t + t_n) + \tilde{r}, \xi(t + t_n) + \sqrt{t + t_n}\right].$$

Hence the estimates of Remark 3.6 apply and yields

$$A\left(x \cdot e_\infty + a^-\right) e^{-\frac{c^*}{2} x \cdot e_\infty} \leq u_\infty(t, c^* t e_\infty + x) \leq B\left(x \cdot e_\infty + a^+\right) e^{-\frac{c^*}{2} x \cdot e_\infty},$$

wherein we have set $a^- = \liminf a_n$ and $a^+ = \limsup a_n$ and

$$A = \alpha e^{\frac{-c^* a^+}{2}} \text{ and } B = \beta e^{-\frac{c^* a^-}{2}}.$$

Hence increasing $\widehat{r}$ and modifying A and B completes the proof Claim 4.3. ■

We now complete the proof of Theorem 1.3 and let us recall that $F'(1) < 0$ is assumed here. To do so, let us state the following lemma:

Lemma 4.4 *Let u_∞ be an entire solution of* (3) *satisfying $u_\infty(t,x) \in [0,1]$ for all $(t,x) \in \mathbb{R}^2$ and such that* (50) *and* (51) *holds true. Then there exists $h \in \mathbb{R}$ such that*

$$u_\infty(t,x) \equiv U\left(x \cdot e_\infty - c^* t + h\right),$$

where U is the travelling front defined as the solution of (7)*.*

This proof of this lemma is similar to the one of Lemma 8.2 in [19] making use of Theorem 3.5 in [4].

Now due to Claim 4.2 and Claim 4.3, one concludes that there exists some constant $h \in \mathbb{R}$ such that

$$u_\infty(t,x) \equiv U(x \cdot e_\infty - c^* t + h),$$

Now recalling that $u_\infty(0,0) = \frac{1}{2}$ and that $U(0) = \frac{1}{2}$, one obtains that $h = 0$. This leads us to $u_\infty(0, x_\infty) = U(x_\infty \cdot e_\infty)$. This contradicts (49) and completes the proof of Theorem 1.3. ■

5 Proof of Theorem 1.6

This section is devoted to the proof of Theorem 1.6. The arguments we shall use in this section are very close to the ones developed in Section 3 by taking into account the non-integrable perturbation of the equation (see (8)). Before dealing with lower and upper estimates of the solution let us first state that the solution of (8) enjoys the following asymptotic speed of spread properties

Lemma 5.1 *Let $\lambda \in \mathbb{R}$ be given. Let $u \equiv u(t,x)$ be the solution of* (8)*. It satisfies the following properties:*

(i) For each $0 < c < c' < c^*$ one has

$$\lim_{t\to\infty} \sup_{ct \le \|x\| \le c't} |1 - u(t,x)| = 0.$$

(ii) The function u satisfies for each $\sigma > 0$:

$$\lim_{t\to\infty} \sup_{\|x\| \ge c^* t + \sigma\sqrt{t}} u(t,x) = 0.$$

Proof. The first item follows from Lemma 3.1. It remains to prove (ii). To this aim, note that, due to KPP assumption, the function u satisfies on the set $\|x\| \ge c^* t$:

$$L[u](t,x) := u_t - \Delta u - u\left[F'(0) + \frac{|\lambda|}{1 + c^* t}\right] \le 0.$$

Let $e \in \mathbb{S}^{N-1}$ be given. Consider, for some given constant $K > 0$, the map

$$\overline{u}(t,x) = Ke^{\frac{|\lambda|}{c^*}\ln(1+c^*t)}e^{-\frac{c^*}{2}(x\cdot e - c^*t)}.$$

Then since $L[\overline{u}](t,x) = 0$, if we choose $K > 0$ sufficient large, one gets from the comparison principle that for all $t \geq 0$ and $\|x\| \geq c^*t$:

$$u(t,x) \leq K(1+c^*t)^{\frac{|\lambda|}{c^*}}e^{-\frac{c^*}{2}(\|x\|-c^*t)},$$

and the result follows. ∎

We shall now focus on deriving a lower estimate of the solution $u \equiv u(t,x)$ of (8).
From now on, we assume that $\lambda \geq 0$ is given and fixed. First note that, due to this constraint, the solution $u = u(t,x)$ of (8) satisfies

$$0 < u(t,x) < 1, \ \ \forall t > 0, \ x \in \mathbb{R}^N.$$

Let $\eta \in (0,1)$ be given. Consider a function $\tilde{f}_\eta : [0,1] \to \mathbb{R}$ of the class C^1 as defined in (39) and assume furthermore that there exists $\kappa > 0$ such that

$$\tilde{f}_\eta(u) - \frac{F(u)}{u} \leq -\kappa u, \ \forall u \in (0,1]. \tag{52}$$

Using these notations, our first estimate reads as

Lemma 5.2 *Let $u \equiv u(t,x)$ be the solution of (8). There exists $\varrho > 0$ such that for each $\eta \in (0,1)$ and each $c \in (0,c^*)$, there exist $h_\eta > 0$ large enough and $t_\eta > 0$ large enough such that for all $t \geq t_\eta$ and $\|x\| \in \left[ct, c^*t + \varrho\sqrt{t}\right]$:*

$$u(t,x) \geq \widetilde{U}_\eta\left(\|x\| - c^*t + \left(\frac{N+2}{c^*} + \frac{\lambda}{(c^*)^2}\right)\ln t + h_\eta\right),$$

wherein $\widetilde{U}_\eta$ is defined in (40).

The proof of this lower estimate follows from some similar arguments as in the proof of Lemma 3.3. Since the perturbation is now non-integrable, some technical changes are needed. The proof of this lemma is thus detailed in the following.
Similarly as in the proof of Lemma 3.3 we first claim that:

Claim 5.3 *There exists $\varrho_0 > 0$ such that function $u \equiv u(t,x)$, the solution of (8), satisfies for each $\varrho \in (0,\varrho_0)$:*

$$\liminf_{t\to\infty} t^{\frac{N+1}{2}+\frac{\lambda}{c^*}}e^{\frac{\varrho c^*\sqrt{t}}{2}} \inf_{\|x\|=c^*t+\varrho\sqrt{t}} u(t,x) > 0.$$

Before proving this claim let us first complete the proof of Lemma 5.2.
Proof of Lemma 5.2. Let $\eta \in (0,1)$ be given and fixed. For notational simplicity we omit the dependence with respect to η during this proof. We also

fix $c \in (0, c^*)$. In addition to the behaviour (41) for $\widetilde{U}$, let us also recall the following behaviour:

$$\lim_{z\to\infty} \frac{\widetilde{U}'(z)}{\widetilde{U}(z)} = -\frac{c^*}{2}. \tag{53}$$

Recalling that $\lambda \geq 0$, let us fix $c_1 > c$ such that

$$\frac{N-1}{N+2}c^* \leq \frac{2\lambda + c^*(N-1)}{2\lambda + c^*(N+2)}c^* < c_1 < c^*. \tag{54}$$

Next for each $h \in \mathbb{R}$ let us define

$$\underline{w}^h(t,x) := \widetilde{U}\left(\|x\| - c^*t + \left(\frac{(N+2)}{c^*} + \frac{2\lambda}{(c^*)^2}\right)\ln(t) + h\right),$$

Here recall that $\widetilde{U}$ is defined as the solution of (40). Next the function u satisfies

$$L[u](t,x) \geq 0 \text{ with } L[u] := u_t - \Delta u - u\left[\frac{F(u)}{u} - \frac{\lambda}{1+\|x\|}\right].$$

Next we claim that:

Claim 5.4 *There exists $\widehat{t} > 0$ large enough such that for all $h \in \mathbb{R}$ one has*

$$L\left[\underline{w}^h\right](t,x) \leq 0, \quad \forall t \geq \widehat{t},\ \|x\| \geq c_1 t.$$

Proof of Claim 5.4. Let $h \in \mathbb{R}$ be given. Then one has:

$$\begin{aligned} L\left[\underline{w}^h\right](t,x) = &\left[\frac{(N+2)}{c^*t} + \frac{2\lambda}{(c^*)^2 t} - \frac{N-1}{\|x\|}\right]\widetilde{U}'(Z) \\ &+ \widetilde{U}(Z)\left[\tilde{f}(\widetilde{U}(Z)) - \frac{F\left(\widetilde{U}(Z)\right)}{\widetilde{U}(Z)}\right] + \frac{\lambda}{1+\|x\|}\widetilde{U}(Z), \end{aligned}$$

wherein we have set $Z = \|x\| - c^*t + \left[\frac{(N+2)}{c^*} - \frac{2\lambda}{(c^*)^2}\right]\ln(t) + h$.
Since $\widetilde{U}' < 0$ and $\|x\| \geq c_1 t$ one obtains using (52) that:

$$\widetilde{U}(Z)^{-1} L\left[\underline{w}^h\right](t,x) \leq \left[\frac{(N+2)}{c^*t} + \frac{2\lambda}{(c^*)^2 t} - \frac{N-1}{c_1 t}\right]\frac{\widetilde{U}'(Z)}{\widetilde{U}(Z)} - \kappa\widetilde{U}(Z) + \frac{\lambda}{1+c_1 t}.$$

Before completing the proof of Claim 5.4, let us first notice that due to (54) one has:

$$\frac{(N+2)}{c^*} + \frac{2\lambda}{(c^*)^2} - \frac{N-1}{c_1} > 0.$$

We now split the estimate into three zones: $Z \leq 1$, $1 \leq Z \leq \alpha \ln t$ and $Z \geq \alpha \ln t$ for some $\alpha > 0$ that will chosen latter on.
Since $\widetilde{U}' < 0$, for all $Z \leq 1$ one has

$$\widetilde{U}(Z)^{-1} L\left[\underline{w}^h\right](t,x) \leq -\kappa\widetilde{U}(1) + \frac{\lambda}{1+c_1 t}.$$

Hence there exists $t_1 > 0$ large enough such that $L\left[\underline{w}^h\right](t,x) \leq 0$ for all $t \geq t_1$, $\|x\| \geq c_1 t$ and $Z \leq 1$.
Next note that there exists $\beta > 0$ such that

$$\widetilde{U}(z) \geq \beta e^{-\frac{c^*}{2}z}, \ \ \forall z \geq 1$$

Hence for each $Z \geq 1$, one obtains:

$$\widetilde{U}(Z)^{-1} L\left[\underline{w}^h\right](t,x) \leq -\kappa\beta e^{-\frac{c^*}{2}Z} + \frac{\lambda}{1+c_1 t}.$$

Hence for each $1 \leq Z \leq \frac{1}{c^*}\ln t$, one obtains

$$L\left[\underline{w}^h\right](t,x) \leq -\kappa\beta t^{-1/2} + \frac{\lambda}{1+c_1 t}.$$

and there exists $t_2 \geq t_1$ such that $L\left[\underline{w}^h\right](t,x) \leq 0$ for all $t \geq t_2$, $\|x\| \geq c_1 t$ and $1 \leq Z \leq \frac{1}{c^*}\ln t$.
Next recalling (53), for each $\varepsilon > 0$, there exists $t_\varepsilon \geq t_2$ such that for all $t \geq t_\varepsilon$, $\|x\| \geq c_1 t$ and $Z \geq \frac{1}{c^*}\ln t$:

$$t\widetilde{U}(Z)^{-1} L\left[\underline{w}^h\right](t,x) \leq \left(\varepsilon - \frac{c^*}{2}\right)\left[\frac{(N+2)}{c^*} + \frac{2\lambda}{(c^*)^2} - \frac{N-1}{c_1}\right] + \frac{\lambda}{c_1} + \varepsilon,$$

The result follows by recalling that due to the choice of c_1 in (54), one has:

$$-\frac{c^*}{2}\left[\frac{(N+2)}{c^*} + \frac{2\lambda}{(c^*)^2} - \frac{N-1}{c_1}\right] + \frac{\lambda}{c_1} < 0.$$

This completes the proof of Claim 5.4. ■

We are now able to complete the proof Lemma 5.2. Indeed note that due to Lemma 5.1 (i) and recalling that $0 < c < c_1 < c^*$, one knows that

$$\lim_{t\to\infty} \sup_{ct \leq \|x\| \leq c_1 t} |1 - u(t,x)| = 0. \tag{55}$$

Hence since $\widetilde{U}$ is decreasing, $\underline{w}^h \leq \eta < 1$ and there exists $t_1 > \widehat{t}$ large enough such that for all $h \in \mathbb{R}$

$$\sup_{\|x\| = c_1 t} \underline{w}^h(t,x) \leq \inf_{ct \leq \|x\| \leq c_1 t} u(t,x), \ \ \forall t \geq t_1.$$

Now recalling (41) as well as Claim 5.3, there exists $h_0 > 0$ large enough such that

$$\liminf_{t\to\infty} \frac{\inf_{\|x\| = c^t + \varrho\sqrt{t}} u(t,x)}{\sup_{\|x\| = c^* t + \varrho\sqrt{t}} \underline{w}^h(t,x)} > 1,$$

uniformly with respect to $h > h_0$ (indeed function $\widetilde{U}$ is non-increasing so is function $\underline{w}^h(t,x)$ with respect to h). This means there exists $t_2 > t_1$ large enough such that

$$u(t,x) \geq \underline{w}^h(t,x), \ \forall t \geq t_2, \ \forall h > h_0, \ \forall \|x\| = c^* t + \varrho\sqrt{t}.$$

Finally, since $\widetilde{U}(z) \to 0$ as $z \to \infty$, let us choose h large enough such that

$$u\left(\widehat{t}, x\right) \geq \underline{w}^h\left(\widehat{t}, x\right) \text{ for all } c_1 t_2 \leq \|x\| \leq c^* t_2 + \varrho\sqrt{t_2}.$$

Then recalling Claim 5.4, the comparison principle applies and ensures that there exists $\tilde{t} > 0$ large enough such that

$$u(t,x) \geq \underline{w}^{h_0}(t,x), \ \ \forall t \geq \tilde{t}, \ \forall c_1 t \leq \|x\| \leq c^* t + \varrho\sqrt{t}.$$

Once again recall that $\widetilde{U} \leq \eta$. Hence because of (55) the lower estimate of Lemma 5.2 follows, possibly by increasing $\tilde{t}$ if necessary. ■

To conclude this lower estimate it remains to prove Claim 5.3.

Proof of Claim 5.3. The proof of this claim follows similar ideas and arguments as the ones in the proof of Claim 3.4. We shall construct a suitable radially symmetric sub-solution for u. Let $r_0 > \widehat{r}_0$ be given large enough wherein $\widehat{r}_0$ is provided by Lemma 2.3 with $c = c^*$. We shall construct such a sub-solution on the set $\|x\| \geq c^* t + r_0$ and $t > 0$. To this aim, let us notice that there exists some constant $\theta > 0$ such that

$$\frac{F(u)}{u} \geq F'(0) - \theta u, \ \ \forall u \in (0,1].$$

Hence the function u satisfies on the set $\|x\| \geq c^* t + r_0$:

$$u_t - \Delta u - u\left[F'(0) - \theta u - \frac{\lambda}{1 + \|x\|}\right] \geq 0.$$

Consider the map $\widehat{u}(t,x) = u(t+1,x)$. Then it satisfies for all $t \geq 0$ and $\|x\| \geq c^* t + \tilde{r}_0$ with $\tilde{r}_0 = r_0 + c^*$:

$$L[\widehat{u}](t,x) := \widehat{u}_t - \Delta\widehat{u} - \widehat{u}\left[F'(0) - \theta\widehat{u} - \frac{\lambda}{1 + \|x\|}\right] \geq 0.$$

Let us chose a non-trivial radially symmetric function $v_0 \geq 0$ smooth and compactly supported such that

$$\widehat{u}(0,x) = u(1,x) \geq v_0(\|x\|), \ \forall \|x\| \geq r_0.$$

Then we look for a sub-solution of the parabolic operator L of the form

$$\underline{w}(t,x) = \alpha(t)v(t,r), \ r = \|x\|,$$

where v is a solution of the linear problem

$$\begin{cases} v_t = v_{rr} + \frac{N-1}{r}v_r + \left[\frac{(c^*)^2}{4} - \frac{\lambda}{1+r}\right]v, \ r > c^* t + \tilde{r}_0, \\ v\left(t, c^* t + \tilde{r}_0\right) = 0, \\ v(0,r) = v_0(r), \ r > \tilde{r}_0. \end{cases}$$

Let us first notice that due to Lemma 2.3 (ii) there exists some constant $\gamma > 0$ such that

$$v(t,r) \leq \gamma(t+1)^{-\frac{\lambda}{c^*}-1-\frac{N}{2}}, \quad \forall r > c^* t + \tilde{r}_0.$$

Next, recalling that $F'(0) = \frac{(c^*)^2}{4}$, one has

$$\begin{aligned} L\left[\underline{w}\right](t,x) =& \alpha'(t)v + \alpha\left[v_t - v_{rr} - \frac{N-1}{r}v_r - F'(0)v\right] + \theta\alpha(t)^2 v^2 \\ &\leq v\left[\alpha'(t) + \theta\gamma\alpha(t)^2(1+t)^{-\frac{\lambda}{c^*}-1-\frac{N}{2}}\right]. \end{aligned}$$

Hence let us fix $\alpha_0 \in (0,1)$ and consider the map

$$\alpha(t) = \frac{\alpha_0}{1 + \alpha_0 \frac{\theta\gamma}{\frac{N}{2}+\frac{\lambda}{c^*}}\left(1 - (1+t)^{-\frac{\lambda}{c^*}-\frac{N}{2}}\right)}.$$

Since $\lambda \geq 0$, the function α is well defined in $[0,\infty)$ and satisfies $\alpha'(t) + \theta\gamma\alpha(t)^2(1+t)^{-\frac{\lambda}{c^*}-1-\frac{N}{2}} = 0$. Moreover note that one has

$$\frac{\alpha_0}{1 + \alpha_0 \frac{\theta\gamma}{\frac{N}{2}+\frac{\lambda}{c^*}}} \leq \alpha(t) \leq \alpha_0 \leq 1, \; \forall t \geq 0.$$

Together with such a function, one obtains that $L\left[\underline{w}\right](t,x) \leq 0$ for all $t \geq 0$, $\|x\| \geq c^* t + \tilde{r}_0$ and similarly to the proof of Claim 3.4, one obtains using Lemma 2.1 with $\delta = 0$, $c = c^*$ and $\lambda = -\lambda$ that there exists $\varrho_0 > 0$ such that Claim 5.3 holds true. ■

We shall now focus on deriving an upper estimate of the solution of the leading edge. Our upper estimate reads as the following lemma

Lemma 5.5 *Let $u \equiv u(t,x)$ be the solution of* (8). *Set*

$$\xi(t) = c^* t - \left[\frac{N+2}{c^*} + \frac{\lambda}{(c^*)^2}\right] \ln \frac{t+T}{T},$$

where $T > 0$ is given and fixed such that $\xi'(t) > 0$ for all $t \geq 0$. Then there exist some constants $\alpha > 0$, $\beta > 0$, $\tilde{r} > 0$ and a time $\tilde{t} > 0$ large enough such that for all $t \geq \tilde{t}$ and all $\|x\| \in \left[\tilde{r} + \xi(t), \xi(t) + \sqrt{t}\right]$

$$u(t,x) \leq \alpha\left(\|x\| - \xi(t)\right) e^{-\frac{c^*}{2}(\|x\|-\xi(t))}\left[1 + \beta t^{-\frac{1}{4}}\right].$$

The proof of this lemma is straightforward using similar argument as in the proof of Lemma 3.5 and a comparison together with the solution of the linear problem:

$$\begin{cases} v_t = v_{rr} + \frac{N-1}{r}v_r + \left[F'(0) - \frac{\lambda}{1+r}\right]v, \; t \geq 0, \; r \geq \xi(t) + \widehat{r}, \\ v(t, \xi(t) + \widehat{r}) = 0, \end{cases}$$

associated to a suitable initial datum and using the estimates derived in Section 2. The details are left to the reader.

Finally note that coupling Lemma 5.2 and 5.5 ensures the boundedness of the transition zone as well as the behaviour of the $\frac{1}{2}-$level line of the solution as stated in the first part of Theorem 1.6. The convergence part follows from the same arguments as those developed in Section 4. This completes the proof of Theorem 1.6.

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