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Travelling waves for a size and space structured model in population dynamics: Point to sustained oscillating solution connections

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A B S T R A C T

This work is devoted to the study of travelling wave solutions for some size structured model in population dynamics. The population under consideration is also spatially structured and has a nonlocal spatial reproduction. This phenomenon may model the invasion of plants within some empty landscape. Since the corresponding unspatially structured size structured models may induce oscillating dynamics due to Hopf bifurcations, the aim of this work is to prove the existence of point to sustained oscillating solution travelling waves for the spatially structured problem. From a biological view point, such solutions represent the spatial invasion of some species with spatio-temporal patterns at the place where the population is established.

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1. Introduction

We consider a population of plants that can invade a spatial area. Here the production of seeds depends on the size of these plants. When they reach some maturity size, they are able to produce some seeds that can disperse through the spatial domain and the population is able to invade the empty spatial domain. To consider this phenomenon, we will use the following model

$$\begin{cases} \dfrac{\partial u(t,s,x)}{\partial t} + \dfrac{\partial (g(s)u(t,s,x))}{\partial s} = -\mu(s)u(t,s,x), \quad \text{for } s \geqslant 0, \text{ and } x \in \mathbb{R}, \\ g(0)u(t,0,x) = \left(I - d^2\Delta_x\right)^{-1}\left(\alpha h\left(\displaystyle\int_0^\infty \gamma(\theta)u(t,\theta,\cdot)\,d\theta\right)\right)(x), \quad \text{for } x \in \mathbb{R}, \\ u(0,.,.) = u_0 \in L^1\left((0,\infty), L^1_+(\mathbb{R})\right). \end{cases} \tag{1.1}$$

The function $u(t,s,x)$ represents the population density of certain plants with respect to the size s and spatial position x at time $t > 0$, so if $x_1 \leqslant x_2$, and $s_1 \leqslant s_2$ the quantity

$$\int_{s_1}^{s_2}\int_{x_1}^{x_2} u(t,s,x)\,dx\,ds$$

denotes the number of plants with size $s \in [s_1, s_2]$, spatial location $x \in [x_1, x_2]$ and at time $t > 0$. The term $\frac{\partial(g(s)u(t,s,x))}{\partial s}$ represents the average growth rate of individuals so that function $g > 0$ describes the growth velocity. Parameter $d > 0$ describes the dispersal of seeds around the position of the individual by using a Gaussian distribution, while function $\mu \in L^\infty_{\text{loc},+}([0,\infty))$ is the size-specific natural death rate. Parameter $\alpha > 0$ and function h describe the reproduction process while function $\gamma \in L^\infty_+(0,\infty)$ represents the maturity of plants, that is the ability of the plants to reproduce.

When the initial distribution $x \to u_0(s,x)$ is spatially uniform for almost every $s \geqslant 0$, the model reduced to

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} + \dfrac{\partial (g(s)u(t,s))}{\partial s} = -\mu(s)u(t,s), \quad \text{for } s \geqslant 0, \\ g(0)u(t,0) = \alpha h\left(\displaystyle\int_0^\infty \gamma(s)u(t,s)\,ds\right), \\ u(0,.) = u_0 \in L^1_+\left((0,\infty), \mathbb{R}\right). \end{cases} \tag{1.2}$$

This problem has been recently studied by Magal and Ruan [19] (see also Chu, Ducrot, Magal and Ruan [6] for an extension of this result). The authors prove that under some conditions, Hopf bifurcation may occur around some positive equilibrium whenever α is large enough.

The aim of this work is to study some qualitative properties of travelling wave solutions for problem (1.1) and more precisely we shall pay a particular attention of the behaviour of the connexion around to the positive equilibrium. Indeed, in view of the oscillating behaviour of system (1.2) one can expect that the wave solutions of (1.1) also exhibit some sustained oscillations around the positive equilibrium. Similarly to [6], we shall assume, through this work, that the map $h : \mathbb{R} \to \mathbb{R}$ is the so-called Ricker's birth function

$$h(s) = s\exp(-s). \tag{1.3}$$

We refer for instance to [25,26] for more details.

Under some assumptions on functions g and μ that will be explained in Section 2, the travelling wave problem corresponding to (1.1) can be re-written as the following second order infinite delay differential equation, that is: find a wave speed $c > 0$ and some bounded function u satisfying the following equation

$$\frac{1}{c^2}u''(t) = u(t) - \alpha h\left(\int_0^\infty \gamma(s)u(t-s)\,ds\right), \quad t \in \mathbb{R}, \tag{1.4}$$

supplemented with the conditions

$$\begin{aligned} &u(t) \geqslant 0, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \sup_{s\in\mathbb{R}} u(s) < \infty, \\ &\lim_{t\to-\infty} u(t) = 0, \qquad \varliminf_{t\to\infty} u(t) > 0. \end{aligned} \tag{1.5}$$

This kind of nonlocal elliptic equation has been widely studied by several authors. We may refer to Ma [17,18], Trofimchuk et al. [32], Thieme and Zhao [29] (for some results on integral equations) and the references cited therein. We may also mention a lot of interest for the nonlocal logistic equations for which one may refer to Apreutesei et al. [1,2], Berestycki et al. [4] or Gourley [12].

More precise information have been obtained for some similar model than (1.4) with a single delay, that reads

$$\frac{1}{c^2}u''(t) = u'(t) + u(t) - \alpha h\big(u(t-\tau)\big), \quad t \in \mathbb{R}.$$

We refer for instance to [10,11,13,17,18,27,28,30,38] for existence results of wave solutions. We also refer to Trofimchuk et al. [31] (see also the references cited therein for other result for this kind of equation with a single delay) where the authors prove the existence of point to sustained oscillating solution connections.

The aim of this work is to study (1.4) with some infinite distributed delay and to show that, under some conditions, this problem admits some point to sustained oscillating solution connection. To reach this goal we shall specify function γ in order to reduce (1.4) to a system of delay differential equations with a single delay. A convenient form will be the following:

Assumption 1.1. The function γ takes the form

$$\gamma(s) = \begin{cases} \delta(s-\tau)^n e^{-\mu s} & \text{if } s \geqslant \tau, \\ 0 & \text{if } s < \tau, \end{cases}$$

wherein $\tau > 0$, $\mu > 0$, $n \in \mathbb{N}$ are some parameters while $\delta > 0$ is a normalization parameter so that

$$\int_0^\infty \gamma(s)\, ds = 1.$$

One may recall that this specific function has also been used in [6] in order to derive more precise information on the behaviour of (1.2).

Under this assumption, the problem under consideration, namely (1.4) together with Assumption 1.1 reduces to a system of delay differential equation with a single delay. More specifically we shall concentrate on finding a real number $c > 0$ and a componentwized positive and bounded vector valued function $(\phi, \psi_0, \dots, \psi_n) : \mathbb{R} \to \mathbb{R}^{n+2}$ satisfying the system of equations

$$\begin{aligned}
&\frac{1}{c^2}\phi''(t) = \phi(t) - \alpha h\big(\psi_n(t)\big), \quad t \in \mathbb{R},\\
&\psi_j'(t) = -\mu\psi_j(t) + \mu\psi_{j-1}(t), \quad t \in \mathbb{R},\ j = 1, \dots, n,\\
&\psi_0'(t) = -\mu\psi_0(t) + \phi(t - \tau), \quad t \in \mathbb{R},\\
&\lim_{t\to-\infty}\big(\phi(t), \psi_0(t), \dots, \psi_n(t)\big) = 0,\\
&\varliminf_{t\to\infty} \phi(t) > 0, \qquad \varliminf_{t\to\infty} \psi_k(t) > 0, \quad k = 0, \dots, n.
\end{aligned} \tag{1.6}$$

Here $c > 0$ is an unknown parameter, $\alpha > 0$, $\mu > 0$ and $n \geqslant 0$ while function h is defined in (1.3). Let us notice that up to change c and τ respectively by $\frac{c}{\mu}$ and $\mu\tau$, one may assume that $\mu = 1$ and we shall consider the following system of equations

$$\begin{aligned}
&\frac{1}{c^2}\phi''(t) = \phi(t) - \alpha\psi_n(t)e^{\psi_n(t)}, \quad t \in \mathbb{R},\\
&\psi_j'(t) = -\psi_j(t) + \psi_{j-1}(t), \quad t \in \mathbb{R},\ j = 1, \dots, n,\\
&\psi_0'(t) = -\psi_0(t) + \phi(t - \tau), \quad t \in \mathbb{R},\\
&\lim_{t\to-\infty}\big(\phi(t), \psi_0(t), \dots, \psi_n(t)\big) = 0,\\
&\varliminf_{t\to\infty} \phi(t) > 0, \qquad \varliminf_{t\to\infty} \psi_k(t) > 0, \quad k = 0, \dots, n.
\end{aligned} \tag{1.7}$$

While the existence of such solution can be found in the literature (see for instance [9,17,18,27]), the behaviour of the wave solutions when $t \to \infty$ remains unknown for a large class of parameters α and τ. Let us also mention the recent work of Fang and Zhao that deals with uniqueness result of non-monotone wave solutions. Let us first recall some known results on this problem (more detailed results are recalled in Section 4.1 for the sake of completeness). When $\alpha \in (1, e^2]$ then any solution $(\phi, \psi_1, \dots, \psi_n)$ with a wave speed $c > 0$ of system (1.7) converges to some positive equilibrium point when $t \to \infty$. More complex situations may occur when $\alpha > e^2$. Here we summarize in the following result some information on system (1.7).

Theorem 1.2. *Let $\alpha > e^2$ be given such that*

$$\alpha^3 \exp\left(-\frac{\alpha}{e} - \alpha^2 e^{-\frac{\alpha}{e}}\right) > \ln\alpha.$$

Let $n \geqslant 0$ be given. For each $\tau > 0$, let $c^*(\tau) = c^*(\alpha, \tau, n) > 0$ denotes the minimal speed of system (1.7) (*see* Theorem 4.5). *Then the following hold true:*

(i) *Let $\tau > 0$ be given. Let $c > c^*(\tau)$ and $(\phi, \psi_1, \ldots, \psi_n)$ be a solution of* (1.7) *that is non-eventually monotone when $t \to \infty$. Then the map $t \to \phi(t)$ oscillates* (*damped or undamped oscillations*) *around* $\ln\alpha$. *Moreover there exists $\widehat{t} \in \mathbb{R}$ large enough such that for any $t > \widehat{t}$ the equation $\phi(s) = \ln\alpha$ with $s \in [t-\tau, t]$ has at most two solutions.*
As a consequence, if ϕ is a non-converging function when $t \to \infty$ then ϕ exhibits some undamped oscillations around $\ln\alpha$ *when $t \to \infty$.*

(ii) *For each $\lambda > 1$ there exists $\tau^* = \tau^*(n, \alpha, \lambda) > 0$ such that for each $\tau > \tau^*$ and each solution $(\phi, \psi_1, \ldots, \psi_n)$ of* (1.7) *with parameter τ and wave speed $c > \max(c^*(\tau), \frac{\lambda}{\tau}\frac{\pi}{\sqrt{\ln\alpha - 2}})$, the map $t \to \phi(t)$ does not converge when $t \to \infty$ and has undamped oscillations around the equilibrium* $\ln\alpha$ *when $t \to \infty$.*

(iii) *Assume that $e < \alpha e^{-\frac{\alpha}{e}}$. Let $\tau > 0$ be given. Let $c > c^*(\tau)$ and $(\phi, \psi_1, \ldots, \psi_n)$ be a solution of* (1.7)*. Then if ϕ does not converge when $t \to \infty$ then the oscillations of function $t \to \phi(t)$ are ultimately periodic when $t \to \infty$.*

Remark 1.3. Let us notice that under the assumptions of (ii), it remains difficult to prove that the solutions are non-converging for any admissible wave speed, that is for any $c > c^*$. Indeed, for the model under consideration, one can expect to have $c^*(\alpha, \tau, n) \sim \frac{C(\alpha,n)}{\tau}$ when $\tau \to \infty$ for some constant $C(\alpha, n)$. The comparison with the quantity $\frac{\pi}{\sqrt{\ln\alpha - 2}}$ remains complicated. However, a more detailed investigation of this situation seems to be possible due the recent work of Fang and Zhao in [9]. It will be studied in a forthcoming work.

Note now that the property that the equation $\phi(s) = \ln\alpha$ with $s \in [t-\tau, t]$ has at most two solutions when t is large enough indicates that the oscillations of the waves are slow oscillations.

While the existence of solutions for system (1.7) and more generally for system (1.4) has been developed in the literature (see for instance [17,18,27] we also refer to Diekmann [7] and Weinberger [36] for a first use a sub and super solution pair to handle these existence problems), the existence of non-converging solutions when $t \to \infty$ remains to be a difficult question. One may refer to Dunbar [8] and Huang [15] for some results in this direction. One may also mention that these works are based on singular perturbation analysis (related to Fenichel theory in the above mentioned work of Dunbar and on Fredholm property and implicit function theorem in the one of Huang). Finally let us mention that recent results obtained by Fang and Zhao in [9] give some answers to this difficult question in the general context of scalar integral equations by developing some properties of the spreading speed.

In this work, we would like to develop some tools based on the discrete Lyapunov functional for cyclic feedback delay differential systems. This tool allows us to obtain some information on the oscillating behaviour of the solution. The discrete Lyapunov functional was originally developed by Mallet-Paret in [20] and then by Cao [5] and Arino [3] for scalar equations with single delay. (We also refer to [24] for some results for ordinary differential equations.) The extension to the case of systems with delay was done by Mallet-Paret and Sell in [22] with some consequences on Poincaré–Bendixon theory given in [23]. However, this result does not directly applies for system (1.7) because of the properties of function h in (1.3). This difficulty is overcome by looking at some maximal monotonic properties of the solution. Finally after showing that the solutions remains in some region where the discrete Lyapunov functional applies, we need to compare the oscillations of the solutions together with the ones of the eigensolutions of the corresponding linearized equation around some positive equilibrium. According to Mallet-Paret [21] (see also Hupkes and Verduyn Lunel in [16]), to apply such a comparison, we need to overcome the difficulty of the possible existence of the so-called superexponential solutions, namely solution that converges faster than any exponential function. This problem is solved by studying the oscillations of the superexponential solutions. More particularly, we show that such a solution, when exists, has an infinite numbers of oscillations in some lag interval $[t, t+\tau]$ for t large enough.

The paper is organized as follows: Section 2 is devoted to the reformulation of the problem of travelling wave solutions for system (1.1) in term of (1.4) and then (1.7) when Assumption 1.1 is satisfied. Section 3 recalls some known results on oscillations for some delay differential systems and also study the infinite oscillations of superexponential solutions of (1.7). Section 4 is devoted to derive some maximal monotonic properties of the solutions and prove the applicability of the discrete Lyapunov functional for this problem when $t \to \infty$. In Section 5 we prove that under conditions on the some characteristic equations, the existence of point to undamped oscillating connection is ensured, while Section 6 investigates some properties of the characteristics equations and complete the proof of Theorem 1.2. Finally since the results of Theorem 1.2 does not deal with the stability of these waves with respect to the evolution equation (1.1), we supplement this work by giving some numerical simulations of the invasion process of (1.1). The numerical investigations given in Section 6 show that moving patterns may occur for (1.1) and are numerically stable.

2. Travelling wave formulation

In this section we come back to system (1.1) and we shall show, under some conditions on functions g and μ, that the travelling wave solutions for this problem correspond to the solutions of (1.4). To do so, we shall assume that $\mu \in L^\infty_{\text{loc},+}([0,\infty))$ and $g \in C^1_b([0,\infty),\mathbb{R})$, namely g is of the class C^1, and g and g' are bounded. Moreover we assume that

$$\mu(s) + g'(s) \geqslant \mu_0 > 0 \quad \text{and} \quad g(s) \geqslant g_0 > 0, \quad \forall s \geqslant 0,$$

for some constants $\mu_0 > 0$ and $g_0 > 0$. Recall that size structured models has been recently revisited by Webb in [35]. Inspired by this work, define $\Psi(s)$ as the solution of

$$\Psi'(s) = g\big(\Psi(s)\big), \quad \text{for } s \geqslant 0, \quad \text{with} \quad \Psi(0) = 0,$$

and set

$$v(t,s,x) := u\big(t, \Psi(s), x\big).$$

Then v satisfies the equation

$$\frac{\partial v(t,s,x)}{\partial t} + \frac{\partial v(t,s,x)}{\partial s} = -\big(\mu\big(\Psi(s)\big) + g'\big(\Psi(s)\big)\big)u\big(t,\Psi(s),x\big),$$

together with

$$g(0)v(t,0,x) = \big(I - d^2\Delta_x\big)^{-1}\left(\alpha h\left(\int_0^\infty \gamma\big(\Psi(l)\big)v(t,l,\cdot)g\big(\Psi(l)\big)\,dl\right)\right)(x).$$

Setting

$$\widehat{\mu}(s) := \mu\big(\Psi(s)\big) + g'\big(\Psi(s)\big) \quad \text{and} \quad \widehat{\gamma}(s) := \gamma\big(\Psi(s)\big)g\big(\Psi(s)\big),$$

we obtain that v satisfies the age-structured problem

$$\begin{cases} \dfrac{\partial v(t,s,x)}{\partial t} + \dfrac{\partial (v(t,s,x))}{\partial s} = -\widehat{\mu}(s)v(t,s,x), \quad \text{for } s \geqslant 0 \text{ and } x \in \mathbb{R}, \\ v(t,0,x) = \big(I - d^2\Delta_x\big)^{-1}\left(\dfrac{\alpha}{g(0)}h\left(\displaystyle\int_0^\infty \widehat{\gamma}(l)v(t,l,\cdot)\,dl\right)\right)(x), \quad \text{for } x \in \mathbb{R}, \\ v(0,.,.) = v_0 \in L^1\big((0,\infty), L^1_+(\mathbb{R})\big). \end{cases} \tag{2.1}$$

Thus without loss of generality (with the above assumptions on the map g), we can assume that $g \equiv 1$. Next, we set

$$v(t,s,x) = e^{-\int_0^s \widehat{\mu}(l)\,dl} w(t,s,x) \quad \text{and} \quad y = \frac{x}{d},$$

and we obtain that w satisfies

$$\begin{cases} \dfrac{\partial w(t,s,y)}{\partial t} + \dfrac{\partial w(t,s,y)}{\partial s} = 0, \text{ for } s \geqslant 0 \text{ and } y \in \mathbb{R}, \\ w(t,0,y) = (I - \Delta_y)^{-1}\left(\alpha h\left(\displaystyle\int_0^\infty \widetilde{\gamma}(s) w(t,s,\cdot)\,ds\right)\right)(y), \quad \text{for } y \in \mathbb{R}, \\ w(0,.,.) = w_0 \in L^1\big((0,\infty), L^1_+(\mathbb{R})\big), \end{cases} \tag{2.2}$$

wherein we have set

$$\widetilde{\gamma}(s) := \widehat{\gamma}(l) e^{-\int_0^s \widehat{\mu}(l)\,dl}, \quad \text{for almost every } s \geqslant 0.$$

As a consequence, without loss of generality, one may assume that $\widehat{\mu} = 0$ and $d = 1$.

Let us now consider travelling wave solutions for (2.2), that is solutions of the form

$$w(t,s,y) = \widehat{w}(s, y + ct),$$

where $c > 0$ denotes the wave speed and $\widehat{w} \in C^{1,2}(\mathbb{R})$ is some bounded positive function. It follows that $\widehat{w}$ satisfies the following equations

$$\begin{cases} \dfrac{\partial}{\partial s}\widehat{w}(s,y) + c\dfrac{\partial}{\partial y}\widehat{w}(s,y) = 0, \\ \widehat{w}(0,y) = \Delta_y \widehat{w}(0,y) + \alpha h\left(\displaystyle\int_0^\infty \widetilde{\gamma}(s)\widehat{w}(s,y)\,ds\right). \end{cases} \tag{2.3}$$

From the first equation in (2.3), we obtain that $\widehat{w}(s,y) = \Phi(y - cs)$ and therefore we only need to look for such a travelling wave solution which takes the form

$$w(t,s,y) = \Phi\big(y + c(t-s)\big),$$

where $\Phi \in C^2(\mathbb{R}, \mathbb{R}^+)$ is a bounded and positive function. So, from (2.3), we deduce that function Φ satisfies the following second order delay differential equation

$$\Phi''(y) = \Phi(y) - \alpha h\left(\int_0^\infty \widetilde{\gamma}(s)\Phi(y - cs)\right) ds, \quad \forall y \in \mathbb{R}. \tag{2.4}$$

Finally setting $\phi(t) = \Phi(ct)$ we get that ϕ satisfies (1.4) with $\gamma(s) \equiv \widetilde{\gamma}(s)$.

Finally if ϕ is a solution (1.4) and under Assumption 1.1, if we set $k = 0, \dots, n$,

$$\psi_k(t) = \delta_k \int_\tau^\infty (s-\tau)^k e^{-\mu s} \phi(t-s), \quad t \in \mathbb{R},$$

where $\delta_k > 0$ is such that $\delta_k \int_\tau^\infty (s-\tau)^k e^{-\mu s}\, ds = 1$, then $(\phi, \psi_0, \dots, \psi_n)$ satisfies (1.6). This justifies the study we shall fulfill in this work for system (1.7).

We complete this section by giving some notations that will be used through this work. To do so, let us notice that the system (2.4) has at most two positive equilibria. Indeed, 0 is always an equilibrium, and system (2.4) has a positive equilibrium if $\alpha > 1$, denoted by $\bar{x}$ and defined by

$$\bar{x} = \ln \alpha. \tag{2.5}$$

3. Oscillations and small solutions

3.1. Oscillations for delay differential systems

Since the goal of this paper is to study the oscillating behaviour of the solutions of (1.7) around the positive equilibrium $\bar{x}$ when $\alpha > 1$, we shall recall some results that will be useful through this work. Let $c > 0$ be given and $(\phi, \psi_0, \dots, \psi_n)$ be a solution of (1.7). Next we set

$$x^0(t) = \bar{x} - \phi(t), \quad t \in \mathbb{R},$$
$$x^2(t) = \bar{x} - \psi_n(t), \quad \dots, \quad x^{n+2}(t) = \bar{x} - \psi_0(t), \quad t \in \mathbb{R}.$$

We also denote $x^1(t) = -\phi'(t)$, $t \in \mathbb{R}$. Next let us notice that the vector valued function $(x^0, x^1, \dots, x^n)$ satisfies the first order system of delay differential equations:

$$\begin{aligned}
\frac{dx^0(t)}{dt} &= F^0\big(x^0(t), x^1(t)\big),\\
\frac{dx^1(t)}{dt} &= c^2 F^1\big(x^0(t), x^1(t), x^2(t)\big),\\
\frac{dx^k(t)}{dt} &= F^k\big(x^{k-1}(t), x^k(t), x^{k+1}(t)\big), \quad k = 2, \dots, n+1,\\
\frac{dx^{n+2}(t)}{dt} &= F^{n+2}\big(x^{n+1}(t), x^{n+2}(t), x^0(t-\tau)\big).
\end{aligned} \tag{3.1}$$

Here we have set

$$\begin{aligned}
&F^0(u, v) = v, \qquad F^1(u, v, w) = u + f(w),\\
&F^k(u, v, w) = -v + w, \quad k = 2, \dots, n+2,
\end{aligned} \tag{3.2}$$

and

$$f(s) = \alpha h(\bar{x} - s) - \bar{x}. \tag{3.3}$$

In order to consider the number of sign changes, let us introduce the set

$$\mathbb{K} = [-\tau, 0] \cup \{1, \dots, n+2\},$$

and, for each $t \in \mathbb{R}$, consider the map $x_t : \mathbb{K} \to \mathbb{R}$ defined by

$$x_t(\theta) = \begin{cases} x^0(t+\theta), & \theta \in [-\tau, 0], \\ x^\theta(t), & \theta \in \{1, \dots, n+2\}. \end{cases} \tag{3.4}$$

Next we have the following lemma:

Lemma 3.1. *Let $(\phi, \psi_0, \dots, \psi_n)$ be a solution of* (1.7)*, then for each $t \in \mathbb{R}$ we have $x_t \in C(\mathbb{K}) \setminus \{0\}$.*

The proof of this lemma is related to the following property of system (1.7).

Lemma 3.2. *Let $\alpha > 1$ be given. Let $(\phi, \psi_0, \dots, \psi_n)$ be a solution of*

$$\begin{aligned} &\frac{1}{c^2}\phi''(t) = \phi(t) - \alpha\psi_n(t)e^{\psi_n(t)}, \quad t \in \mathbb{R}, \\ &\psi_j'(t) = -\psi_j(t) + \psi_{j-1}(t), \quad t \in \mathbb{R},\ j = 1, \dots, n, \\ &\psi_0'(t) = -\psi_0(t) + \phi(t-\tau), \quad t \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Let us assume that there exists $Y > 0$ such that $\phi(y) \equiv \bar{x}$ on $[Y, \infty)$. Then $\phi(y) \equiv \bar{x}$, $\forall y \in \mathbb{R}$.

Proof. Due to the translation invariance, one may consider a solution $(\phi, \psi_0, \dots, \psi_n)$ of (3.5) such that $\phi(x) = \bar{x}$ for any $x \geqslant 0$. Recalling Assumption 1.1, from (3.5) we have for any $y \geqslant 0$:

$$\bar{x} = \alpha h\left(\int_0^\infty \gamma(s)\phi(y-s)\,ds\right).$$

Since the map $y \to \int_0^\infty \gamma(s)\phi(y-s)\,ds$ is continuous, it is constant on $[0, \infty)$. Moreover since $\phi(y) \to \bar{x}$ when $y \to \infty$ we obtain that

$$\int_0^\infty \gamma(l)\phi(y-l)\,dl = \bar{x}, \quad \forall y \geqslant 0.$$

This equality re-writes

$$\int_0^\infty \gamma(l)1_{y-l\geqslant 0}\phi(y-l)\,dl + \int_0^\infty \gamma(l)1_{y-l\leqslant 0}\phi(y-l)\,dl = \bar{x}\int_0^\infty \gamma(l)\,dl.$$

Since $\phi(x) = \bar{x}$ for any $x \geqslant 0$ we obtain that

$$\int_y^\infty \gamma(l)\phi(y-l)\,dl = \bar{x}\int_y^\infty \gamma(l)\,dl, \quad \forall y \geqslant 0.$$

As a consequence, for any $s \in \mathbb{R}$ we get

$$\int_0^\infty dy\, e^{sy} \int_y^\infty \gamma(l)\phi(y-l)\, dl = \bar{x} \int_0^\infty dy\, e^{sy} \int_y^\infty \gamma(l)\, dl.$$

Since the map ϕ is positive, one can use the Fubini theorem to obtain for any $s > 0$:

$$\int_0^\infty e^{sl}\gamma(l)\, dl \int_{-\infty}^0 e^{sx}\phi(x)\, dx = \int_0^\infty e^{sl}\gamma(l)\, dl\, \frac{\bar{x}}{s}.$$

As a consequence we obtain for any $s \in \mathbb{C}$ such that $\Re s \in (0,1)$ that

$$\int_{-\infty}^0 e^{sx}\phi(x)\, dx = \frac{\bar{x}}{s}.$$

Finally from Laplace inversion formula (see for instance [37]) we obtain that

$$\phi(x) = \bar{x}, \quad \forall x \in (-\infty, 0].$$

This completes the proof of the result. □

Proof of Lemma 3.1. If there exists $t \in \mathbb{R}$ such that $x_t = 0$ then we obtain that $x_s = 0$ for all $s \geqslant t$ and $\phi(s) \equiv \bar{x}$ for any $s \geqslant t$. Lemma 3.2 applies and provides a contradiction together with the behaviour of ϕ when $t \to -\infty$. □

Due to Lemma 3.1 one can introduce the notion of sign changes following the definition given by Mallet-Paret and Sell in [22]. For each $\varphi \in C(\mathbb{K}) \setminus \{0\}$ we consider

$$\mathrm{sc}(\varphi) = \sup\bigl\{k \geqslant 0\colon \exists \bigl\{\theta^i\bigr\}_{i=1}^k \subset \mathbb{K}^k,\ \theta^{i-1} < \theta^i,\ \varphi\bigl(\theta^{i-1}\bigr)\varphi\bigl(\theta^i\bigr) < 0,\ \forall i = 1, \ldots, k\bigr\}.$$

Next let $(\phi, \psi_0, \ldots, \psi_n)$ be a solution of (1.7) and recall that for each $t \in \mathbb{R}$ the map $x_t \in C(\mathbb{K}) \setminus \{0\}$. Next we defined the sign changes (around $\bar{x}$) of this solution at time $t \in \mathbb{R}$ by the quantity $\mathrm{sc}\, x_t$.

Let us now give some important remarks on this sign changes. Assume that $\alpha > e$. Then there exists a unique $x^* \in (0, \bar{x})$ such that

$$\alpha h\bigl(x^*\bigr) = \bar{x}. \tag{3.6}$$

Coming back to definition (3.3) one can check that

$$f(s) \begin{cases} > 0, & 0 < s < \bar{x} - x^*, \\ < 0, & s < 0. \end{cases}$$

As a consequence, function F defined in (3.2) satisfies the following feedback conditions

$$F^0(0, v) = v \begin{cases} \geqslant 0 & \text{if } v \geqslant 0, \\ \leqslant 0 & \text{if } v \leqslant 0 \end{cases}$$

and

$$F^1(u,0,w) = u + f(w) \begin{cases} \geqslant 0 & \text{if } u \geqslant 0 \text{ and } w \in [0, \bar{x} - x^*), \\ \leqslant 0 & \text{if } u \leqslant 0 \text{ and } w \leqslant 0, \end{cases}$$
$$F^k(u,0,w) = w \begin{cases} \geqslant 0 & \text{if } u \geqslant 0 \text{ and } w \geqslant 0, \\ \leqslant 0 & \text{if } u \leqslant 0 \text{ and } w \leqslant 0. \end{cases}$$

Thus this function enters the framework developed by Mallet-Paret and Sell in [22], those results will be extensively used in the sequel. More particularly we have the following lemma (see Theorem 2.1 in [22]):

Lemma 3.3. *Assume that* $\alpha > e$. *If* $(\phi, \psi_0, \ldots, \psi_n)$ *is a solution of* (1.7) *such that there exists* $\widehat{t} \in \mathbb{R}$ *such that*

$$\psi_n(t) > x^*, \quad \forall t > \widehat{t},$$

then the map $t \in (\widehat{t}, \infty) \to V(t) \in \{0, 2, \ldots, \infty\}$ *defined by*

$$V(t) = \begin{cases} \operatorname{sc} x_t & \text{if } \operatorname{sc} x_t \text{ is even or infinite}, \\ \operatorname{sc} x_t + 1 & \text{if } \operatorname{sc} x_t \text{ is odd}, \end{cases}$$

is non-increasing.

Moreover we also have some information when a sign change takes place (see Proposition 2.3 in [22]).

Lemma 3.4. *Under the same assumption as in Lemma* 3.3, *if for some* $t^1 > \widehat{t} - 4\tau$ *we have*

$$x^i(t^1) = 0 \quad \text{and} \quad x^{i-1}(t^1)x^{i+1}(t^1) \geqslant 0, \quad \text{for some } i = 1, \ldots, n+2,$$

then either $V(x_{t^1}) < V(x_{t^1-3\tau})$ *or* $V(x_{t^1}) = \infty$. *Here we have set* $x^{n+3}(t^1) = x^0(t^1 - \tau)$.

3.2. Super-exponentially converging solutions

Let $(\phi, \psi_0, \ldots, \psi_n)$ be a solution of (1.7) such that

$$\lim_{t\to\infty} (\phi, \psi_0, \ldots, \psi_n)(t) = \bar{x}(1, 1, \ldots, 1).$$

Then we shall show the following result:

Theorem 3.5. *Recalling definition* (3.4), *if*

$$\lim_{t\to\infty} V(x_t) < \infty,$$

then $(\phi, \psi_0, \ldots, \psi_n)$ *does not super-exponentially converges to* $\bar{x}(1, 1, \ldots, 1)$. *More particularly, there exist two constants* $0 < K_1 < K_2$ *such that*

$$K_1 \leqslant \frac{\|x_{t+\tau}\|}{\|x_t\|} \leqslant K_2, \quad \forall t \geqslant 0. \tag{3.7}$$

Here we have set for each $\varphi \in C(\mathbb{K})$,

$$\|\varphi\| = \sup_{\theta \in [-\tau, 0]} \left|\varphi(\theta)\right| + \sum_{k=1}^{n+2} \left|\varphi(k)\right|.$$

In order to prove this result, we shall prove (3.7). To do so, let us first state the following upper bound:

Lemma 3.6. *Let $(\phi, \psi_0, \dots, \psi_n)$ be a solution of (1.7), then there exists some constant $K > 0$ such that*

$$\|x_{t+\tau}\| \leqslant K\|x_t\|, \quad \forall t \geqslant 0.$$

Proof. The proof of this result is a direct application of Gronwall inequality. Indeed one can notice that the vector valued map F defined in (3.2) is sub-linear and the result follows. □

In order to prove the lower-bound in (3.7), one will argue by contradiction by assuming that

$$\inf\left\{ \frac{\|x_{t+\tau}\|}{\|x_t\|},\ t \geqslant 0 \right\} = 0, \tag{3.8}$$

and we aim to show that

$$\lim_{t \to \infty} V(x_t) = \infty. \tag{3.9}$$

Due to (3.8) there exists a sequence $\{t_j\}_{j \geqslant 0}$ such that

$$t_j \to \infty \quad \text{and} \quad \frac{\|x_{t_j+\tau}\|}{\|x_{t_j}\|} \to \infty \quad \text{when } j \to \infty.$$

Next for each $j \geqslant 0$, each $t \in \mathbb{R}$ and each $k = 0, \dots, n+2$ we set

$$y_j^k(t) = \frac{x^k(t + t_j)}{\|x_{t_j}\|},$$

as well as the vector

$$y_j(t) = \left(y_j^0(t), \dots, y_j^{n+2}(t)\right).$$

We shall show the following important property:

Lemma 3.7. *Let (3.8) be satisfied. For each $(a, b) \in \mathbb{R}^2$ such that $a < b$ we have*

$$\lim_{j \to \infty} \inf\left\{\left|y_j^k(s)\right|:\ s \in [a, b]\right\} = 0, \quad \forall k = 0, \dots, n+2.$$

Proof. Recalling definition (3.4), for each $m \geqslant 0$ and each $t \in \mathbb{R}$, the map $y_{j,t} \in C(\mathbb{K})$ is well defined and according to Lemma 3.1 we have $y_{j,t} \in C(\mathbb{K}) \setminus \{0\}$ and due to the definition of the sequence $\{t_j\}$, we have

$$\lim_{j\to\infty} \|y_{j,\tau}\| = 0. \tag{3.10}$$

Moreover, for each $j \geqslant 0$, y_j satisfies the system of equations for all $t \in \mathbb{R}$:

$$\begin{aligned}
\frac{dy_j^0(t)}{dt} &= y_j^1(t),\\
\frac{1}{c^2}\frac{dy_j^1(t)}{dt} &= y_j^0(t) + f^*\big(x^2(t+t_j)\big)y_j^2(t),\\
\frac{dy_j^k(t)}{dt} &= -y_j^k(t) + y_j^{k+1}(t), \quad k = 2, \ldots, n+1,\\
\frac{dy_j^{n+2}(t)}{dt} &= -y_j^{n+2}(t) + y_j^0(t-\tau),
\end{aligned} \tag{3.11}$$

where we have set

$$f^*(s) = \begin{cases} \frac{f(s)}{s}, & s \neq 0,\\ f'(0), & s = 0. \end{cases} \tag{3.12}$$

Next, due to Lemma 3.6, one can notice that the sequence $\{y_j^0\}$ is locally bounded on $[-\tau, \infty)$ while for each $k \geqslant 1$, the sequences $\{y_j^k\}$ are locally bounded on $[0, \infty)$. Moreover for each $j \geqslant 0$ there exists $\theta_j \in [-\tau, 0]$ such that

$$\big|y_j^0(\theta_j)\big| + \sum_{k=1}^{n+2}\big|y_j^k(0)\big| = 1, \quad \forall j \geqslant 0. \tag{3.13}$$

Due to system (3.11), for each $k = 0, \ldots, n+2$, the sequences $t \to y_j^k(t)$ are locally bounded in $C^1_{\mathrm{loc}}([0,\infty), \mathbb{R})$ while $t \to y_j^0(t)$ is locally bounded in $C_{\mathrm{loc}}([-\tau,\infty), \mathbb{R})$. Possibly along a subsequence, the sequence $\{y_j\}$ converges locally uniformly towards $y_* = (y_{0,*}, \ldots, y_{n+2,*})$ on $[0,\infty)$ while y_j^0 converges for the weak star topology of $L^\infty(-\tau, 0)$ towards $\gamma \in L^\infty(-\tau, 0)$. Note that y_* and γ satisfy the linear system of delay differential equations

$$\begin{aligned}
\frac{dy_{0,*}(t)}{dt} &= y_{1,*}(t), \quad t \geqslant 0,\\
\frac{dy_{1,*}(t)}{dt} &= c^2\big(y_{0,*}(t) + f'(0)y_{2,*}(t)\big), \quad t \geqslant 0,\\
\frac{dy_{k,*}(t)}{dt} &= -y_{k,*}(t) + y_{k+1,*}(t), \quad k = 2, \ldots, n+1,\ t \geqslant 0,\\
\frac{dy_{n+2,*}(t)}{dt} &= -y_{n+2,*}(t) + y_{0,*}(t-\tau), \quad t \geqslant \tau,\\
\frac{dy_{n+2,*}(t)}{dt} &= -y_{n+2,*}(t) + \gamma(t-\tau), \quad \text{a.e. } t \in (0,\tau).
\end{aligned} \tag{3.14}$$

Due to (3.10), we obtain that $(y_*)_\tau \equiv 0$ on $\mathbb{K}$ so that $(y_*)_t \equiv 0$ for any $t \geqslant \tau$. Due to (3.14) we obtain that for each $k = 0, \dots, n+2$, $y_{k,*}(t) \equiv 0$ for $t \geqslant 0$ and that $\gamma \equiv 0$. As a consequence, since $\gamma = 0$ we obtain that for each $a \neq b$ such that $[a,b] \subset [-\tau, 0]$,

$$\lim_{j\to\infty} \inf_{\theta\in[a,b]} \left|y_j^0(\theta)\right| = 0.$$

Next since for each $j \geqslant 0$ and each $t \in \mathbb{R}$ the sequence $\{y_j\}$ satisfies (3.11), for each $\varphi \in D(-\tau, \infty)$ we obtain that

$$\int_{-\tau}^{\infty} \varphi(s) y_j^1(s)\, ds = -\int_{-\tau}^{0} \varphi'(s) y_j^0(s)\, ds \to 0, \quad j \to \infty.$$

By induction we obtain for each $k = 0, \dots, n+2$ and for each $\varphi \in D(-\tau, \infty)$ that

$$\int_{-\tau}^{\infty} \varphi(s) y_j^k(s)\, ds \to 0, \qquad \int_{-\tau}^{\infty} \varphi(s) y_j^0(s-\tau)\, ds \to 0, \quad j \to \infty.$$

Thus we obtain that for each $\varphi \in D(-\tau, \infty)$, $\psi \in D(-2\tau, \infty)$,

$$\int_{-\tau}^{\infty} \varphi(s) y_j^0(s)\, ds \to 0, \qquad \int_{-2\tau}^{\infty} \psi(s) y_j^0(s)\, ds \to 0, \quad j \to \infty. \tag{3.15}$$

Therefore we obtain that for each $-\tau < a \neq b$ and each $-2\tau < a' < b'$,

$$\lim_{j\to\infty} \inf_{\theta\in[a,b]} \left|y_j^k(\theta)\right| = 0, \quad k = 1, \dots, n+2,$$
$$\lim_{j\to\infty} \inf_{\theta\in[a',b']} \left|y_j^0(\theta)\right| = 0.$$

Next using the same argument as above we have for each $\varphi \in D(-2\tau, \infty)$,

$$\int_{-2\tau}^{\infty} \varphi(s) y_j^1(s)\, ds = -\int_{-2\tau}^{\infty} \varphi'(s) y_j^0(s)\, ds \to 0, \quad j \to \infty. \tag{3.16}$$

Next we obtain for each $\varphi \in D(-2\tau, \infty)$,

$$-\int_{-2\tau}^{\infty} \varphi'(s) y_j^1(s)\, ds = c^2 \int_{-2\tau}^{\infty} y_j^0 \varphi\, ds + \int_{-2\tau}^{\infty} f^*\big(x^2(s+t_j)\big) y_j^2 \varphi\, ds,$$

and we infer from (3.15)–(3.16) that

$$\lim_{j\to\infty} \int_{-2\tau}^{\infty} y_j^2 \varphi\, ds = 0, \quad \forall \varphi \in D(-2\tau, \infty).$$

We may repeat the argument to obtain that

$$\lim_{j\to\infty}\int_{-2\tau}^{\infty} y_j^k\varphi\,ds=0,\quad \forall\varphi\in D(-2\tau,\infty),\ \forall k=0,\dots,n+2.$$

Once again one may repeat the argument on each interval $[-(p+1)\tau,-p\tau]$ to get the expected result. □

Next we derive the following lemma:

Lemma 3.8. *Let $N\geqslant 1$ be given, define for each interval $[a,b]\subset[-N\tau,\tau]$ and each $k=0,\dots,n+2$,*

$$h_k^j\big([a,b]\big)=\inf_{\theta\in[a,b]}\big|y_j^k(\theta)\big|,\quad j\in\mathbb{N},$$

as well for each $d\in(0,\tau]$ define

$$H_k^j(d)=\inf\big\{h_k^j\big([a,b]\big)\colon\ [a,b]\subset[-N\tau,\tau],\ b-a=d\big\}.$$

Then we have

$$\lim_{j\to\infty}H_k^j(d)=0,\quad \forall k=0,\dots,n+2,\ \forall d\in(0,\tau].$$

The proof is similar to the one of Lemma 2.4 in [5].

Finally we complete this section by proving Theorem 3.5.

Proof of Theorem 3.5. To prove this result, let us argue by contradiction by assuming that (3.9) does not hold true, that is

$$\lim_{t\to\infty}V(x_t)=N<\infty,$$

for some N. Next, as a consequence of (3.13), we obtain that

$$\lim_{j\to\infty}\big|y_j^k(0)\big|=0,\quad \forall k=1,\dots,n+2,$$

so that for each $\eta>0$, there exists $j_0\geqslant 1$ such that

$$\big|y_j^0(\theta_j)\big|>1-\eta,\quad \forall j\geqslant j_0. \tag{3.17}$$

Next we set $T=(N+2)\tau$ and $\delta\in(0,1)$ such that

$$2(N+1)\delta<\tau,\ \delta<(n+2)\tau,\quad \frac{e^{-2T}}{\mu M}\frac{1}{2(N+1)\delta}>1$$

and recalling definition (3.12) we define $M>0$ by

$$M=\big\|f^*\big\|_\infty.$$

Next we fix $\varepsilon \in (0,1)$ such that

$$e^T \varepsilon < e^{-T}\frac{1}{M}, \quad \frac{e^{-T}}{2(N+1)\delta}\left(\frac{e^{-T}}{M} - e^T\varepsilon\right) > 1, \tag{3.18}$$

we fix $j_0 \geqslant 1$ such that

$$H_k^{j_0}(d) \leqslant \varepsilon, \quad \forall k,\ \forall d \in \left[\frac{\delta}{n+2}, \tau\right]. \tag{3.19}$$

Up to increase j_0 one may assume that there exists $m > 0$ such that

$$m \leqslant f^*\big(x^2(t+t_j)\big) \leqslant M, \quad \forall t \geqslant -T,$$

and

$$V(x_{t+t_{j_0}}, \mathbb{K}) = \lim_{t\to\infty} V(x_t, \mathbb{K}), \quad \forall t \geqslant -T. \tag{3.20}$$

In the sequel we will omit the dependence with respect to j_0 and we shall write y_k for each $k = 0, \ldots, n+2$ instead of $y_{j_0}^k$.

Recall that according to the normalization condition (3.17), up to change the vector y by γy for some constant $\gamma > 0$, one may assume (for notional simplicity) that there exists $\theta_0 \in [-\tau, 0]$ such that

$$\big|y_0(\theta_0)\big| > 1.$$

One shall prove by induction (on k) that for each $k = 1, \ldots, N+1$, there exist

$$(\mathcal{P}_k) \quad \begin{cases} \tau_k - k\delta < t_1^k < \theta_1^k < t_2^k < \theta_2^k < \cdots < \theta_k^k < t_{k+1}^k < \tau_k + k\delta, \\ y_0\big(\theta_i^k\big) = 0, \quad i = 1, \ldots, k, \qquad \big|y_0\big(t_p^k\big)\big| > 1, \quad p = 1, \ldots, k+1, \end{cases} \tag{3.21}$$

wherein we have set

$$\tau_k = \theta_0 - k\tau, \quad k \in \mathbb{N}. \tag{3.22}$$

The case $k = 1$: We shall prove that the induction property holds true for $k = 1$. Due to (3.19), there exist $\gamma_1 \in (\theta_0 - \frac{\delta}{n+2}, \theta_0)$ and $\gamma_2 \in (\theta_0, \theta_0 + \frac{\delta}{n+2})$ such that

$$\big|y_0(\gamma_i)\big| \leqslant \varepsilon.$$

As a consequence, there exists $s_1 \in (\theta_0 - \frac{\delta}{n+2}, \theta_0 + \frac{\delta}{n+2})$ such that

$$y_0'(s_1) = 0, \qquad y_0''(s_1)y_0(s_1) \leqslant 0, \qquad \big|y_0(s_1)\big| > 1.$$

Recalling that y_0 satisfies the second order equation

$$\frac{1}{c^2}y_0''(t) - y_0(t) = f^*(t+t_j)y_2(t), \quad t \in \mathbb{R},$$

one obtains that

$$y_0^2(s_1) + f^*(s_1 + t_j) y_0(s_1) y_2(s_1) \leqslant 0,$$

that yields to

$$\left|y_2(s_1)\right| > \frac{1}{M}.$$

Next due to (3.19), we have

$$\inf\left\{\left|y_2(s)\right|,\ s \in \left[\theta_0 - \frac{2\delta}{n+2}, \theta_0 - \frac{\delta}{n+2}\right]\right\} \leqslant \varepsilon,$$
$$\inf\left\{\left|y_2(s)\right|,\ s \in \left[\theta_0 + \frac{\delta}{n+2}, \theta_0 + \frac{2\delta}{n+2}\right]\right\} \leqslant \varepsilon.$$

As a consequence, there exists $s_2 \in (\theta_0 - \frac{2\delta}{n+2}, \theta_0 + \frac{2\delta}{n+2})$ such that

$$y_2'(s_2) = 0, \qquad \left|y_2(s_2)\right| > \frac{1}{M}.$$

This yields to $|y_3(s_2)| > \frac{1}{M}$. By induction, for each $p = 2, \dots, n+2$ there exists $s_{p-1} \in (\theta_0 - \frac{(p-1)\delta}{n+2}, \theta_0 + \frac{(p-1)\delta}{n+2})$ such that

$$\left|y_p(s_{p-1})\right| > \frac{1}{M}.$$

Setting

$$z_p(t) = e^t y_p(t), \quad p = 2, \dots, n+2,\ t \geqslant -T,$$

we obtain that

$$\sup_{[\theta_0 - \frac{n+1}{n+2}\delta, \theta_0 + \frac{n+1}{n+2}\delta]} |z_{n+2}| \geqslant e^{-T} \frac{1}{M}.$$

Next due to (3.19) the map $|y_{n+2}|$ takes some values less that ε on each interval $[\theta_0 - \delta, \theta_0 - \frac{n+1}{n+2}\delta]$ and $[\theta_0 + \frac{n+1}{n+2}\delta, \theta_0 + \delta]$ and the map $|z_{n+2}|$ takes some values less that $e^T \varepsilon$ on each of these intervals. As a consequence, due to (3.18), there exists some point $\theta_0' \in [\theta_0 - \delta, \theta_0 + \delta]$ such that $z_n'(\theta_0') = 0$, that is $y_0(\theta_0' - \tau) = 0$. Moreover there exist $t_1 \in [\theta_0 - \delta, \theta_0']$ and $t_2 \in [\theta_0', \theta_0 + \delta]$ such that

$$\left|z_{n+2}'(t_i)\right| \geqslant \frac{1}{\delta}\left(e^{-T} \frac{1}{M} - e^T \varepsilon\right).$$

Due to the definition of z_{n+2}, recall that it satisfies the equation

$$z_{n+2}'(t) = e^t y_0(t - \tau), \quad t \geqslant -T,$$

and we obtain due to (3.18) that

$$\left|y_0(t_i-\tau)\right| \geqslant e^{-t_i}\left|z'_{n+2}(t_i)\right| > 1.$$

By setting $\theta_1^1=\theta'_0-\tau$ and $t_i^1=t_i-\tau$ for $i=1,2$ we obtain that

$$\begin{aligned}
&-\tau+\theta_0-\delta \leqslant t_1^1<\theta_1^1<t_2^1 \leqslant -\tau+\theta_0+\delta,\\
&y_0\left(\theta_1^1\right)=0, \qquad \left|y_0\left(t_1^i\right)\right|>1, \quad i=1,2.
\end{aligned}$$

This completes the proof of the induction property for $k=1$.

From k to $k+1$: Let us assume that the induction property holds true for some $k\in\{1,\dots,N\}$ and we shall show that it also holds true for $k+1$. For that purpose we shall also argue by induction by showing that for each $p=2,\dots,n+2$, there exist

$$\begin{aligned}
&\tau_k-\left(k+\frac{p+1}{n+2}\right)\delta<r_1^p<s_1^p<r_2^p<s_2^p<\cdots<s_k^p<r_{k+1}^p<\tau_k+\left(k+\frac{p+1}{n+2}\right)\delta,\\
&y_p\left(s_i^p\right)=0, \quad i=1,\dots,k, \qquad \left|y_p\left(r_l^p\right)\right|>\frac{1}{M}, \quad l=1,\dots,k+1.
\end{aligned} \tag{3.23}$$

Let us first prove that this property holds true for $p=2$. To do so, let us recall that due to (3.19), there exist $\theta_0^k\in[\tau_k-k\delta-\frac{1}{n+2}\delta,\tau_k-k\delta]$ and $\theta_{k+1}^k\in[-\tau_k+k\delta,\tau_k+k\delta+\frac{1}{n+2}\delta]$ such that

$$\left|y_0\left(\theta_0^k\right)\right|<\varepsilon, \left|y_0\left(\theta_{k+1}^k\right)\right|<\varepsilon.$$

First notice that due to (3.20) and Proposition 2.3 in [22] we have

$$y'_0\left(\theta_i^k\right)\neq 0, \quad \forall i=1,\dots,k.$$

As a consequence of the induction property, for each $i=0,\dots,k$, there exists

$$\theta_i^k<\xi_i^+\leqslant\gamma_i\leqslant\xi_{i+1}^-<\theta_{i+1}^k,$$

such that

$$\begin{aligned}
&y'_0(\gamma_i)=0, \qquad y''_0(\gamma_i)y_0(\gamma_i)\leqslant 0, \qquad \left|y_0(\gamma_i)\right|>1, \quad \forall i=0,\dots,k,\\
&y'_0\left(\xi_i^-\right)=0=y'_0\left(\xi_i^+\right), \qquad y_0\left(\xi_i^-\right)y_0\left(\xi_i^+\right)<0, \quad \forall i=1,\dots,k,\\
&\left|y'_0(s)\right|>0, \quad \forall s\in\left(\xi_i^-,\xi_i^+\right),\ i=1,\dots,k.
\end{aligned} \tag{3.24}$$

Again using Proposition 2.3 in [22] as well as (3.20), one obtains that

$$y_0\left(\xi_i^\pm\right)y_2(\xi_i\pm)<0.$$

Since $y_0(\xi_i^-)y_0(\xi_i^+)<0$, $\forall i=1,\dots,k$, one obtains that for each $i=1,\dots,k$, there exist $s_1^2,\dots,s_k^2$ such that

$$\xi_i^-<s_i^2<\xi_i^+ \quad \text{and} \quad y_2\left(s_i^2\right)=0, \quad \forall i=1,\dots,k.$$

By setting $r_i^2 = \gamma_{i-1}$ for any $i = 1, \ldots, k+1$, one obtains due to the equation for y_0 that

$$\left|y_2\left(r_i^2\right)\right| > \frac{1}{M}, \quad \forall i = 1, \ldots, k+1.$$

This completes the proof of the induction property (3.23) for $p = 2$.

Let us now assume that the induction property (3.23) holds true for some $p \in \{2, \ldots, n+1\}$. First of all let us notice that due to Proposition 2.3 in [22] as well as (3.20) we have

$$y_p'\left(s_i^p\right) \neq 0, \quad \forall i = 1, \ldots, k.$$

Indeed due to the equation for y_p, if it is zero for some i one obtains that $y_p(s_i^p) = y_{p+1}(s_i^p)$ which contradicts (3.20). Here again due to (3.19), there exist s_0^p and s_{k+1}^p such that

$$\begin{aligned}
&\tau_k - k\delta - \frac{p+2}{n+2} < s_0^p < \tau_k - k\delta - \frac{p+1}{n+2}, \\
&\tau_k + k\delta + \frac{p+1}{n+2} < s_{k+1}^p < \tau_k + k\delta + \frac{p+2}{n+2}, \\
&\left|y_p\left(s_0^k\right)\right| > \varepsilon, \qquad \left|y_p\left(s_{k+1}^k\right)\right| > \varepsilon.
\end{aligned}$$

As a consequence, for each $i = 0, \ldots, k$, there exists

$$s_i^p < \xi_i^+ \leqslant \gamma_i \leqslant \xi_{i+1}^- < s_{i+1}^p,$$

such that

$$\begin{aligned}
&y_p'(\gamma_i) = 0, \qquad \left|y_p(\gamma_i)\right| > \frac{1}{M}, \quad \forall i = 0, \ldots, k, \\
&y_p'\left(\xi_i^-\right) = 0 = y_p'\left(\xi_i^+\right), \qquad y_p\left(\xi_i^-\right) y_p\left(\xi_i^+\right) < 0, \quad \forall i = 1, \ldots, k, \\
&\left|y_p'(s)\right| > 0, \quad \forall s \in \left(\xi_i^-, \xi_i^+\right),\ i = 1, \ldots, k.
\end{aligned} \tag{3.25}$$

Due to the equation for y_p we get that

$$y_p\left(\xi_i^\pm\right) = y_{p+1}\left(\xi_i^\pm\right), \quad \forall i = 1, \ldots, k.$$

As a consequence, for each $i = 1, \ldots, k$, there exist $s_1^{p+1}, \ldots, s_k^{p+1}$ such that

$$\xi_i^- < s_i^{p+1} < \xi_i^+ \quad \text{and} \quad y_{p+1}\left(s_i^{p+1}\right) = 0, \quad \forall i = 1, \ldots, k.$$

Next, setting $r_i^{p+1} = \gamma_{i-1}$ for any $i = 1, \ldots, k+1$, one obtains, due to the equation for y_p, that

$$\left|y_{p+1}\left(r_i^{p+1}\right)\right| > \frac{1}{M}, \quad \forall i = 1, \ldots, k+1.$$

This completes the proof of the induction property (3.23) for $p+1$.

We conclude that the induction property (3.23) holds true for $p = n+2$, that re-writes

$$\tau_k - k\delta - \frac{n+1}{n+2}\delta < r_1 < s_1 < r_2 < s_2 < \cdots < s_k < r_{k+1} < \tau_k + k\delta + \frac{n+1}{n+2}\delta,$$
$$y_{n+2}(s_i) = 0, \quad i = 1, \ldots, k, \qquad \left|y_{n+2}(r_l)\right| > \frac{1}{M}, \quad l = 1, \ldots, k+1.$$

Due to (3.19), there exist s_0 and s_{k+1} such that

$$\tau_k - (k+1)\delta < s_0 < \tau_k - k\delta - \frac{n+1}{n+2}\delta, \qquad \tau_k + k\delta + \frac{n+1}{n+2}\delta < s_{k+1} < \tau_k + (k+1)\delta,$$
$$\left|y_{n+2}(s_0)\right| > \varepsilon, \qquad \left|y_{n+2}(s_{k+1})\right| > \varepsilon.$$

As a consequence of (3.18), for each $i = 0, \ldots, k$, there exists t_i such that

$$s_i < t_i < s_{i+1}, \qquad z'_{n+2}(t_i) = 0,$$
$$\left|z_{n+2}(t_i)\right| > \frac{e^{-T}}{M},$$
$$\left|z_{n+2}(s_l)\right| \leqslant e^T \varepsilon, \quad \forall l = 0, \ldots, k+1.$$

Thus for each $i = 0, \ldots, k+2$ there exists r_i such that

$$s_0 \leqslant r_0 < t_0 < r_1 < t_1 < \cdots < r_{k+1} < t_{k+1} < r_{k+2} < s_{k+1},$$
$$\left|z'_{n+2}(r_i)\right| > \frac{1}{t_i - s_i}\left(\frac{e^{-T}}{M} - e^T \varepsilon\right).$$

Due to the definition of z_{n+2}, recall that it satisfies the equation

$$z'_{n+2}(t) = e^t y_0(t-\tau), \quad t \geqslant -T,$$

and we obtain due to (3.18) that

$$y_0(t_i - \tau) = 0, \quad \forall i = 0, \ldots, k,$$
$$\left|y_0(r_i - \tau)\right| \geqslant e^{-t_i}\left|z'_{n+2}(r_i)\right| > 1, \quad \forall i = 0, \ldots, k+2.$$

Setting $\theta_i^{k+1} = t_{i-1} - \tau$ for $i = 1, \ldots, k+1$ and $t_l^{k+1} = r_{l-1} - \tau$ for $l = 1, \ldots, k+3$ we obtain that (3.21) holds true for $k+1$. This completes the proof of Theorem 3.5. □

4. General properties

The aim of this section is to derive some properties of the solutions of (1.7). To do that we first introduce the characteristic equation of system (1.7) around the equilibrium 0. It reads

$$\Delta^-(c, \lambda) = \left(\frac{\lambda^2}{c^2} - 1\right)(\lambda+1)^{n+1} + \alpha e^{-\tau\lambda}. \tag{4.1}$$

Here we have explicitly written down the dependence with respect to $c > 0$ because it is also an unknown number of the problem (1.7). A first basic property of Δ^- which follows from some concavity property is the following:

Lemma 4.1. *Let us assume that $\alpha > 1$. Let $\tau > 0$ and $n \geqslant 0$ be given. There exists a unique $c^* = c^*(\alpha, \tau, n) > 0$ such that the following hold*:

(i) *For each $c \in (0, c^*)$ we have $\inf_{\lambda \geqslant 0} \Delta^-(c, \lambda) > 0$.*
(ii) *For $c = c^*$, the equation $\Delta^-(c^*, \lambda) = 0$ with $\lambda \geqslant 0$ has a unique solution λ of multiplicity two.*
(iii) *For each $c > c^*$ the equation $\Delta^-(c^*, \lambda) = 0$ with $\lambda \geqslant 0$ has two (simple) solutions $0 < \lambda_1(c) < \lambda_2(c)$.*

Then one can give the following definition:

Definition 4.2. Let $\tau > 0$ and $n \geqslant 0$ be given. We assume that $\alpha > 1$, and we consider $c^* = c^*(\alpha, \tau, n) > 0$ provided by Lemma 4.1 which will be referred as the minimal speed.

4.1. Bound of the solutions and existence result

Through this section we shall assume that

Assumption 4.3. We assume that $\alpha > 1$.

Let us introduce the following quantities:

$$\bar{x}^+ := \alpha h(S_M) = \frac{\alpha}{e}, \qquad \bar{x}^- := \alpha h\big(\bar{x}^+\big) = \frac{\alpha^2}{e} e^{-\frac{\alpha}{e}}. \tag{4.2}$$

Let us notice that $\alpha h([\bar{x}^-, \bar{x}^+]) \subset [\bar{x}^-, \bar{x}^+]$. According to [32] we obtain the following result:

Lemma 4.4. *Let $c > 0$ be given. Assume that system (1.7) has a solution $(\phi, \psi_0, \ldots, \psi_n)$ then we have*

$$\bar{x}^- \leqslant \varliminf_{t \to \infty} \phi(t) \leqslant \sup_{\mathbb{R}} \phi \leqslant \bar{x}^+,$$

and also

$$\bar{x}^- \leqslant \varliminf_{t \to \infty} \psi_k(t) \leqslant \sup_{t \in \mathbb{R}} \psi_k(t) \leqslant \bar{x}^+, \quad \forall k = 0, \ldots, n.$$

The proof is similar to the one of Theorem 3 in [32].

Let us now recall some existence results. To do that let us recall that for each $\alpha > 1$ the value $c^* > 0$ is defined in Definition 4.2.

Theorem 4.5. *Assume that $\alpha > 1$. Then for any $c > c^*$, (1.7) has a unique (up to translation) positive and bounded solution satisfying*

$$\bar{x}^- \leqslant \liminf_{y \to \infty} \phi(y) \leqslant \limsup_{y \to \infty} \phi(y) \leqslant \bar{x}^+.$$

Moreover system (1.7) does not have any solution for $0 < c < c^*$.

Assume that $\alpha \in (1, e^2]$ and let $c > c^*$ be given. Let $(\phi, \psi_0, \ldots, \psi_n)$ be a solution of (1.7)*. Then this function converges at $t \to \infty$, that is*

$$\lim_{t \to \infty} (\phi, \psi_0, \ldots, \psi_n)(t) = \bar{x}(1, \ldots, 1).$$

Moreover, for each $c > c^*$, when $\alpha \in (1, e]$ then ϕ is non-decreasing.

The proof of this result on relies on the reduction of (1.7) or more generally system (1.4) under some integral formulation and use Theorems 2.2 and 3.1 in [9]. We also refer to Ma [17,18], So et al. [27] and Wu and Zou [38] for some existence results of travelling wave solutions for some delayed reaction–diffusion equations (see Thieme et al. [29] for integral equations). And we refer to Thieme et al. [29], Trofimchuk et al. [30] and Trofimchuk et al. [32] for other non-existence results.

4.2. Monotonicity properties

In order to study some monotonicity properties of the leading edge, we need some more precise properties of the characteristic equation Δ^- defined in (4.1). We shall assume the following:

Assumption 4.6. Let $\alpha > 1$ be given and let $c > c^*$ be given. We set

$$\Omega = \{\lambda \in \mathbb{C}\colon\ \Delta^-(c, \lambda) = 0,\ \Re\lambda \geqslant 0\} \setminus \{\lambda_1(c), \lambda_2(c)\},$$

wherein $\lambda_i(c)$ are defined in Lemma 4.1. Then we assume that

$$\Re\lambda < \lambda_1(c) < \lambda_2(c), \quad \forall \lambda \in \Omega.$$

Then we shall show the following result:

Proposition 4.7 *(Maximal monotonicity). Let Assumption* 4.6 *be satisfied and let $(\phi, \psi_0, \ldots, \psi_n)$ be a solution of* (1.7)*. There exists a maximal interval $(-\infty, \sigma)$ for some $\sigma \in (-\infty, \infty]$ such that $\phi'(t) > 0$ on this interval $(-\infty, \sigma)$.*

Proof. We shall argue by contradiction by assuming that there exists a sequence $\{t_m\}_{m \geqslant 0}$ such that $t_m \to -\infty$ and $\phi'(t_m) = 0$ for each $m \geqslant 0$. Next we set

$$\zeta(t) = \frac{\alpha h(\psi_n(t))}{\psi_n(t)} = \alpha e^{-\psi_n(t)},$$

and

$$u_m(t) = \frac{\phi(t + t_m)}{\phi(t_m)}, \qquad v_m^i(t) = \frac{\psi_i(t + t_m)}{\phi_i(t_m)}.$$

Since $\phi \to 0$ as well as ψ_i when $t \to -\infty$, one may assume that

$$\phi(t + t_m) \leqslant \phi(t_m), \quad \forall t \leqslant 0. \tag{4.3}$$

Next we shall show the following result:

Lemma 4.8. *The sequence of functions $\{u_m, v_m^0, \ldots, v_m^n\}_{n \geqslant 0}$ is locally bounded on $\mathbb{R}$.*

Proof. Let us first notice that (4.3) implies that $u_m(t) \leqslant 1$ for each $t \leqslant 0$. Next, due to the definition of ψ_0 we obtain that

$$\psi_0(t+t_m) = \int_{-\infty}^{0} e^l \phi(l+t+t_m-\tau)\,dl.$$

Therefore for each $t \leqslant \tau$ we have

$$\psi_0(t+t_m) \leqslant \phi(t_m).$$

Next due to the definition of ψ_1 we get that

$$\psi_1(t+t_m) = \int_{-\infty}^{0} e^l \psi_0(l+t+t_m)\,dl.$$

Therefore for each $t \leqslant \tau$ we obtain that

$$\psi_1(t+t_m) \leqslant \phi(t_m).$$

Finally by induction we get that

$$\psi_j(t+t_m) \leqslant \phi(t_m), \quad \forall m, j,\ \forall t \leqslant \tau.$$

From this, one will show that $\{u_m\}$ is also bounded on $[0,\tau]$. Indeed due to the definition of $\{t_m\}$ we have $u_m(0)=1$ while $u_m'(0)=0$. As a consequence we get that

$$u_m(t) = \frac{e^t+e^{-t}}{2} + \int_0^t \left(e^{s-t}-e^{t-s}\right)\zeta(s+t_m)v_m^0(s)\,ds.$$

Since $v_m(t) \leqslant 1$ for each $t \in [0,\tau]$ we obtain that u_m is uniformly bounded on $(-\infty,\tau]$. Repeating the above arguments on $[0,2\tau]$, $[0,3\tau]$, ..., we obtain the expected result. □

As a consequence of the above lemma, up to a subsequence, one may assume that these functions converges to some positive functions $(u, v^0, \ldots, v^n)$ locally uniformly on $\mathbb{R}$ that satisfy the linear problem on $\mathbb{R}$:

$$\begin{aligned}
&\frac{1}{c^2}u''(t) = u(t) - \alpha v^n(t),\\
&v^{j'}(t) + v^j(t) = v^{j-1}(t), \quad j = 1,\ldots,n,\\
&v^{0'}(t) + v^0(t) = u(t-\tau).
\end{aligned}$$

Moreover we have

$$u'(0)=0, \qquad u(0)=1, \tag{4.4}$$

and $0 \leqslant u(t) \leqslant 1$ and $0 \leqslant v^j(t) \leqslant 1$ for $t \leqslant 0$.

Next the following lemma holds true.

Lemma 4.9. *For any $a \in \mathbb{R}$, the linear problem*

$$\begin{aligned} &\frac{1}{c^2}u''(t) = u(t) - \alpha v^n(t), \\ &v^{j'}(t) + v^j(t) = v^{j-1}(t), \quad j = 1, \ldots, n, \\ &v^{0'}(t) + v^0(t) = u(t-\tau), \end{aligned} \tag{4.5}$$

does not have any solution on $(-\infty, a]$ converging to zero when $t \to -\infty$ faster than any exponential functions.

The proof of this result is similar to some step of the proof of Lemma 6 in [31]. The details are left to the reader.

According to Assumption 4.6, the characteristic equation has two positive real solution $0 < \lambda_2(c) < \lambda_1(c)$. Since the solution is positive, it does not oscillate around zero when $t \to -\infty$. Thus we obtain that

$$\begin{pmatrix} u(t) \\ v^0(t) \\ \cdots \\ v^n(t) \end{pmatrix} = e^{\lambda_1(c)t}A_1 + e^{\lambda_2(c)t}A_2 + \xi(t),$$

where ξ is a small solution and A_1, A_2 are some real constant vectors. According to Lemma 4.9 we have $\xi \equiv 0$ and due to (4.4) we obtain that

$$u(t) = \frac{-\lambda_2(c)e^{\lambda_1(c)t} + \lambda_1(c)e^{\lambda_2(c)t}}{\lambda_1(c) + \lambda_2(c)}.$$

Since $0 < \lambda_2(c) < \lambda_1(c)$, we obtain a contradiction with the positivity of u (when $t \to \infty$). □

We are now able to state the following result:

Theorem 4.10. *Let Assumption 4.6 be satisfied and let $(\phi, \psi_0, \ldots, \psi_n)$ be a solution of (1.7). Let $\sigma \in (-\infty, \infty]$ be defined by*

$$\sigma = \sup\{t \in \mathbb{R}\colon \phi'(s) > 0, \ \forall s \leqslant t\}.$$

If $\sigma < \infty$ then $\psi_j(\sigma + \tau) > \bar{x}$ for any $j = 0, \ldots, n$ while $\phi(\sigma) > \bar{x}$.

Note that $\sigma > -\infty$ because of Proposition 4.7.

In order to prove Theorem 4.10, we shall use the following computational lemmas those proofs are similar to the ones of Lemmas 8 and 10 in [32].

Lemma 4.11. *Let $z(t) = \phi'(t)$ and set $\Gamma(t) = \alpha h(\psi_n(t))$. Then the $t \to \Gamma(t)$ is of bounded variation. If z satisfies $z(a) = z_0$ and $z(0) = 0$ then we have*

$$z(t) = \frac{e^{-t} - e^t}{e^{-a} - e^a}\left\{z_0 + \frac{c}{2}\int_a^t \left(e^{u-a} - e^{a-u}\right) d\Gamma(u)\right\} + \frac{c}{2}\left(e^{t-a} - e^{a-t}\right)\int_t^0 \frac{e^{-u} - e^u}{e^{-a} - e^a}\, d\Gamma(u),$$

$$z'(0) = \frac{2}{e^a - e^{-a}} \left\{ z_0 + \frac{c}{2} \int_a^0 \left(e^{u-a} - e^{a-u}\right) d\Gamma(u) \right\},$$

$$z'(a) = \frac{-e^{-a} - e^a}{e^{-a} - e^a} z_0 + c \int_a^0 \frac{e^{-u} - e^u}{e^{-a} - e^a} \, d\Gamma(u).$$

Lemma 4.12. *If $z(t) = \phi'(t)$ satisfies $z(-\infty) = z(0) = 0$ then*

$$z(t) = \frac{c}{2} \left\{ \left(e^{-t} - e^t\right) \int_{-\infty}^t e^s \, d\Gamma(s) + e^t \int_t^0 \left(e^{-t} + e^t\right) d\Gamma(s) \right\}$$

$$= \frac{c}{2} \left\{ \left(e^{-t} - e^t\right) \int_{-\infty}^0 e^s \, d\Gamma(s) + e^t \int_t^0 \left(e^{s-t} + e^{t-s}\right) d\Gamma(s) \right\}.$$

To prove Theorem 4.10 let us first prove the following lemma:

Lemma 4.13. *Let us assume that $\sigma < \infty$. Then the following hold true*:

(i) $\psi_n(\sigma + \tau) > S_M$.
(ii) *For each $j = 0, \ldots, n$ we have $\psi_j'(t) > 0$ for any $t \in (-\infty, \sigma + \tau]$.*

Proof. Let us assume that $\sigma = 0$ and that $\phi'(0) = 0$. One will show that $\psi_n(\tau) > S_M$. To do so we shall argue by contradiction by assuming that $\psi_n(\tau) \leqslant S_M$. Since ϕ is non-decreasing on $(-\infty, 0]$, one can notice that ψ_j is non-decreasing on $(-\infty, \tau]$. In view of Lemma 4.12 we obtain that $\phi'(t) < 0$ on $(0, \tau]$. Consider

$$\sigma_1 = \sup\left\{t \geqslant \tau \colon \phi'(s) < 0, \ \forall s \in (0, t]\right\}.$$

Let us notice that $\sigma_1 < \infty$. Indeed if not, ϕ is decreasing on $(0, \infty)$ and less that S_M and therefore converges to zero at $t = \infty$, a contradiction with $\underline{\lim}_{t \to \infty} \phi(t) > 0$. As a consequence $\sigma_1 > \tau$ and

$$\phi'(\sigma_1) = 0, \qquad \phi''(\sigma_1) \geqslant 0, \qquad \phi(\sigma_1) < \phi(\sigma_1 - \tau).$$

Moreover since ϕ is non-increasing on $(0, \sigma_1)$ and ψ_0 satisfies the equation

$$\psi_0'(t) = -\psi_0(t) + \phi(t - \tau),$$

we obtain that ψ_0 is non-increasing on $(\tau, \tau + \sigma_1)$. Therefore we obtain that $\phi(\sigma_1 - \tau) < \psi_0(\sigma_1)$. Next ψ_1 satisfies

$$\psi_1'(t) = -\psi_1(t) + \psi_0(t),$$

so that ψ_1 is non-increasing on $(\tau, \sigma_1 + \tau)$, that leads to $\psi_0(\sigma_1) < \psi_1(\sigma_1)$. By induction we obtain that $\phi(\sigma_1 - \tau) < \psi_n(\sigma_1)$. Due to the equation for ϕ as well as the definition of σ_1 we get that

$$\alpha h\left(\psi_n(\sigma_1)\right) \leqslant \phi(\sigma_1) < \phi(\sigma_1 - \tau) < \psi_n(\sigma_1).$$

On the other hand, let us recall that $\psi_n(\tau) \leqslant S_M$ so that $\psi_n(\sigma_1) \leqslant S_M$. Thus

$$\psi_n(\sigma_1) \leqslant S_M, \qquad \alpha h\big(\psi_n(\sigma_1)\big) < \psi_n(\sigma_1),$$

a contradiction. This completes the proof of the result. □

Here let us recall that for each $t \leqslant \sigma + \tau$ we have

$$\psi_n(t) \leqslant \cdots \leqslant \psi_1(t) \leqslant \psi_0(t) \leqslant \phi(t-\tau).$$

As a consequence of this remark, to prove Theorem 4.10 it is sufficient to show that $\psi_n(\tau) > \bar{x}$.

Proof of Theorem 4.10. Let us assume that $\sigma = 0$ and that $\phi'(0) = 0$. Next from Lemma 4.13 we have $S_M < \psi_n(\tau)$. Moreover let us recall that for each $t \leqslant \tau$ and each $j = 1, \ldots, n$ we have

$$\psi_j(t) \leqslant \phi(t-\tau).$$

Therefore $S_M < \phi(0)$. We shall argue by contradiction by assuming that $\psi_n(\tau) \in (S_M, \bar{x}]$. Next there exists a unique $t^* < \tau$ such that $\psi_n(t^*) = S_M$. We claim that

Claim 4.14. *There exists $\sigma_2 > 0$ such that $\sigma_2 > t^*$ and*

$$\phi'(\sigma_2) = 0, \qquad \phi''(\sigma_2) \geqslant 0, \qquad \phi'(t) < 0 \quad \text{on } (0, \sigma_2).$$

Before proving this claim, let us complete the proof of the result. First note that, due to this claim, we get

$$\alpha h\big(\psi_n(\sigma_2)\big) \leqslant \phi(\sigma_2).$$

Next ψ_0 is non-decreasing on $(-\infty, \tau)$ and non-increasing on $(\tau, \tau + \sigma_2)$, so that $\psi_0'(\tau) = 0$ and we obtain that

$$\psi_0(\tau) = \phi(0).$$

By induction we obtain that for each $j = 1, \ldots, n$, the map ψ_j is non-decreasing on $(-\infty, \tau)$ and non-increasing on $(\tau, \tau + \sigma_2)$. Thus $\psi_j'(\tau) = 0$ so that $\psi_j(\tau) = \psi_{j-1}(\tau)$. Therefore we get that

$$\psi_n(\tau) = \phi(0).$$

Due to this property we get that $\psi_n(\sigma_2) \leqslant \bar{x}$.

We now discuss the argument according to the location of σ_2 with respect to τ.

If $\sigma_2 \geqslant \tau$ then $\phi(\sigma_2) < \phi(\sigma_2 - \tau)$. Moreover since for any $j = 1, \ldots, n$ $\psi_j'(\sigma_2) \leqslant 0$ we obtain that $\phi(\sigma_2 - \tau) \leqslant \psi_n(\sigma_2)$. As a consequence we get

$$\alpha h\big(\psi_n(\sigma_2)\big) \leqslant \phi(\sigma_2) < \phi(\sigma_2 - \tau) \leqslant \psi_n(\sigma_2).$$

This implies $\psi_n(\sigma_2) > \bar{x}$, a contradiction.

If $\sigma_2 < \tau$ then since $\sigma_2 > t^*$ we obtain that $\psi_n(\sigma_2) > S_M$. Here again we split the argument into two parts $\phi(\sigma_2) \leqslant \psi_n(\sigma_2)$ and $\phi(\sigma_2) > \psi_n(\sigma_2)$.

When $\phi(\sigma_2) \leqslant \psi_n(\sigma_2)$ then

$$\psi_n(\sigma_2) \geqslant \phi(\sigma_2) \geqslant g\big(\psi_n(\sigma_2)\big) \quad \text{and} \quad \psi(\sigma_2) < \bar{x},$$

a contradiction.

When $\phi(\sigma_2) > \psi_n(\sigma_2)$, then since $\sigma_2 > t^*$ we obtain that $\psi_n(\sigma_2) > S_M$ and therefore

$$h\big(\phi(\sigma_2)\big) < h\big(\psi_n(\sigma_2)\big).$$

Since $\phi(\sigma_2) \geqslant \alpha h(\psi_n(\sigma_2))$ we obtain that $\phi(\sigma_2) \leqslant \bar{x}$ and

$$\phi(\sigma_2) > \alpha h\big(\phi(\sigma_2)\big),$$

a contradiction.

It remains to prove Claim 4.14: We now split our argument into two parts:

First: Assume that $t_* > 0$. Then since $\phi'(0) = 0$, $\phi(0) > 0$ and the map $g(\psi_n)$ is non-decreasing on $[0, t_*]$ we obtain from Lemma 4.12 that $\phi'(t) < 0$ on $(0, t_*]$. Therefore, there exists $\sigma_2 > t^*$ such that

$$\phi'(\sigma_2) = 0, \qquad \phi''(\sigma_2) \geqslant 0, \qquad \phi'(t) < 0 \quad \text{on } (0, \sigma_2).$$

Second: Assume now that $t_* \leqslant 0$. Then the map $t \to g(\psi_n(t))$ is non-increasing on $[0, \tau]$. Therefore since $\phi(0) > 0$ and $\phi'(0) = 0$ we obtain from Lemma 4.11 that $\phi'(t) < 0$ for all $t \in (0, \tau]$. As a consequence, there exists $\sigma_2 > \tau$ such that

$$\phi'(\sigma_2) = 0, \qquad \phi''(\sigma_2) \geqslant 0 \quad \text{and} \quad \phi(\sigma_2) < \phi(0) \leqslant \bar{x}.$$

This completes the proof of Claim 4.14 and therefore the proof of Theorem 4.10. □

Theorem 4.15. *If $\sigma < \infty$ then $\operatorname{sc} x_{\sigma+\tau} \in \{0, 1, 2\}$. Moreover we have*

$$\begin{aligned} &\phi(s) \in \big[\bar{x}^-, \bar{x}^+\big] \quad \textit{for all } s \geqslant \sigma, \\ &\psi_k(s) \in \big[\bar{x}^-, \bar{x}^+\big], \quad \forall s \geqslant \sigma + \tau,\ k = 0, \ldots, n. \end{aligned} \tag{4.6}$$

Proof. Without loose of generality we shall assume that $\sigma = 0$. We shall split to proof into two parts. First we shall prove

$$\operatorname{sc} x_{\sigma+\tau} \in \{0, 1, 2\} \tag{4.7}$$

and then that (4.6) holds true.

According to Theorem 4.10, there exists a unique $t^* < \tau$ such that $\psi_n(t_*) = S_M$. In order to prove (4.7) we shall discuss the location of t^* with respect to $\sigma = 0$.

First case: Assume that $\phi'(t) < 0$ on $(0, \tau]$ and set $\tau_2 = \sup\{t\colon\ \phi'(s) < 0,\ \forall s \in (0, t)\} > \tau$. Then is clear that $\operatorname{sc} x_\tau \in \{1, 2\}$. If $\tau_2 = \infty$ then $\phi(t) \geqslant \bar{x}$ for all $t \geqslant 0$ and the result follows. Assume that τ_2 is finite. Then $\phi'(\tau_2) = 0$ and $\phi''(\tau_2) \geqslant 0$. We obtain that

$$\frac{1}{c^2}\phi''(\tau_2) = \phi(\tau_2) - \alpha h\big(\psi_n(\tau_2)\big) \geqslant 0.$$

Thus

$$\phi(\tau_2) \geqslant \alpha h\big(\psi_n(\tau_2)\big).$$

Since $\psi_0'(t) > 0$ on $(-\infty, \tau)$ and $\psi_0'(t) < 0$ on $(\tau, \tau + \tau_2)$. Thus since $\tau_2 > \tau$ we get $\psi_0'(\tau_2) = -\psi_0(\tau_2) + \phi(\tau_2 - \tau) \leqslant 0$, that is

$$\phi(\tau_2 - \tau) \leqslant \psi_0(\tau_2) \leqslant \cdots \leqslant \psi_n(\tau_2).$$

Recall that $\phi(\tau_2) < \phi(\tau_2 - \tau) \leqslant \psi_n(\tau_2)$. This implies that

$$\phi(\tau_2) < \psi_n(\tau_2) \quad \text{and} \quad \alpha h\big(\psi_n(\tau_2)\big) \leqslant \phi(\tau_2).$$

Thus $\psi_n(\tau_2) > \bar{x}$.

Next

$$\phi(\tau_2) = \frac{1}{c^2}\phi''(\tau_2) + \alpha h\big(\psi_n(\tau_2)\big) \geqslant \alpha h\big(\psi_n(\tau_2)\big) \geqslant \bar{x}^-.$$

Thus $\phi(t) \geqslant \bar{x}^-$ for $t \in [0, \tau]$.

Second case: We assume that $t^* \leqslant 0$ and $\phi'(a) = 0$ for some $a \in (0, h]$. Since $\Gamma(t) = \alpha h(\psi_n(t))$ is strictly decreasing on $[0, a]$, we obtain from Lemma 4.11 that $\phi'(t) < 0$ for all $t \in (0, a)$ and $\phi''(a) = z'(a) > 0$. Hence ϕ has at most one critical point $a \in (0, \tau]$ and $\phi'(t) < 0$ on $t \in (0, a)$ and $\phi'(t) > 0$ on $t \in (a, \tau]$. Thus $\operatorname{sc} x_\tau \in \{0, 1, 2\}$ and we set $\tau_2 = a$. Next since $t^* \leqslant 0$ then $\psi_n(\tau_2) > S_M$ and we have

$$\phi(\tau_2) = \frac{1}{c^2}\phi''(\tau_2) + \alpha h\big(\psi_n(\tau_2)\big) \geqslant \alpha h\big(\psi_n(\tau_2)\big) \geqslant \bar{x}^-,$$

and the result is proved.

Third case: We assume that $t^* \leqslant 0$ and $\phi'(t) > 0$ on $(0, \tau]$. We set

$$c := \sup\big\{t\colon \phi'(s) > 0,\ s \in (0, t)\big\}.$$

Since $\phi(0) > \bar{x}$, we obtain that c is finite, $\phi'(c) = 0$, $\phi''(c) \leqslant 0$ and $\Gamma(t) = \alpha h(\psi_n(t))$ is strictly decreasing on $(0, c)$. From Lemma 4.11, we get that $\phi''(c) = z'(c) > 0$, a contradiction.

Last case: We assume that $t^* > 0$. Then the map $t \to \Gamma(t)$ is strictly increasing on $(-\infty, t^* + \tau]$ and $\phi'(t) < 0$ for $t \in (0, t_*]$. Let us assume that there exists $a \in (t^*, \tau]$ such that $\phi'(a) = 0$. Then Lemma 4.11 shows that $\phi''(a) > 0$ and ϕ has at most one critical point on $(0, h]$ and $\operatorname{sc} x_\tau \in \{0, 1, 2\}$.

If $\phi'(t) < 0$ on $(0, \tau]$ we obtain the first case. If there exists $a \in (0, \tau]$ such that $\phi'(a) = 0$ we conclude as in the second cases.

This completes the proof of (4.7).

Let us now prove (4.6). Since the upper bound is obvious, we first prove that $\phi(s) \geqslant \bar{x}^-$ for all $s \geqslant 0$. We shall argue by contradiction by assuming that there exists $t_0 \geqslant 0$ such that $\phi(t_0) < \bar{x}^-$. Let us denote by $\widehat{\tau} > 0$ the first point where

$$\phi(\widehat{\tau}) < \bar{x}^-, \qquad \phi'(\widehat{\tau}) = 0, \qquad \phi''(\widehat{\tau}) \geqslant 0.$$

Therefore we obtain that

$$\alpha h\big(\psi_n(\widehat{\tau})\big) \leqslant \phi(\widehat{\tau}) < \bar{x}^-.$$

On the other hand one has

$$\phi(s - \tau) \geqslant \phi(\widehat{\tau}), \quad \forall s \in [\tau, \widehat{\tau} + \tau].$$

Since for each $k = 0, \dots, n$, $\psi_k(\tau) > \bar{x} > \bar{x}^-$, we obtain that

$$\psi_k(s) > \phi(\widehat{\tau}), \quad \forall s \in [\tau, \widehat{\tau} + \tau].$$

This implies in particular that

$$\psi_n(\widehat{\tau}) > \phi(\widehat{\tau}).$$

We finally obtain that

$$\alpha h\big(\psi_n(\widehat{\tau})\big) \leqslant \phi(\widehat{\tau}) < \bar{x}^- \quad \text{and} \quad \psi_n(\widehat{\tau}) > \phi(\widehat{\tau}).$$

While the first inequality implies that $\psi_n(\widehat{\tau}) < \bar{x}^-$, the inequality $\alpha h(\psi_n(\widehat{\tau})) \leqslant \psi_n(\widehat{\tau})$ implies that $\psi_n(\widehat{\tau}) > \bar{x}$, a contradiction. Therefore we obtain

$$\phi(s) \geqslant \bar{x}^- \quad \text{for all } s \geqslant 0.$$

We now complete the proof of the result. To do so note that we have

$$\psi_0'(t) = -\psi_0(t) + \phi(t - \tau) \geqslant -\psi_0(t) + \bar{x}^-, \quad \forall t \geqslant \tau.$$

Since $\psi_0(\tau) > \bar{x}^-$, the result follows for ψ_0. The proof for ψ_k with $k = 1, \dots, n$ is similar and (4.6) holds true. The completes the proof of the result. □

As a direct consequence of Lemma 3.3 (see Theorem 2.1 in [22] for more details) and Theorem 4.15 we obtain that:

Lemma 4.16. *Let us assume that $x^* < \bar{x}^-$ and $\sigma < \infty$. Then for each $t \geqslant \sigma + \tau$ we have*

$$\operatorname{sc} x_t \in \{0, 1, 2\}.$$

5. Applications

We aim of this section is to study some properties of the solution of (1.7) when $t \to \infty$. Most of these results will be related to the properties of the characteristic equation associated to (1.7) around the equilibrium point $\bar{x}$ when it exists, that is when $\alpha > 1$. This characteristic equation reads

$$\Delta^+(c, \lambda) = \left(\frac{\lambda^2}{c^2} - 1\right)(1 + \lambda)^{n+1} + (1 - \ln \alpha)e^{-\tau\lambda}. \tag{5.1}$$

5.1. Non-monotone solutions

Since the result of Theorem 4.15 are related to the assumption $\sigma < \infty$, it is important to give some conditions that ensure that such a property holds true. We shall derive the following result:

Proposition 5.1. *Let $c > c^*$ be given and let $(\phi, \psi_0, \dots, \psi_n)$ be a converging and non-oscillating solution of* (1.7) *when $t \to \infty$,*

$$\lim_{t \to \infty} (\phi, \psi_0, \dots, \psi_n)(t) = \bar{x}(1, 1, \dots, 1).$$

Then there exists $\lambda_0 \in (-\infty, 0]$ such that $\Delta^+(c, \lambda_0) = 0$.

Proof. In order to deal with functions tending to zero at infinity we set

$$x_0(t) = \bar{x} - \phi(t), \qquad x_1(t) = x_0'(t),$$
$$x_k(t) = \bar{x} - \psi_{k-2}(t), \quad k = 2, \dots, n+2.$$

Recall that it satisfies (3.11). Since $t \to x(t)$ does not oscillate around $(0, 0, \dots, 0)$, Theorem 3.5 implies that $t \to x(t)$ does not converge superexponentially to zero. Let $\rho > 3\tau$ be given. Using Corollary 24 in [31], there exist a sequence $t_j \to \infty$ and some constant $d > 1$ such that the map $f(t) := \|x_t\|$ satisfies

$$f(t_j) = \max_{s \geqslant t_j} f(s), \qquad \max_{t_j - \rho \leqslant s \leqslant t_j} f(s) \leqslant d f(t_j).$$

Next we set the vector valued function $y_j(t) = (y_{0,j}(t), \dots, y_{n+2,j}(t))$ defined by

$$y_{k,j}(t) = \frac{x_k(t + t_j)}{\|x_{t_j}\|}, \quad j \in \mathbb{N},\ k = 0, \dots, n+2.$$

Note that we have

$$\left| y_{k,j}(t) \right| \leqslant d, \quad \forall j \geqslant 0,\ k = 0, \dots, n+2,\ t \geqslant -2\tau.$$

Let us notice that y_j satisfies the system of equations

$$\begin{aligned}
\frac{dy_{0,j}(t)}{dt} &= y_{1,j}(t), \\
\frac{dy_{1,j}(t)}{dt} &= c^2\big(y_{0,j}(t) + f^*\big(x^2(t+t_j)\big) y_{2,j}(t)\big), \\
\frac{dy_{k,j}(t)}{dt} &= -y_{k,j}(t) + y_{k+1,j}(t), \quad k = 2, \dots, n+1, \\
\frac{dy_{n+2,j}(t)}{dt} &= -y_{n+2,j}(t) + y_{0,j}(t - \tau),
\end{aligned} \tag{5.2}$$

where we have set

$$f^*(s) = \begin{cases} \frac{f(s)}{s}, & s \neq 0, \\ f'(0), & s = 0. \end{cases}$$

Moreover for each $j \geqslant 0$ there exists $\theta_j \in [-\tau, 0]$ such that

$$\left\| y_j(\theta_j) \right\| = 1, \quad \forall j \geqslant 0.$$

Due to system (5.2), the sequence $t \to y_j(t)$ is bounded in $C^1([-\tau, \infty), \mathbb{R}^{n+2})$. Up to a subsequence, one may assume that y_j converges towards $y_* = (y_{0,*}, \dots, y_{1,*})$ locally uniformly and $\theta_j \to \theta_0 \in [-\tau, 0]$. Moreover passing to the limit in (5.2) along a subsequence if necessary, we obtain that y_* satisfies the linear system of equations on $[-\tau, \infty)$:

$$\begin{aligned}
\frac{dy_{0,*}(t)}{dt} &= y_{1,*}(t), \\
\frac{dy_{1,*}(t)}{dt} &= c^2\big(y_{0,*}(t) + f'(0)y_{2,*}(t)\big), \\
\frac{dy_{k,*}(t)}{dt} &= -y_{k,*}(t) + y_{k+1,*}(t), \quad k = 2, \ldots, n+1, \\
\frac{dy_{n+2,*}(t)}{dt} &= -y_{n+2,*}(t) + y_{0,*}(t-\tau),
\end{aligned} \tag{5.3}$$

together with the normalization condition

$$\big\| y_*(\theta_0) \big\| = 1.$$

Since the characteristic equation associated to system (5.3) reads as Δ^+ and since this complex map is of type τ, we conclude using Theorem 3.1 in [14] (see also Theorem 4.4 in [34] or [33]), that y_* is not a small solution so that its asymptotic behaviour when $t \to \infty$ is driven by its eigenvectors. Finally if the equation $\Delta(c, \lambda) = 0$ does not have any real and negative solution, then according to Proposition 7.2 in [21] (see also Proposition 2.2 in [16]), we conclude that y_* oscillates around zero, that is contradiction with the definition of y_* and the non-oscillating behaviour of x at $t = \infty$. This completes the proof of the result. □

Corollary 5.2. *Let us assume that $x^* < \bar{x}^-$. Let $c > c^*$ be given and let $(\phi, \psi_0, \ldots, \psi_n)$ be a non-eventually monotone solution of system* (1.7) *when $t \to \infty$. Then there exists $\widehat{t} \in \mathbb{R}$ large enough such that for each $t \geqslant \widehat{t}$,*

$$\operatorname{sc} x_t \in \{1, 2\},$$

and ϕ oscillates around the positive equilibrium $\bar{x}$.

The proof of the result is straightforward by noticing that when ϕ is not eventually monotone when $t \to \infty$ then ϕ' oscillates around zero.

Remark 5.3. Let us notice that this result has some consequences. Indeed assume that $x^* < \bar{x}^-$ and, let $c > c^*$ and $(\phi, \psi_0, \ldots, \psi_n)$ be a non-converging solution of system (1.7) when $t \to \infty$. Then ϕ has undamped oscillations around $\bar{x}$ when $t \to \infty$.

5.2. Point to oscillating solution connection

We shall now study the oscillating properties of the solutions around $\bar{x}$. We first investigate some properties of converging non-small solutions in the following lemma:

Lemma 5.4. *Let $c > c^*$ be given and $(\phi, \psi_0, \ldots, \psi_n)$ a non-small solution of* (1.7) *such that*

$$\lim_{t \to \infty} (\phi, \psi_0, \ldots, \psi_n)(t) = \bar{x}(1, 1, \ldots, 1).$$

Let us assume that the equation $\Delta^+(c, \lambda) = 0$ does not have any solution in the set

$$S_{00} = \left\{ \lambda \in \mathbb{C}\colon\ \Re\lambda \leqslant 0 \text{ and } \Im\lambda \in \left[-\frac{2\pi}{\tau}, \frac{2\pi}{\tau} \right] \right\}.$$

Then, recalling definition (3.4), *for each $\widehat{t}$, there exists $t > \widehat{t}$ such that $V(x_t) \geqslant 3$.*

Proof. The proof of this result will follow the same steps as the proof of Lemma 5.1. In order to deal with functions tending to zero at infinity we set

$$x_0(t) = \bar{x} - \phi(t), \qquad x_1(t) = x_0'(t),$$
$$x_k(t) = \bar{x} - \psi_{k-2}(t), \quad k = 2, \dots, n+2.$$

Recall that it satisfies (3.11). Since the map $t \to x(t)$ does not converge superexponentially to zero, there exist a sequence $t_j \to \infty$ and some $d > 1$ such that the map $f(t) := \|x_t\|$ satisfies

$$f(t_j) = \max_{s \geqslant t_j} f(s), \qquad \max_{t_j - 3\tau \leqslant s \leqslant t_j} f(s) \leqslant d f(t_j).$$

Next using the same arguments as in the proof of Lemma 5.1, the sequence of vector valued function $y_j(t) = (y_{0,j}(t), \dots, y_{n+2,j}(t))$ defined by

$$y_{k,j}(t) = \frac{x_k(t + t_j)}{\|x_{t_j}\|}, \quad j \in \mathbb{N}, \ k = 0, \dots, n+2,$$

converges (up to a subsequence) towards $y_* = (y_{0,*}, \dots, y_{1,*})$ locally uniformly on $[-\tau, \infty)$ where y_* is a nontrivial solution of the linear system of Eqs. (5.3) on $[-\tau, \infty)$ Recall that y_* is not a small solution so that for each sufficiently large $|\nu|$, with $\nu > 0$ we have

$$y_*(t) = v(t) + 0\big(e^{-\nu t}\big), \quad t \to \infty,$$

where v is a nonempty finite sum of eigensolutions associated to the eigenvalues in $F = \{\lambda \in \mathbb{C}\colon \Delta^+(c, \lambda) = 0, \ -\nu < \Re\lambda \leqslant 0\}$. Since we have

$$F \cap \{\lambda\colon -2\pi \leqslant \tau \Im\lambda \leqslant 2\pi\} = \emptyset,$$

we obtain that for each $\widehat{t} \in \mathbb{R}$ there exists $\beta > \widehat{t}$ large enough such that the map y_* changes its sign at least three times on the interval $(\beta, \beta + \tau)$. Recalling that the sequence y_j converges uniformly to y_* on the interval $[\beta, \beta + \tau]$, we conclude that y_j also changes its sign at least three times on this interval when j is large enough. This completes the proof of the result. □

We shall now give some conditions on the characteristic equation at $\bar{x}$ that ensure the non-existence of a 0 to $\bar{x}$ connection for system (1.7), and thus, due to Remark 5.3, the existence of point to sustained oscillating connection for this problem. To do so we shall assume that:

Assumption 5.5. Let $c > c^*$ be given such that the equation $\Delta^+(c, \lambda) = 0$ does not have any solution in the strip S_{00} defined by

$$S_{00} = \left\{\lambda \in \mathbb{C}\colon \Re\lambda \leqslant 0 \text{ and } \Im\lambda \in \left[-\frac{2\pi}{\tau}, \frac{2\pi}{\tau}\right]\right\}.$$

We complete this section by the following result.

Theorem 5.6. *Recalling definition* (3.6), *let us assume that* $x^* < \bar{x}^-$. *Let* $c > c^*$ *be given such that Assumptions* 4.6 *and* 5.5 *are satisfied. Then any solution* $(\phi, \psi_0, \dots, \psi_n)$ *of* (1.7) *with the speed* c *has non-decaying sustained oscillations at* $t = \infty$.

Remark 5.7. Note that the assumption $x^* < \bar{x}^-$ can be re-written in term of a condition on $\alpha > e$ as

$$\alpha^3 \exp\left(-\frac{\alpha}{e} - \alpha^2 e^{-\frac{\alpha}{e}}\right) > \ln\alpha,$$

that corresponds to the assumption of Theorem 1.2.

Proof. Let $(\phi, \psi_0, \ldots, \psi_n)$ be a solution of (1.7). In view of Corollary 5.2, it is sufficient to show that $(\phi, \psi_0, \ldots, \psi_n)$ does not converge to $\bar{x}(1, 1, \ldots, 1)$ when $t \to \infty$. Let us argue by contradiction by assuming that

$$\lim_{t\to\infty} (\phi, \psi_0, \ldots, \psi_n)(t) = \bar{x}(1, 1, \ldots, 1).$$

Since $x^* < \bar{x}^-$, Lemma 4.16 applies and show that $\lim_{t\to\infty} V(x_t) \in \{0, 1, 2\}$. As a consequence of Theorem 3.5, $(\phi, \psi_0, \ldots, \psi_n)$ is not a small solution at $t = \infty$. Thus due to Lemma 5.4 we get that

$$\lim_{t\to\infty} V(x_t) \geqslant 3,$$

a contradiction. This completes the proof of the result. □

Corollary 5.8. *Let us assume that $S_M < \bar{x}^-$. Let $c > c^*$ be given such that Assumptions* 4.6 *and* 5.5 *are satisfied. Then any solution $(\phi, \psi_0, \ldots, \psi_n)$ with the speed c is ultimately periodic at $t = \infty$.*

The proof of this result is a direct consequence of Poincaré–Bendixon theorem provided by Mallet-Paret and Sell in [23].

6. Further results on the characteristic equation

This section is devoted to given some conditions that ensure that Assumption 4.6 as well as Assumption 5.5 hold true.

6.1. On Assumption 4.6

In order to study some properties of the equation $\Delta^-(c, \lambda) = 0$, we introduce $\varepsilon = \frac{1}{c^2}$ and $\varepsilon^* = \frac{1}{c^{*2}}$ and we introduce the map

$$\Psi(\varepsilon, \lambda) = \left(\varepsilon\lambda^2 - 1\right)(\lambda + 1)^{n+1} + \alpha e^{-\tau\lambda},$$

and as well as the map

$$\Phi(\lambda) = \Psi(0, \lambda) = -(\lambda + 1)^{n+1} + \alpha e^{-\tau\lambda}.$$

Note that we have

$$\Psi(\varepsilon, \lambda) = \Phi(\lambda) + \varepsilon P(\lambda),$$

wherein we have set

$$P(\lambda) = \lambda^2(\lambda + 1)^{n+1}.$$

First we have the following lemma:

Lemma 6.1. *Let $\alpha > 1$ and $\tau > 0$ be given. Then the equation $\Phi(\lambda) = 0$ with $\Re\lambda \geqslant 0$ has only one real root $0 < z_1 < \alpha^{\frac{1}{n+1}} - 1$. Moreover, all other roots z_j, $j = 2, 3, \ldots$, are simple and can be enumerated in such a way that*

$$0 \leqslant \cdots \leqslant \Re z_3 = \Re z_2 < z_1.$$

Proof. The equation $\Phi(\lambda) = 0$ with $\lambda \geqslant 0$ reads as finding $\lambda \geqslant 0$ such that

$$(\lambda + 1)^{n+1} = \alpha e^{-\tau\lambda}.$$

Since $\alpha > 1$ the existence and uniqueness of z_1 follows.

We now consider a root z such that $\Re z \geqslant 0$ and $z \notin [0, \infty)$. Then there exists $k = 0, \ldots, n + 1$ such that

$$z = -1 + pe^{-hz} e^{\frac{i2k\pi}{n+1}}, \quad \text{with} \quad p = \alpha^{\frac{1}{n+1}}, \quad h = \frac{\tau}{n+1}.$$

Since z is not real, we obtain that

$$\Re z < -1 + pe^{-h\Re z}$$

and the result follows. □

Lemma 6.2. *Let $\alpha > 1$ and $\tau > 0$ be given. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the equation*

$$\Psi(\varepsilon, \lambda) = 0, \qquad \Re\lambda \geqslant 0,$$

has exactly two real roots $\lambda_1(\varepsilon)$ and $\lambda_\infty(\varepsilon)$ such that

$$0 < z_1 < \lambda_1(\varepsilon) < 2(p - 1) < \varepsilon^{-1/4} < \lambda_\infty(\varepsilon) < \varepsilon^{-1/2}. \tag{6.1}$$

Moreover, all the roots with positive real part, denoted by $\{\lambda_j(\varepsilon)\}_{j \geqslant 1}$, are simple and we have

$$\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = z_j, \quad \forall j \geqslant 1.$$

Proof. Set $p = \alpha^{\frac{1}{n+1}}$ and note that

$$\Psi(\varepsilon, 0) = \Phi(0) > 0, \qquad \Psi(\varepsilon, z_1) = \varepsilon P(z_1) > 0.$$

Next we have for any sufficiently small $\varepsilon > 0$,

$$\Psi(\varepsilon, 2p - 1) = \left(e^{-\tau(2p-1)} - 2^{n+1} + \varepsilon 2^{n+1}(2p - 1)^2\right)\alpha < 0.$$

Next let us notice that $\Psi(\varepsilon, \varepsilon^{-1/2}) > 0$. Finally we have

$$\Psi\left(\varepsilon, \varepsilon^{-1/4}\right) = \left(\varepsilon^{1/2} - 1\right)\left(\varepsilon^{-1/4} + 1\right)^{n+1} + \alpha e^{-\tau\varepsilon^{-1/4}} \sim -\varepsilon^{-(n+1)/4} < 0.$$

This completes the proof of (6.1). To complete the proof we will show that there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the equation $\Psi(\varepsilon, \lambda) = 0$ has unique root in the half plane $\mathcal{P} = \{\lambda \in \mathbb{C}\colon \Re\lambda > 2p - 1\}$. To do so we apply Rouché's theorem and show that when $\varepsilon > 0$ is small enough then the maps $\Psi(\varepsilon, \lambda)$ and the polynomial $\lambda \to (\varepsilon\lambda^2 - 1)(\lambda + 1)^{n+1}$ have the same number of roots in some rectangle $R_\varepsilon = [2p - 1, 2p - 1 + b_\varepsilon] \times [-c_\varepsilon, c_\varepsilon]$, wherein $b_\varepsilon > 0$ and $c_\varepsilon > 0$ are large enough. The details of the proof of this result are similar to the one used for Lemma 13 in [11]. □

As a direct consequence of the above lemmas we obtain the following result:

Lemma 6.3. *Let $\alpha > 1$, $\tau > 0$ and $n \geqslant 0$ be given. Then there exists $\widehat{c} > 0$ such that any $c \geqslant \widehat{c}$ satisfies Assumption* 4.6.

From this lemma we shall show the following result:

Proposition 6.4. *Let $\alpha > 1$, $\tau > 0$ and $n \geqslant 0$ be given. Then any $c > c^*$ satisfies Assumption* 4.6.

Proof. To prove this result we shall argue by continuation. Let us consider $c > c^*$ and consider a root of the equation $\Delta^-(c, \lambda) = 0$ with $\Re\lambda = \lambda_1(c)$ provided by Lemma 4.1. Let us write this root as $\lambda = \lambda_1(c) + i\omega$ with $\omega \geqslant 0$. Then we shall show that $\omega = 0$. To do so let us notice that λ satisfies

$$\left|\frac{\lambda^2}{c^2} - 1\right| \leqslant \alpha \frac{e^{-\tau\lambda_1(c)}}{|1+\lambda|^{n+1}}$$
$$\leqslant \alpha \frac{e^{-\tau\lambda_1(c)}}{|1+\lambda_1(c)|^{n+1}} \leqslant \left|\frac{\lambda_1(c)^2}{c^2} - 1\right|.$$

This re-writes as

$$\omega^2\left(\frac{\omega^2}{c^2} + \frac{2 + 2\lambda_1(c)^2}{c^2}\right) \leqslant 0,$$

so that $\omega = 0$ and the result follows. □

6.2. On Assumption 5.5

We now study Assumption 5.5 and we will state the following result.

Proposition 6.5. *Let $\alpha > e^2$ be given and $n \geqslant 0$ be given. Let $\underline{c} : (0, \infty) \to (0, \infty)$ be a function such that*

$$\liminf_{\tau\to\infty} \tau\underline{c}(\tau) > \Gamma(\alpha) := \frac{\pi}{\sqrt{\ln\alpha - 2}}. \tag{6.2}$$

Then there exists $\tau^* = \tau^*(\alpha, n)$ such that for each $\tau > \tau^*$ and each $c > \underline{c}(\tau)$, the equation

$$\left(\frac{\lambda^2}{\tau^2 c^2} - 1\right)\left(1 + \frac{\lambda}{\tau}\right)^{n+1} = (\ln\alpha - 1)e^{-\lambda}$$

does not have any solution in the strip

$$S = \left\{\lambda \in \mathbb{C}\colon \Re\lambda \leqslant 0 \text{ and } \Im\lambda \in [-2\pi, 2\pi]\right\},$$

Before proving this result, let us show the following lemma:

Lemma 6.6. *Let $\alpha > e^2$ be given. Then for each $\gamma \in (\Gamma(\alpha), \infty)$, the equation*

$$\left(\frac{\lambda^2}{\gamma^2} - 1\right) = (\ln\alpha - 1)e^{-\lambda},$$

does not have any solution in S.

Proof. To prove this result, let us first notice that for each $\gamma > 0$, the equation

$$\left(\frac{\lambda^2}{\gamma^2} - 1\right) = (\ln\alpha - 1)e^{-\lambda},$$

does not have any solution $\lambda = -\mu \pm 2\pi i$ for some $\mu > 0$. Next it is easily checked that this equation has a solution $\lambda = i\omega$ with $\omega \in [-2\pi, 2\pi]$ if and only if $\gamma = \Gamma(\alpha)$. Finally let us notice that when $\gamma \to \infty$, this equation does not have any solution in S. Indeed, let $\{\gamma_m\}_{m\geqslant 0}$ be a given sequence tending to ∞ and assume by contradiction that there exists for each $m \geqslant 0$ a solution $\lambda_m \in S$ of the equation

$$\left(\frac{\lambda_m^2}{\gamma_m^2} - 1\right) = (\ln\alpha - 1)e^{-\lambda_m}.$$

Let us first notice that $\{\lambda_m\}$ is unbounded because since $\alpha > e^2$, the equation $-1 = (\ln\alpha - 1)e^{-\lambda}$ does not have any solution in S. Thus, possibly up to a subsequence, one may assume that

$$\mu_m := -\Re\lambda_m \to \infty, \qquad \nu_m = \Im\lambda_m \to \nu_\infty \in [-2\pi, 2\pi].$$

As a consequence we obtain that for each $m \geqslant 0$,

$$\left(\frac{1}{\gamma_m^2} - \frac{1}{\lambda_m^2}\right) = (\ln\alpha - 1)\frac{e^{-\lambda_m}}{\lambda_m^2}.$$

By letting $m \to \infty$, the left-hand side of the above equality remains bounded while the right-hand side is unbounded. This leads to a contradiction.

The proof of the result now directly follows from standard results on holomorphic maps. □

Proof of Proposition 6.5. Assume by contradiction, that there exists a sequence $\{\tau_m \to \infty\}_{m\geqslant 0}$, a sequence $\{c_m\}_{m\geqslant 0} \subset [0, \infty)$ such that $c_m \geqslant \underline{c}(\tau_m)$ and a sequence $\{\lambda_m\}_{m\geqslant 0} \subset S$ such that for each $m \geqslant 0$,

$$\frac{\lambda_m^2(1 + \frac{\lambda_m}{\tau_m})^{n+1}}{\tau_m^2 c_m^2} = \left(1 + \frac{\lambda_m}{\tau_m}\right)^{n+1} - (1 - \ln\alpha)e^{-\lambda_m}.$$

We first claim that $\{\lambda_m\}_{m\geqslant 0}$ is unbounded. Indeed if it is not unbounded, up to a subsequence, one may suppose that $\lambda_m \to \lambda_\infty \in S$. Note also that due to (6.2) one gets that (up to a subsequence) $c_m\tau_m \to \gamma \in (\Gamma(\alpha), \infty]$. First assume that $\gamma = \infty$. Then passing to the limit $m \to \infty$ one obtains that

$$1 = (1 - \ln\alpha)e^{-\lambda_\infty},$$

a contradiction with $\lambda_\infty \in S$.

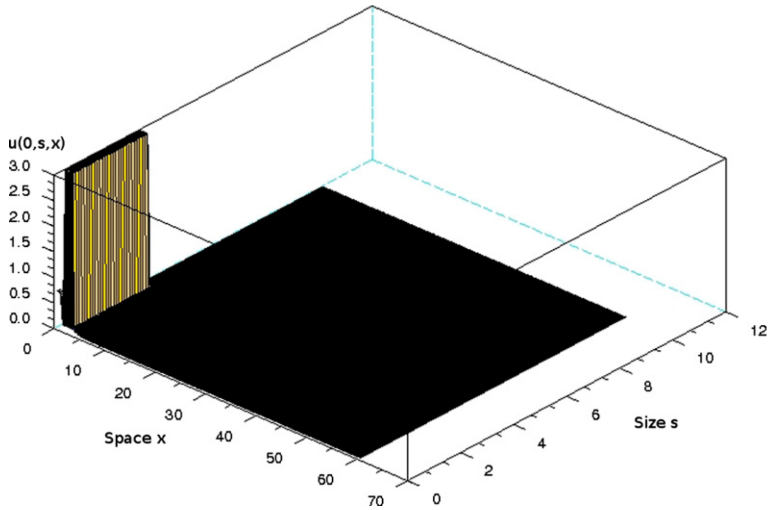

Fig. 1. Initial distribution of the population $u(0, s, x)$.

Next assume that $\gamma \in (\Gamma(\alpha), \infty)$. Then, passing to the limit one obtains that $\lambda_\infty \in S$ satisfies

$$\left(\frac{\lambda_\infty^2}{\gamma^2} - 1\right) = (\ln \alpha - 1)e^{-\lambda_\infty}.$$

Due to Lemma 6.6, one can observe that the above equation does not have any solution in S. This leads us to a contradiction and proves the claim.

Using this result, up to a subsequence, one may assume that

$$\Re\lambda_m := -\mu_m \to -\infty, \qquad \Im\lambda_m \to \nu_0 \in [-2\pi, 2\pi].$$

As a consequence, one obtains that for each $m \geqslant 0$,

$$\left(\frac{1}{\tau_m^2 c_m^2} - \frac{1}{\lambda_m^2}\right)\left(\frac{1}{\lambda_m} + \frac{1}{\tau_m}\right)^{n+1} = (\ln \alpha - 1)\frac{e^{-\lambda_m}}{\lambda_m^{n+3}}.$$

The right-hand side of the above expression goes to infinity in modulus while the left-hand side is bounded. This leads us to a contradiction that completes the proof of the result. □

To conclude this section and the proof of Theorem 1.2, let us notice that (i) follows from Corollary 5.2. The point (ii) follows from Theorem 5.6 together with Propositions 6.4 and 6.5 while (iii) is a consequence of Corollary 5.8. This completes the proof of Theorem 1.2.

7. Numerical simulations

In this section we come back to the evolution problem (1.1) and fulfill some numerical simulations of this model for various values of the parameter α in order to illustrate Theorem 1.2 as well as Theorem 4.5. For that purpose we choose $g(s) \equiv 1$, $\mu(s) \equiv 0$ and γ given in Assumption 1.1 (with $\mu = 1$). Initially the population we shall consider is given in Fig. 1. Moreover we choose $\tau = 3$ and $n = 1$.

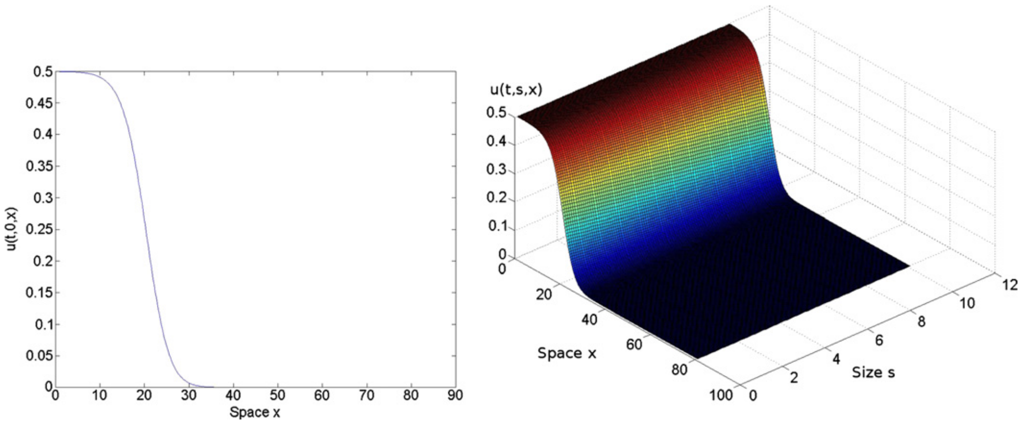

Fig. 2. Distribution of the population $u(t,s,x)$ at some time t for $\alpha = e^{0.5}$ with $s = 0$ on the left.

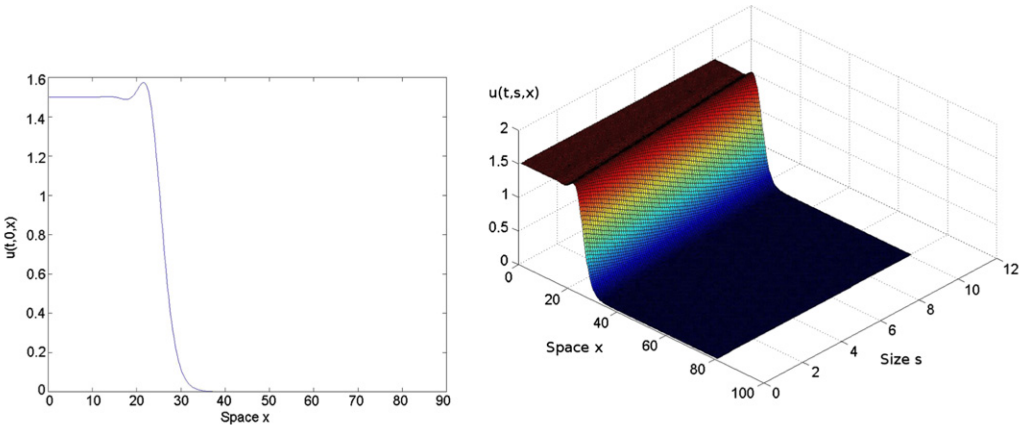

Fig. 3. Distribution of the population $u(t,s,x)$ at some time t for $\alpha = e^{1.5}$ with $s = 0$ on the left.

In order to deal with bounded domains in the size variable and also in space, one considers that a maximal size is $S_{max} = 10$ and we supplement the Laplace operator arising in the birth process of Eq. (1.1) together with homogeneous Neumann boundary conditions. Finally, system (1.1) is solved by using a numerical integration along the characteristic curves for the size variable while the spatial dispersal is solved by using a finite difference approximation. As shown in the figures the population invades the empty landscape where α is large enough, namely $\alpha > 1$. After settling some place, the population reaches either some constant state (with monotone invasion) (see Fig. 2), either exhibits some damped oscillations (or outbreaks in the population density) (see Fig. 3) or sustained oscillations giving rise to (numerically stable) spatio-temporal pattern formation and propagation (see Figs. 4 and 5). One can notice that the two first possibilities are explained by Theorem 4.5 while the third situation is partially explained by Theorem 1.2. Here one can notice that the conditions on α given in Theorem 1.2 re-writes as $\alpha \in (e^2, \alpha_M)$ with $\alpha_M \approx e^{2.83}$. However, as shown in Fig. 5, this parameter restriction seems to be a technical one. Indeed the parameter value $\alpha = e^5$ does not match with the assumptions of this theoretical result but the corresponding evolution problem also numerically exhibits some spatio-temporal pattern propagation.

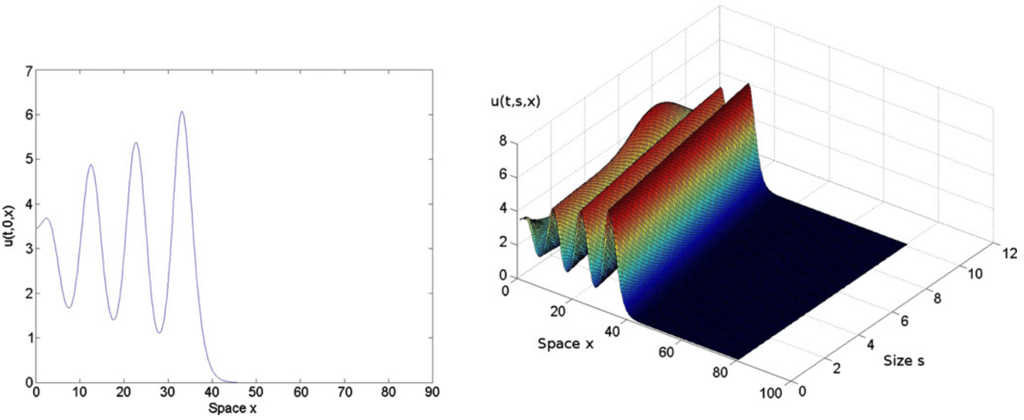

Fig. 4. Distribution of the population $u(t, s, x)$ at some time t for $\alpha = e^{2.8}$ with $s = 0$ on the left.

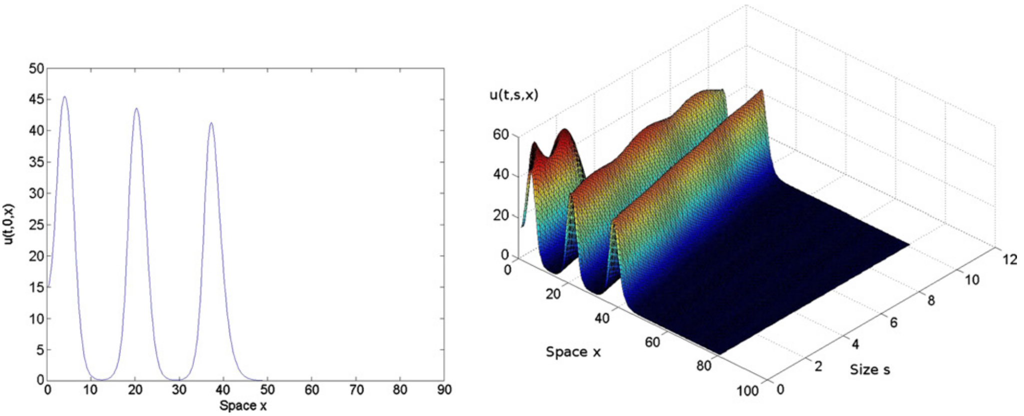

Fig. 5. Distribution of the population $u(t, s, x)$ at some time t for $\alpha = e^{5}$ with $s = 0$ on the left.

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