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A singular reaction–diffusion system modelling prey–predator interactions: Invasion and co-extinction waves

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ABSTRACT

We consider a singular reaction–diffusion system arising in modelling prey–predator interactions in a fragile environment. Since the underlying ODEs system exhibits a complex dynamics including possible finite time quenching, one first provides a suitable notion of global travelling wave weak solution. Then our study focusses on the existence of travelling waves solutions for predator invasion in such environments. We devise a regularized problem to prove the existence of travelling wave solutions for predator invasion followed by a possible co-extinction tail for both species. Under suitable assumptions on the diffusion coefficients and on species growth rates we show that travelling wave solutions are actually positive on a half line and identically zero elsewhere, such a property arising for every admissible wave speeds.

1. Introduction

In this work we consider a singular diffusive prey–predator system

$$\begin{aligned}
&\frac{\partial B(t,x)}{\partial t} - d\Delta B(t,x) = B(t,x)g\big(B(t,x)\big) - C(t,x),\\
&\frac{\partial C(t,x)}{\partial t} - \Delta C(t,x) = rC(t,x)\left(1 - \frac{C(t,x)}{B(t,x)}\right),
\end{aligned} \tag{1}$$

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posed for $t > 0$, $x \in \mathbb{R}^N$. The underlying spatially homogeneous problem was derived in Courchamp and Sugihara [6] to model prey–predator interactions in fragile (insular) environments. The spatially structured System (1) supplemented with initial data and no-flux boundary conditions was introduced in Gaucel and Langlais [9]. Herein $B(t,x)$ (resp. $C(t,x)$) denotes the density of prey (resp. predator) at time t located at $x \in \mathbb{R}^N$. Parameter $d > 0$ represents the normalized diffusion coefficient, namely the ratio between the actual diffusivities of prey and predators. Function g stands for the intrinsic growth rate of prey while $r > 0$ is the growth rate of predators. In the original model [6] $Bg(B)$ is a logistic growth function.

Note that System (1) can also be viewed as a special case of the so-called Holling–Tanner prey–predator system, see [11] or [12],

$$\begin{aligned} &\frac{\partial B(t,x)}{\partial t} - d\Delta B(t,x) = B(t,x)g\big(B(t,x)\big) - \frac{B(t,x)C(t,x)}{\gamma + B(t,x)}, \\ &\frac{\partial C(t,x)}{\partial t} - \Delta C(t,x) = rC(t,x)\left(1 - \frac{C(t,x)}{B(t,x)}\right). \end{aligned} \tag{2}$$

From a formal point of view, System (1) is a specific cases of System (2) with $\gamma = 0$. However, as it is proved in [9], System (1) may exhibit finite time quenching (that is not the case for System (2)) so that the above mentioned formal limit turns out to be a singular like limit.

The aim of this work is to look for the existence of travelling wave solutions of predator invasion for System (1). Numerical simulations in Burie, Ducrot and Langlais (work in progress) suggested that travelling wave solutions correspond to the typical numerical response of the system to introducing a spatially localized perturbation of predators within a homogeneous population of prey resting at its carrying capacity.

Due to the finite time quenching property of the evolution system under consideration, namely System (1), one may expect that under some suitable circumstances, after the predator invasion wave both populations may vanish, that is, the solution decreases to $B = C = 0$ at a finite spatio-temporal location. The aim of this work is to understand such a qualitative property of the predator invasion waves, that is the existence of so-called sharp travelling waves. We refer to [2] and [3] for first results on the existence of sharp travelling waves in the context of degenerate reaction–diffusion equations. One also refers to [13], [14], [16] or [17] for other results and discussions on this topic in the context of degenerate Fisher–KPP equations.

Such singular travelling waves problems also arise for reaction–diffusion (and convection) equations exhibiting a finite time blow-up. These waves are sometimes called semi-finite waves. See for instance the monograph of [10] and references therein.

Let us also mention that, according to our knowledge, only little work has been done for sharp waves or semi-finite waves for reaction–diffusion systems without comparison principle. We refer to [15] for results on the existence of travelling waves for degenerate reaction–diffusion systems without comparison principle modelling bacterial pattern formation.

Note that in this work System (1) does not exhibit any diffusion degeneracy but a singular reaction term. The method we shall develop is a regularization procedure. To be more precise, one shall first look at travelling wave solutions for the Holling–Tanner model (2) and then pass to the limit as $\gamma \to 0$ to get suitable (weak) travelling wave solutions to (1). Qualitative properties of these weak solutions are carefully studied to find sufficient conditions on the parameter set ensuring the existence of either everywhere positive waves or of sharp waves that vanish at some finite spatial location.

Our work is organized as follows. Section 2 is devoted to listing our main assumptions and to stating our main results. In Section 3 the existence of travelling wave solution for the regularized Holling–Tanner system is analyzed. Section 4 is concerned with the proof of some non-existence results. Section 5 deals with passing to the limit in the regularization to provide the existence of mild solutions. Finally, Sections 6–8 are concerned with the proofs of qualitative properties of travelling wave solutions.

2. Assumptions and main results

We will first give a definition of travelling wave solutions for (1). Let $e \in S^{N-1}$ be a given unit vector. The usual definition of a travelling wave solution in the direction e is an entire solution with a constant profile travelling at constant speed c. For a formal point of view, plugging this definition into (1) leads to the following computations: Let $U:\mathbb{R} \to \mathbb{R}_+$ and $V:\mathbb{R} \to \mathbb{R}_+$ be a predator invading constant profile and $c \in \mathbb{R}$ be a speed of propagation so that $B(\tau, x) = U(x.e - c\tau)$ and $C(\tau, x) = V(x.e - c\tau)$ is an entire solution of (1). Setting $t = x.e - c\tau$, (U, V) satisfies the formal system of equations:

$$\begin{aligned} & dU''(t) + cU'(t) + U(t)g\big(U(t)\big) - V(t) = 0, \\ & V''(t) + cV'(t) + rV(t)\left(1 - \frac{V(t)}{U(t)}\right) = 0, \\ & \lim_{t\to\infty} \big(U(t), V(t)\big) = (1, 0). \end{aligned} \tag{3}$$

Definition 1 *(Weak travelling wave solution).* A triplet (U, V, c) lying in $C^1(\mathbb{R}) \times C^0(\mathbb{R}) \times \mathbb{R}$ is a (predator invading) travelling wave solution with speed c of System (1) provided the following set of conditions holds true:

(i) $c > 0$, $0 \leqslant U(t) \leqslant 1$, $0 \leqslant V(t) \leqslant 1$ for all $t \in \mathbb{R}$; $U'' \in L^1_{loc}(\mathbb{R}; \mathbb{R})$ and

$$\lim_{t\to\infty} (U, V)(t) = (1, 0), \quad V \not\equiv 0.$$

(ii) $dU(t)U''(t) + cU(t)U'(t) + U(t)^2 g(U(t)) - U(t)V(t) = 0$, a.e. $t \in \mathbb{R}$.
(iii) For each $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \big((\varphi U)''(\xi) - c(\varphi U)'(\xi) + r\varphi(\xi)\big(U(\xi) - V(\xi)\big)\big)V(\xi)\,d\xi = 0.$$

Let us make more precise the assumption on the prey growth rate, g.

Assumption 1. Function $g : [0, \infty) \to R$ is continuous and satisfies

$$g(s) \geqslant 0, \quad \forall s \in [0, 1], \qquad g(s) \leqslant 0, \quad \forall s \in [1, \infty).$$

The above assumption implies in particular that $g(1) = 0$.

A typical shape for g corresponds to a logistic growth for the prey population, that is $g(s) = r_B(1 - s)$ with $r_B \geqslant 0$. As it is numerically shown in Burie, Ducrot and Langlais (work in progress) when $r_B > 0$ the dynamics of the weak solutions to System (1) may be very complex including pattern formation after the predator invasion front.

For this reason most of our results are stated under a stronger assumption.

Assumption 2. $g(s) \equiv 0$.

This assumption means that in a predator-free environment the prey population remains at a spatially homogeneous equilibrium.

Our first result deals with the existence of travelling wave solutions for System (1) under the general Assumption 1.

Theorem 1. *Let Assumption* 1 *be satisfied. Let $d > 0$ and $r > 0$ be given. For each $c > c^* := 2\sqrt{r}$, there exists a pair of functions $(U_c, V_c) : \mathbb{R} \to [0, 1] \times [0, 1]$ such that*:

(i) *function U_c belongs to $W^{2,p}_{loc}(\mathbb{R})$ for each $p \in [1, \infty)$, function V_c belongs to $W^{1,q}_{loc}(\mathbb{R})$ for each $q \in [1, \infty)$ and*

$$\frac{V_c^2}{U_c}\chi_{\{U_c>0\}} \in L^1_{loc}(\mathbb{R}),$$

wherein χ_A denotes the characteristic function of a set A;

(ii) *triplet (U_c, V_c, c) is a travelling wave solution according to Definition* 1.

Let us now state non-existence results as well as first basic properties.

Theorem 2. *Let Assumption* 1 *be satisfied. Let $d > 0$ and $r > 0$ be given. If the triplet (U, V, c) is a travelling wave solution to System* (1) *with wave speed $c > 0$ then $c \geqslant 2\sqrt{r}$.*

If there exists $\alpha < \beta$ such that $U(x) = 0$ for all $x \in [\alpha, \beta]$ then $V(x) = 0$ for each $x \in [\alpha, \beta]$.

This result shows that the minimal predator invading wave speed is given by the Fisher–KPP minimal wave speed. Actually ahead of the predator invading front the prey population density is approximately a constant equal to one. From a heuristic view point, ahead the front, the equation for the predator is given by the so-called Fisher equation,

$$\frac{\partial C(x,t)}{\partial t} \approx \frac{\partial^2 C(x,t)}{\partial x^2} + rC(x,t)\big(1 - C(x,t)\big).$$

This yields a heuristic explanation for the coincidence between the minimal speed and minimal Fisher wave.

In the case of Assumption 2 one gets the following improvement.

Theorem 3 *(Existence). Let Assumption* 2 *be satisfied. Let $d > 0$ and $r > 0$ be given. For each $c > c^*$, there exists a pair of functions $(U_c, V_c) : \mathbb{R} \to [0, 1] \times [0, 1]$ such that*:

(i) *function U_c is increasing and of class C^2 on $\mathbb{R}$,*
(ii) *function V_c is locally Lipschitz continuous and $V \in L^1(\mathbb{R})$,*
(iii) *triplet (U_c, V_c, c) is a travelling wave solution to System* (1) *and satisfies the following co-extinction behaviour*

$$\lim_{\xi \to -\infty} \big(U(\xi), V(\xi)\big) = (0, 0).$$

In addition function (U_c, V_c, c) satisfies

$$U''(\xi) + cU'(\xi) - V(\xi) = 0, \quad \forall \xi \in \mathbb{R}.$$

This proves the existence of a predator invading front followed by the co-extinction of the two populations due to the singularity of the carrying capacity for the predator. The aim of the next results is to provide some information on the sharpness of the travelling waves, that is on the co-extinction behaviour after the invasion front.

Theorem 4 *(Sharp travelling wave). Let Assumption* 2 *be satisfied. Let us assume that $d \geqslant 1$ and $dr \in (0, 1)$. Let (U, V, c) be a travelling wave solution to System* (1) *with wave speed $c > 0$. Then the solution is a sharp travelling wave, that is that there exists $\bar{x} \in R$ such that*

$$U(s) = 0, \qquad V(s) = 0, \quad \forall s \leqslant \bar{x},$$

and

$$U(s) > 0, \qquad V(s) > 0, \quad \forall s > \bar{x}.$$

Theorem 5 *(Everywhere positive wave). Let Assumption* 2 *be satisfied. Let us assume that $d \leqslant 1$ and $rd > 1$. Then for each $c > 2\sqrt{r}$, there exists a travelling wave solution to System* (1) *such that*

$$U(t) > 0, \quad \forall t \in \mathbb{R}.$$

Theorem 6 *(Everywhere positive wave (bis)). Let Assumption* 2 *be satisfied. Assume that $d \leqslant 1$ and $rd > 4$. Let (U, V, c) be a travelling wave solution to System* (1)*, then*

$$U(t) > 0, \quad \forall t \in \mathbb{R}.$$

Remark 1. In the limiting case $d = 1$, the sharpness of the solutions is determined by the location of r with respect to one. Small predator growth rate (namely r small) leads to finite time extinction while large enough r implies global persistence of both species.

3. Proof of Theorem 1

The proof goes through several steps. One first constructs travelling wave solutions for ε approximated problems. Then we pass to the limit, $\varepsilon \to 0$.

3.1. An ε approximated problem

Let $\varepsilon > 0$ be given. Consider the problem of finding a pair of positive and bounded functions $(U, V) : \mathbb{R} \to [0, \infty)$ and a constant $c > 0$ satisfying the following system of equations:

$$\begin{gathered} dU''(x) + cU'(x) + U(x)g\big(U(x)\big) - \frac{U(x)V(x)}{U(x) + \varepsilon} = 0, \\ V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x) + \varepsilon}\right) = 0, \\ \lim_{x \to \infty} (U, V)(x) = (1, 0). \end{gathered} \tag{4}$$

The latter problem corresponds to the travelling wave problem for the Holling–Tanner system of Eqs. (2) (with $\gamma = \varepsilon$). According to our knowledge, this problem seems not to be documented in the literature.

Theorem 7. *Let Assumption* 1 *be satisfied. Let $d > 0$ and $r > 0$ be given. For each $c > 2\sqrt{r}$ and each $\varepsilon > 0$, there exists a pair of functions (U, V) of class C^2 on $\mathbb{R}$ such that*:

(i) *$0 < U(x) < 1$, $0 < V(x) \leqslant 1 + \varepsilon$, for each $x \in \mathbb{R}$,*
(ii) *(U, V) is a classical solution of* (4)*.*

When Assumption 2 *is satisfied, then function U is increasing.*
Moreover if one assumes that $d \leqslant 1$ and $dr > 1$ then

$$V(x) \leqslant \frac{dr}{dr - 1}\big(U(x) + \varepsilon\big), \quad \forall x \in \mathbb{R}. \tag{5}$$

The proof relies on several steps and the construction of a suitable sub- and super-solution pair. We refer to [5] and [7] for more details in related systems. Let $\varepsilon > 0$ be given and fixed. Let $c > 2\sqrt{r}$ be given and set $\lambda > 0$,

$$\lambda = \frac{c - \sqrt{c^2 - 4r}}{2}. \tag{6}$$

Consider the map

$$\overline{w}(x) = e^{-\lambda x}, \quad x \in \mathbb{R}.$$

It satisfies the linear equation

$$w''(x) + cw'(x) + rw(x) = 0, \quad \forall x \in \mathbb{R}.$$

Lemma 1. *Let $\gamma > 0$ and $\beta > 0$ be given with*

$$\gamma < \min\left(\lambda, \frac{c}{d}\right), \qquad \beta \geqslant \left(\frac{1}{c\gamma - d\gamma^2}\right)^{\frac{\gamma}{\lambda}}. \tag{7}$$

Then the map $\underline{q}(x) = 1 - \beta e^{-\gamma x}$ satisfies

$$-d\underline{q}''(x) - c\underline{q}'(x) \leqslant -e^{-\lambda x}, \tag{8}$$

on the set $\{x \in \mathbb{R}\colon 1 - \beta e^{-\gamma x} \geqslant 0\}$.

Proof. Let $\gamma > 0$ and $\beta > 0$ be given as in (7). Then (8) is equivalent to

$$\beta e^{-\gamma x}\left(d\gamma^2 - c\gamma\right) \leqslant -e^{-\lambda x}, \quad \forall x \geqslant \frac{1}{\gamma}\ln\beta.$$

This also reads

$$\beta\left(c\gamma - d\gamma^2\right) \geqslant e^{(\gamma-\lambda)x}, \quad \forall x \geqslant \frac{1}{\gamma}\ln\beta.$$

Thus, due to the first condition in (7), it is sufficient to have

$$\beta\left(c\gamma - d\gamma^2\right) \geqslant \beta^{1-\frac{\lambda}{\gamma}},$$

that holds true because of the second condition in (7). $\square$

Lemma 2. *Let $\beta > 0$ and $\gamma > 0$ be as in* (7)*. Let $\eta \in (0, \lambda)$ be such that*

$$-A(\eta) := (\lambda + \eta)^2 - c(\lambda + \eta) + r < 0. \tag{9}$$

Let $k > 0$ be chosen so that

$$k \geqslant \beta^{\frac{\eta}{\gamma}},$$
$$k^{-\frac{\lambda}{\eta}} + \frac{A(\eta)}{r}\beta k^{-\frac{\gamma}{\eta}} \leqslant \frac{A(\eta)}{r}. \tag{10}$$

Then the map $u(x) = e^{-\lambda x} - ke^{-(\lambda+\eta)x}$ *satisfies*

$$-ru + r\frac{e^{-\lambda x}u}{1-\beta e^{-\gamma x}} \leqslant u''(x) + cu'(x), \tag{11}$$

on the set $\{x \in \mathbb{R}\colon\ u(x) \geqslant 0\}$.

Remark 2. From the choice of λ in (6) such a small enough $\eta > 0$ is feasible.

Proof. Let us first notice that due to the first condition in (10) one has

$$\frac{1}{\eta}\ln k \geqslant \frac{1}{\gamma}\ln \beta,$$

that implies that for each $x \in \{t \in \mathbb{R}\colon\ u(t) \geqslant 0\}$, $1 - \beta e^{-\gamma x} > 0$.

Next (11) is equivalent to: for each $x \geqslant \frac{1}{\eta}\ln k$,

$$r\frac{e^{-\lambda x}(1-ke^{-\eta x})}{1-\beta e^{-\gamma x}} \leqslant -ke^{-\eta x}\big((\lambda+\eta)^2 - c(\lambda+\eta) + r\big).$$

Introducing $A(\eta) > 0$ from (9), this is equivalent to

$$e^{(\eta-\lambda)x} + \frac{k}{r}A(\eta)\beta e^{-\gamma x} \leqslant \frac{k}{r}A(\eta) + ke^{-\lambda x}, \quad \forall x \geqslant \frac{1}{\eta}\ln k.$$

Thus it is sufficient to have

$$k^{1-\frac{\lambda}{\eta}} + \frac{k}{r}A(\eta)\beta k^{-\frac{\gamma}{\eta}} \leqslant \frac{k}{r}A(\eta),$$

while the above condition holds true because of the second condition in (10). $\square$

Let $\gamma > 0$ and $\beta > 0$ be as in (7), $\eta > 0$ be as in (9) and $k > 0$ be as in (10). Set

$$\begin{gathered}\overline{U}(x) \equiv 1, \qquad \underline{U}(x) = \max\big(0, 1-\beta e^{-\gamma x}\big),\\ \overline{V_\varepsilon}(x) = \min\big(1+\varepsilon, e^{-\lambda x}\big), \qquad \underline{V}(x) = e^{-\lambda x}\max\big(0, 1-ke^{-\eta x}\big).\end{gathered} \tag{12}$$

Let $a_0 > 0$ be given such that

$$\begin{gathered}\underline{U}(-a) = 0, \qquad \underline{V}(-a) = 0, \quad \forall a \geqslant a_0,\\ \underline{U}(a_0) > 0, \qquad \underline{V}(a_0) > 0.\end{gathered} \tag{13}$$

Let $a > a_0$ be given and consider the approximated problem posed on $[-a, a]$,

$$\begin{gathered}dU''(x) + cU'(x) + U(x)g\big(U(x)\big) - \frac{U(x)V(x)}{U(x)+\varepsilon} = 0,\\ V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x)+\varepsilon}\right) = 0,\\ (U, V)(\pm a) = (\underline{U}, \underline{V})(\pm a).\end{gathered} \tag{14}$$

Consider the set

$$\mathbb{X}_{a,\varepsilon} = \left\{(U, V) \in C\big([-a, a]\big)^2\colon\ \underline{U} \leqslant U \leqslant \overline{U},\ \underline{V} \leqslant V \leqslant \overline{V_\varepsilon}\right\}.$$

Lemma 3. *For each $\varepsilon > 0$ and each $a > a_0$ Problem* (14) *has a solution $(U, V) \in \mathbb{X}_{a,\varepsilon}$. Moreover if $g(s) \equiv 0$ then U is increasing on $[-a, a]$.*

Proof. Let $\varepsilon > 0$ and $a > a_0$ be given. Consider the map $\mathcal{T} : \mathbb{X}_{a,\varepsilon} \to C([-a, a])^2$ defined by

$$\mathcal{T}\begin{pmatrix}\widehat{U}\\ \widehat{V}\end{pmatrix} = \begin{pmatrix}U\\ V\end{pmatrix},$$

where (U, V) are defined as the solution to

$$\begin{gathered}
dU''(x) + cU'(x) - \Lambda_b U(x) = \frac{\widehat{U}(x)\widehat{V}(x)}{\widehat{U}(x) + \varepsilon} - \widehat{U}(x)\big(g\big(\widehat{U}(x)\big) + \Lambda_b\big),\\
V''(x) + cV'(x) - r\Lambda_c V(x) = r\frac{\widehat{V}(x)^2}{\widehat{V}(x) + \varepsilon} - r(1 + \Lambda_c)\widehat{V}(x),\\
(U, V)(\pm a) = (\underline{U}, \underline{V})(\pm a),
\end{gathered}$$

where $\Lambda_b > 0$ and $\Lambda_c > 0$ will be chosen latter on.

Claim 1. *For large enough $\Lambda_b > 0$ and $\Lambda_c > 0$ one has $\mathcal{T}(\mathbb{X}_{a,\varepsilon}) \subset \mathbb{X}_{a,\varepsilon}$.*

Let us postponed the proof of this claim and complete the proof of Lemma 3. From elliptic estimates the non-linear map $\mathcal{T}$ is completely continuous so that Schauder fixed theorem provides the existence of $(U, V) \in \mathbb{X}_{a,\varepsilon}$ satisfying

$$\mathcal{T}\begin{pmatrix}U\\ V\end{pmatrix} = \begin{pmatrix}U\\ V\end{pmatrix};$$

the first part of the result follows.

It remains to prove that U is increasing when $g(s) \equiv 0$. Since $U(-a) = 0$ and $U \geqslant 0$ one has $U'(-a) \geqslant 0$. Moreover since $V \geqslant 0$ one gets

$$U''(x) + cU'(x) \geqslant 0,$$

and the result follows. This completes the proof of Lemma 3. □

Now it remains to prove Claim 1.

Proof of Claim 1. Let $(\widehat{U}, \widehat{V}) \in \mathbb{X}_{a,\varepsilon}$ be given. Recalling definition (12), consider $a^* \in (-a_0, a_0)$ (see definition (13)) defined by

$$a^* = \frac{1}{\gamma}\ln\beta. \tag{15}$$

Recall that is satisfies $\underline{U}(a^*) = 0$.

Note that if Λ_b is chosen so that $\Lambda_b \geqslant \frac{1+\varepsilon}{\varepsilon}$ then

$$\frac{\widehat{V}(x)}{\widehat{U}(x)+\varepsilon} - g\big(\widehat{U}(x)\big) \leqslant \Lambda_b.$$

It follows from the maximum principle that $U(x) \geqslant 0$.

Along the same lines if $\Lambda_c + 1$ is chosen so that $\Lambda_c + 1 \geqslant \frac{1+\varepsilon}{\varepsilon}$ then

$$r\frac{\widehat{V}(x)^2}{\widehat{U}(x)+\varepsilon} - r(1+\Lambda_c)\widehat{V}(x) \leqslant 0.$$

The maximum principle implies that $V(x) \geqslant 0$. Next choose Λ_b large enough so that the map $s \mapsto s(g(s)+\Lambda_b)$ is increasing on $[0,1]$. Then one has

$$dU''(x) + cU'(x) - \Lambda_b U(x) \geqslant -\big(g(1)+\Lambda_b\big) = -\Lambda_b.$$

Then the comparison principle applies and provides that $U(x) \leqslant 1$.

Moreover one also has

$$V''(x) + cV'(x) - r\Lambda_c V(x) \geqslant r\widehat{V}\left(\frac{\widehat{V}(x)}{1+\varepsilon} - (1+\Lambda_c)\right).$$

Let us chose Λ_c large enough so that the map $s \to rs(\frac{s}{1+\varepsilon} - (1+\Lambda_c))$ is decreasing on $[0, 1+\varepsilon]$. Since $\widehat{V} \leqslant 1+\varepsilon$, one gets

$$r\widehat{V}\left(\frac{\widehat{V}}{1+\varepsilon} - (1+\Lambda_c)\right) \geqslant -r(1+\varepsilon)\Lambda_c,$$

so that,

$$V''(x) + cV'(x) - r\Lambda_c V(x) \geqslant -r\Lambda_c(1+\varepsilon).$$

From the maximum principle one gets $V(x) \leqslant 1+\varepsilon$.

Next, recalling (15), one has for each $x \in (a^*, a)$:

$$\begin{aligned} dU''(x) + cU'(x) - \Lambda_b U(x) &= \frac{\widehat{U}(x)\widehat{V}(x)}{\widehat{U}(x)+\varepsilon} - \widehat{U}(x)\big(g\big(\widehat{U}(x)\big)+\Lambda_b\big) \\ &\leqslant e^{-\lambda x} - \big(g\big(\underline{U}(x)\big)+\Lambda_b\big)\underline{U}(x) \\ &\leqslant e^{-\lambda x} - \Lambda_b \underline{U}(x) \\ &\leqslant d\underline{U}''(x) + c\underline{U}'(x) - \Lambda_b \underline{U}(x). \end{aligned}$$

Since $U(a^*) \geqslant \underline{U}(a^*) = 0$ and $U(a) = \underline{U}(a)$, the elliptic maximum principle implies that $U \geqslant \underline{U}$ on $[a^*, a]$. Finally since $U \geqslant 0$ on the interval $[-a, a]$, one concludes that

$$\underline{U}(x) \leqslant U(x), \quad \forall x \in [-a, a].$$

The remaining of the proof can be easily checked by using analogous comparison arguments. Details are left to the reader. □

Lemma 4. *Assume that $g(s) \equiv 0$, $d \leqslant 1$ and $dr > 1$. Let $\varepsilon > 0$ be given. For each $a > a_0$, let (U_a, V_a) be a solution of* (14)*. Then one has*

$$V_a(x) \leqslant \max\left(\frac{dr}{dr-1}, \frac{\underline{V}(a)}{\underline{U}(a)+\varepsilon}\right)\big(U_a(x)+\varepsilon\big), \quad \forall x \in [-a, a].$$

Proof. Consider the map $P_a(x) = \frac{V_a(x)}{U_a(x)+\varepsilon}$. It satisfies

$$P'' + \left(c + 2\frac{U_a'}{U_a+\varepsilon}\right)P' + rP + \frac{1}{d}\left(\frac{U_a}{U_a+\varepsilon} - rd\right)P^2 = c\frac{1-d}{d}\frac{PU_a'}{U_a+\varepsilon},$$

together with

$$P_a(-a) = 0, \qquad P_a(a) = \frac{\underline{V}(a)}{\underline{U}(a)+\varepsilon}.$$

Now since $U_a' \geqslant 0$ and $d \leqslant 1$ one has

$$P'' + \left(c + 2\frac{U_a'}{U_a+\varepsilon}\right)P' + rP + \frac{1}{d}\left(\frac{U_a}{U_a+\varepsilon} - rd\right)P^2 \geqslant 0.$$

Let $x_0 \in [-a, a]$ be such that $P_a(x_0) = \max_{[-a,a]} P_a$. Since $P_a(-a) = 0$ then $x_0 \in (-a, a]$. If $x_0 = a$ then the result holds true and if $x_0 \in (-a, a)$ then

$$P_a''(x_0) \leqslant 0, \qquad P_a'(x_0) = 0,$$

that leads to

$$dr + \left(\frac{U_a(x_0)}{U_a(x_0)+\varepsilon} - rd\right)P_a(x_0) \geqslant 0.$$

Since $\frac{U_a(x_0)}{U_a(x_0)+\varepsilon} \leqslant 1$ and $rd > 1$, the result follows. $\square$

We now complete the proof of Theorem 7 by passing to the limit $a \to \infty$.

Proof of Theorem 7. Let $\varepsilon > 0$ be given and fixed. Consider $\{U_a, V_a\}$ a solution of (14) for each $a > a_0$. Consider an increasing sequence $\{a_n\}_{n\geqslant 0}$ such that $a_n \to \infty$ when $n \to \infty$ and denote by $U_n = U_{a_n}$ and $V_n = V_{a_n}$ for each $n \geqslant 0$. Since for each $n \geqslant 1$, $(U_n, V_n) \in \mathbb{X}_{a_n,\varepsilon}$, one obtains, due to elliptic regularity, that, possibly up to a subsequence, the sequence $\{U_n, V_n\}$ converges to some functions U and V for the topology of $C^1_{loc}(\mathbb{R})$ and (U, V) satisfies

$$\begin{gathered} U'(x) \geqslant 0, \quad \forall x \in \mathbb{R} \text{ if } g(s) \equiv 0, \\ \underline{U}(x) \leqslant U(x) \leqslant \overline{U}(x), \qquad \underline{V}(x) \leqslant V(x) \leqslant \overline{V_\varepsilon}(x), \quad \forall x \in \mathbb{R}. \end{gathered} \tag{16}$$

Moreover (U, V) satisfies the system of equations:

$$\begin{gathered} dU''(x) + cU'(x) + U(x)g\big(U(x)\big) - \frac{U(x)V(x)}{U(x)+\varepsilon} = 0, \quad x \in \mathbb{R}, \\ V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x)+\varepsilon}\right) = 0, \quad x \in \mathbb{R}. \end{gathered}$$

Due to (16), one obtains that $V(x) \not\equiv 0$ and

$$\lim_{x\to\infty}(U,V)(x) = (1,0).$$

Finally it remains to prove (5) that directly follows from Lemma 4 using definition (12). $\square$

3.2. Passing to the limit $\varepsilon \to 0^+$

Let $c > 2\sqrt{r}$ be given and fixed. Let us consider a decreasing sequence $\{\varepsilon_n\}_{n\geqslant 0}$ such that

$$\varepsilon_n > 0, \quad \forall n \geqslant 0 \quad \text{and} \quad \lim_{n\to\infty}\varepsilon_n = 0.$$

Let us denote for each $n \geqslant 0$, (U_n, V_n) a solution of (4) provided by Theorem 7 with $\varepsilon = \varepsilon_n$. We aim to pass to the limit (up to a subsequence) $n \to \infty$. To do so, let us first notice that the sequence $\{U_n\}_{n\geqslant 0}$ is uniformly bounded as well as the sequence $\{dU_n'' + cU_n'\}_{n\geqslant 0}$. This implies that $\{U_n\}$ is bounded in $W^{2,p}_{loc}(\mathbb{R})$ for each $p \in (0,\infty)$. Up to a subsequence, one may assume that

$$U_n \to U \quad \text{for the topology of } C^1_{loc}(\mathbb{R}),$$

where U is a positive increasing function of the class C^1 and such that $U'' \in L^p_{loc}(\mathbb{R})$ for any $p \in (1,\infty)$. On the other hand, the sequence $\{V_n\}_{n\geqslant 0}$ is uniformly bounded. Then one shall prove the following lemma:

Lemma 5. *For each $R > 0$ there exists a constant $C_R > 0$ such that for all $n \geqslant 0$,*

$$\|V_n\|_{W^{2,1}(-R,R)} + \int_{-R}^{\infty} \frac{V_n^2(x)}{U_n(x)+\varepsilon_n}\,dx \leqslant C_R.$$

Proof. Let $R > 0$ be given. Recall that for each $n \geqslant 0$, $(U_n, V_n)(x) \to (1,0)$ when $x \to \infty$. From elliptic regularity, this implies that $V_n'(x) \to 0$ when $x \to \infty$ for each $n \geqslant 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a positive and smooth function such that $\operatorname{supp}\varphi \subset [-R,\infty)$ and such that $\varphi(x) = 1$ for all $x \geqslant -R+1$. Then multiplying the V-equation by φ and integrating on $\mathbb{R}$ leads us to

$$\int_{-\infty}^{\infty} \big(\varphi''(x) - c\varphi(x) + r\varphi(x)\big)V_n(x)\,dx = r\int_{-\infty}^{\infty} \varphi(x)\frac{V_n(x)^2\,dx}{U(x)+\varepsilon_n}.$$

Since $V_n \leqslant \overline{V_{\varepsilon_n}}$ for each $n \geqslant 0$, the estimate on $\int_{-R}^{\infty} \frac{V_n^2(x)}{U_n(x)+\varepsilon_n}\,dx$ follows. Then integrating the V-equation on (x,∞), one obtains that V_n' is uniformly (with respect to n) bounded on each interval of the form $[-R,\infty)$ and once again, due to the V-equation, the result follows. $\square$

As a consequence of the above estimate, up to a subsequence (still denoted by $\{\varepsilon_n\}_{n\geqslant 0}$), the sequence $\{V_n\}_{n\geqslant 0}$ converges towards some function V strongly for the topology of $W^{1,q}_{loc}(\mathbb{R})$ for each $q \in [1,\infty)$.

It follows from the above extraction procedures that function $U \in W^{2,p}_{loc}(\mathbb{R})$ for each $p \in [1,\infty)$ and satisfies

$$dU(x)U''(x) + cU'(x)U(x) + U(x)^2 g\big(U(x)\big) - V(x)U(x) = 0 \quad \text{a.e. } x \in \mathbb{R},$$

and

$$\underline{U}(x) \leqslant U(x) \leqslant \overline{U}(x), \quad \forall x \in \mathbb{R}.$$

On the other hand function $V \in W^{1,q}_{loc}(\mathbb{R})$ for each $q \in [1,\infty)$ and satisfies for each $\varphi \in C^1_c(\mathbb{R})$:

$$-\int_{-\infty}^{\infty} (\varphi U)'(x)V'(x) + \int_{-\infty}^{\infty} \varphi(x)\big(cU(x)V'(x) - rV(x)\big(U(x) - V(x)\big)\big)\,dx = 0.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let (U, V, c) be a travelling wave solution to (1) with speed $c > 0$.

Lemma 6. *Assume that there exists $\alpha < \beta$ such that $U(x) = 0$ for each $x \in [\alpha, \beta]$. Then $V(x) = 0$ for all $x \in [\alpha, \beta]$.*

Proof. Let $\eta > 0$ be given such that $\alpha + \eta < \beta - \eta$. Choose a map $\varphi \in \mathcal{D}_+(\mathbb{R})$ such that

$$\varphi(x) = 1, \quad x \in [\alpha - \eta, \beta - \eta], \qquad \varphi(x) = 0, \quad x \notin (\alpha, \beta).$$

Due to Definition 1 (iii), one obtains

$$\int_{-\infty}^{\infty} \varphi(x)V(x)^2\,dx = 0,$$

and $V(x) = 0$ for each $x \in [\alpha + \eta, \beta - \eta]$. The result follows by continuity. □

Let $c > 0$ be given and let (U, V, c) be a travelling wave solution to (1). In order to complete the proof of Theorem 2, it remains to prove that $c \geqslant c^*$. To do so, since U converges to one when $x \to \infty$, there exists $\bar{x} \in \mathbb{R}$ such that $U(x) > \frac{1}{2}$ for all $x > \bar{x}$. Therefore (U, V) becomes a classical solution on $(\bar{x}, \infty)$ of the system

$$\begin{aligned} dU'' + cU' + Ug(U) &= V, \\ V'' + cV' + rV\left(1 - \frac{V}{U}\right) &= 0. \end{aligned}$$

Without lost of generality, one may assume that $\bar{x} = 0$. This implies that the map $\frac{V}{U}$ is bounded on $[0, \infty)$. This property will allow us to obtain that there exists a constant $M > 0$ such that

$$\left|V'(x)\right| \leqslant MV(x), \quad \forall x \in [2, \infty). \tag{17}$$

The proof of the above estimate relies on elliptic Harnack inequality coupled with interior elliptic regularity (we refer to the monograph of Gilbarg and Trudinger [8] for more details). Indeed, since $(1 - \frac{V}{U})$ is a bounded function on $[0, \infty)$, elliptic Harnack inequality applies to the V-equation and one concludes that there exists some constant $\widehat{K} > 0$ such that for each $x_0 \geqslant 2$:

$$\max_{s \in [x_0-1, x_0+1]} V(s) \leqslant \widehat{K} \min_{s \in [x_0-1, x_0+1]} V(s) \leqslant \widehat{K} V(x_0).$$

On the other hand, elliptic regularity implies that there exists some constant $\widehat{M} > 0$ such that for each $x_0 \geqslant 2$:

$$\|V\|_{W^{1,\infty}(x_0-\frac{1}{2}, x_0+\frac{1}{2})} \leqslant \widehat{M} \|V\|_{L^\infty(x_0-1, x_0+1)}.$$

As a consequence of the two above estimates, one obtains that for each $x_0 \geqslant 2$:

$$\left|V'(x_0)\right| \leqslant \widehat{M}\widehat{K} V(x_0).$$

Therefore setting $M = \widehat{K}\widehat{M}$, (17) follows.

Introduce $\Lambda \in \mathbb{R}$ defined by

$$\Lambda = \liminf_{x \to \infty} \frac{V'(x)}{V(x)}.$$

Note that since $V(x) \to 0$ when $x \to \infty$ then $\Lambda \leqslant 0$. Consider an increasing sequence $\{x_n\}_{n \geqslant 0}$ such that

$$x_n \to \infty \quad \text{when } n \to \infty, \qquad \lim_{n \to \infty} \frac{V'(x_n)}{V(x_n)} = \Lambda. \tag{18}$$

Next consider the sequence of map $\{w_n\}_{n \geqslant 0}$ defined by

$$w_n(x) = \frac{V(x + x_n)}{V(x_n)}, \quad n \geqslant 0,\ x \geqslant -x_n.$$

Due to Harnack inequality for the V-equation, one gets that $\{w_n\}_{n \geqslant 0}$ is locally bounded. Indeed if $h > 0$ is given. Since $x_n \to \infty$ as $n \to \infty$, there exists $n_h \in \mathbb{N}$ such that

$$h + x_n \geqslant 1, \quad \forall n \geqslant n_h.$$

Since $\frac{V}{U}$ is globally bounded over $[0, \infty)$, Harnack inequality for the V-equation provides the existence of some constant $M = M(h) > 0$ such that for each $n \geqslant n_h$:

$$\max_{x \in [x_n - h, x_n + h]} V(x) \leqslant M(h) V(x_n), \quad \forall n \geqslant n_h.$$

This implies that for each $h > 0$, there exist some constant $M(h) > 0$ and $n_h \in \mathbb{N}$ such that

$$w_n(x) \leqslant M(h), \quad \forall n \geqslant n_h,\ \forall x \in [-h, h].$$

Furthermore function w_n satisfies

$$w_n'' + c w_n' + r w_n \left(1 - \frac{V(x + x_n)}{U(x + x_n)}\right) = 0.$$

Since $(U, V)(x) \to (1, 0)$ as $x \to +\infty$, up to a subsequence, one may assume that $\{w_n\}_{n\geqslant 0}$ converges to some function $w \geqslant 0$ locally uniformly that satisfies

$$w'' + cw' + rw = 0 \quad \text{and} \quad w(0) = 1.$$

Thus $w(x) > 0$ for all $x \in \mathbb{R}$. One the other hand, the map w_n' satisfies

$$w_n'(x) = \frac{V'(x + x_n)}{V(x + x_n)} w_n(x).$$

Next from the definition of $\{x_n\}$ given in (40), one has

$$w'(0) = \Lambda w(0), \qquad w'(x) \geqslant \Lambda w(x).$$

This implies that the map $w_* = w' - \Lambda w$ satisfies

$$w_*''(x) + cw_*'(x) + rw_*(x) = 0, \quad \forall x \in \mathbb{R},$$
$$w_*(x) \geqslant 0, \quad \forall x \in \mathbb{R}, \qquad w_*(0) = 0.$$

This provides that $w_*(x) \equiv 0$ so that

$$w(x) = w(0)e^{\Lambda x}, \quad \forall x \in \mathbb{R},$$

and

$$\Lambda^2 + c\Lambda + r = 0, \quad \Lambda \leqslant 0,$$

and this implies that

$$c^2 - 4r \geqslant 0.$$

This completes the proof of the result.

5. Proof of Theorem 3

We use solutions to the approximated problem (4) taking into account that under Assumption 2 the latter problem admits a solution with an increasing U-component (see Theorem 7). Let $\{\varepsilon_n\}_{n\geqslant 0}$ be a given sequence of positive number tending to zero when $n \to \infty$. Let $c > 2\sqrt{r}$ be given and fixed. Let us denote for each $n \geqslant 0$ by (U_n, V_n, c) a solution of (4) provided by Theorem 7 with $\varepsilon = \varepsilon_n$ and such that $x \mapsto U_n(x)$ is increasing for each $n \geqslant 0$. As explained before, possibly up to a subsequence, one may assume that $\{U_n\}_{n\geqslant 0}$ converges to some function U for the topology of $C^1_{loc}(\mathbb{R})$ and where function U is a positive increasing function of the class C^1 and such that $U'' \in L^p_{loc}(\mathbb{R})$ for any $p \in (1, \infty)$. Moreover, due to the construction of U_n given in the proof of Theorem 7, one has

$$\underline{U}(x) \leqslant U(x) \leqslant \overline{U}(x), \quad \forall x \in \mathbb{R}.$$

We will now split our arguments into two parts:

Case 1: $U(x) > 0$ for all $x \in \mathbb{R}$.
Case 2: There exists $\widehat{x} \in \mathbb{R}$ such that $U(\widehat{x}) = 0$ and, since U is increasing and tends to one when $x \to +\infty$, there exists $x_0 \geqslant \widehat{x}$ such that

$$U(x) \begin{cases} = 0 & \text{if } x \leqslant x_0, \\ > 0 & \text{if } x > x_0. \end{cases}$$

5.1. Case 1

Together with Case 1, the sequence $\{V_n\}_{n \geqslant 0}$ is also bounded in $W^{2,p}_{loc}(\mathbb{R})$ for each $p \in (1, \infty)$ and thus, possibly up to a subsequence, one may assume

$$V_n \to V \quad \text{for the topology of } C^1_{loc}(\mathbb{R}),$$

for some function V of class C^1. As a consequence (U, V) is a solution to the system of equations:

$$U(x) > 0, \quad \forall x \in \mathbb{R},$$
$$U(x) \to 1 \quad \text{when } x \to \infty \text{ and } U \text{ increasing},$$
$$dU''(x) + cU'(x) - V(x) = 0,$$
$$V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x)}\right) = 0.$$

For each $n \geqslant 0$ one has $\underline{V} \leqslant V_n \leqslant \overline{V_{\varepsilon_n}}$, one gets that function V satisfies

$$\underline{V} \leqslant V \leqslant \overline{V},$$

wherein we have set

$$\overline{V}(x) = \min\bigl(1, e^{-\lambda x}\bigr), \quad x \in \mathbb{R}.$$

Thus $V(x) \to 0$ as $x \to \infty$.

Now let us notice that for each $n \geqslant 0$ and each $\varphi \in \mathcal{D}(\mathbb{R})$ one has

$$\int_{-\infty}^{\infty} \varphi(x) \frac{U_n(x)V_n(x)}{U_n(x) + \varepsilon_n}\, dx = \int_{-\infty}^{\infty} \bigl(d\varphi'' - c\varphi'\bigr)(x) U_n(x)\, dx. \tag{19}$$

Passing to the limit as $n \to \infty$ yields

$$\int_{-\infty}^{\infty} \varphi(x)V(x)\, dx = \int_{-\infty}^{\infty} \bigl(d\varphi'' - c\varphi'\bigr)(x)U(x)\, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}). \tag{20}$$

Consider now a map $\varphi \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant \varphi(x) \leqslant 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in [-1, 1], \\ 0 & \text{if } |x| \geqslant 2, \end{cases}$$

and consider for each $R>0$ the map $\varphi_R \in \mathcal{D}(\mathbb{R})$ defined by

$$\varphi_R(x)=\varphi\left(\frac{x}{R}\right), \quad x \in \mathbb{R}. \tag{21}$$

Then from (20) with $\varphi \equiv \varphi_R$ one gets: for each $R>0$,

$$\begin{aligned}\int_{-R}^{R} V(x)\,dx &\leqslant \int_{-\infty}^{\infty} U(x)\left[\frac{d}{R^2}\varphi''\left(\frac{x}{R}\right)-\frac{1}{R}\varphi'\left(\frac{x}{R}\right)\right]dx \\ &\leqslant \frac{d}{R}\|\varphi''\|_{L^1(\mathbb{R})}+c\|\varphi'\|_{L^1(\mathbb{R})}.\end{aligned}$$

Letting $R \to \infty$ implies that

$$V \in L^1(\mathbb{R}). \tag{22}$$

Next for each $n \geqslant 0$ and each $\varphi \in \mathcal{D}(\mathbb{R})$ one has

$$\int_{-\infty}^{\infty} \varphi(x)\frac{V_n(x)^2}{U_n(x)+\varepsilon_n}\,dx=\int_{-\infty}^{\infty} \varphi(x)V_n(x)\,dx+\frac{1}{r}\int_{-\infty}^{\infty} \big(\varphi''(x)-c\varphi'(x)\big)V_n(x)\,dx.$$

Passing to the limit $n \to +\infty$ in the above equality yields

$$\int_{-\infty}^{\infty} \varphi(x)\frac{V(x)^2}{U(x)}\,dx=\int_{-\infty}^{\infty} \varphi(x)V(x)\,dx+\frac{1}{r}\int_{-\infty}^{\infty} \big(\varphi''(x)-c\varphi'(x)\big)V(x)\,dx,$$

for each $\varphi \in \mathcal{D}(\mathbb{R})$. Choosing $\varphi=\varphi_R$ defined in (21) as a test function in the above equality one gets: for each $R>0$

$$\int_{-R}^{R} \frac{V(x)^2}{U(x)}\,dx \leqslant \int_{R} \varphi_R(x)V(x)\,dx+\frac{1}{r}\int_{-\infty}^{\infty}\left(\frac{1}{R^2}\varphi''\left(\frac{x}{R}\right)-\frac{c}{R}\varphi'\left(\frac{x}{R}\right)\right)V(x)\,dx.$$

Since $0 \leqslant V(x) \leqslant 1$ and $V \in L^1(\mathbb{R})$ (see (22)) it follows that

$$\frac{V^2}{U} \in L^1(\mathbb{R}).$$

This last property together with $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ implies that V' is bounded and therefore

$$\lim_{x \to -\infty} V(x)=0 \quad \text{and} \quad \lim_{x \to -\infty} V'(x)=0.$$

It remains to show

$$\lim_{x \to -\infty} U(x)=0.$$

Since U is increasing set $l = \lim_{x\to-\infty} U(x) \in [0,1]$. We argue by contradiction by assuming that $l > 0$. Then the map $x \to \frac{V(x)}{U(x)}$ is globally bounded on $\mathbb{R}$. Therefore elliptic Harnack inequality applies to the V-equation and provides for each $h > 0$, the existence of some constant $\widehat{K}_h > 0$ such that for each $x_0 \in \mathbb{R}$:

$$\max_{s\in[x_0-h,x_0+h]} V(s) \leqslant \widehat{K}_h \min_{s\in[x_0-k,x_0+h]} V(s) \leqslant \widehat{K}_h V(x_0). \tag{23}$$

Similarly to the proof of Theorem 2, using the interior elliptic regularity, one concludes that the map $\frac{V'}{V}$ is globally bounded on $\mathbb{R}$. Since $V(x) \to 0$ as $x \to -\infty$ let us define $\Lambda \in [0,\infty)$ as

$$\Lambda = \limsup_{x\to-\infty} \frac{V'(x)}{V(x)}.$$

Consider a decreasing sequence $\{x_n\}_{n\geqslant 0}$ such that

$$x_n \to -\infty \quad \text{and} \quad \lim_{n\to\infty} \frac{V'(x_n)}{V(x_n)} = \Lambda.$$

Next consider the sequence of map $\{w_n\}_{n\geqslant 0}$ defined by

$$w_n(x) = \frac{V(x+x_n)}{V(x_n)}, \quad n \geqslant 0,\ x \in \mathbb{R}.$$

Due to (23), one obtains that $\{w_n\}_{n\geqslant 0}$ is locally bounded and satisfy the equations

$$w_n'' + cw_n' + rw_n\left(1 - \frac{V(x+x_n)}{U(x+x_n)}\right) = 0.$$

Since $l > 0$, the map $\frac{V}{U}$ is bounded. Using elliptic estimates, up to a subsequence, $\{w_n\}_{n\geqslant 0}$ converges to some function $w \geqslant 0$ locally uniformly, a solution to

$$w'' + cw' + rw = 0 \quad \text{and} \quad w(0) = 1.$$

Thus $w(x) > 0$ for all $x \in \mathbb{R}$. One the other hand, the map w_n' satisfies

$$w_n'(x) = \frac{V'(x+x_n)}{V(x+x_n)} w_n(x).$$

Next from the definition of $\{x_n\}_{n\geqslant 0}$ one gets

$$w'(0) = \Lambda w(0), \qquad w'(x) \leqslant \Lambda w(x).$$

This implies that the map $w_* = w' - \Lambda w$ is a solution to

$$w_*''(x) + cw_*'(x) + rw_*(x) = 0, \quad \forall x \in \mathbb{R},$$
$$w_*(x) \leqslant 0, \quad \forall x \in \mathbb{R}, \qquad w_*(0) = 0.$$

This provides that $w_*(x) \equiv 0$ so that

$$w(x) = w(0)e^{\Lambda x}, \quad \forall x \in \mathbb{R},$$

and

$$\Lambda^2 + c\Lambda + r = 0, \quad \Lambda \geqslant 0,$$

that leads us to a contradiction.

Finally we get that

$$\lim_{x \to -\infty} U(x) = 0.$$

Moreover let us notice that in this Case 1, function U and V are both C^∞ in $\mathbb{R}$. This completes the proof of Theorem 3 in Case 1.

5.2. Case 2

Let us now assume that there exists $\bar{x} \in \mathbb{R}$ such that

$$U(x) \begin{cases} = 0 & \text{if } x \leqslant \bar{x}, \\ > 0 & \text{if } x > \bar{x}. \end{cases}$$

Then since $\{U_n\}$ locally uniformly converges to U, the sequence $\{\frac{V_n}{U_n + \varepsilon_n}\}$ is bounded on $(\bar{x} + \eta, \infty)$ for each $\eta > 0$. Moreover, since the sequence $\{V_n\}$ is uniformly bounded, due to elliptic estimates, one may assume that $\{V_n\}$ converges to some function $\widehat{V} : (\bar{x}, \infty) \to [0, 1]$ for the C^1-topology on $[\bar{x} + \eta, \bar{x} + \gamma]$ for each $0 < \eta < \gamma$. In addition $(U, \widehat{V})$ is a solution to

$$\begin{gathered} U(x) > 0, \quad \forall x > \bar{x}, \\ U(x) \to 1 \quad \text{when } x \to \infty \text{ and } U \text{ increasing}, \\ dU''(x) + cU'(x) - \widehat{V}(x) = 0, \quad x > \bar{x}, \\ \widehat{V}''(x) + c\widehat{V}'(x) + r\widehat{V}(x)\left(1 - \frac{\widehat{V}(x)}{U(x)}\right) = 0, \quad x > \bar{x}. \end{gathered} \tag{24}$$

Since for each $n \geqslant 0$ one has $\underline{V} \leqslant V_n \leqslant \overline{V_{\varepsilon_n}}$, one also gets

$$\underline{V}(x) \leqslant \widehat{V}(x) \leqslant \overline{V}(x), \quad \forall x > \bar{x}. \tag{25}$$

Next note that $\widehat{V}(x) > 0$ for all $x > \bar{x}$. Indeed, due to lower estimates in (25), $\widehat{V} \geqslant 0$ on $(\bar{x}, \infty)$ and non-zero. Since for each $\tau > 0$, the map $\frac{\widehat{V}}{U}$ is bounded on $[\bar{x} + \tau, \infty)$, the strong maximum principle applies and provides that $\widehat{V}(x) > 0$ for all $x > \bar{x}$.

We claim that:

Claim 2. *Function $\widehat{V}$ satisfies*

$$\lim_{x \searrow \bar{x}} \widehat{V}(x) = 0, \tag{26}$$

and there exists $v \in \mathbb{R}$ such that

$$\lim_{x \searrow \bar{x}} \widehat{V}'(x) = v. \tag{27}$$

Before proving this claim, let us complete the proof of Theorem 3.

Consider the map $V : \mathbb{R} \to \mathbb{R}$ defined by

$$V(x) = \begin{cases} \widehat{V}(x) & \text{if } x > \bar{x}, \\ 0 & \text{if } x \leqslant \bar{x}. \end{cases} \tag{28}$$

Due to Claim 2 the map V is Lipschitz continuous on $\mathbb{R}$. Moreover since U is of class C^1 on $\mathbb{R}$ and a solution to

$$dU'' + cU' = V \quad \text{for all } x > \bar{x},$$

one obtains by letting $x \to \bar{x}$ that U is of class C^2 on $\mathbb{R}$, that is $U''(\bar{x}) = 0$. As a consequence (U, V) satisfies

$$dU''(x) + cU'(x) = V(x) \quad \text{for all } x \in \mathbb{R}.$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$ be given. From (24) and (28) for each $\eta > 0$ one gets

$$\int_{\bar{x}+\eta}^{\infty} \varphi(x)\big(U(x)V''(x) + cU(x)V'(x) + rV(x)\big(U(x) - V(x)\big)\big)\,dx = 0.$$

Integrating by parts yields

$$\big(-\varphi UV' + (\varphi U)'V - c\varphi UV\big)(\bar{x}+\eta) + \int_{\bar{x}+\eta}^{\infty} \big((\varphi U)'' - c(\varphi U)'\big)V\,dx + \int_{\bar{x}+\eta}^{\infty} \varphi rV(U - V)\,dx = 0.$$

Using Claim 2 and passing to the limit $\eta \searrow 0$ one obtains that (U, V) satisfies

$$\int_{\bar{x}}^{\infty} \big((\varphi U)'' - c(\varphi U)'\big)V\,dx + \int_{\bar{x}}^{\infty} \varphi rV(U - V)\,dx = 0.$$

Since $V(x) = 0$ for any $x \leqslant \bar{x}$ one gets that for each $\varphi \in \mathcal{D}(\mathbb{R})$

$$\int_{-\infty}^{\infty} \big((\varphi U)'' - c(\varphi U)'\big)V\,dx + \int_{-\infty}^{\infty} \varphi rV(U - V)\,dx = 0.$$

This show that (U, V) is a travelling wave solution according to Definition 1 of (1). To complete the proof of Theorem 3 it remains to prove Claim 2.

Proof of Claim 2. In order to prove this claim, let us first prove that for each $\tau > 0$:

$$\frac{\widehat{V}^2}{U} \in L^1(\bar{x}, \bar{x}+\tau). \tag{29}$$

To prove the above fact, let $\tau > 0$ be given and consider $\varphi \in \mathcal{D}(\mathbb{R})$ such that

$$\varphi(x) \geqslant 0, \quad \forall x \in \mathbb{R} \quad \text{and} \quad \varphi(x) = 1, \quad \forall x \in [\bar{x}, \bar{x} + \tau]. \tag{30}$$

Let us notice that for each $n \geqslant 0$ one has

$$\int_{-\infty}^{\infty} \varphi(x) \frac{V_n(x)^2}{U_n(x) + \varepsilon_n} \, dx = \int_{-\infty}^{\infty} \varphi(x) V_n(x) \, dx + \frac{1}{r} \int_{-\infty}^{\infty} \big(\varphi''(x) - c\varphi'(x)\big) V_n(x) \, dx.$$

Recalling that the sequence $\{V_n\}_{n \geqslant 0}$ is uniformly bounded and using (30), one obtains that there exists some constant $K > 0$ such that

$$\int_{\bar{x}}^{\bar{x}+\tau} \frac{V_n(x)^2}{U_n(x) + \varepsilon_n} \, dx \leqslant K, \quad \forall n \geqslant 0.$$

Recalling that

$$\frac{V_n^2}{U_n + \varepsilon_n} \to \frac{\widehat{V}^2}{U} \quad \text{a.e. } x \in (\bar{x}, \infty),$$

Fatou lemma applies and provides (29).

To complete the proof of Claim 2, note that (24) implies that $\widehat{V}$ satisfies

$$\widehat{V}'' + c\widehat{V}' := f \in L^1_{loc}[\bar{x}, \infty).$$

Thus both $\widehat{V}$ and $\widehat{V}'$ possess limits when $x \searrow \bar{x}$. Since $U(\bar{x}) = U'(\bar{x}) = 0$ and $\frac{\widehat{V}^2}{U} \in L^1_{loc}[\bar{x}, \infty)$ one finds

$$\lim_{x \searrow \bar{x}} \widehat{V}(x) = 0.$$

This completes the proof of Claim 2. $\square$

6. Remarks on the travelling waves of (1)

Through this section, one shall assume that Assumption 2 is satisfied. Let (U, V, c) be a travelling wave of (1) according to Definition 1.

Lemma 7. *The map U is increasing on $\mathbb{R}$.*

Proof. Let us first show that if $U(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then $U(x) = 0$ for all $x \leqslant x_0$. Assume by contradiction that there exists two points $x_1 < x_2$ such that $U(x_1) = U(x_2) = 0$ and $U(x) > 0$ for each $x \in (x_1, x_2)$. Then almost everywhere in (x_1, x_2), U satisfies

$$dU''(x) + cU'(x) = V(x).$$

As a consequence, U belongs to C^2 on (x_1, x_2) and at a maximum value $x_3 \in (x_1, x_2)$, since $V(x) \geqslant 0$, one gets that $V(x_3) = 0$. On the other hand, V becomes a classical solution to

$$V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x)}\right) = 0, \quad x \in (x_1, x_2).$$

The strong maximum principle implies $V(x) \equiv 0$ on (x_1, x_2). As a consequence $U(x) \equiv 0$ on (x_1, x_2), a contradiction and the claim follows.

As a consequence of this claim, there exists some value $\bar{x} \in [-\infty, \infty)$ such that $U(x) > 0$ for all $x > \bar{x}$ and $U(x) = 0$ for each $x \leqslant \bar{x}$. Next on $(\bar{x}, \infty)$, function U satisfies $dU''(x) + cU'(x) > 0$ and cannot achieve a local minimum value. Since $U(x) \leqslant 1$ and $U(x) \to 1$ when $x \to \infty$, the result follows. □

Lemma 7 allows us to introduce a quantity $\bar{x} \in \mathbb{R} \cup \{-\infty\}$,

$$\bar{x} = \sup\bigl\{\xi \in \mathbb{R}\colon\ U(s) = 0,\ \forall s < \xi\bigr\}. \tag{31}$$

Due to Lemma 7, we obtain in the case where $\bar{x} = -\infty$ that $U(x) > 0$ for all $x \in \mathbb{R}$, and, in the cases where $\bar{x} \in \mathbb{R}$ that

$$U(x) \begin{cases} = 0 & \text{if } x \leqslant \bar{x}, \\ > 0 & \text{if } x > \bar{x}. \end{cases}$$

As a consequence of this property as well as of Definition 1 function V is of class C^2 on the interval $(\bar{x}, \infty)$. In other words, (U, V) becomes a classical solution to the following problem posed on the interval $(\bar{x}, \infty)$,

$$\begin{gathered} dU''(x) + cU'(x) = V(x), \\ V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x)}\right) = 0. \end{gathered}$$

Lemma 8. *Let (U, V, c) be a travelling wave solution to* (1)*. Let $\bar{x}$ be as in* (31)*. One has*:

(i) $\lim_{x \searrow \bar{x}} U'(x) = 0$.
(ii) *Let $P : (\bar{x}, \infty) \to [0, \infty)$ be the function defined by $P(.) = \frac{V(.)}{U(.)}$. If $\liminf_{x \searrow \bar{x}} P(x) > 1$ then there exists $x_0 \in \mathbb{R}$ and $x_0 > \bar{x}$ such that*

$$V'(x) \geqslant 0, \quad \forall x \in (\bar{x}, x_0), \tag{32}$$

and

$$\lim_{x \searrow \bar{x}} V(x) = 0. \tag{33}$$

Moreover the following estimates hold true

$$\frac{d}{2}\bigl|U'(x)\bigr|^2 \leqslant U(x)V(x), \quad \forall x \in (\bar{x}, x_0). \tag{34}$$

Proof. First, (i) directly follows from U of class C^1 when $\bar{x} \in \mathbb{R}$. When $\bar{x} = -\infty$, recall Lemma 7 applies and function U has some limit $l \geqslant 0$ when $x \to -\infty$. Then (i) follows from elliptic estimates since V is bounded.

Let us now prove (ii). Assume that $\liminf_{x \searrow \bar{x}} P(x) > 1$. Then there exists $x_0 \in \mathbb{R}$ such that $x_0 > \bar{x}$ and

$$P(x) \geqslant 1, \quad \forall x \in (\bar{x}, x_0].$$

Thus,

$$V''(x) + cV'(x) = rV(x)\big(1 - P(x)\big) \geqslant 0, \quad \forall x \in (\bar{x}, x_0]. \tag{35}$$

We now split the argument into two parts.

Let us first assume that $\bar{x} \in \mathbb{R}$. Then using Theorem 2, one has $V(x) = 0$ for all $x \leqslant \bar{x}$ and therefore $\lim_{x\to\bar{x}} V(x) = 0$ that proves (33). Since $V \geqslant 0$ and $\lim_{x\to\bar{x}} V(x) = 0$, there exists a decreasing sequence $\{x_n\}_{n\geqslant 0}$ such that $x_n < x_0$ for any $n \geqslant 0$ and $x_n \to \bar{x}$ and such that $V'(x_n) > 0$ for any $n \geqslant 0$. Then for each $n \geqslant 0$ and each $x_n < y < x_0$ one obtains

$$V'(y)e^{cy} \geqslant V'(x_n)e^{cx_n} > 0.$$

This implies that $V'(x) > 0$ for any $x \in (\bar{x}, x_0)$ and the result follows.

Let us now assume that $\bar{x} = -\infty$. In this case, we claim that function has a limit $m \geqslant 0$ when $x \to -\infty$. Indeed, we have:

(a) either $V'(x) \leqslant 0$ in some interval $(-\infty, x_0)$ (up to reduce x_0), or,
(b) there exists a decreasing sequence $\{x_n\}_{n\geqslant 0}$ such that $\bar{x} < x_n < x_0$ for any $n \geqslant 0$ and $x_n \to \bar{x}$ and such that $V'(x_n) > 0$ for any $n \geqslant 0$.

Note that in case (b), the same argument as above allows us to obtain that $V'(x) > 0$ for all $x \in (-\infty, x_0)$. Thus the claim follows. If one consider

$$m := \lim_{x\to-\infty} V(x),$$

then we claim that $m = 0$. To see this, recall that since function U is increasing, it possesses some limit $l \geqslant 0$ when $x \to -\infty$. Next consider a sequence $\{x_n\}_{n\geqslant 0}$ tending to $-\infty$ as $n \to \infty$ as well as the sequence of map

$$U_n(x) = U(x + x_n), \qquad V_n(x) = V(x + x_n).$$

Then since $\{V_n\}$ is uniformly bounded, elliptic estimates provide that $\{U_n\}$ is bounded in $W^{2,p}(-1,1)$ for each $p \in (1,\infty)$. Therefore, recalling that $U(x) \to l$ when $x \to -\infty$, up to a subsequence, one may assume that

$$U_n(x) \to l, \qquad U_n'(x) \to 0, \qquad U_n''(x) \to 0 \quad \text{weakly in } L^2(-1,1).$$

On the other hand, since $V(x) \to m$ when $x \to -\infty$, one obtains that

$$V_n(x) \to m \quad \text{in } L^2(-1,1).$$

Finally noticing that

$$dU_n''(x) + cU_n'(x) = V_n(x), \quad x \in (-1,1),\ n \geqslant 0,$$

leads us to $m = 0$ by passing to the limit $n \to \infty$. As a consequence we obtain that

$$\lim_{x\to-\infty} V(x) = 0.$$

This completes the proof of (33). To prove (32), it sufficient to notice that since $V \geqslant 0$ and $V(x) \to 0$ as $x \to -\infty$, there exists decreasing sequence $\{x_n\}_{n\geqslant 0}$ such that $\bar{x} < x_n < x_0$ for any $n \geqslant 0$ and $x_n \to \bar{x}$ and such that $V'(x_n) > 0$ for any $n \geqslant 0$. Then (32) follows from (35) using the same arguments as before.

It remains to prove (34). Set $J : (\bar{x}, x_0] \to \mathbb{R}$,

$$J(x) = \frac{d}{2}\left|U'(x)\right|^2 - U(x)V(x).$$

It satisfies $J(x) \to 0$ when $x \to \bar{x}$. Moreover, for each $\bar{x} < x \leqslant x_0$ one has

$$\begin{aligned} J'(x) &= dU'(x)U''(x) - U'(x)V(x) - U(x)V'(x) \\ &= U'(x)\big(-cU'(x) + V(x)\big) - U'(x)V(x) - U(x)V'(x) \\ &= -c\left|U'(x)\right|^2 - U(x)V'(x) \leqslant 0, \end{aligned}$$

so that $J(x) \leqslant 0$ for any $x < x_0$ and the result follows. □

Lemma 9. *Let* (U, V, c) *be a travelling wave of* (1)*. Let* $\bar{x}$ *be as in* (31) *and assume that* $\bar{x} \in \mathbb{R}$*. Then the following hold true*:

(i) *Function* $P : (\bar{x}, \infty) \to [0, \infty)$ *defined by* $P(.) = \frac{V(.)}{U(.)}$ *is unbounded.*
(ii) *If there exists* $h > 0$ *such that the map* $x \mapsto V(x)$ *is increasing on* $(\bar{x}, \bar{x} + h)$ *then*

$$U(x) \leqslant \frac{(x - \bar{x})^2}{2d} V(x), \quad \forall x \in (\bar{x}, \bar{x} + h).$$

Proof. Let us first prove (i). We will argue by contradiction by assuming that P is bounded. Assume there exists some constant $M > 0$ such that

$$P(x) \leqslant M, \quad \forall x > \bar{x}.$$

Then since $U' \geqslant 0$ for each $x > \bar{x}$ one has

$$U(x) \leqslant \frac{M}{d} \int_{\bar{x}}^{x} e^{\frac{c}{d}(x-t)} \int_{\bar{x}}^{t} U(s)\, ds\, dt \leqslant \frac{M}{d} U(x) \int_{\bar{x}}^{x} (t - \bar{x}) e^{\frac{c}{d}(x-t)}\, dt.$$

Thus if we set $\delta(x) = \int_{\bar{x}}^{x} (t - \bar{x}) e^{\frac{c}{d}(x-t)}\, dt$ one gets

$$1 \leqslant \frac{M}{d} \delta(x), \quad \forall x > \bar{x}.$$

Since $\delta(x) \to 0$ when $x \to \bar{x}$ we obtain a contradiction. This completes the proof of (i).

Next, since $U'(x) \geqslant 0$ for each $x \in (\bar{x}, \infty)$ one has

$$dU''(x) \leqslant V(x),$$

so that, for each $x \in (\bar{x}, \infty)$ one gets

$$dU(x) \leqslant \int_{\bar{x}}^{x} \int_{\bar{x}}^{t} V(s)\,ds.$$

Next since V is assumed to be increasing on $(\bar{x}, \bar{x}+h)$ one obtains

$$U(x) \leqslant \frac{(x-\bar{x})^2}{2d} V(x), \quad \forall x \in (\bar{x}, \bar{x}+h),$$

and the result follows. □

7. Proof of Theorem 4

Let $c > 0$ be given and consider (U, V, c) a travelling wave solution to (1). Let $\bar{x}$ be as in (31). We aim to show that $\bar{x} \in \mathbb{R}$. We will argue by contradiction assuming that

$$U(x) > 0, \quad \forall x \in \mathbb{R}. \tag{36}$$

Together with this assumption, function (U, V) becomes a classical and bounded solution on $\mathbb{R}$ to the system of equations

$$\begin{gathered} dU''(x) + cU'(x) = V(x), \\ V''(x) + cV'(x) + rV(x)\left(1 - \frac{V(x)}{U(x)}\right) = 0, \\ \lim_{x\to\infty} (U, V)(x) = (1, 0). \end{gathered}$$

Let $P : \mathbb{R} \to \mathbb{R}$ be defined by $P(x) = \frac{V(x)}{U(x)}$; it satisfies the equation

$$P'' + \left(2\frac{U'}{U} + c\right)P' + \frac{1-rd}{d}P^2 + rP + c\frac{d-1}{d}\frac{U'P}{U} = 0. \tag{37}$$

Lemma 10. *Function P is decreasing and $\lim_{x\to-\infty} P(x) = \infty$. Furthermore function U satisfies*

$$\lim_{x\to-\infty} V(x) = 0 \quad \textit{and} \quad \lim_{x\to-\infty} \big(U, U'\big)(x) = (0, 0). \tag{38}$$

Proof. Let us first show that P is unbounded. Indeed if P is bounded then function U satisfies

$$U''(x) + cU'(x) = P(x)U(x), \quad \forall x \in \mathbb{R}.$$

Here again, coupling elliptic Harnack inequality together with interior elliptic regularity leads us to the existence of some constant $M > 0$ such that (see the proof of Theorem 2)

$$\left|\frac{U'(x)}{U(x)}\right| \leqslant M, \quad \forall x \in \mathbb{R}. \tag{39}$$

Consider the map $W(t, x) = P(x - ct)$. It is a solution to

$$\partial_t W - \partial_x^2 W = 2\frac{U'(x-ct)}{U(x-ct)}\partial_x W + \frac{1-rd}{d}W^2 + rW + c\frac{d-1}{d}\frac{U'(x-ct)}{U(x-ct)}W(t,x),$$

for each $(t,x) \in \mathbb{R}^2$.

Since $d \geqslant 1$ and $U'(x) \geqslant 0$ for any $x \in \mathbb{R}$ one obtains using (39) that W satisfies the following differential inequality for all $(t,x) \in \mathbb{R}^2$:

$$\partial_t W - \partial_x^2 W + 2M|\partial_x W| - \frac{1-rd}{d}W^2 \geqslant 0.$$

However, since $1 - rd > 0$, any solution w of the problem

$$\partial_t w - \partial_x^2 w = -2M|\partial_x w| + \frac{1-rd}{d}w^2$$

has a finite time blow-up. We refer to [1] for such a result and also to the monograph [18]. This implies that W also has a finite time blow-up. This yields a contradiction together with the definition of W. Thus P is unbounded. As a consequence, there exists a decreasing sequence $\{x_n\}_{n\geqslant 0}$ such that

$$x_n \to -\infty \quad \text{and} \quad P'(x_n) < 0 \quad \text{for any } n \geqslant 0. \tag{40}$$

Next, note that due to (37), P satisfies the inequality

$$P'' + \left(2\frac{U'}{U} + c\right)P' \leqslant 0, \quad \forall x \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}$ be given such that $P'(x_0) < 0$. Set

$$z = \sup\bigl\{s \geqslant x_0\colon\ P'(t) < 0,\ t \in [x_0, s]\bigr\}. \tag{41}$$

We aim to show that $z = \infty$. We shall argue by contradiction by assuming that $z < \infty$ so that $P'(z) = 0$. Let $M = \sup\{(2\frac{U'(t)}{U(t)} + c) > 0,\ t \in [x_0, z]\}$. Then since $P'(s) \leqslant 0$ on $[x_0, z]$ one gets

$$P''(t) + MP'(t) \leqslant 0, \quad t \in [x_0, z],$$

that leads to

$$P'(z)e^{Mz} \leqslant P'(x_0)e^{Mx_0} < 0,$$

a contradiction to the definition of z in (41). Thus $z = \infty$.

Recalling (40), one obtains that for each $n \geqslant 0$, $P'(s) < 0$ for all $s \in [x_n, \infty)$. Since $x_n \to -\infty$ as $n \to \infty$, one concludes that $P'(x) < 0$ for all $x \in \mathbb{R}$. This completes the proof of the properties of function P.

It remains to prove (38). Let us first notice that the limit of function V follows from Lemma 8 (ii). Next, recalling Lemma 7 and Lemma 8 (i) one has

$$U'(x) \geqslant 0, \quad \forall x \in \mathbb{R} \quad \text{and} \quad \lim_{x\to-\infty} U'(x) = 0,$$

and there exists some constant $l \geqslant 0$ such that

$$\lim_{x\to-\infty} U(x) = l.$$

Note that if $l > 0$ then since V is bounded and $U \geqslant l > 0$ on the whole line $\mathbb{R}$, function P is bounded. That is a contradiction together with the first part of the lemma. This completes the proof of the lemma. □

Let us now recall that due to Lemma 10, Lemma 8 (ii) applies and provides the existence of an $x_0 \in \mathbb{R}$ such that

$$V'(x) \geqslant 0, \quad \forall x \in (-\infty, x_0] \quad \text{and} \quad \frac{d}{2}\left|U'(x)\right|^2 \leqslant U(x)V(x), \quad \forall x \in (-\infty, x_0]. \tag{42}$$

Lemma 11. *There exists $k > 0$ and $r < \alpha < 1$ such that*

$$kV(x) \geqslant U^{\alpha}(x), \quad \forall x \leqslant x_0.$$

Proof. Since $dr \in (0, 1)$, let $\alpha \in (dr, 1)$ be given with

$$r + \frac{\alpha}{d} - 2\frac{\alpha^2}{d} \leqslant 0.$$

Let $k > 0$ be given and fixed such that

$$kV(x_0) > U^{\alpha}(x_0). \tag{43}$$

Consider the map $w : \mathbb{R} \to \mathbb{R}$ defined by

$$w(x) = kV(x) - U^{\alpha}(x), \quad x \in \mathbb{R}.$$

Then w is a solution to the equation

$$w'' + cw' - r\frac{V}{U}w = c\alpha\frac{1-d}{d}U^{\alpha-1}U' - krV + \frac{V}{U}\left(r - \frac{\alpha}{d}\right)U^{\alpha} + \alpha(1-\alpha)U^{\alpha-2}\left|U'\right|^2. \tag{44}$$

On the other hand, (42) leads us to the following inequality for all $x \in (-\infty, x_0]$

$$\begin{aligned}\frac{V}{U}U^{\alpha}\left(r - \frac{\alpha}{d}\right) + \alpha(1-\alpha)U^{\alpha-2}\left|U'\right|^2 &\leqslant \frac{V}{U}U^{\alpha}\left(r - \frac{\alpha}{d}\right) + 2\frac{\alpha}{d}(1-\alpha)U^{\alpha-2}UV \\ &\leqslant VU^{\alpha-1}\left(r + \frac{\alpha}{d} - 2\frac{\alpha^2}{d}\right) \leqslant 0.\end{aligned}$$

We therefore infer from (44) that function w satisfies

$$w'' + cw' - r\frac{V}{U}w \leqslant c\alpha\frac{1-d}{d}U^{\alpha-1}U', \quad \forall x \leqslant x_0. \tag{45}$$

Recalling that $d \geqslant 1$ and U is increasing one obtains that

$$w''(x) + cw'(x) - r\frac{V(x)}{U(x)}w(x) \leqslant 0, \quad \forall x \in (-\infty, x_0). \tag{46}$$

From the choice of k in (43), one gets $w(x_0) > 0$. On the other hand, using $V \geqslant 0$ as well as the behaviour of U close to $x = -\infty$ provided in Lemma 10, one obtains that

$$\liminf_{x \to -\infty} w(x) \geqslant 0.$$

We aim to show that $w(x) \geqslant 0$ for all $x \in (-\infty, x_0)$. Assume by contradiction that there exists $x_1 \in (-\infty, x_0)$ such that

$$w'(x_1) = 0, \qquad w(x_1) < 0 \quad \text{and} \quad w''(x_1) \geqslant 0,$$

and one gets a contradiction with (46). Therefore $w(x) \geqslant 0$ for all $x \leqslant x_0$ and the result follows. $\square$

End of the proof of Theorem 4. Let us notice that function U satisfies the following equation on $(-\infty, x_0)$,

$$dU'' + cU' = V \geqslant \frac{1}{k} U^\alpha. \tag{47}$$

Next by using the possibility of the formation of a dead core for reaction–diffusion equation with $\alpha \in (0, 1)$, we shall show such a nonnegative function U tending to zero at $x = -\infty$ cannot exist. This will supply a contradiction and complete the proof of Theorem 4.

Recall now that there exists $\beta > 0$, $T > 0$ and a regular map $W : [0, T) \times [-1, 1] \to [0, \infty)$ with

$$\begin{cases} \partial_t W - d\partial_x^2 W + \frac{1}{k} W^\alpha = 0, \\ W(0, x) > 0, \qquad W(t, \pm 1) = \beta, \\ \lim\limits_{t \to T^-} \min\limits_{x \in [-1,1]} W(t, x) = 0. \end{cases}$$

This finite time dead core formation is proved in [4].

Now, since $U(x) \to 0$ when $x \to -\infty$ (see Lemma 10), let $x_1 < x_0$ be given such that

$$U(x) \leqslant \beta, \quad x \leqslant x_1 + 1, \qquad U(s + x_1) \leqslant W(0, s), \quad s \in [-1, 1].$$

Then consider the map

$$u(t, x) = U(x + x_1 - ct), \quad x \in [-1, 1], \ t \geqslant 0. \tag{48}$$

Recalling (47), it satisfies the following inequality for $t \geqslant 0$ and $x \in [-1, 1]$,

$$\begin{cases} \partial_t u - d\partial_x^2 u + \frac{1}{k} u^\alpha \leqslant 0, \\ u(t, \pm 1) \leqslant \beta, \quad t \geqslant 0, \\ u(0, x) \leqslant W(0, x), \quad x \in [-1, 1]. \end{cases}$$

Then a comparison principle applies yielding

$$u(t, x) \leqslant W(t, x), \quad \forall x \in [-1, 1], \ \forall t \in [0, T).$$

Due to the finite time quenching property of W one obtains

$$\min_{x\in[-1,1]} u(T,x) \leqslant 0.$$

Finally, since U is increasing, recalling (48), the above inequality leads to

$$U(-1+x_1-cT) \leqslant 0,$$

which is a contradiction with (36). This completes the proof of Theorem 4. □

8. Proof of Theorem 5

Assume now that $d \leqslant 1$ and $dr > 1$. Let us consider a decreasing sequence $\{\varepsilon_n\}_{n\geqslant 0}$ such that

$$\varepsilon_n > 0, \quad \forall n \geqslant 0 \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n = 0.$$

Let us denote for each $n \geqslant 0$, (U_n, V_n) the solution of (4) provided by Theorem 7 with $\varepsilon = \varepsilon_n$ and such that

$$V_n(x) \leqslant \frac{dr}{dr-1} U_n(x), \quad \forall x \in \mathbb{R},\ \forall n \geqslant 0.$$

Using the same argument as in Section 3, we may assume that

$$U_n \to U \quad \text{for the topology of } C^1_{loc}(\mathbb{R}).$$

In order to show that $U(x) > 0$, $x \in \mathbb{R}$, we argue by contradiction assuming that

$$U(x) \begin{cases} = 0 & \text{if } x \leqslant 0, \\ > 0 & \text{if } x > 0. \end{cases}$$

Using the same arguments than the ones used in Section 3, Case 2, the sequence $\{V_n\}$ converges to some function V for the topology of $C^1_{loc}(0,\infty)$. Thus one obtains that

$$V(x) \leqslant \frac{dr}{dr-1} U(x), \quad \forall x > 0.$$

Then, since one has

$$dU'' + cU' = V, \quad \forall x > 0,$$

one gets, due to the increasing property of function U that

$$U(x) \leqslant \frac{r}{dr-1} \int_0^x e^{\frac{c}{d}(x-t)} \int_0^t U(s)\,ds\,dt \leqslant \frac{r}{dr-1} U(x) \int_0^x t e^{\frac{c}{d}(x-t)}\,dt.$$

Thus if we set $\delta(x)=\int_0^x te^{\frac{c}{d}(x-t)}\,dt$ it follows that

$$1\leqslant \frac{r}{dr-1}\delta(x),\quad \forall x>0.$$

Since $\delta(x)\to 0$ when $x\to 0$ we reach a contradiction. This completes the proof of Theorem 5.

9. Proof of Theorem 6

Consider the case

$$d\leqslant 1\quad\text{and}\quad rd>1.$$

Let $c>0$ be given and let us argue by contradiction by assuming that there exist (U,V,c) a travelling wave solution to (1) such that

$$U(x)\begin{cases}=0 & \text{if } x\leqslant 0,\\ >0 & \text{if } x>0.\end{cases}$$

Lemma 12. *One has*

$$\lim_{x\to 0^+}P(x)=\infty,$$

and P is decreasing on some interval of the form $(0,x_0)$.

Proof. From Lemma 9 (i) we know that function P is unbounded so that

$$\limsup_{x\to 0^+}P(x)=\infty.$$

We first aim to show that $\lim_{x\to 0}P(x)=\infty$. To prove this we argue by contradiction by assuming that

$$\liminf_{x\to 0}P(x)<\infty.$$

Then there exists a decreasing sequence $\{x_n\}_{n\geqslant 0}$ such that

$$\lim_{n\to\infty}x_n=0,$$
$$P''(x_n)\leqslant 0,\qquad P'(x_n)=0,\qquad P(x_n)\to\infty.$$

From the following equation satisfied by P,

$$P''+\left(2\frac{U'}{U}+c\right)P'+\frac{1-rd}{d}P^2+rP+c\frac{d-1}{d}\frac{U'P}{U}=0,$$

one obtains that

$$0\leqslant\frac{1-rd}{d}P^2(x_n)+rP(x_n),$$

a contradiction.

Now P satisfies for x close to zero,

$$P'' + \left(2\frac{U'}{U} + c\right)P' \geqslant 0,$$

and the result follows. □

We are now able to prove Theorem 6. Function P satisfies the equation

$$P'' + \left(2\frac{U'}{U} + c\right)P' + \frac{1-rd}{d}P^2 + rP + c\frac{d-1}{d}\frac{U'P}{U} = 0,$$

for each $x > 0$. This implies that

$$P'' + \frac{1-rd}{d}P^2 + rP \geqslant 0, \quad \forall x > 0.$$

Let $0 < \varepsilon < rd - 1$ be given. Then there exists $0 < x_\varepsilon < x_0$ such that $\frac{1-rd}{d}P^2 + rP \leqslant \frac{1-rd+\varepsilon}{d}P^2$ on $(0, x_\varepsilon)$. Thus setting $k_\varepsilon = -\frac{1-rd+\varepsilon}{d}$ one gets that

$$P''(x) - k_\varepsilon P^2(x) \geqslant 0, \quad \forall x \in (0, x_\varepsilon).$$

Since P is decreasing on $(0, x_\varepsilon)$ one obtains

$$P'(x)P''(x) - k_\varepsilon P'(x)P^2(x) \leqslant 0, \quad \forall x \in (0, x_\varepsilon).$$

Integrating this equation yields

$$\frac{1}{2}\left(P'(x_\varepsilon)^2 - P'(x)^2\right) \leqslant \frac{k_\varepsilon}{3}\left(P^3(x_\varepsilon) - P^3(x)\right), \quad \forall x \in (0, x_\varepsilon).$$

Let $0 < \eta < k_\varepsilon$ be given. Since $P(x) \to \infty$ when $x \to 0^+$, then there exists $0 < y_\eta < x_\varepsilon$ such that

$$P'(x)^2 \geqslant 2\frac{k_\varepsilon - \eta}{3}P^3(x), \quad \forall x \in (0, y_\eta).$$

It follows from this latter differential inequality that

$$P'(x)P^{-3/2}(x) \leqslant -\sqrt{2\frac{k_\varepsilon - \eta}{3}}, \quad \forall x \in (0, y_\eta).$$

Integrating the above inequality from 0 to x one finds

$$2P^{-1/2}(x) \geqslant \sqrt{2\frac{k_\varepsilon - \eta}{3}}x, \quad \forall x \in (0, y_\eta).$$

Therefore for any sufficiently close to zero values of $x > 0$ one obtains

$$\frac{6}{k_\varepsilon - \eta} \geqslant x^2 P(x).$$

Next, from Lemma 9 (ii) for each $x > 0$ sufficiently small one has

$$x^2 P(x) \geqslant 2d.$$

This implies that

$$\frac{6}{k_\varepsilon - \eta} \geqslant 2d.$$

Letting η (resp. ε) go to zero one gets $3 \geqslant rd - 1$. This completes the proof of Theorem 6.

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