Convergence to a pulsating travelling wave for an epidemic reaction-diffusion system with non-diffusive susceptible population

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Abstract

In this work we study the asymptotic behaviour of the Kermack-McKendrick reaction-diffusion system in a periodic environment with non-diffusive susceptible population. This problem was proposed by Kallen et al. as a model for the spatial spread for epidemics, where it can be reasonable to assume that the susceptible population is motionless. For arbitrary dimensional space we prove that large classes of solutions of such a system have an asymptotic spreading speed in large time, and that the infected population has some pulse-like asymptotic shape. The analysis of the one-dimensional problem is more developed, as we are able to uncover a much more accurate description of the profile of solutions. Indeed, we will see that, for some initially compactly supported infected population, the profile of the solution converges to some pulsating travelling wave with minimal speed, that is to some entire solution moving at a constant positive speed and whose profile's shape is periodic in time.

Key words Kermack-McKendrick reaction-diffusion system, periodic environment, pulsating travelling wave, asymptotic behaviour.

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1 Introduction

The goal of this work is to study the spatio-temporal dynamics of the following spatially heterogeneous reaction-diffusion system

$$\begin{cases} \partial_t v(t,x) &= -\sigma(x)v(t,x)u(t,x), \\ \partial_t u(t,x) &= \Delta u(t,x) + \sigma(x)v(t,x)u(t,x) - \rho(x)u(t,x), \end{cases} \tag{1.1}$$

posed for $(t,x) \in (0,+\infty) \times \mathbb{R}^N$. Here functions $\sigma : \mathbb{R}^N \to \mathbb{R}$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ are assumed to be positive and of the class $C^{1,\gamma}$ for some $\gamma > 0$. System (1.1) is supplemented together with some nonnegative initial data

$$v(0,x) = v_0(x) \not\equiv 0, \;\; u(0,x) = u_0(x) \not\equiv 0, \;\; \forall x \in \mathbb{R}^N. \tag{1.2}$$

Problems of the form (1.1) are a particular case of the so-called Kermack-McKendrick reaction-diffusion system where the $v-$component is assumed to be motionless (that is without diffusing effect for this component). We refer to the original paper of Kermack-McKendrick [23] for the modelling of the kinetic part. In the context of epidemiology, (1.1) models the spatio-temporal dynamics of transmission of a disease within a spatially distributed population. Here $v(t,x)$ (resp. $u(t,x)$) denotes the density of susceptible (resp. infected) individuals at time $t > 0$ and spatial location $x \in \mathbb{R}^N$, while $\sigma(x)$ denotes the contamination rate and $\rho(x)$ corresponds to the additional mortality (or removed) rate due to the infection. This problem has been proposed by Kallen et al. in [21] (we also refer to the monograph Murray [25]), with constant functions $\sigma(x) \equiv \sigma$ and $\rho(x) \equiv \rho$, to model the spatial spread of rabies epizootic in foxes across Europe. Let us recall that rabies is a viral infection of the central nervous system that is transmitted by direct contact between animals. While raccoons are one of the main vector of the disease in north America, foxes have been identified as the main vector for Europe.

As suggested by Kallen et al [21] the spatial spread of rabies across Europe is essentially due to the migration of rabid foxes that exhibit an erratic movement induced by the disease. As explained by Murray et al in [22] the time scale for the colonisation of new territories by susceptible foxes is very slow with respect to the motion of rabid foxes. Hence (1.1) was proposed by Kallen et al in [21] to describe the spatial spread of rabies using the above assumptions. Let us also notice that in (1.1) the vital dynamics of foxes is omitted by assuming that the time scale for the propagation of the disease is fast with respect to natural birth and death processes. Recall that the average time of disease is about one month (including clinical symptoms and incubation period) while birth rate is about one per year (see [1]). As described by Kallen et al this model reproduces the spatial spread of a primary outbreak of the disease. The latter assumption has been weakened by Murray et al in [22] where the authors considered logistic vital dynamics as well as latency period (namely non-diffusive exposed class). Then, in addition to the description of the primary outbreak, because of the vital dynamics, secondary outbreaks appear behind this primary wave.

In this work we shall focus on the description of the spatial spread of this primary outbreak by studying (1.1). Let us notice that systems similar to (1.1) appear in many other applications. We refer to Britton [10] and the references cited therein for other applications of such reaction-diffusion systems in the context of chemical reactions, diffusing fungal growth over a motionless resource as well as simple phytoplankton-zooplankton interactions. Let us also mention a model of species invasion proposed by Murray [25] where the population is split into dispersers and non-dispersers. The corresponding mathematical model reads as a reaction-diffusion equation coupled together with an ordinary differential equation.

Coming back to (1.1), as explained above, in this work we will focus on the description of the spatial spread of the outbreak of the disease induced by the introduction of the localized amount of infectious individuals. For this reason, we will restrict our analysis to a special class of initial data where the initial distribution of susceptibles $v_0 = v_0(x)$ is positive everywhere, while the initial density of infected $u_0 = u_0(x)$ is a nonnegative compactly supported function. As conjectured by Kallen in [20], in such a context the asymptotic behaviour of (1.1) is expected to be given by the travelling wave associated to (1.1) with the minimal wave speed. In this work, we will shed some light on this question for the one-dimensional problem including spatially periodic heterogeneities.

Indeed, this study will focus upon the spatially periodic framework. We will assume that there exists a cell $C = (0, L_1) \times ... \times (0, L_N)$, with $L_i > 0$ for each $i = 1, .., N$ such that functions σ and ρ are both $C-$periodic, namely they satisfy for any $x = (x_1, ..., x_N) \in \mathbb{R}^N$ and $(k_1, ..., k_N) \in \mathbb{Z}^N$,

$$\sigma(x_1 + k_1 L_1, ..., x_N + k_N L_N) = \sigma(x_1, ..., x_N),$$
$$\rho(x_1 + k_1 L_1, ..., x_N + k_N L_N) = \rho(x_1, ..., x_N).$$

As mentioned above, the asymptotic behaviour of solutions of (1.1) is related to the existence of travelling waves. The existence of such solutions has been investigated by Kallen in [20] for system (1.1) in a homogeneous environment. Let us also mention that the existence as well as qualitative properties of travelling waves have been widely studied for the two diffusive species system, for particular geometrical situations. Most of the literature on this topic has been concerned with either the one dimensional homogeneous case or the cylindrical case, and exhibited very similar results to the standard scalar KPP equation [3, 15, 18, 19, 20]. Let us also mention the works [2, 13, 14] where the existence of travelling waves has been investigated for age structured Kermack-McKendrick like problems. Finally, note that very little is known about the spreading properties of the corresponding Cauchy problem, which is due to the lack of a comparison principle in the general case.

The aim of this work is to derive some spreading properties for the one diffusive species problem (1.1). We will prove that under suitable conditions, this system has a spreading speed property. This includes the persistence of the disease as well as a pulse-shaped profile of the infection. We also give a more precise study of the one dimensional problem. In such a framework, convergence to a pulsating travelling wave is proved. The rest of this section is devoted to some preliminary notations, our assumptions and the statement of our main results. Section 2 is devoted to the proof of the spreading speed properties, while the convergence of the profile of the solution to the pulsating travelling wave with minimal speed is investigated in Section 3. Our proof will use a result of independent interest, namely the convergence to a travelling wave in a periodic environment with some local perturbation.

1.1 Preliminaries and assumptions

The basic idea to study (1.1) follows from the works of Diekmann [11, 12] and Thieme [28, 29] (see also [2, 9, 27]). We will introduce a new unknown function that will satisfy an equation with similar properties to the well knwon Fisher-KPP reaction-diffusion equation. Let us consider a solution (u, v) of the above system, namely (1.1)-(1.2). Then the integration of the first equation yields

$$v(t, x) = v_0(x)e^{-\sigma(x)\int_0^t u(s,x)ds},$$

so that the second equation re-writes as

$$\partial_t u = \Delta u(t, x) - \rho(x)u(t, x) - v_0(x)\partial_t \left(e^{-\sigma(x)\int_0^t u(s,x)ds}\right).$$

Next by setting $w(t, x) = \int_0^t u(s, x)ds$, one gets that w satisfies for $t > 0$ and $x \in \mathbb{R}^N$:

$$\begin{cases} \partial_t w = \Delta w(t, x) + u_0(x) + f(x, w(t, x)), \\ w(0, x) = 0, \end{cases} \tag{1.3}$$

wherein we have set $f : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ defined by

$$f(x, s) = v_0(x)\left(1 - e^{-\sigma(x)s}\right) - \rho(x)s. \tag{1.4}$$

Here, the function w represents the history of the infection at each spatial location $x \in \mathbb{R}^N$.

As announced, if one ignores the term u_0, the above equation (namely (1.3)) becomes a single reaction-diffusion equation

$$\partial_t w = \Delta w(t, x) + f(x, w(t, x)). \tag{1.5}$$

Recalling (1.4), the initial condition v_0 plays a role in the reaction term and, in order to use the standard tools on the Cauchy problem above, we will make the following additional assumption:

Assumption 1.1 *We assume that*

(i) The function $v_0 \geq 0$ and

$$v_0 \in C^1\left(\mathbb{R}^N\right) \setminus \{0\} \text{ and is } C\text{- periodic}, \tag{1.6}$$

while the function $u_0 \in C^0\left(\mathbb{R}^N\right) \setminus \{0\}$ is nonnegative and bounded.

(ii) Furthermore, the function u_0 is compactly supported.

Remark 1.2 *Periodicity of v_0 is a reasonable hypothesis in the purely periodic framework. Indeed, as explained before, the vital dynamics is omitted so that v_0 can be considered as the state of the population at equilibrium before the introduction of the disease. Such an equilibrium typically shares the periodicity of the environment [5, 7]. However, we will comment on this assumption and its biological relevance again in the last section.*

Let us now check that f defined in (1.4) satisfies the standard Fisher-KPP hypothesis. Notice that

$$s \mapsto \frac{f(x,s)}{s} \text{ is decreasing and } f(x,s) \leq b(x)s \;\; \forall (x,s) \in \mathbb{R}^N \times [0,\infty), \tag{1.7}$$

wherein we have set $b(x) := v_0(x)\sigma(x) - \rho(x) = \frac{\partial f}{\partial s}(x,0)$. Because of Assumption 1.1, problem (1.5) is a usual Fisher-KPP equation posed in a periodic environment, while (1.3) is a localized spatial perturbation of the latter.

Before stating our main results, let us recall some well-known properties about the Fisher-KPP equation (1.5). The existence of a non-trivial $C-$periodic stationary state for (1.5) relies on the linear unstability of the trivial stationary state $w \equiv 0$, that is on the sign of the principal periodic eigenvalue for Problem (1.5) linearized around 0, which we will denote by $\mu_0 \in \mathbb{R}$. More precisely, the number μ_0 is the principal eigenvalue associated with the following elliptic problem:

$$\begin{cases} -\Delta\phi(x) - b(x)\phi(x) = \mu_0\phi(x), \;\; x \in \mathbb{R}^N, \\ \phi(x) > 0, \;\; \forall x \in \mathbb{R}^N \text{ and } \phi \text{ is } C-\text{periodic.} \end{cases} \tag{1.8}$$

Now, as a consequence of Theorem 2.1 and 2.4 proved by Berestycki et al. in [5], the following holds true: if $\mu_0 < 0$ then the elliptic equation

$$\Delta q(x) + f(x, q(x)) = 0, \;\; x \in \mathbb{R}^N,$$

has exactly two nonnegative bounded solutions, the trivial solution 0 and a positive and $C-$periodic function that we denote by p.

Since the operator is self-adjoint, the eigenvalue μ_0 can be expressed thanks to the Rayleigh formula:

$$\mu_0 = \min_{\phi \in H^1_{per}, \phi \not\equiv 0} \frac{\int_C \left[|\nabla\phi(x)|^2 - (v_0(x)\sigma(x) - \rho(x))\, \phi^2(x) \right] dx}{\int_C \phi^2(x) dx}. \tag{1.9}$$

In particular, although it is not an optimal condition, it is negative when $\int_C [v_0(x)\sigma(x) - \rho(x)]\, dx > 0$. In the homogeneous case, that is when $v_0(x) = v_0$, $\sigma(x) \equiv \sigma$ and $\rho(x) \equiv \rho$, the condition $\mu_0 > 0$ re-writes as

$$R_0 := \frac{v_0\sigma}{\rho} < 1.$$

Besides, the left hand side of the above inequality corresponds to the so-called basic reproduction rate of the Kermack-McKendrick model [23]. As it will be seen later, the quantity μ_0 will play the role of an epidemic threshold, similarly to the basic reproduction number in the homogeneous case.

Spreading properties associated to (1.5) are also well studied. In the multidimensional setting, one needs to perform directional analysis for each direction e in the unit sphere $S^{N-1} \subset \mathbb{R}^N$. To do so, let us introduce for each $\lambda \geq 0$ and each $e \in S^{N-1}$ the principal eigenvalue, denoted by $\mu_e(\lambda) \in \mathbb{R}$, of the following problem

$$\begin{cases} -\Delta\phi_\lambda(x) - 2\lambda e\cdot\nabla\phi_\lambda(x) - b(x)\phi_\lambda(x) = \mu_e(\lambda)\phi_\lambda(x), \quad x\in\mathbb{R}^N, \\ \phi_\lambda(x) > 0, \quad \forall x\in\mathbb{R}^N \text{ and } \phi_\lambda \text{ is } C\text{-periodic.} \end{cases} \tag{1.10}$$

Recalling the definition of μ_0 in (1.8), note that for each $e \in S^{N-1}$ one has $\mu_e(0) = \mu_0$.

Assuming now that $\mu_0 < 0$, then one may consider for each $e \in S^{N-1}$:

$$c_e^* := \inf\left\{c\in\mathbb{R} \mid \exists\lambda > 0,\ \lambda^2 - c\lambda = \mu_e(\lambda)\right\} \in (0,+\infty).$$

Using the results of [6] (see also the references cited therein), it is known that for each $e \in S^{N-1}$ equation (1.5) admits pulsating travelling waves with speed c in the direction $e \in S^{N-1}$, that is particular entire solutions W of (1.5) of the form

$$\begin{cases} W\left(t + \dfrac{k\cdot e}{c}, x+k\right) = W(t,x),\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N,\ \forall k\in\displaystyle\prod_{i=1}^N L_i\mathbb{Z}, \\ \displaystyle\lim_{x\cdot e\to\infty} W(t,x) = 0, \quad \lim_{x\cdot e\to-\infty} W(t,x) = p(x), \end{cases}$$

if and only if $c \geq c_e^*$.

Furthermore, the asymptotic speed of spread (also referred to as spreading speed) in the direction $e \in S^{N-1}$ of the solution of (1.5) together with a given compactly supported initial data, has been proved [4, 31] to be equal to

$$\omega_e^* = \min_{\xi\in S^{N-1},\ e\cdot\xi>0} \left(\frac{c_\xi^*}{e\cdot\xi}\right).$$

Roughly speaking, this means that one moving through the domain at any speed less than ω_e^* in the direction e will see the solution w approaching the positive equilibrium p, while one moving at any speed more than ω_e^* will only see the solution w stay close to 0.

Note that it is clear that, in the one dimensional framework, the asymptotic speed of spread coincides with the minimal speed of pulsating travelling waves. In such a one-dimensional setting, the link between asymptotic speed of spread and travelling waves is known to be even stronger. It is indeed classical that the profile of propagation of solutions of the homogeneous Fisher-KPP equation, associated to some fast decaying initial data (in particular, compactly supported), converges to the profile of the travelling wave with minimal speed [8, 24, 30]. This result has been generalized very recently to spatially periodic environments [16, 17].

Since (1.3) only differs from (1.5) by a localized perturbation, one expects that the large time behaviour of solutions will still be strongly related to this Fisher-KPP single equation (1.5). We will indeed state in the next section that the spreading speed of our system is equal to w_e^* defined above, and that in the one dimensional setting, the profile of the solution also converges to the pulsating travelling wave with minimal speed.

1.2 Statement of the results

In this section we will state the main results of this work, whose proofs will follow from the above discussion recalling that equation (1.3) is (in some sense) close to the Fisher-KPP periodic one (1.5).

We will first check the existence and uniqueness of some positive steady state for (1.3).

Lemma 1.3 *Let Assumption 1.1 (i) be satisfied and assume furthermore that $\mu_0 < 0$, μ_0 being defined from (1.8). Then the elliptic equation*

$$\Delta q(x) + u_0(x) + f(x, q(x)) = 0, \quad x \in \mathbb{R}^N,$$

admits a unique positive and bounded solution $p_0 > p$.

Furthermore, if Assumption 1.1 (ii) is satisfied then $p_0(x) - p(x) \to 0$ as $x \to \pm\infty$.

Although this lemma is only concerned with stationary solutions of (1.3), it allows us to expect, from the way w was introduced, to look at the pair $u = \partial_t p_0 = 0$ and $v = v_0 e^{-\sigma p_0}$ as a candidate for the limiting state toward which propagation does occur for (1.1).

Our first main theorem deals with the spatial spread phenomena for (1.1) when the initial infected population u is compactly supported.

Theorem 1.4 (Asymptotic speed of spread) *Let Assumption 1.1 be satisfied. Assume furthermore that $\mu_0 < 0$ (see (1.8)). Then the solution (u, v) of (1.1)-(1.2) spreads at the speed ω_e^* for each $e \in S^{N-1}$ in the sense that:*

(i) for each $c > \omega_e^*$

$$\lim_{t\to\infty} \sup_{\beta \geq c} u\,(t, \beta te) = 0, \quad \lim_{t\to\infty} \sup_{\beta \geq c} |v(t, \beta te) - v_0(\beta te)| = 0;$$

(ii) for each $0 < c < \omega_e^*$:

$$\lim_{t\to\infty} \sup_{0\leq\beta\leq c} u(t, \beta te) = 0, \quad \lim_{t\to\infty} \sup_{0\leq\beta\leq c} \left| v(t, \beta te) - v_0(\beta te) e^{-\sigma(\beta te) p_0(\beta te)} \right| = 0.$$

(iii) (Uniform persistence of the disease) The function u satisfies

$$\liminf_{t\to\infty} \sup_{x\in\mathbb{R}^N} u(t, x) > 0.$$

Remark 1.5 *The condition $\mu_0 < 0$ is almost optimal for spreading to occur. Indeed, one can easily check that $v \leq v_0$ for any positive time, and that $\|u_0\|_\infty e^{-\mu_0 t}$ is a super-solution for the equation (1.1) satisfied by u, for any $v \leq v_0$. It immediately follows that, if $\mu_0 > 0$, extinction occurs.*

Remark 1.6 *Let us recall that due* (1.9) *when* $\int_C v_0(x)\sigma(x)dx < \int_C \rho(x)dx$ *then* $\mu_0 > 0$ *and extinction of the disease occurs.*

In the same way if one assume that the initial distribution of susceptible v_0 *depends upon a parameter* $\alpha > 0$ *as* $\alpha v_0(x)$ *then the corresponding eigenvalue* $\mu_0(\alpha)$ *(defined as in* (1.9) *with* v_0 *replaced by* αv_0*) is strictly decreasing from* $\mu_0(0) > 0$ *to* $-\infty$ *when* $\alpha \to \infty$*. Hence there exists a threshold value* $\alpha_0 > 0$ *such that extinction occurs when* $\alpha \in (0, \alpha_0)$ *while disease invasion takes place for* $\alpha > \alpha_0$*.*

This remark shows that a favourable environment for the disease invasion requires sufficiently large (in some sense) density of susceptible individuals.

In the one dimensional framework, one can perform a more detailed analysis of the dynamics of the solution (u, v) to describe its profile of propagation. We show that it converges to a pulsating travelling wave, whose definition in the framework of (1.1) can easily be derived from the classical notion that we recalled above for (1.5).

Theorem 1.7 (Convergence to pulsating travelling wave) *Let us assume that* $N = 1$*, that Assumption 1.1 is satisfied and furthermore that* $\mu_0 < 0$*. Let* $W^*(t, x; v_0)$ *be a pulsating travelling wave solution of* (1.5) *associated to the minimal speed* c^**.*

Then there exists a function $t \mapsto m(t)$ *defined for all* t *large enough with* $m(t) = o(t)$ *such that for any* $\delta > 0$*, one has as* $t \to +\infty$ *that the solution* (u, v) *of* (1.1)-(1.2) *satisfies*

$$\|u(t,\cdot) - \partial_t W^*(t - m(t), \cdot; v_0)\|_{L^\infty(\delta t, +\infty)} \to 0,$$

and

$$\left\| v(t,\cdot) - v_0(\cdot)e^{-\sigma(\cdot)W^*(t-m(t),\cdot;v_0)} \right\|_{L^\infty(\delta t, +\infty)} \to 0.$$

Remark 1.8 *Note that the previous theorem, Theorem 1.4* (*ii*)*, already describes what happens between* 0 *and* δt *for any* $0 < \delta < c^*$*, so that this latter convergence result gives a complete picture of the dynamics of the propagation.*

Remark 1.9 *The delay* $m(t)$ *should be of logarithmic order, more precisely, it should satisfy*

$$m(t) \sim \frac{3}{2\lambda_*} \log(t) \text{ as } t \to +\infty,$$

where λ_* *is the unique positive solution of* $\lambda_*^2 - c^*\lambda_* = \mu(\lambda_*)$*.*

Indeed, this was shown in [17] for the unperturbed single periodic equation (1.5)*, under the additionnal assumption* $p \equiv 1$ *(mostly to simplify the computations). Since the main idea of our proof will be to trap the solution* w *of* (1.3) *between two solutions of* (1.5)*, we get the same equivalent delay here for the system with non-diffusive susceptible population.*

The pair $(\partial_t W^*, v_0 e^{-\sigma W^*})$ can clearly be seen as a pulsating travelling wave solution of (1.1), as it satisfies the same time-space periodicity than W^*, which is the main feature of classical pulsating travelling waves. One could conversely check that integrating the infected population component U of any pulsating travelling wave (U, V) of (1.1) gives a pulsating travelling wave of (1.5). This is clear from an immediate formal computation, although a rigorous proof would first require to check that U is integrable near $t = -\infty$ (using, for instance, the same method as in Lemma 3.1 in [6]). From these observations, Theorem 1.7 means, as announced, that the profile of the solution of (1.1) with initial data satisfying Assumption 1.1 does converge to the pulsating travelling wave with minimal speed.

One should note that the initial datum u_0 plays an important role in the stationary state p_0 described in Lemma 1.3 that is reached in the fixed frame. In other words, how many susceptibles remain after the propagation is directly related to the number of infected that were initially introduced in the medium. However, it does not play any role on the speed and shape of the propagation in the suitable moving frame, whose dynamics are accurately described by p and W^* which only depend on the initial susceptible population v_0.

2 Spreading speed in the Cauchy problem

This section is concerned with the proofs of Lemma 1.3 and Theorem 1.4 in the multidimensional whole space $\mathbb{R}^N$.

2.1 Stationary states: proof of Lemma 1.3

In this section we prove Lemma 1.3. From the positivity of ρ, there exists some constant $M > 0$ such that for each $x \in \mathbb{R}^N$ and each $s \geq M$ then $u_0(x) + f(x, s) < 0$. Then, from the standard elliptic theory and using the fact that 0 and M are respectively strict sub- and super-solutions of (1.3), we already get the existence of a positive and bounded stationary solution of (1.3).

We now prove that any such solution p_0 lies above p. Let us consider w the solution of

$$\partial_t w = \Delta w + f(x, w), \ t > 0 \text{ and } x \in \mathbb{R}^N,$$

supplemented together with the initial datum $w(0, .) = p_0$. Since $u_0 \geq 0$, p_0 becomes a super-solution of the above equation, therefore w is decreasing in time, and it converges to some stationary solution $w_\infty \leq p_0$. On the other hand, from the well-known spreading property in the KPP periodic case, it is clear that $w_\infty \geq p$. As p is the only positive stationary solution, this means that $w_\infty \equiv p \leq p_0$. From the strong maximum principle, this last inequality is in fact strict. In particular, the infimum of p_0 is positive.

It remains to show that it is unique. Let p_0 and q_0 be two positive and bounded stationary solutions of (1.3). Since $\inf p_0 \geq \inf p > 0$ and q_0 is bounded, one has that $\theta p_0 > q_0$ for θ large enough. We introduce

$$\theta^* = \inf\{\theta \mid \theta p_0 > q_0\}.$$

Since $\inf q_0 > 0$ and p_0 is bounded, we have that $\theta^* > 0$. Assume by contradiction that $\theta^* > 1$. Note then that $\inf(\theta^* p_0 - p_0) > 0$. From our KPP hypothesis,

$$\theta^* f(x, p_0) \geq f(x, \theta^* p_0) + \varepsilon.$$

Besides, one has

$$\Delta(\theta^* p_0 - q_0) + (\theta^* - 1)u_0 + \theta^* f(x, p_0) - f(x, q_0) = 0.$$

Thus,

$$\Delta(\theta^* p_0 - q_0) + f(x, \theta^* p_0) - f(x, q_0) \leq -\varepsilon < 0.$$

As f is C^1, we can rewrite this equation as

$$\Delta(\theta^* p_0 - q_0) + g(t, x)(\theta^* p_0 - q_0) \leq -\varepsilon < 0,$$

where g is a bounded function. From the construction of θ^*, it is clear that $\theta^* p_0 \geq q_0$ and $\inf(\theta^* p_0 - q_0) = 0$. This is a contradiction with a strong maximum principle (as stated in Lemma 2.1 in [7]), and we conclude that $p_0 \geq q_0$. By reversing the role of the two functions, we also get that $q_0 \geq p_0$, hence $q_0 \equiv p_0$.

Lastly, when u_0 is compactly supported, one can easily check that

$$p_0(x) - p(x) \to 0$$

as $|x| \to +\infty$, from the uniqueness of a positive stationary solution of (1.5). This ends the proof of Lemma 1.3.

2.2 Inner spreading speed

We recall that we have set $w(t, x) = \int_0^t u(s, x)ds$. Note that w converges locally uniformly to p_0 as $t \to +\infty$. Indeed, by positivity of $u = \partial_t w$ and since $w \leq p_0$ from the parabolic maximum principle, it is clear that w converges to a positive stationary solution of (1.3), thus to p_0.

Let us now consider a moving frame in the direction $e \in S^{N-1}$ with speed $0 < c < \omega^*(e)$. We show that $w(t, cte)$ converges to $p(cte)$. Let us introduce $\underline{w}$ the solution of the Cauchy problem

$$\partial_t \underline{w} = \Delta \underline{w} + f(x, \underline{w}), \;\; \underline{w}(0, x) = w(1, x), \;\; x \in \mathbb{R}^N.$$

It then follows from the comparison principle that, for any $t \geq 0$,

$$\underline{w}(t, cte) \leq w(t + 1, cte) \leq p_0(cte). \tag{2.11}$$

On one hand, it is clear, thanks to the diffusion, that $w(1, x) > 0$ for all $x \in \mathbb{R}^N$. Therefore, it is known that $\underline{w}$ asymptotically spreads at least with the speed $\omega^*(e)$ in the direction e (see [4, 31]), so that for any $c < \omega^*(e)$:

$$\lim_{t \to +\infty} \sup_{0 \leq \beta \leq c} |\underline{w}(t, \beta te) - p(\beta te)| = 0.$$

On the other hand, we know that

$$\limsup_{|x|\to+\infty} |p_0(x) - p(x)| = 0.$$

Now let $\varepsilon > 0$ be given. Choose $D > 0$ sufficiently large such that for any $|x| \geq D$, $|p_0(x) - p(x)| \leq \frac{\varepsilon}{2}$, and $T > 0$ large enough such that $|w(t,x) - p_0(x)| \leq \varepsilon$ for any $|x| \leq D$ and $t \geq T$. Then, from (2.11), we easily get that for any $c < \omega^*(e)$:

$$\lim_{t\to+\infty} \sup_{0\leq\beta\leq c} |w(t,\beta te) - p_0(\beta te)| \leq \varepsilon.$$

One can then conclude the proof of statement (ii) of Theorem 1.4 by using standard parabolic estimates (the function w is locally bounded in $W^{2,p}$ for any $p \geq 1$, hence the convergence to any ω-limit must hold in $C^{1,\beta}$ for $0 \leq \beta < 1$) and the fact that for any $(t,x) \in (0,\infty) \times \mathbb{R}^N$,

$$v(t,x) = v_0(x)e^{-\sigma(x)w(t,x)} \text{ and } u(t,x) = \partial_t w(t,x).$$

2.3 Outer spreading speed

Let $e \in S^{N-1}$ be a given direction and let $c > \omega^*(e)$ be given. Then the inequality $c > \omega^*(e)$ implies that there exists $\xi_0 \in S^{N-1}$ such that $e \cdot \xi_0 > 0$ and $c_0 := ce \cdot \xi_0 > c^*_{\xi_0}$.

We now choose $c'_0 \in (c^*_{\xi_0}, c_0)$. One can then find a super-solution moving in the direction e with speed c'_0. Indeed, by definition of $c^*_{\xi_0}$, one can easily check that there exists $\lambda > 0$ such that $\lambda^2 - c'_0\lambda = \mu_e(\lambda)$, and then that $(t,x) \mapsto e^{-\lambda(x\cdot\xi_0 - c'_0 t)}\phi_\lambda(x)$ is a super-solution of (1.5). Furthermore, since u_0 is compactly supported, one can choose some constant $M > 0$ such that the map

$$(t,x) \mapsto \inf\left\{p_0(x), Me^{-\lambda\left(x\cdot\xi_0 - c'_0 t\right)}\right\},$$

is a super-solution of (1.3). From the parabolic comparison principle, one gets that for all $t > 0$ and $x \in \mathbb{R}^N$,

$$w(t,x) \leq Me^{-\lambda\left(x\cdot\xi_0 - c'_0 t\right)}. \tag{2.12}$$

With $\beta \geq c$ and $x = \beta te$, one obtains $x \cdot \xi_0 = t\beta e \cdot \xi_0 \geq tc_0$, so that

$$\lim_{t\to+\infty} \sup_{\beta\geq c} w(t,\beta te) = 0.$$

As above, we immediately get that

$$\lim_{t\to+\infty} \sup_{\beta\geq c} |v(t,\beta te) - v_0(\beta te)| = 0 \text{ and } \lim_{t\to+\infty} \sup_{\beta\geq c} u(t,\beta te) = 0,$$

that completes the proof of statement (i) in Theorem 1.4.

2.4 Uniform persistence of the disease

In order to prove the last statement of Theorem 1.4, let us argue here by contradiction by assuming that there exists a sequence $\{t_k\}_{k\geq 0}$ such that

$$\lim_{k\to\infty} t_k = +\infty \text{ and } \lim_{k\to\infty} \sup_{x\in\mathbb{R}} u(t_k, x) = 0. \tag{2.13}$$

Then due to the spreading speed property in the direction $e_1 \in S^{N-1}$ of the first coordinate, there exists a sequence $r_k \to +\infty$ as $k \to \infty$ and ε such that

$$\varepsilon < w(t_k, r_k e_1) < \inf_{x\in\mathbb{R}^N} p(x) - \varepsilon, \quad \forall k \geq 0.$$

One can write $r_k = n_k L_1 + s_k$ where n_k is an integer and $s_k \in (0, L_1)$ converges, up to extraction of some subsequence, to s_∞. Next consider the sequence $w_k(x) = w(t_k, x + n_k L_1 e_1)$. It satisfies

$$u(t_k, x + n_k L_1 e_1) = \Delta w_k + u_0(x + n_k L_1 e_1) + f(x, w_k), \quad \forall x \in \mathbb{R}^N.$$

It follows from standard elliptic estimates that w_k is bounded in $W^{2,p}_{loc}(\mathbb{R}^N)$ for all $1 \leq p < +\infty$. Hence, up to extraction of a subsequence, one may assume that w_k converges to some function w_∞ weakly in $W^{2,p}_{loc}(\mathbb{R}^N)$ for all $1 \leq p < +\infty$ and strongly in $C^{1,b}_{loc}(\mathbb{R}^N)$ for all $0 \leq b < 1$. Then $\varepsilon < w_\infty(s_\infty e_1) < p(s_\infty e_1) - \varepsilon$ and, due to (2.13) and the fact that u_0 is compactly supported, the function w_∞ satisfies

$$0 = \Delta w_\infty + f(x, w_\infty), \quad w_\infty(0) \in \left(0, \inf_{x\in\mathbb{R}^N} p(x)\right).$$

However, we know that the above equation only admits a unique positive solution, that is p. Therefore, we have reached a contradiction and we can conclude that statement (*iii*) of Theorem 1.4 holds.

3 Convergence to a pulsating wave

From now on, we will always assume that we are in a one dimensional domain, that is $x \in \mathbb{R}$. We will first give some estimate on the behaviour of w ahead of the propagation, namely that it has very fast decay. This will allow us to use a result of independent interest, that will be stated in Theorem 3.2.

3.1 Decay of the profile for any given time

In this section, we briefly prove the following lemma:

Lemma 3.1 *Let (u, v) be the solution of (1.1) with an initial datum (u_0, v_0) satisfying Assumption 1.1. Then $w(t, x) = \int_0^t u(s, x)ds$ satisfies, for any given time t and $\alpha > 0$:*

$$\limsup_{x\to+\infty} e^{\alpha x} \left(w(t, x) + |\partial_x w(t, x)|\right) = 0.$$

Proof. One can first check using the expression of the solution by convolution with the heat kernel together with Gronwall's lemma, that function $w(t,x)$ satisfies for each $t \geq 0$ and $x \in \mathbb{R}$:

$$w(t,x) \leq \int_0^t \frac{e^{\max_{x\in\mathbb{R}} \partial_w f(x,0)s}}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4s}} u_0(y)dyds.$$

Since u_0 is compactly supported, it immediately follows that for each $t > 0$ and $\alpha > 0$, one has

$$w(t,x) = O\left(e^{-\alpha x}\right) \text{ as } x \to \infty.$$

Using this first estimate, and the fact that $\partial_x w$ satisfies

$$\partial_t(\partial_x w(t,x)) = \partial_x^2(\partial_x w(t,x)) + \partial_x f(x, w(t,x)) + \partial_x w(t,x) \times \partial_u f(x, w(t,x)),$$

one can apply similar arguments to obtain the exponential decay of the $x-$derivative (recall that $f \in C^{1,\gamma}$ for some positive γ, hence $\partial_x f(x, w(t,x))$ decays faster than any exponential as $x \to +\infty$). ∎

3.2 Convergence to a pulsating wave in a locally perturbed medium

Consider w a solution of the equation

$$\partial_t w - \partial_x^2 w = u_0(x) + f(x,w), \tag{3.14}$$

for any $t > 0$ and $x \in \mathbb{R}$, where u_0 is a nonzero, nonnegative, continuous and compactly supported function while $f(x,w)$ is a periodic KPP nonlinearity (see (1.7)). Let us assume that $w(t=0,\cdot) \leq p_0(\cdot)$ is non-negative, non trivial and decays faster than any exponential to 0 as $x \to +\infty$.

According to Lemma 1.3, let us recall that the equation above admits a positive stationary solution p_0, while the same equation without the perturbation u_0 admits a unique positive and bounded stationary solution $p < p_0$, which is periodic, as well as a pulsating travelling wave W^* with minimal speed $c^* > 0$. We also remind the reader that $p(x) - p_0(x) \to 0$ as $x \to +\infty$.

Theorem 3.2 *Under the assumptions above, the function w satisfies, for any $\delta > 0$,*

$$w(t,x) - W^*(t - m(t), x) \to 0$$

as $t \to +\infty$ uniformly on $(\delta t, +\infty)$ for some function $m(t) = o(t)$.

Remark 3.3 *This is a result of independent interest. Convergence to pulsating travelling waves was previously known in the classical periodic KPP reaction-diffusion equation for fast decaying initial data [16, 17] (we will actually use such a result here), as well as for some perturbed reaction terms, which did not include a zero-th order term as here.*

Proof. Let any $\varepsilon > 0$, and $x_1 > 0$ large enough so that $(1+\frac{\varepsilon}{2})p(x) \geq p_0(x)$ for any $x \geq x_1$.We also assume, without loss of generality, that

$$support\ (u_0) \subset (-\infty, x_1).$$

From our spreading theorem (see Theorem 1.4) and the fact that $w \leq p_0$ and $p < p_0$, there exists some $t_1 > 0$ such that for any time $t \geq t_1$,

$$p(x_1) \leq w(t, x_1) \leq p_0(x_1) \leq \left(1 + \frac{\varepsilon}{2}\right) p(x_1).$$

Our goal is to look for a sub-solution and a super-solution on the right of x_1, which will both converge to a pulsating travelling wave. By $\tilde{w}$ we denote the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \tilde{w} = \partial_x^2 \tilde{w} + f\left(x, \tilde{w}\right), \\ \tilde{w}(0,x) = \min\{p(x)\ ,\ w(t_1,x)\}. \end{cases}$$

Then $\underline{w} := (1-\varepsilon)\tilde{w}$ satisfies

$$\begin{cases} \partial_t \underline{w} = \partial_x^2 \underline{w} + (1-\varepsilon) f\left(x, \dfrac{w}{1-\varepsilon}\right), \\ \underline{w}(0,x) = (1-\varepsilon)\min\{p(x)\ ,\ w(t_1,x)\}, \end{cases}$$

and $\overline{w} := (1+\varepsilon)\tilde{w}$ satisfies

$$\begin{cases} \partial_t \overline{w} = \partial_x^2 \overline{w} + (1+\varepsilon) f\left(x, \dfrac{\overline{w}}{1+\varepsilon}\right), \\ \overline{w}(0,x) = (1+\varepsilon)\min\{p(x)\ ,\ w(t_1,x)\}, \end{cases} \tag{3.15}$$

It follows from the KPP hypothesis (see (1.7)) and the fact that $support(u_0) \subset (-\infty, x_1)$ that these two functions are respectively sub- and super-solutions for equation (3.14) satisfied by w on the domain $\{(t,x) \mid x \geq\ x_1\}$.

Furthermore, it is clear that $w(t_1,x) \geq \underline{w}(0,x)$. One can also easily verify that $w(t_1,x) \leq \overline{w}(0,x)$ for any $x \geq x_1$ (since $w \leq p_0 \leq (1+\frac{\varepsilon}{2})p$ on this part of the domain). We now only need to check that $w(t+t_1,x_1)$ stays between the two in order to use a parabolic comparison principle.

We already know that $w(t+t_1,x_1)$ stays above $(1-\varepsilon)p(x_1)$ hence above $\underline{w}(t,x_1)$ for any $t>0$. On the other hand, we claim that

$$\overline{w}(t,x_1) \geq \left(1+\frac{\varepsilon}{2}\right) p(x_1) \ \text{ for any } t \geq 0. \tag{3.16}$$

Let us first note that $\overline{w}(0,x)$ lies above $p(x)\chi_{(0,x_1)}(x)$, where χ denotes the characteristic function. Since the solution of (3.15) associated with the initial datum $p(x)\chi_{(0,x_1)}(x)$ is known to converge locally uniformly to $(1+\varepsilon)p$, we get by comparison that there exists $T>0$ such that

$$\overline{w}(t,x_1) \geq \left(1+\frac{\varepsilon}{2}\right) p(x_1)$$

for any $t \geq T$. It remains to show that the same inequality also holds for any $0 \leq t \leq T$. Let

$$M = \sup_{x \in \mathbb{R}, w \in (0, 2\|p\|_\infty)} |\partial_w f(x,w)|,$$

and x_2 large enough so that

$$e^{-\sqrt{M}(x_2 - x_1 - 2\sqrt{M}T)} \leq \frac{\varepsilon}{2} \frac{p(x_1)}{1+\varepsilon}.$$

The following function

$$(t,x) \mapsto (1+\varepsilon)\left(p(x) - e^{\sqrt{M}(x - x_2 + 2\sqrt{M}t)} - e^{-\sqrt{M}(x + x_2 - 2\sqrt{M}t)}\right)$$

is a sub-solution for (3.15). Without loss of generality, one can assume up to increasing t_1 that $\overline{w}(0,x)$ lies above this sub-solution for any $|x| \leq x_2$ (since w converges to $p_0 > p$, one can choose t_1 such that $w(t_1, x) \geq p(x)$ for any $|x| \leq x_2$). From the parabolic maximum principle, it follows that for any $0 \leq t \leq T$:

$$\begin{aligned} \overline{w}(t, x_1) &\geq (1+\varepsilon)\left(p(x_1) - e^{\sqrt{M}(x_1 - x_2 + 2\sqrt{M}T)} - e^{-\sqrt{M}(x_1 + x_2 - 2\sqrt{M}T)}\right) \\ &\geq (1+\varepsilon)\left(p(x_1) - 2e^{\sqrt{M}(x_1 - x_2 + 2\sqrt{M}T)}\right) \\ &\geq \left(1 + \frac{\varepsilon}{2}\right) p(x_1), \end{aligned}$$

where the last inequality holds thanks to our choice of x_2. This completes the proof of (3.16). It immediately follows than $w(t+t_1, x_1) \leq \overline{w}(t, x_1)$ for any $t \geq 0$.

We can now apply the comparison principle to get that, for any $t \geq t_1$ and $x \geq x_1$,

$$(1-\varepsilon)\tilde{w}(t - t_1, x) = \underline{w}(t - t_1, x) \leq w(t,x) \leq \overline{w}(t - t_1, x) = (1+\varepsilon)\tilde{w}(t - t_1, x).$$

Besides, we know that $\tilde{w}(t,x)$ converges to W^*: indeed, we have assumed that $\tilde{w}(0,x)$ has a fast decay, so that we can apply results from [16] on the convergence to pulsating travelling waves. More precisely, there exists $\tilde{m}(t) = o(t)$ such that

$$|\tilde{w}(t - t_1, x) - W^*(t - t_1 + \tilde{m}(t - t_1), x)| \to 0,$$

where the convergence holds as $t \to +\infty$ uniformly with respect to $x \geq 0$.

Let now $m_1(t)$ such that $t + m_1(t)$ is the first time when w reaches the value $W^*(t, c^* t) < p(c^* t)$ at the point $c^* t$. One can check from our spreading theorem (which dealt with a particular case of w but can easily be extended to any w considered here with the same method) that $m_1(t) = o(t)$. Up to extraction of some subsequence $t_n \to +\infty$, the function $w(t + t_n + m_1(t_n), x + c^* t_n - y)$ converges to some entire solution $w_\infty(t,x)$ of (1.5) such that $w_\infty(0,y) = W^*(y/c^*, y)$ (where $y \in [0, L)$ is such that $c^* t_n \to y$ modulo the period L).

From all the above, it is clear that for any $\varepsilon > 0$, there exists $m(t;\varepsilon)$ such that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$:

$$(1-\varepsilon)W^*(t+m(t;\varepsilon),x) \leq w_\infty(t,x) \leq (1+\varepsilon)W^*(t+m(t;\varepsilon),x).$$

Note that $m(t;\varepsilon)$ depends only on t and ε, as x_1 and t_1 depended on ε only. Besides, $m(0;\varepsilon)$ stays bounded with respect to small ε, otherwise we would get a contradiction with the fact that $0 < w_\infty(0,y) < p(y)$. By passing to the limit as $\varepsilon \to 0$, we get that

$$w_\infty(0,x) \equiv W^*\left(\frac{y}{c^*},x\right),$$

Therefore, from the strong maximum principle, $w_\infty(t,x) \equiv W^*\left(t+\frac{y}{c^*},x\right)$. Besides, it follows from the definition of a pulsating travelling wave and the choice of y that

$$W^*\left(t+\frac{y}{c^*},x+y\right) = \lim_{n\to+\infty} W^*\left(t+t_n,x+c^*t_n\right).$$

Thus, as $\frac{y}{c^*} = O(1)$, one can get that

$$w(t+m(t),x+c^*t) - W^*\left(t,x+c^*t\right) \to 0$$

locally uniformly with respect to x as $t \to +\infty$, and with $m(t) = o(t)$ as $t \to +\infty$.

It now only remains to prove that the convergence is in fact uniform, for any $\delta > 0$, with respect to $x \geq \delta t$. We use again the fact that we have trapped w between two functions converging to pulsating travelling waves. Indeed, let again any $\varepsilon > 0$, and for t larger than some t_1 and x larger than some x_1:

$$(1-\varepsilon)\tilde{w}(t-t_1,x) \leq w(t,x) \leq (1+\varepsilon)\tilde{w}(t-t_1,x).$$

From the convergence of $\tilde{w}$ to W^*, we get for $t \geq t_1$ large enough and $x \geq \delta t - c^*t \geq x_1 - c^*t$:

$$(1-\varepsilon)W^*(t+\alpha(t),x+c^*t) \leq w(t+m(t),x+c^*t) \leq (1+\varepsilon)W^*(t+\alpha(t),x+c^*t).$$

Similarly as before, the shift $\alpha(t)$ stays bounded. Therefore, there exists some $D > 0$ such that for any t large enough and x, if $x \geq D$ and $x + c^*t \geq \delta t$, then

$$0 \leq w(t+m(t),x+c^*t) \leq 2\varepsilon,$$

while if $x \leq -D$ and $x + c^*t \geq \delta t$, then

$$(1-2\varepsilon)p(x+ct) \leq w(t+m(t),x+c^*t) \leq (1+2\varepsilon)p(x+ct),$$

Up to increasing D and from the asymptotics of W^*, the same inequalities hold with w replaced by $W^*(t,x+c^*t)$, so that the difference $|w - W^*|$ is less than $2\varepsilon + 4\varepsilon\|p\|_\infty$ on the same parts of the domain. Finally, using the locally uniform convergence proved above, one can easily conclude that

$$\|w(t+m(t),\cdot) - W^*(t,\cdot)\|_{L^\infty(\delta t,+\infty)} \to 0$$

as $t \to +\infty$. This concludes the proof of Theorem 3.2. ■

3.3 Profile of rabies propagation

From the two previous subsections, the proof of Theorem 1.7 is now very straightforward. Indeed, the fact that $w(t,x) = \int_0^s u(s,x)ds$ converges to some shift of the pulsating travelling wave W^* immediately follows from Lemma 3.1 and Theorem 3.2 applied to $w(t+1,x)$. One can easily deduce the convergence result for $v(t,x) = v_0(x)e^{-\sigma(x)w(t,x)}$, and for $u(t,x) = \partial_t w(t,x)$ using standard parabolic estimates.

4 Discussion

Our results accurately depict the long time behaviour of the solutions of the epidemic reaction-diffusion system with non-diffusive susceptible population (1.1). First, we exhibit some sharp threshold, in terms of the principal eigenvalue μ_0, between only two reasonable possible outcomes: extinction of the disease, or propagation of the outbreak with some constant speed in any direction. This parameter is of course related to the death and infection rates, but also depends upon the initial density of susceptible: the more there are susceptible individuals, the more likely is the disease to propagate and, even, the faster it will spread.

Indeed, we are also able to compute the spreading speeds $w^*(e)$ in any direction e, at least implicitly, through the family of elliptic eigenvalues $\{\mu_e(\lambda)\}_{\lambda\in\mathbb{R}}$. Unfortunately, unlike μ_0, those are in general the principal eigenvalues of non self-adjoint operators, so that the Rayleigh formula is not available. However, we refer the reader to the work of Nadin [26] who derived some similar formulae in this setting. These were already used to investigate the influence of some of the parameters, such as of the heterogeneity of the environment. In particular, in [26], the author proved that concentrated favourable areas led to faster propagation (see also Remark 1.6 for other discussions). As we have shown our system behaves like the single KPP equation, so that such observations immediately extend to our framework.

On the other hand, whether propagations occurs does not depend on the initial number of infected. It is also true for the spreading speed (and, in one dimension, for the asymptotic shape of the front), at least as long the initial datum is compactly supported. It could easily be proved, using once again results from the single KPP equation, that a slow decay of $u_0(x)$ as $|x| \to +\infty$ may lead to faster spreading speeds, hence different profiles (involving, of course, faster travelling waves). In other words, the evolution of the disease does not depend on the strength of the initial local outbreak. This is of course related to how u and v interact in our model.

Furthermore, our results highlight how travelling waves capture the large time behaviour of the solutions for large classes of initial data. This fact is made even clearer in the one dimensional case, where the convergence of the profile of the front to that of the travelling wave with minimal speed was shown. Even when the exact shape of such a wave cannot be computed, this gives stronger insight on the shape of the solution, where the front is located precisely, and how sharp it is. In the multidimensional setting, the spreading speed in some direction e may not be the same as the pulsating travelling wave with minimal speed in the same direction. This is because the asymptotic shape of solutions is no longer circular but elliptic. Thus, when looking in the moving frame in the direction e, we should expect to retrieve the profile of a travelling wave in the direction perpendicular to the level set, which is not e in general. Unfortunately, such results are not available yet, even in the single equation case, so that it is not in the reach of this paper. However, should such a result be proved in the future, an argument similar to ours in Section 3 could certainly be used to extend Theorem 1.7 to higher dimensions.

However, travelling waves fail to capture the fact that the limiting state for the susceptible population depend on the initial infected density, at least in the reference frame. As was noted before, this is the only part where u_0, or in other words, the strength of the initial outbreak of infection, plays a role in the asymptotic picture. This is related to our Assumption 1.1 and the subsequent remark. Within such considerations, it is implicitly assumed that the outbreak has occurred in a fast time scale, allowing u_0 to become non trivial while its effect on v is still negligible. For instance, this can hold if the initial infected population has been imported from some other location, which may be considered as independent of the domain of our model. Even if the outbreak occurs on a slower time scale, it should still be assumed that v_0 only locally differs from a periodic (equilibrium) function, so that similar spreading properties should still hold using similar arguments.

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