A multi-dimensional bistable nonlinear diffusion equation in a periodic medium

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Abstract

In this work we study the existence of wave solutions for a scalar reaction-diffusion equation of bistable type posed in a multi-dimensional periodic medium. Roughly speaking our result states that bistability ensures the existence of waves for both balanced and unbalanced reaction term. Here the term wave is used to describe either pulsating travelling wave or standing transition solution. As a special case we study a two-dimensional heterogeneous Allen–Cahn equation in both cases of slowly varying medium and rapidly oscillating medium. We prove that bistability occurs in these two situations and we conclude to the existence of waves connecting $u = 0$ and $u = 1$. Moreover in a rapidly oscillating medium we derive a sufficient condition that guarantees the existence of pulsating travelling waves with positive speed in each direction.

Key words: Bistable reaction-diffusion equation, periodic heterogeneities, pulsating travelling waves, standing transition.

1 Introduction

In this work we consider a nonlinear diffusion equation of the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(A(x)\nabla u\right) = F(x,u). \tag{1}$$

This equation is posed on the whole space $x \in \mathbb{R}^N$, where N is some given positive integer, and the spatial heterogeneities are assumed to be $\mathbb{Z}^N-$periodic. To be more precise we denote by $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ the $N-$dimensional torus. We assume that $A : \mathbb{T}^N \to \mathcal{S}_N$ is a symmetric matrix valued function of the class $C^{1+\gamma}$ for some exponent $\gamma \in (0,1)$, that is furthermore assumed to be uniformly elliptic in the sense that there exists some constant $\alpha > 0$ such that

$$\alpha\|\xi\|^2 \le \xi^T A(x)\xi \le \alpha^{-1}\|\xi\|^2,\ \ \forall (x,\xi) \in \mathbb{T}^N \times \mathbb{R}^N.$$

Moreover we assume that the function $F : \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}$ is continuous, C^γ in x uniformly with respect to $u \in \mathbb{R}$, of the class C^1 in u uniformly with respect to $x \in T^N$ while the partial derivative F_u is continuous on $\mathbb{T}^N \times \mathbb{R}$. In this work we shall focus on Problem (1) under the so-called bistable assumption. In order to state our main assumption, let us introduce the following initial data parabolic problem endowed together periodic boundary conditions, namely posed on the torus $\mathbb{T}^N$:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(A(x)\nabla u\right) = F(x,u),\ t>0,\ x \in \mathbb{T}^N, \\ u(0,.) = u_0 \in C\left(\mathbb{T}^N\right). \end{cases} \tag{2}$$

Then our main bistability assumption is stated below and it is related to the dynamical properties of Problem (2).

Assumption 1.1 (Bistable assumption) *We assume that:*

(i) System (2) *has two stable stationary states* $\psi^- < \psi^+$ *with* $\psi^\pm \in C^2\left(\mathbb{T}^N\right)$.

(ii) If $\mathcal{E}$ *denotes the set of stationary solutions of* (2) *in* $C^2\left(\mathbb{T}^N\right)$ *between* ψ^- *and* ψ^+, *then all points in* $\mathcal{E} \setminus \{\psi^\pm\}$ *are unstable with respect to* (2).

In such a case, we say (2) *is bistable between* ψ^- *and* ψ^+.

Let us comment the above assumptions (i), (ii). Here stable (resp. unstable) means linearly stable (resp. unstable). To formally expressed this assumption, let us denote for each function $q \equiv q(x) \in L^\infty\left(\mathbb{T}^N\right)$ the quantity $\Lambda(q) \in \mathbb{R}$ defined as the principle periodic eigenvalue associated to the elliptic operator $\operatorname{div}(A(x)\nabla\cdot) + q(x)\cdot$. Let us furthermore recall that due to Krein-Rutmann theorem, such a principle periodic eigenvalue is simple and associated to a positive eigenvector. Hence Assumption 1.1 re-writes as

$$\Lambda(q_\psi) < 0,\ \forall \psi \in \{\psi^\pm\} \text{ and } \Lambda(q_\psi) > 0,\ \forall \psi \in \mathcal{E} \setminus \{\psi^\pm\},$$

wherein we have set $q_\psi(x) := F_u\left(x, \psi(x)\right)$ for any $\psi \in \mathcal{E}$.

Before going further let us recall that for $q \in L^\infty(\mathbb{T}^N)$, the real number $\Lambda(q)$ can be expressed in term of Rayleigh quotient as follows:

$$\begin{aligned} -\Lambda(q) &= \inf_{\phi \in C^1(\mathbb{T}^N)\setminus\{0\}} \frac{\int_{\mathbb{T}^N} \left[\nabla\phi A(x)\nabla\phi - q(x)\phi^2\right] dx}{\int_{\mathbb{T}^N} \phi^2 dx} \\ &= \inf_{\phi \in C^1_c(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{T}^N} \left[\nabla\phi A(x)\nabla\phi - q(x)\phi^2\right] dx}{\int_{\mathbb{T}^N} \phi^2 dx}, \end{aligned}$$

wherein C^1_c denotes the set of C^1-functions with compact support. Note that this number can also be expressed in term of generalized eigenvalue λ_1 and we refer to [2, 8] and the references therein for more details.

The aim of the work is to study the existence of waves for Problem (1) connecting the two stable stationary states $u \equiv \psi^-$ and $u \equiv \psi^+$. To reach this goal let us first recall the definition of a pulsating travelling wave that generalises the usual notion of travelling wave for homogeneous reaction-diffusion equation. Following [29, 30, 31] we define pulsating travelling waves for Problem (1) as follows:

Definition 1.2 (Pulsating wave) *A pair* (U, c) *with* $U : \mathbb{R} \times \mathbb{T}^N \to \mathbb{R}$ *and* $c \in \mathbb{R}$ *is said to be a* **pulsating travelling wave** *of Problem* (1) *with effective speed* c *in the direction* $e \in \mathbb{S}^{N-1}$ *connecting* $\psi^\pm$ *if the two following conditions are satisfied:*

(i) The map $u(t, x) := U(x \cdot e - ct, x)$ *is an entire (classical) solution of the parabolic problem* (1).

(ii) The profile U *satisfies*

$$\lim_{s \to \pm\infty} U(s, x) = \psi^\pm(x) \text{ uniformly for } x \in \mathbb{T}^N.$$

In the sequel we shall say that a pulsating travelling wave (U, c) *of Problem* (1) *is:*

(a) **a standing pulsating wave** *if* $c = 0$*;*

(b) **a moving pulsating wave** *if* $c \neq 0$.

Let us first notice that if (U, c) is a pulsating travelling wave of (1) in the direction $e \in \mathbb{S}^{N-1}$ then it satisfies the limit condition (ii) in the above definition as well as the semi-linear degenerate elliptic equation

$$\mathcal{L}_0 U(s, x) + c\partial_s U + F(x, U) = 0, \;\; \forall (s, x) \in \mathbb{R} \times \mathbb{T}^N, \tag{3}$$

wherein $\mathcal{L}_0$ denotes the degenerate elliptic operator

$$\mathcal{L}_0 U = (e\partial_s + \nabla_x)^T \left[A(x) \left(e\partial_s + \nabla_x\right) U\right].$$

Note also that the above definition of moving pulsating wave is equivalent to the notion of pulsating waves introduced in [26] and further developed in [4] (see also the references therein). To precise this let us recall that according to [4, 26] an entire solution $u \equiv u(t, x)$ of Problem (1) is said be a pulsating travelling wave solution (of (1)) in the direction $e \in \mathbb{S}^{N-1}$ and effective speed $c \neq 0$ if the two following conditions are satisfied

(i) $u\left(t + \frac{k \cdot e}{c}, x\right) = u(t, x + k), \; \forall (t, x, k) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{Z}^N$.

(ii) If we denote by $e^\perp = \{y \in \mathbb{R}^N : \; y \cdot e = 0\}$, then

$$\lim_{r \to \pm\infty} |u(t, re + y) - \psi^\pm(re + y)| = 0,$$

wherein the above limits holds locally uniformly for $t \in \mathbb{R}$ and uniformly with respect to $y \in e^\perp$.

Now observe that if (U, c) is a moving pulsating wave (in the direction $e \in \mathbb{S}^{N-1}$) then $u(t, x) := U(x \cdot e - ct, x)$ becomes a pulsating wave with speed c in the sense of [4, 26]. Reciprocally if $u \equiv u(t, x)$ is a pulsating wave in the direction $e \in \mathbb{S}^{N-1}$ and effective speed $c \neq 0$ then $U(s, x) := u\left(\frac{x \cdot e - s}{c}, x\right)$ is a moving pulsating wave (with speed c in the direction e) in the sense of Definition 1.2.

As it will be noticed below (see Remark 1.7 below) Problem (1) may admit a stationary solution that is not a standing pulsating wave. For that reason we introduce a weaker notion of standing wave that we call **standing transition**. The precise definition of such a solution is detailed below.

Definition 1.3 (Standing transition) *A function $u \equiv u(x)$ is said to be a* **standing transition** *for Problem* (1) *in the direction $e \in \mathbb{S}^{N-1}$ if $u \in C^2(\mathbb{R}^N)$ is a stationary solution of* (1) *that satisfies the following asymptotic behaviour:*

$$\lim_{r \to \pm\infty} \sup_{y \in e^\perp} |u(re + y) - \psi^{\pm}(re + y)| = 0.$$

Remark 1.4 *Let us observe that if U is a standing pulsating wave profile – in the sense of Definition 1.2 – in the direction $e \in \mathbb{S}^{N-1}$ then the function $u(x) := U(x \cdot e, x)$ is a standing transition for Problem* (1)*.*

Note that when the medium is independent of $x \in \mathbb{T}^N$, namely $A(x) = I_N$ and $F(x, u) = f(u)$, then up to normalisation, bistable equation corresponds to a reaction term $f = f(u)$ such that $f(0) = f(\theta) = f(1) = 0$ for some value $\theta \in (0, 1)$ with $f < 0$ on $(0, \theta)$, $f > 0$ on $(\theta, 1)$ and $f'(0), f'(1) < 0$ while $f'(\theta) > 0$. In such a case existence and uniqueness of travelling wave solution is known since the works of Aronson and Weinberger [3] and Fife and McLeod [17]. We also refer to the monograph of Volpert et al [28] for more results about scalar equations but also monotone systems. A numerous number of results have also been obtained in higher dimensions. We refer for instance to Berestycki and Nirenberg [7] for travelling wave on cylinders, Hamel and Omrani [19] and Volpert and Volpert [27] for multistable nonlinearities on cylinders.

For explicit $x-$periodic dependence, only few results has been obtained in the bistable case. We may refer to the works of Xin [29, 30, 31] who used refined perturbation arguments to obtain the existence of waves when $F(x, u) = f(u)$ and the diffusivity matrix is close to identity. Recently Fang and Zhao [15] prove the existence of one-dimensional pulsating wave under a bistability assumption. Let us stress that this result is obtained as a special case of their Banach lattice valued results. Such a bistablity assumption has recently been investigated by Ding et al in [13]. The authors derived some conditions for bistability to hold true for a one-dimensional Allen–Cahn equation. Let us mention that existence of one-dimensional pulsating travelling waves with positive wave speed has been obtained by Ducrot et al [14] using intersection number arguments under a bistability assumption and a weak attractiveness property of one stationary state. Finally let us mention the work of Le Guilcher [11] who proves the existence of pulsating travelling waves for a class of one-dimensional reaction-diffusion equations using the construction of plane-like solutions and some intersection number arguments. As mentioned before, for the multi-dimensional problem, the works of Xin mentioned above seem to be the only works dealing with multi-dimensional meda. Here we would like to obtain a result in the spirit of Fang and Zhao's results [15] for one-dimensional reaction-diffusion equations in the multi-dimensional context. Roughly speaking we shall prove that bistability ensures the existence wave, in the sense that either moving pulsating wave or standing transition do exist. In order to state our main results we introduce the quantity $\mathcal{I}$ defined by

$$\mathcal{I} = \int_{\mathbb{T}^N} dx \int_{\psi^-(x)}^{\psi^+(x)} F(x,s)ds. \tag{4}$$

The reaction term is said to be balanced (resp. unbalanced) between ψ^- and ψ^+ when $\mathcal{I} = 0$ (resp. $\mathcal{I} \neq 0$).
Our first result is concerned with the balanced case and deals with the existence of standing transition. It reads as follows.

Theorem 1.5 (Case $\mathcal{I} = 0$) *Let Assumption 1.1 be satisfied. Assume that $\mathcal{I} = 0$. Then for each direction $e \in \mathbb{S}^{N-1}$ Problem (1) has a standing transition solution u_e in the direction e according to Definition 1.3. Moreover these profiles u_e satisfy the following almost monotonicity property: for each $x \in \mathbb{R}^N$ and each $k, k' \in \mathbb{Z}^N$:*

$$k \cdot e < k' \cdot e \;\Rightarrow\; u_e(x+k) < u_e(x+k'). \tag{5}$$

The almost monotonicity described above is also refereed as the so-called Birkhoff property. We refer to [12] and the references therein. Concerning this result, we would like to mention the existence of standing transition (also called plane-like solution) for periodic Allen–Cahn equations with double well potential. Such solutions have been obtained as energy functional minimisers in some suitable spaces. We refer to [12] and the references cited therein for more details. Note that these results are not based on a bistability hypothesis. Actually if we consider a $\mathbb{T}^N-$periodic potential $W = W(x,u)$ ($\mathbb{T}^N-$periodic in x) such that $W(x,\pm 1) = 0$ and $W(x,u) > 0$ for all $x \in \mathbb{T}^N$ and $u \in (-1,1)$, we do not know if Problem (2) with $F(x,u) = W'_u(x,u)$ satisfies the bistable assumption between -1 and 1. A typical example is given by the usual balanced Allen–Cahn potential $W(x,u) = Q(x)(1-u^2)^2$ with $Q : \mathbb{T}^N \to (0,\infty)$. As it will be seen in the applications below, for large diffusion coefficient this problem turns out to satisfy the bistable assumption at least in space dimension one and two.

Next our second result is concerned with the unbalanced case, namely $\mathcal{I} \neq 0$, and it reads as the following dichotomy result.

Theorem 1.6 (Case $\mathcal{I} \neq 0$) *Let Assumption 1.1 be satisfied. Assume that $\mathcal{I} \neq 0$. Then for each direction $e \in \mathbb{S}^{N-1}$ Problem (1) has either:*

1. *A moving pulsating travelling wave solution (U_e, c_e) in the direction e according to Definition 1.2. Moreover this travelling wave profile is non-decreasing with respect to its first variable and one has* $\operatorname{sign}(c_e) = -\operatorname{sign}(\mathcal{I})$*; Or*

2. *An almost monotonic (see (5)) standing transition solution $u_e \equiv u_e(x)$ in the direction e according to Definition 1.3.*

As it will be recalled in the next remark, the unbalanced condition is not sufficient to ensure that the speed is non-zero. However some results in this direction has been obtained in [13] in the one-dimensional framework. For higher dimensional problem such a question remains largely open. A very specific example is discussed in Remark 1.11 below. We also refer to Corollary 1.12 below.

Remark 1.7 *Xin in [32] (see Section 4 of that paper) described an example of one-dimensional unbalanced problem admitting a standing transition solution connecting* $u = 0$ *to* $u = 1$. *This example reads as the one-dimensional problem* (1) *with*

$$A(x) = 1 + \delta a(x) \text{ and } F(x,u) = \mu u(1-u)\left(u - \frac{1}{2} + \delta\right),$$

wherein $a(x)$ *is a given suitable periodic function while* $\mu > 0$ *and* $0 < \delta << 1$ *are suitable parameters.*
However using the profile equation as described in (3) *(see also Remark 2.2) it is easy to check that the existence of standing pulsating wave in the sense of Definition 1.2 implies that the nonlinearity is balanced. As a consequence the standing transition constructed by Xin is not a standing pulsating wave.*

Before going to the proof of these results, let us recall that if $(U \equiv U(s,x), c)$ is a pulsating travelling wave solution for (1) in the direction $e \in \mathbb{S}^{N-1}$ and speed $c \in \mathbb{R}$ then it satisfies the semi-linear degenerate elliptic equation (3) and this profile equation is supplemented together with the following limit behaviour at $s = \pm\infty$:

$$\lim_{s\to\pm\infty} U(s,x) = \psi^{\pm}(x) \text{ uniformly w.r.t. } x \in \mathbb{T}^N. \tag{6}$$

In order to deal with such an equation and overcome the lack of strict ellipticity of operator $\mathcal{L}_0$ we shall first study a regularized problem defined for each $\varepsilon > 0$ by

$$\mathcal{L}_\varepsilon U + c\partial_s U + F(x,U) = 0, \quad \forall (s,x) \in \mathbb{R} \times \mathbb{T}^N, \tag{7}$$

with $\mathcal{L}_\varepsilon := \varepsilon\partial_{ss} + \mathcal{L}_0$. The regularized profile equation is supplemented with the limit behaviour (6). Such a regularisation procedure has been widely used to study pulsating front and we refer for instance to [4]. Let us now observe that when (U,c) is a solution (7) then the function $v(t,z,x) := U(z-ct,x)$ becomes a usual travelling wave solution with speed c of the following uniformly parabolic problem on the cylinder $\mathbb{R} \times \mathbb{T}^N$:

$$\partial_t v = \left[\varepsilon\partial_{zz} + (e\partial_z + \nabla_x)^T \left[A(x)\left(e\partial_z + \nabla_x\right)\right]\right] v + F(x,v). \tag{8}$$

Hence our strategy to prove Theorems 1.5 and 1.6 is to prove first the existence of travelling wave solutions for the bistable parabolic equation (8) on the cylinder $\mathbb{R} \times \mathbb{T}^N$ and then to pass to the limit $\varepsilon \to 0$ to conclude that Theorems 1.5 and 1.6 hold true.
After proving the existence of travelling wave solution for (8) (see Proposition 2.1), a main step is to obtain a uniform bound independent of $\varepsilon \in (0,1]$ for

the family of corresponding wave speed c^ε. This point is reached in Lemma 2.4 in Section 2. From a technical point of view passing to the limit $\varepsilon \to 0$ will require some compactness estimate. This step is reached using bounded variation estimates as well as Helly type compactness theorem.

Remark 1.8 *Let us mention that our proofs can be extended to more general problems of the form*

$$\partial_t u - \operatorname{div}(A(x)\nabla u) + V(x) \cdot \nabla u = F(x, u),$$

where $V : \mathbb{T}^N \to \mathbb{R}^N$ is a given smooth advection term. In that case, under bistability assumption, the above equation has either a moving pulsating wave or a standing transition. However because of the advection term, balanced nonlinearity does not necessarily ensure the existence of a standing transition. The characterization of periodic heterogeneities leading to the existence of such stationary solution remains an open problem (See [13, 32] and Remark 1.7 above for examples without advection term).

As an application of the above results we consider the two-dimensional parabolic problem

$$\partial_t u = d^2 \Delta u + f(x, u) \text{ with } f(x, u) = r(x)u\,(u - a(x))\,(1 - u)\,, \qquad (9)$$

with $t > 0$ and $x \in \mathbb{R}^2$. Here $d^2 > 0$ denotes the diffusion coefficient while $a, r \in C^\gamma(\mathbb{T}^2)$ (for some $\gamma \in (0, 1)$) are given functions such that $0 < a(x) < 1$ and $r(x) > 0$. This problem corresponds to the so-called Allen–Cahn equation arising in mathematical physics and describing phase fields separation. This equation is also called Nagumo equation and we refer the reader to [24]. Such an equation also arises in Ecology and it is refereed in this context to the so-called strong Allee-effect.

For the above problem we shall provide some sufficient conditions for bistability to hold true between the two stable stationary states $u = 0$ and $u = 1$. This study will be performed using two types of asymptotic: small diffusion coefficient $d << 1$ and large diffusion coefficient $d >> 1$. The condition $d << 1$ corresponds to a slowly varying heterogeneity while $d >> 1$ corresponds to the homogenization limit. To see this if we set $v(t, x) = u(t, xd)$ then the function v satisfies

$$\frac{\partial v}{\partial t} = \Delta v + f(dx, v),$$

and the heterogeneity slowly varies when $d << 1$ and rapidly oscillates when $d >> 1$ according the rescaled variable dx.

In this context our first result reads as follows:

Theorem 1.9 (Slowly varying medium $d << 1$) *Under the above assumptions, let us furthermore assume that $0 < a(x) < \frac{1}{2}$ for all $x \in \mathbb{T}^2$, then there exists $d_0 > 0$ small enough such that for all $d \in (0, d_0)$ Problem (9) admits in each direction either a moving pulsating travelling wave or a standing transition connecting 0 and 1.*

Note that the condition $a(x) < \frac{1}{2}$ ensures that the nonlinearity $f(x,u)$ is never balanced, in the sense that one has

$$\min_{x \in \mathbb{T}^2} \int_0^1 f(x,u)du > 0.$$

For the one-dimensional problem, Ding et al [13] conclude from such a non-balanced assumption that the wave speed is non-zero. This is based on the complete classification for the entire solutions of the ODE problem $p''(x) + f(x_\infty, p(x)) = 0$ for some fixed heterogeneity $x_\infty \in \mathbb{R}$. In the two dimensional context that we consider here, we are not able to conclude that the wave speed is non-zero.
When function $a = a(x)$ crosses the valued $\frac{1}{2}$ then stable sharp layered stationary solutions may appear and in that case, we expect that Problem (9) exhibits a multistable dynamics. We refer to the work of Angenent et al [1] for a detailed description of the stable solutions for the one-dimensional problem equipped with Neumann boundary conditions.

Our next result is concerned with large diffusion asymptotic and reads as:

Theorem 1.10 (Homogenization limit $d >> 1$**)** *There exists* $d^* > 0$ *large enough such that for each* $d > d^*$ *Problem* (9) *admits in each direction either a moving pulsating travelling wave or a standing transition connecting* 0 *and* 1*.*

The proof of Theorem 1.9 and Theorem 1.10 will be given in Section 5. Here we would like to mention that our proof is not based on a perturbation argument of travelling wave for some "limit" problem but we will prove a persistence property for bistability. This property will not follow from the classification of the solutions of the elliptic problem $0 = \Delta p(x) + f(x_\infty, p(x))$ for some fixed heterogeneity but on the classification of a smaller class of solutions called weakly stable and provided by Dancer in [9].

Remark 1.11 *As a specific case of* (9) *one can look at row structure heterogeneities, namely* $r(x) = r(x_2)$ *and* $a(x) = a(x_2)$ *for all* $x = (x_1, x_2) \in \mathbb{T}^1 \times \mathbb{T}^1$*. In that case one can improve Theorem 1.9 in the sense that no standing transition between* 0 *and* 1 *exists in the direction of the rows, that is in the direction of the vector* $e_0 = (1,0) \in \mathbb{S}^1$*.*
Under such a row structure Theorem 1.10 can also be improved. Indeed if we furthermore assume that the homogeneized nonlinearity is unbalanced, that reads as

$$\int_{\mathbb{T}^1} r(x_2)a(x_2)dx_2 \neq \frac{1}{2}\int_{\mathbb{T}^1} r(x_2)dx_2,$$

then no standing transition between 0 *and* 1 *exists in the direction of the rows. In these specific situations Theorems 1.9 and 1.10 ensure the existence of moving pulsating wave solutions in the direction of the rows respectively for small and large diffusion asymptotic. That point will be discussed in Appendix B. Under this framework we suspect that for each direction the wave solutions provided by Theorems 1.9 and 1.10 are always moving pulsating wave solutions. This remains to be an open problem.*

The above remark provides simple examples of Problem (9) which admit a moving pulsating wave solution at least in some particular direction. Below we shall provide an other example that admits moving pulsating wave solutions in each direction. Our precise result reads as follows.

Corollary 1.12 *Let $a, r \in C^\gamma(\mathbb{T}^2)$ (for some $\gamma \in (0,1)$) be two given functions such that $0 < a(x) < 1$, $r(x) > 0$ and*

$$\overline{\theta} := \frac{\int_{\mathbb{T}^2} r(x)a(x)dx}{\int_{\mathbb{T}^2} r(x)dx} < \frac{1}{2}. \tag{10}$$

Then there exists $d_0 \geq 1$ large enough such that for each $d \geq d_0$, Problem (9) admits moving pulsating waves between $u = 0$ and $u = 1$ in any given direction $e \in \mathbb{S}^1$.

Remark 1.13 *In order to prove this corollary we shall prove that under the above set of assumptions Problem (9) does not admit any almost monotonic stationary transition between $u = 0$ and $u = 1$ in any given direction. Hence the dichotomy result stated in Theorem 1.10 leads to the existence of moving pulsating wave solutions in each direction.*

This work is organized as follows. In Section 2 we investigate the existence of travelling wave solution for the regularized parabolic problem (8) with periodic boundary conditions. In Section 3 we derive compactness estimates and we pass to the limit $\varepsilon \to 0$ to conclude to Theorem 1.5 and a part of Theorem 1.6 while the proof of Theorem 1.6 is completed in Section 4. Finally Section 5 is concerned with the proof of Theorem 1.9, Theorem 1.10 and its corollary.

2 Travelling wave solutions for (8)

As explained above we shall start by studying travelling wave solutions for the parabolic problem (8) posed on the cylinder $\mathbb{R} \times \mathbb{T}^N$ for some given and fixed $\varepsilon > 0$. Concerning this problem, our result reads as follows.

Proposition 2.1 *Let Assumption 1.1 be satisfied. Let $\varepsilon > 0$ be given. Then Problem (7) and (6) has a solution $(U^\varepsilon, c^\varepsilon) \in C^2(\mathbb{R} \times \mathbb{T}^N) \times \mathbb{R}$ such that*

$$\partial_s U^\varepsilon(s, x) > 0, \ \forall (s, x) \in \mathbb{R} \times \mathbb{T}^N.$$

Remark 2.2 (Sign of the wave speed) *Let us notice that multiplying (7) by $\partial_s U^\varepsilon$ and integrating over the cylinder $\mathbb{R} \times \mathbb{T}^N$ yields, recalling (4),*

$$c^\varepsilon \int_{\mathbb{R} \times \mathbb{T}^N} |\partial_s U^\varepsilon|^2 ds dx + \mathcal{I} = 0. \tag{11}$$

This means that two cases occur depending upon the sign of the quantity $\mathcal{I}$:

Case 1: if $\mathcal{I} \neq 0$ then $\operatorname{sign} c^\varepsilon = -\operatorname{sign}(\mathcal{I})$ for all $\varepsilon > 0$.

Case 2: If $\mathcal{I}=0$ then $c^\varepsilon=0$ for all $\varepsilon>0$.

In order to prove this proposition we shall make use of the deep results proved by Fang and Zhao in [15] about the existence of bistable travelling wave for monotone semiflows. We shall apply the aforementioned work to Problem (8). To that aim consider the Banach lattice $\mathcal{X}=C(\mathbb{T}^N)$ of all continuous functions on the torus $\mathbb{T}^N$ endowed with its usual order generated by its positive cone $\mathcal{X}^+$ consisting of all nonnegative continuous functions. Note that since $\mathbb{T}^N$ is compact then one can identify $C(\mathbb{R};C(\mathbb{T}^N))$ and $C(\mathbb{R}\times\mathbb{T}^N)$ when these two spaces are endowed with the open compact topology. Then we consider the convex sets

$$\begin{aligned}
&\mathcal{C}_\pm=\{\phi\in C(\mathbb{R};\mathcal{X}):\ \psi^-(\cdot)\leq\phi(z)(\cdot)\leq\psi^+(\cdot),\ \forall z\in\mathbb{R}\},\\
&\mathcal{X}_\pm=\{\psi\in\mathcal{X}:\ \psi^-(\cdot)\leq\psi(\cdot)\leq\psi^+(\cdot)\}.
\end{aligned}$$

Recalling that $\psi^\pm$ are both stationary solutions of (8), the strong parabolic comparison principle ensures that (8) generates a strongly monotone semiflow $\{T(t)\}_{t\geq 0}$ on $\mathcal{C}_\pm$ such that $T(t)\mathcal{X}_\pm\subset\mathcal{X}_\pm$ for all $t\geq 0$. In the sequel of this section we shall write

$$v(t,z,x;\phi)=\left[T(t)\phi\right](z,x),\ \forall(z,x)\in\mathbb{R}\times\mathbb{T}^N,\ \phi\in\mathcal{C}_\pm.$$

Next due to parabolic regularity, for each $t>0$, the nonlinear operator $T(t)$ is compact on $\mathcal{C}_\pm$ with respect to the open compact topology. Furthermore the map $(t,\psi)\mapsto T(t)\psi$ is continuous from $[0,\infty)\times\mathcal{C}_\pm$ to $\mathcal{C}_\pm$ (endowed together with the open compact topology). Finally let us notice that since all the coefficients in (8) are independent of z, the semiflow T is translation invariant with respect to all translation with respect to z. Now note that for each $\psi\in\mathcal{X}_\pm$, the map $t\to T(t)\psi\in\mathcal{X}_\pm$ corresponds to the solution of (2) with initial data ψ.

It remains to check the so-called counter propagation property . In order to recall this important property, let us fix $\psi\in\mathcal{E}\setminus\{\psi^\pm\}$ and consider the sets

$$\mathcal{C}^-(\psi)=\left\{\phi=\phi(z,x)\in\mathcal{C}_\pm:\ \begin{cases}\phi(z,.)=\psi\text{ for }z<-1\\ \limsup\limits_{z\to\infty,\ x\in\mathbb{T}^N}(\phi(z,x)-\psi(x))<0\end{cases}\right\},$$

$$\mathcal{C}^+(\psi)=\left\{\phi=\phi(z,x)\in\mathcal{C}_\pm:\ \begin{cases}\phi(z,.)=\psi\text{ for }z>1\\ \limsup\limits_{z\to-\infty,\ x\in\mathbb{T}^N}(\phi(z,x)-\psi(x))>0\end{cases}\right\}.$$

Next according to [20, 21] we define the so-called leftward and rightward spreading speed, respectively denoted by $c_*^-(\psi)$ and $c_*^+(\psi)$, by

$$c_*^-(\psi)=\sup\left\{c\in\mathbb{R}:\ \lim_{t\to\infty}\sup_{z\geq -ct,x\in\mathbb{T}^N}|v(t,z,x;\phi)-\psi^-(x)|=0,\ \forall\phi\in\mathcal{C}^-(\psi)\right\},$$

$$c_*^+(\psi)=\sup\left\{c\in\mathbb{R}:\ \lim_{t\to\infty}\sup_{z\leq ct,x\in\mathbb{T}^N}|v(t,z,x;\phi)-\psi^+(x)|=0,\ \forall\phi\in\mathcal{C}^+(\psi)\right\}.$$

Then the main property we will check to apply the result of Fang and Zhao [15] reads as follows:

Claim 2.3 *The following holds true:*

$$c_*^-(\psi) + c_*^+(\psi) > 0,\ \forall \psi \in \mathcal{E} \setminus \{\psi^\pm\}.$$

Note that this claim ends the proof of Proposition 2.1 using the results of Fang and Zhao [15].
The proof of Claim 2.3 is similar to the proof of Lemma 2.9 in [13]. The details of the proof are omitted. Let us mention that the key point argument is that the set $\mathcal{E} \setminus \{\psi^\pm\}$ is totally unordered because of Assumption 1.1 and the strong comparison principle.

We now derive a uniform estimate for the family of wave speeds with respect to ε small enough.

Lemma 2.4 (Uniform estimate for the wave speed) *The following estimate holds true: There exists some constant $K > 0$ such that for each $\varepsilon \in (0,1]$ one has $|c^\varepsilon| \leq K$.*

Proof. We shall only prove a lower bound for the wave speed. The upper bound follows from similar arguments.

Let $\varepsilon \in (0,1]$ be given. Let $\psi \in \mathcal{E} \setminus \{\psi^\pm\}$ be given. Consider a function $\phi_0 \in \mathcal{C}_\pm$ such that

$$\begin{aligned}&\psi \leq \phi_0 \leq \psi^+,\\ &\phi_0(z,\cdot) = \psi \text{ if } z \leq 0 \text{ and } \phi_0(z,\cdot) = \psi^+ \text{ if } z \geq 1.\end{aligned}$$

Hence up to shift (with respect to z) one may assume that

$$\phi_0(z,x) \geq U^\varepsilon(z,x),\ \ \forall (z,x) \in \mathbb{R} \times \mathbb{T}^N.$$

Hence the comparison principle applies and yields

$$v(t,z,x;\phi_0) \geq U^\varepsilon(z - c^\varepsilon t, x). \tag{12}$$

However the function v can be re-written as follows $v(t,z,x;\phi_0) = \psi(x) + w(t,z,x)$ where $w \geq 0$ is the solution of

$$\partial_t w = \left[\varepsilon \partial_{zz} + (e\partial_z + \nabla_x)^T A(x)\,(e\partial_z + \nabla_x)\right] w + G(x,w),$$

with $G(x,w) = F(x,\psi(x)+w) - \psi(x)$ and supplemented together with the initial data $w(0,z,x) = w_0(z,x) := \phi_0(z,x) - \psi(x) \geq 0$. Moreover there exists $M > 0$ large enough such that for all $0 \leq w \leq \|\psi^+ - \psi\|_\infty$:

$$G(x,w) \leq Mw,\ \forall x \in \mathbb{T}^N.$$

Therefore one obtains that $0 \leq w \leq \overline{w}$ where $\overline{w}$ denotes the solution of the linear parabolic equation:

$$\begin{aligned}&\partial_t \overline{w} = \left[\varepsilon \partial_{zz} + (e\partial_z + \nabla_x)^T A(x)\,(e\partial_z + \nabla_x)\right] \overline{w} + M\overline{w},\\ &\overline{w}(0,z,x) = w_0(z,x).\end{aligned} \tag{13}$$

To complete the proof of the lower bound, we claim that

Claim 2.5 *There exists $c > 0$ large enough and independent of $\varepsilon \in (0,1]$ such that*

$$\lim_{t\to\infty} \overline{w}(t,-ct,x) = 0 \text{ uniformly for } x \in \mathbb{T}^N.$$

Assuming for a while that this claim holds true, let us observe that (12) ensures that

$$\limsup_{t\to\infty} U^\varepsilon(-ct - c^\varepsilon t, x) \le \psi(x), \ \ \forall x \in \mathbb{T}^N.$$

The behaviour at $z = \infty$ of U^ε implies that

$$-c \le c^\varepsilon, \ \forall \varepsilon \in (0,1],$$

and the uniform lower bound follows.
Now it remains to prove Claim 2.5. To that aim we look for a super-solution of (13) of the form:

$$W(t,z,x) = Ke^{\mu(z+\kappa t)}\theta(x),$$

where $K > 0$, $\mu > 0$, $\kappa > 0$ and the function θ will be chosen latter on.

Consider for each $\lambda \in \mathbb{R}^N$ the elliptic operator $L_\lambda : C^2(\mathbb{T}^N) \to C(\mathbb{T}^N)$ defined by

$$L_\lambda \phi = \text{div}\,(A(x)\nabla\phi) + 2\lambda \cdot A(x)\nabla\phi + [\text{div}\,(A(x)\lambda) + \lambda \cdot A(x)\lambda]\,\phi. \tag{14}$$

Let us now consider for each $\mu > 0$ the principle eigenvalue problem:

$$\begin{cases} \Phi_\varepsilon(\mu)\theta_\mu = L_{\mu e}\theta_\mu + \left(M + \mu^2\varepsilon\right)\theta_\mu, \\ \theta_\mu \in C^2\left(\mathbb{T}^N\right),\ \theta_\mu > 0. \end{cases}$$

Then according to Nadin in [23] one obtains the following variational representation for this principle eigenvalue:

$$-\Phi_\varepsilon(\mu) = \min_{\alpha\in\mathcal{A}} \left[\int_{\mathbb{T}^N} \nabla\alpha A(x)\nabla\alpha - M - \mu^2\left(\varepsilon + D_e(\alpha)\right)\right], \tag{15}$$

with $\mathcal{A} = \{\alpha \in C^1(\mathbb{T}^N) : \int_{\mathbb{T}^N} \alpha^2 = 1\}$ and

$$D_e(\alpha) = \min_{\chi\in C^1(\mathbb{T}^N)} \int_{\mathbb{T}^N} \left(e + \nabla\chi(x)\right) A(x) \left(e + \nabla\chi(x)\right) dx.$$

From this representation formula one obtains that $\Phi_\varepsilon \le \Phi_1$ for all $\varepsilon \in (0,1]$ while

$$\lim_{\mu\to\infty} \frac{\Phi_1(\mu)}{\mu} = \infty.$$

Now let us fix $\mu > 0$ such that $\Phi_1(\mu) > 0$ and fix $\kappa > \frac{\Phi_1(\mu)}{\mu}$. Consider $\theta = \theta_{\mu,\varepsilon} > 0$ the eigenvector associated to $\Phi_\varepsilon(\mu)$ normalized such that $0 < \theta_{\mu,\varepsilon}(x) \le 1$ for all $x \in \mathbb{T}^N$. Consider now $K = K_\varepsilon > 0$ large enough such that

$$w_0(z,x) \le Ke^{\mu z}\theta_{\mu,\varepsilon}(x), \ \ \forall (z,x) \in \mathbb{R} \times \mathbb{T}^N.$$

Then note that one has

$$\begin{aligned}&\partial_t W - \left[\varepsilon\partial_{zz} + (e\partial_z + \nabla_x)A(x)(e\partial_e + \nabla_x)\right]W - MW\\ =&Ke^{\mu(z+\kappa t)}\theta_{\mu,\varepsilon}(x)\left[\mu\kappa - \Phi_\varepsilon(\mu)\right]\\ \geq&Ke^{\mu(z+\kappa t)}\theta_{\mu,\varepsilon}(x)\left[\mu\kappa - \Phi_1(\mu)\right] > 0.\end{aligned}$$

Thus the comparison principle implies that

$$\overline{w}(t,z,x) \leq Ke^{\mu(z+\kappa t)}\theta_{\mu,\varepsilon}(x),$$

and Claim 2.5 follows by choosing $c = -\kappa - 1$.

■

Because of this uniform bound we shall split the proof of Theorem 1.4 and 1.5 into two parts according to the following alternative: either

$$\lim_{\varepsilon\to 0} c^\varepsilon = 0, \tag{16}$$

or

$$\limsup_{\varepsilon\to 0} |c^\varepsilon| \in (0,\infty). \tag{17}$$

These two cases will lead to the following results that will be proved in the next sections.

Proposition 2.6 *Under Assumption 1.1, if* (16) *is satisfied then Problem* (1) *has an almost monotonic standing transition in the given direction $e \in \mathbb{S}^{N-1}$.*

Note that because of Remark 2.2, this proposition proves both Theorem 1.5 and the second part of Theorem 1.6.
On the other hand, in the case where (17) is satisfied, the following proposition holds true:

Proposition 2.7 *Under Assumption 1.1, if* (17) *is satisfied then Problem* (1) *has a non-decreasing - moving - pulsating wave solution (U_e, c_e) in the given direction $e \in \mathbb{S}^{N-1}$. Moreover one has* $\operatorname{sign}(c_e) = -\operatorname{sign}(\mathcal{I})$.

Note that this proposition proves the first part of Theorem 1.6.
Hence, to conclude the proof of both Theorem 1.5 and 1.6, it is sufficient to focus on the proof of the above two propositions. Proposition 2.6 will be proved in the next section (Section 3) while Proposition 2.7 will be proved in Section 4.

3 Proof of Proposition 2.6

This section is devoted to proof of Proposition 2.6. Recall that $e \in \mathbb{S}^{N-1}$ is given and fixed and we assume throughout this section that the family of travelling wave solutions $(U^\varepsilon, c^\varepsilon)$ provided by Proposition 2.1 satisfies (16). Here recall that the travelling wave profile $U^\varepsilon \equiv U^\varepsilon(s,x)$ satisfies the equation on $\mathbb{R}\times\mathbb{T}^N$

$$\begin{cases}\varepsilon\partial_{ss}U^\varepsilon + c^\varepsilon\partial_s U^\varepsilon + (e\partial_s + \nabla_x)^T A(x)\,(e\partial_s + \nabla_x) + F(x,U^\varepsilon) = 0,\\ \lim_{s\to\pm\infty} U^\varepsilon(s,x) = \psi^\pm(x) \text{ uniformly for } x\in\mathbb{T}^N.\end{cases}$$

Next let us introduce the function $u^\varepsilon \equiv u^\varepsilon(t,x)$ defined by

$$u^\varepsilon(t,x) := U^\varepsilon(x \cdot e + t, x), \tag{18}$$

and note that u^ε satisfies the problem:

$$\varepsilon \partial_{tt} u^\varepsilon + c^\varepsilon \partial_t u^\varepsilon + \operatorname{div}\left(A(x)\nabla u^\varepsilon\right) + F(x, u^\varepsilon) = 0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{19}$$

Now we aim at deriving compactness properties for the family of functions $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$. This is summarized in the following lemma.

Lemma 3.1 (Compactness) *Let $\{\varepsilon_n\}_{n\geq 0} \subset (0,\infty)$ be a given sequence tending to 0 as $n \to \infty$. Let $\{t_n\}_{n\geq 0} \subset \mathbb{R}$ be a given time sequence. Consider the sequence of function $u_n \equiv u_n(t,x)$ defined by $u_n(t,x) = u^{\varepsilon_n}(t+t_n, x)$. Then there exists a subsequence $\{n_k\}_{k\geq 0}$ and a function $u \equiv u(t,x) \in L^\infty(\mathbb{R} \times \mathbb{R}^N)$, increasing with respect to t, such that*

(i) the map $t \mapsto u(t,\cdot)$ is right continuous from $\mathbb{R}$ to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and has at most a countable set of discontinuities, denoted by D.

(ii) $\psi^-(\cdot) \leq u(t,\cdot) \leq \psi^+(\cdot)$ for all $t \in \mathbb{R}$ and for all $t \in \mathbb{R}$, $k \in \mathbb{Z}^N$:

$$u(t + k \cdot e, \cdot) = u(u, \cdot + k) \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{20}$$

(iii) The following convergences hold true as $k \to \infty$:

$$u_{n_k}(t,x) \to u(t,x) \text{ strongly in } L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^N),$$

and for each $t \in \mathbb{R} \setminus D$ one has:

$$u_{n_k}(t,\cdot) \rightharpoonup u(t,\cdot) \text{ weakly in } L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

(iv) The function u satisfies for all $t \in \mathbb{R}$, $u(t,\cdot) \in \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and for all $t \in \mathbb{R}$

$$\operatorname{div}\left(A(x)\nabla u(t,x)\right) + F(x, u(t,x)) = 0 \text{ a.e. } x \in \mathbb{R}^N. \tag{21}$$

There exists $\left(\psi, \tilde{\psi}\right) \in \mathcal{E}^2$ with $\psi \leq \tilde{\psi}$ such that

$$\lim_{t\to -\infty} u(t,\cdot) = \psi \text{ and } \lim_{t\to\infty} u(t,\cdot) = \tilde{\psi} \text{ locally uniformly in } \mathbb{R}^N.$$

(v) If we set $e^\perp = \{y \in \mathbb{R}^N :\ e \cdot y = 0\}$ then for each given $t \in \mathbb{R}$ one has

$$\lim_{r\to-\infty} \sup_{y \in e^\perp} |u(t, re+y) - \psi(re+y)| = \lim_{r\to\infty} \sup_{y\in e^\perp} |u(t, re+y) - \tilde{\psi}(re+y)| = 0.$$

The proof of the above compactness lemma is based on bounded variation estimates as well as Banach lattice valued Helly theorem.

Proof. In order to proceed to the proof of this lemma, let us first recall the definition of the bounded variation of a L^1−function. Let $\Omega \subset \mathbb{R}^p$ be a bounded open set and let $v \in L^1(\Omega)$ be a given function. The bounded variation (or total variation) of v in Ω is defined by

$$V(v;\Omega) := \sup\left\{\int_\Omega v \operatorname{div}\Phi : \ \Phi \in C_c^1(\Omega) \text{ and } \|\Phi\|_\infty \leq 1\right\}.$$

Using this definition we first claim that:

Claim 3.2 *For each bounded open set $\Omega \subset \mathbb{R}\times\mathbb{R}^N$, there exists some constant $M(\Omega) > 0$ such that one has*

$$V(u_n;\Omega) \leq M(\Omega),\ \forall n \geq 0.$$

In order to prove this claim, we shall make use of two key ingredients. First we shall obtain a uniform estimate of $\|\nabla_x u^\varepsilon\|_\infty$. Then we complete the derivation of such an estimate by using the monotonicity of u_n with respect to t. As mentioned above, let us first notice that Theorem 1.6 in [5] applies to (19) and this ensures that there exists some constant $K > 0$ such that

$$\|\nabla_x u_n\|_\infty \leq K,\ \forall n \geq 0.$$

Let $\Omega \subset \mathbb{R}\times\mathbb{R}^N$ be given. Now let $a < b$ and $r > 0$ be given such that $\Omega \subset (a,b)\times B(0,r)$ where $B(0,r)$ denotes the ball centred at the origin with radius r in $\mathbb{R}^N$. Let $\Phi = \left(\Phi_0, \tilde{\Phi}\right) \in C^1(\Omega)\times C_c^1(\Omega)^N$ be given. Then one has:

$$\begin{aligned}\left|\int_\Omega u_n(t,x)\operatorname{div}_{t,x}\Phi(t,x)dtdx\right| \leq &\left|\int_\Omega \partial_t u_n(t,x)\Phi_0(t,x)dtdx\right| \\ &+\left|\int_\Omega \nabla_x u_n(t,x)\cdot\tilde{\Phi}(t,x)dtdx\right|.\end{aligned}$$

Hence since $\partial_t u_n > 0$ and $\psi^- \leq u_n \leq \psi^+$ one obtains:

$$\begin{aligned}\left|\int_\Omega u_n(t,x)\operatorname{div}_{t,x}\Phi(t,x)dtdx\right| &\leq \|\Phi_0\|_\infty \int_{B(0,r)}\int_a^b \partial_t u_n dtdx + K\|\tilde{\Phi}\|_\infty|\Omega| \\ &\leq \|\Phi\|_\infty\left\{K|\Omega| + \int_{B(0,r)}[\psi^+ - \psi^-]dx\right\}.\end{aligned}$$

This completes the proof of Claim 3.2.

Now let us recall that for each open ball $B(0,\kappa) \subset R^N$ with radius $\kappa > 0$, the space $L^1(B(0,\kappa))$ is a Banach lattice with order continuous norm (see for instance [22, 25]). Now let us recall that for each $\kappa \in \mathbb{N}\setminus\{0\}$ and each $n \geq 0$ the function $u_n^\kappa : t \mapsto u_n(t,\cdot)|_{B(0,\kappa)}$ in increasing with value in $L^1(B(0;\kappa))$. Hence applying the characterisation of Banach lattice with order continuous norm derived in [10] (see Theorem 1), one obtains that for $\kappa = 1$ that there exists a subsequence $u^1_{\varphi_1(n)}$ and a right continuous function $u^1 : \mathbb{R} \to L^1(B(0,1))$ with at most a countable set of discontinuities D^1 such that

$$u^1_{\varphi_1(n)}(t,\cdot) \rightharpoonup u^1(t,\cdot) \text{ weakly in } L^1(B(0,1)),\ \forall t \in \mathbb{R}\setminus D^1.$$

Still applying this result with $\kappa = 2$, there exists a subsequence $u^2_{\varphi_1\circ\varphi_1(n)}$ and a right continuous function $u^2 : \mathbb{R} \to L^1(B(0,2))$ with at most a countable set of discontinuities D^2 such that

$$u^2_{\varphi_2\circ\varphi_1(n)}(t,\cdot) \rightharpoonup u^2(t,\cdot) \text{ weakly in } L^1(B(0,2)),\ \forall t \in \mathbb{R}\setminus D^2.$$

In addition one gets $u^2(t,\cdot)|_{B(0;1)} \equiv u^1(t,\cdot)$ and $D^1 \subset D^2$. Continuing such a procedure and using a diagonal extraction process, one obtains that there exists a subsequence $\{n_k\}_{k\geq 0}$ and a right continuous function $u : \mathbb{R} \to L^1_{\text{loc}}(\mathbb{R}^N)$ with a most a countable set of discontinuities D such that for each $\varrho > 0$

$$u^{n_k}(t,\cdot)|_{B(0,\varrho)} \rightharpoonup u(t,\cdot)|_{B(0,\varrho)} \text{ weakly in } L^1(B(0,\varrho)),\ \forall t \in \mathbb{R}\setminus D.$$

Now since $\psi^- \leq u^\varepsilon \leq \psi^+$ for all $\varepsilon > 0$, one gets

$$\psi^- \leq u(t,\cdot) \leq \psi^+,\ \forall t \in \mathbb{R}\setminus D,$$

and due to the right continuity, the above inequality holds for all $t \in \mathbb{R}$. The same argument applies to prove the equality (ii).

Next one can notice that due to Claim 3.2 and the usual Helly compactness theorem, the sequence $\{u_{n_k}\}_{k\geq 0}$ is relatively compact for the strong topology of $L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^N)$. This complete the proof of $(i)-(iii)$.

We now prove (iv). To that aim let $\varphi = \varphi(t) \in \mathcal{D}(\mathbb{R})$ and $\phi = \phi(x) \in \mathcal{D}(\mathbb{R}^N)$ be given. Then multiplying (19) with $\varepsilon = \varepsilon_{n_k}$ by $\varphi(t)\phi(x)$ and integrating over $\mathbb{R}\times\mathbb{R}^N$ yields for each $k \geq 0$

$$\varepsilon_{n_k}\int_{\mathbb{R}\times\mathbb{R}^N}\varphi''(t)\phi(x)u^{\varepsilon_{n_k}}(t,x)dtdx + c^{\varepsilon_{n_k}}\int_{\mathbb{R}\times\mathbb{R}^N}\varphi'(t)\phi(x)u^{\varepsilon_{n_k}}(t,x)dtdx =$$
$$\int_{\mathbb{R}\times\mathbb{R}^N}\varphi(t)\left[-\text{div}\left(A(x)\nabla\phi\right)u^{\varepsilon_{n_k}} + \phi(x)F\left(x,u^{\varepsilon_{n_k}}\right)\right]dtdx.$$

Passing to the limit $k \to \infty$ yields

$$\int_{\mathbb{R}\times\mathbb{R}^N}\varphi(t)\left[-\text{div}\left(A(x)\nabla\phi\right)u + \phi(x)F\left(x,u\right)\right]dtdx = 0,\ \forall(\varphi,\phi) \in \mathcal{D}(\mathbb{R})\times\mathcal{D}(\mathbb{R}^N).$$

Hence since $t \mapsto u(t,\cdot)$ is right continuous into $L^1_{\text{loc}}(\mathbb{R}^N)$, we obtain that

$$\int_{\mathbb{R}^N}\left[-\text{div}\left(A(x)\nabla\phi\right)u + \phi(x)F\left(x,u\right)\right]dx = 0,\ \forall\phi \in \mathcal{D}(\mathbb{R}^N),\ \forall t \in \mathbb{R}.$$

As a consequence of elliptic regularity (see for instance [18]) and since for each $t \in \mathbb{R}$, $F(\cdot,u(t,\cdot)) \in L^\infty(\mathbb{R}^N)$, one obtains that for each $t \in \mathbb{R}$, $u(t,\cdot) \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ for all $p > 1$ and it satisfies for all $t \in \mathbb{R}$:

$$\text{div}\left(A(x)\nabla u(t,x)\right) + F(x,u(t,x)) = 0 \ a.e.\ x \in \mathbb{R}^N.$$

Let us finally prove the asymptotic behaviour of u as $t \to \pm\infty$. Let us first observe that since the map $t \mapsto u(t,\cdot)$ is increasing, there exists two functions $\left(\psi, \tilde{\psi}\right) \in C^2(\mathbb{R}^N)^2$ solution of (21) with $\psi^- \leq \psi \leq \tilde{\psi} \leq \psi^+$ such that

$$\lim_{t \to -\infty} u(t,\cdot) = \psi \text{ and } \lim_{t \to \infty} u(t,\cdot) = \tilde{\psi} \text{ locally uniformly.}$$

To complete the proof of (iv), it remains to prove that these limit functions are $\mathbb{Z}^N-$periodic. However this property directly follows from (20). Indeed for instance for ψ, one has for each given $k \in \mathbb{Z}^N$ and $x \in \mathbb{R}^N$:

$$\lim_{t \to -\infty} u(t, x+k) = \lim_{t \to -\infty} u(t + k \cdot e, x) = \psi(x+k) = \psi(x).$$

The same holds true for $\tilde{\psi}$ and this completes the proof of (iv).
It remains to prove (v). This is also a direct consequence of (20). To see this we only consider the first case (namely $r \to -\infty$), the other case can be handled similarly. Let us argue by contradiction by assuming that

$$\limsup_{r \to -\infty} \sup_{y \in e^\perp} |u(t, re+y) - \psi(re+y)| > 0.$$

Then there exists a sequence $\{r_n\}_{n\geq 0} \subset (-\infty, 0)$ with $r_n \to -\infty$ and a sequence $\{y_n\}_{n\geq 0} \subset e^\perp$ such that

$$\limsup_{n \to \infty} |u(t, r_n e + y_n) - \psi(r_n e + y_n)| > 0.$$

We now write $r_n e = k_n + x_n$ and $y_n = k_n' + x_n'$ with $k_n, k_n' \in \mathbb{Z}^N$ and $x_n, x_n' \in [0,1]^N$. Since $r_n \to -\infty$ then $k_n \cdot e \to -\infty$ while $y_n \in e^\perp$ implies that

$$|k_n' \cdot e| = |x_n' \cdot e| \leq 1, \ \forall n \geq 0.$$

Now due to (20) and since ψ is $\mathbb{Z}^N-$periodic one obtains that for each $n \geq 0$:

$$|u(t, r_n e + y_n) - \psi(r_n e + y_n) = |u(t + k_n \cdot e + k_n' \cdot e, x_n + xn') - \psi(x_n + x_n')|.$$

However since $t + k_n \cdot e + k_n' \cdot e \to \infty$ as $n \to \infty$ and the sequence $\{x_n + x_n'\}_{n\geq 0}$ is bounded, this contradicts the local uniform convergence as stated in (iv). This completes the proof of (v). ∎

We are now able to complete the proof of Proposition 2.6.

Proof of Proposition 2.6. Let us first notice that due to Assumption 1.1 and the strong maximum principle for elliptic equation, the stationary states ψ^- and ψ^+ are isolated in $\mathcal{E}$. This means that there exist two constants $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} &\forall \psi \in \mathcal{E}, \ (\psi^- < \psi) \ \Rightarrow \ \psi > \delta_0 + \psi^-, \\ &\forall \psi \in \mathcal{E}, \ (\psi < \psi^+) \ \Rightarrow \ \psi + \delta_1 < \psi^+. \end{aligned} \tag{22}$$

Next let us recall that for each $\varepsilon > 0$, the function u^ε defined in (18) satisfies

$$\lim_{s \to \pm\infty} \int_{(0,1)\times\mathbb{T}^N} u^\varepsilon(t+s, x) dt dx = \int_{\mathbb{T}^N} \psi^\pm(x) dx.$$

Hence consider a family of normalisation time $\{t_\varepsilon\}_{\varepsilon\in(0,1]} \subset \mathbb{R}$ such that

$$\int_{(0,1)\times\mathbb{T}^N} u^\varepsilon(t+t_\varepsilon,x)dtdx = \int_{\mathbb{T}^N} \psi^+(x)dx - \delta_1,\ \forall\varepsilon\in(0,1]. \tag{23}$$

Consider now a sequence $\{\varepsilon_n\}_{n\geq 0} \subset (0,1]$ tending to 0 as $n\to\infty$ and let us set $u_n(t,x) := u^{\varepsilon_n}(t+t_{\varepsilon_n},x)$, so that (23) re-writes as

$$\int_{(0,1)\times\mathbb{T}^N} u_n(t,x)dtdx = \int_{\mathbb{T}^N} \psi^+(x)dx - \delta_1,\ \forall n\geq 0.$$

Then because of Lemma 3.1, up to a subsequence one may assume that

$$u_n(t,x) \to u(t,x) \text{ in } L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^N),$$

where the limit function $u \equiv u(t,x)$ satisfies all the properties described in Lemma 3.1 as well as the normalisation condition

$$\int_{(0,1)\times\mathbb{T}^N} u(t,x)dtdx = \int_{\mathbb{T}^N} \psi^+(x)dx - \delta_1. \tag{24}$$

In particular, let us observe that $U(x) := u(0,x) \not\equiv \psi^+(x)$. Indeed if $U(x) \equiv \psi^+(x)$ then because the $t-$increasing property and $u(t,\cdot)\leq\psi^+$, (24) would be violated. Hence $U(x) < \psi^+(x)$. Furthermore because the $t-$increasing property of function u and (24), the function $\tilde{\psi}(x) := \lim_{t\to\infty} u(t,x)$ satisfies

$$\tilde{\psi} \leq \psi^+ \text{ and } \int_{\mathbb{T}^N} \tilde{\psi}(x)dx \geq \int_{\mathbb{T}^N} \psi(x)dx - \delta_1.$$

Hence due to (22) we deduce that $\tilde{\psi}\equiv\psi^+$. As a consequence of Lemma 3.1 we obtain that the function U is a solution of (21) and satisfies

$$\begin{aligned}
&\psi^-(x) \leq U(x) \leq \psi^+(x),\ \forall x\in\mathbb{R}^N,\\
&U(x)\not\equiv\psi^+(x) \text{ and } \lim_{r\to\infty}\sup_{y\in e^\perp} |U(re+y)-\psi^+(re+y)| = 0.
\end{aligned}$$

To complete the proof of the existence of standing transition, we shall investigate the behaviour of function U at infinity. To do so let us consider the function $\psi(x) := \lim_{t\to-\infty} u(t,x) \leq U(x)$ and it is sufficient prove that $\psi\equiv\psi^-$. In order to prove such a property, let us argue by contradiction by assuming that $\psi^-(x) < \psi(x)$. Hence the function $V := U-\psi > 0$ becomes a stationary solution of the following problem

$$\partial_t V - \operatorname{div}(A(x)\nabla V) = G(x,V) \text{ with } G(x,V) = F(x,V+\psi) - F(x,\psi), \tag{25}$$

and satisfying

$$\lim_{r\to-\infty} V(re) = 0. \tag{26}$$

To reach a contradiction, we will show, using the asymptotic speed of spread property coupled together with the instability of ψ, that such a solution cannot

exist.
Let us consider a non-zero compactly supported initial data $V_0 \geq 0$ such that $V_0(x) \leq V(x)$. Consider $V \equiv V(t,x)$ the corresponding solution of (25) equipped together with the initial data $V(0,\cdot) = V_0$. Then the comparison principle implies that

$$0 \leq V(t,x) \leq V(x), \ \forall t \geq 0, \ x \in \mathbb{R}^N. \tag{27}$$

Let us first observe that due to Assumption 1.1 (iii), the stationary solution ψ is unstable with respect to (2). Following Theorem 1.13 in [6] (see also the references therein), let us consider for each $\lambda \in \mathbb{R}^N$, the real number $k_\lambda \in \mathbb{R}$ defined as the principle periodic eigenvalue of the following elliptic problem

$$(L_\lambda + q)\,\phi = -k_\lambda \phi \text{ with } \phi \in C^2(\mathbb{T}^N) \text{ and } \phi > 0,$$

wherein the operator L_λ is defined in (14) while $q \equiv q(x)$ is given by $q(x) = F_u\,(x, \psi(x))$. Now note that due to the representation formula derived in [23] one obtains that

$$k_\lambda \leq \Lambda(q) < 0, \ \forall \lambda \in \mathbb{R}^N.$$

As a consequence of Theorem 1.13 in [6], there exists $\omega^*(e) > 0$ such that for all $\omega \in [0, \omega^*(e))$

$$\liminf_{t\to\infty} V(t, -\omega t e) > 0.$$

However (27) implies that for each $0 \leq \omega < \omega^*(e)$

$$\liminf_{t\to\infty} V(-\omega t e) > 0,$$

that contradicts (26). As a consequence one obtains that $\psi \equiv \psi^+$ and this completes the proof of Proposition 2.6. ■

4 Proof of Proposition 2.7

The aim of this section is to prove Proposition 2.7. Recall that $e \in \mathbb{S}^{N-1}$ is a given and fixed direction while the family of travelling wave solutions $(U^\varepsilon, c^\varepsilon)$ satisfies (17).

Consider for each $\varepsilon \in (0,1]$ the function $u^\varepsilon : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$u^\varepsilon(t,x) = U^\varepsilon\left(x \cdot e + c^\varepsilon t, x\right), \tag{28}$$

wherein $(U^\varepsilon, c^\varepsilon)$ is the travalling profile provided by Proposition 2.1. Next note that due to Remark 2.2, since $\mathcal{I} \neq 0$, then $c^\varepsilon \neq 0$ so that function u^ε satisfies the equation on $\mathbb{R} \times \mathbb{R}^N$

$$\frac{\varepsilon}{\left(c^\varepsilon\right)^2} \partial_{tt} u^\varepsilon - \partial_t u^\varepsilon + \operatorname{div}\left(A \nabla u^\varepsilon\right) + F\left(x, u^\varepsilon\right) = 0, \tag{29}$$

as well as the relation

$$u^\varepsilon\left(t + \frac{k \cdot e}{c^\varepsilon}, x\right) = u^\varepsilon(t, x+k), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \ \forall k \in \mathbb{Z}^N. \tag{30}$$

In order to prove Proposition 2.7 consider a sequence $\{\varepsilon_n\}_{n\geq 0}$ tending to 0 and $c\neq 0$ with $\text{sign}\,(c)=-\text{sign}\,(\mathcal{I})$ such that

$$\lim_{n\to\infty} c^{\varepsilon_n}=c. \tag{31}$$

Before going to the proof of Proposition 2.7 and similarly to the above section, we first derive a compactness lemma. It reads as:

Lemma 4.1 *Let $\{t_n\}_{n\geq 0}\subset\mathbb{R}$ be a given time sequence. Consider the sequence of function $u_n\equiv u_n(t,x)$ defined by $u_n(t,x)=u^{\varepsilon_n}(t+t_n,x)$. Then there exists a subsequence $\{n_k\}_{k\geq 0}$ and a function $u\equiv u(t,x)\in C^{1,2}(\mathbb{R}\times\mathbb{R}^N)$, increasing (resp. decreasing) with respect to t when $c>0$ (resp. $c<0$), such that*

(i) $\psi^-(x)\leq u(t,x)\leq\psi^+(x)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and for all $k\in\mathbb{Z}^N$:

$$u\left(t+\frac{k\cdot e}{c},x\right)=u(t,x+k),\ \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N. \tag{32}$$

(ii) The following converge holds true as $k\to\infty$:

$$u_{n_k}(t,x)\to u(t,x) \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}\times\mathbb{R}^N).$$

(iii) The function u satisfies for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$

$$\partial_t u=\text{div}\,(A(x)\nabla u)+F(x,u). \tag{33}$$

There exists $\left(\psi,\tilde{\psi}\right)\in\mathcal{E}^2$ with $\psi\leq\tilde{\psi}$ if $c>0$ (and the reverse inequality if $c<0$) such that

$$\lim_{t\to-\infty} u(t,x)=\psi(x) \text{ and } \lim_{t\to\infty} u(t,x)=\tilde{\psi}(x) \text{ locally uniformly in } \mathbb{R}^N.$$

(iv) The following convergence properties hold true locally uniformly with respect to $t\in\mathbb{R}$

$$\lim_{r\to-\infty}\sup_{y\in e^\perp}|u(t,re+y)-\psi(re+y)|=\lim_{r\to\infty}\sup_{y\in e^\perp}|u(t,re+y)-\tilde{\psi}(re+y)|=0.$$

Proof. Let us first remark that (11) re-writes as for each $n\geq 0$:

$$\int_{\mathbb{R}\times\mathbb{T}^N}|\partial_t u_n|^2 dtdx\leq -c^{\varepsilon_n}\mathcal{I}\leq K|\mathcal{I}|, \tag{34}$$

wherein $K>0$ denotes the uniform estimate of the wave speed provided by Lemma 2.4. On the other hand, note that since $c\neq 0$ then $\frac{\varepsilon_n}{(c_n^\varepsilon)^2}\to 0$. Hence applying the Bernstein type estimates derived by Berestycki and Hamel in [5] to (29) one obtains that there exists some large enough constant $M>0$ such that:

$$\|\nabla_x u_n\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}\leq M,\ \forall n\geq 0. \tag{35}$$

Coupling these two bounds, namely (34) and (35), together with the uniform estimate $\psi^- \leq u_n \leq \psi^+$ ensure that the sequence $\{u_n\}_{n\geq 0}$ is bounded in $H^1_{\rm loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$.
Thus there exists a subsequence $\{n_k\}_{k\geq 0}$ such that

$$u_n(t,x)\to u(t,x),$$

for the strong topology of $L^2_{\rm loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$ and weakly in $H^1_{\rm loc}(\mathbb{R}\times\mathbb{R}^N)$. Moreover due to parabolic regularity, the limit function $u\equiv u(t,x)$ becomes a classical solution of the parabolic problem (we refer to the proof of Proposition 5.10 in [4])

$$\partial_t u = \operatorname{div}\left(A(x)\nabla u\right) + F(x,u),\ t\in\mathbb{R},\ x\in\mathbb{R}^N. \tag{36}$$

Since u^{ε_n} is non-decreasing with respect to t when $c^\varepsilon>0$ (and non-increasing when $c^\varepsilon<0$), the function u becomes increasing with respect to t when $c>0$ (and decreasing when $c<0$) and bounded between ψ^- and ψ^+.
Finally note that (32) directly follows from (30) using the above convergence. It remains to investigate the asymptotic behaviour of the limit function u. To investigate such properties we assume without loss of generality that $c>0$ so that u is time increasing. Hence there exist two stationary solutions $u^\pm$ of (33) such that

$$\psi^-\leq u^-\leq u^+\leq\psi^+ \text{ and } \lim_{t\to\pm\infty} u(t,x)=u^\pm(x) \text{ locally uniformly.}$$

Because of (32) one obtains that $u^\pm\in\mathcal{E}$ so that (iii) follows. Finally the proof of (iv) also follows from (32) using similar arguments as in the proof of Lemma 3.1 (v). This completes the proof of Lemma 4.1. ■

Equipped with this lemma, we are now able to complete the proof of Proposition 2.7.

Proof of Proposition 2.7. Here recall that the sequence $\{\varepsilon_n\}_{n\geq 0}$ is given while $c\neq 0$ is defined in (31). Without loss of generality we assume during this proof that $c>0$. The case $c<0$ is similar.

In that case, namely $c>0$, the map u^ε is non-decreasing in time. Let us consider a normalisation time sequence $\{t_n\}_{n\geq 0}$ such that

$$\int_{(0,1)\times\mathbb{T}^N} u^{\varepsilon_n}(t+t_n,x)dtdx = \int_{\mathbb{T}^N}\psi^-(x)dx+\delta_0,\ \forall n\geq 0. \tag{37}$$

Here $\delta_0>0$ is defined in (22). We set $u_n(t,x)=u^{\varepsilon_n}(t+t_n,x)$ and due to Lemma 4.1, possibly along a subsequence, one may assume that

$$u_n(t,x)\to u(t,x) \text{ strongly in } L^2_{\rm loc}(\mathbb{R}\times\mathbb{R}^N),$$

and where the limit function u is time increasing and satisfies all the properties described in Lemma 4.1. Moreover passing to the limit $n\to\infty$ into (37) yields

$$\int_{(0,1)\times\mathbb{T}^N} u(t,x)dtdx = \int_{\mathbb{T}^N}\psi^-(x)dx+\delta_0. \tag{38}$$

Next we consider $u^{\pm} \in \mathcal{E}$ with $u^- \leq u^+$ defined in Lemma 4.1 as

$$\lim_{t \to \pm\infty} u(t,x) = u^{\pm}(x) \text{ locally uniformly.}$$

Since u is time increasing (38) yields

$$u^- \in \mathcal{E} \text{ and } \int_{\mathbb{T}^N} u^-(x) dt dx \leq \int_{\mathbb{T}^N} \psi^-(x) dx + \delta_0.$$

Hence the definition of δ_0 in (22) ensures that $u^-(x) \equiv \psi^-(x)$. In addition note that (38) ensures that $u(t,x) \not\equiv \psi^-(x)$ and $\psi^-(x) < u^+(x)$. Therefore in order to complete the proof of Proposition 2.7 it is sufficient to prove that $u^+(x) \equiv \psi^+(x)$.

To perform such an analysis we argue by contradiction by assuming that $u^+(x) \not\equiv \psi^+(x)$. Note that this means that $u^+ \in \mathcal{E}$ satisfies $u^+ < \psi^+$. Next recalling the definition of δ_0 and δ_1 in (22), we obtain that $u^+ + \delta_1 \leq \psi^+$. Then we fix $\eta_1 > 0$ small enough such that

$$\delta_0 + \eta_1 < \int_{\mathbb{T}^N} \left(\psi^+ - \psi^-\right) dx \text{ and } \eta_1 < \delta_1,$$

and we consider an other time sequence normalisation $\{\tilde{t}_n\}_{n \geq 0}$ such that for each $n \geq 0$, $\tilde{t}_n \geq t_n$ and

$$\int_{(0,1)\times\mathbb{T}^N} u^{\varepsilon_n}\left(t + \tilde{t}_n, x\right) dt dx = \int_{\mathbb{T}^N} \psi^+(x) dx - \eta_1. \tag{39}$$

Now we set $v_n(t,x) := u^{\varepsilon_n}(t + \tilde{t}_n, x)$ and Lemma 4.1 applies and ensures that, possibly along a subsequence, one may assume that it converges in strongly $L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^N)$ to a time increasing function $v \equiv v(t,x)$, satisfying all the properties stated in Lemma 4.1 as well as

$$\int_{(0,1)\times\mathbb{T}^N} v\left(t,x\right) dt dx = \int_{\mathbb{T}^N} \psi^+(x) dx - \eta_1. \tag{40}$$

Similarly as above we consider $v^{\pm} \in \mathcal{E}$ defined as

$$v^{\pm}(x) := \lim_{t \to \pm\infty} v(t,x),$$

and (40) together with (22) ensures that $v^+(x) \equiv \psi^+(x)$. Note that (40) also implies that $v(t,x) \not\equiv \psi^+(x)$ so that $v^-(x) < \psi^+(x)$.

In order to reach a contradiction, we first claim that the following ordering property holds true:

Claim 4.2 *One has*

$$\psi^-(x) < u^+(x) \leq v^-(x) < \psi^+(x). \tag{41}$$

The proof of this claim is postponed and we first reach a contradiction to complete the proof of Proposition 2.7.

Since the set $\mathcal{E} \setminus \{\psi^{\pm}\}$ is totally unordered, one obtains that $u^+ = v^-$. This point is denoted by $\psi \in \mathcal{E} \setminus \{\psi^{\pm}\}$. Hence the pair (v, c) is a time increasing pulsating wave (in the sense of [4, 26]) of (1) connecting ψ and ψ^+ and such that

$$\psi(x) < v(t,x) < \psi^+(x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Therefore according to Lemma 6.5 in [4], there exists $\nu > 0$ such that

$$\Phi\left(\frac{\nu}{c}\right) = \nu > 0,$$

wherein we have set for each $\mu \in \mathbb{R}$, $\Phi(\mu) \in \mathbb{R}$ the principle periodic eigenvalue of the problem

$$\left(L_{\mu e} + q_\psi\right)\phi = \Phi(\mu)\phi,\ \phi \in C^2(\mathbb{T}^N) \text{ and } \phi > 0,$$

where operator L_λ is defined in (14) and $q_\psi(x) = F_u(x, \psi(x))$. However this inequality is impossible because for all $\mu \in \mathbb{R}$, one has

$$\Phi(\mu) \le \Phi(0) = -\Lambda(q_\psi) < 0.$$

As a consequence we have reached a contradiction and $u^+ = \psi^+$. This completes the proof of Proposition 2.7 up to the proof of Claim 4.2 that is detailed below. ∎

Proof of Claim 4.2. In order to prove Claim 4.2 we will first notice that the sequence $\{\tilde{t}_n - t_n\}_{n\ge 0} \subset [0,\infty)$ is unbounded. Indeed if such a sequence would be bounded, then one may assume that $\tilde{t}_n - t_n \to \tau_\infty \in [0,\infty)$. Then from (39) we get

$$\int_{(0,1)\times\mathbb{T}^N} v_n(t,x)dtdx = \int_{(\tilde{t}_n - t_n, \tilde{t}_n - t_n + 1)\times\mathbb{T}^N} u_n(t,x)dtdx = \int_{\mathbb{T}^N} \psi^+(x)dx - \eta_1.$$

Thus since u is time increasing this yields

$$\int_{\mathbb{T}^N} \psi^+(x)dx - \eta_1 \le \int_{\mathbb{T}^N} u^+(x)dx,$$

and (22) ensures that $u^+(x) \equiv \psi^+(x)$, a contradiction. As a first conclusion the sequence $\{\tilde{t}_n - t_n\}_{n\ge 0}$ is unbounded and, up to a subsequence one may assume that $\tilde{t}_n - t_n \to \infty$ as $n \to \infty$. Now since u^ε is time increasing one obtains that for each $s \ge 0$ there exists $n_s \ge 0$ large enough such that $2s \le \tilde{t}_n - t_n$ for all $n \ge n_s$ and this yields for all $n \ge n_s$ and all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$:

$$u_n(t+s,x) = u^{\varepsilon_n}(t+s+t_n,x) \le u^{\varepsilon_n}(t+s+\tilde{t}_n,x) = v_n(t-s,x).$$

Therefore for each continuous test function $\varphi \ge 0$ with compact support, one gets for all $s \ge 0$ and $n \ge n_s$:

$$\int_{\mathbb{R}\times\mathbb{T}^N} \varphi(t-s,x)u_n(t,x)dtdx \le \int_{\mathbb{R}\times\mathbb{T}^N} \varphi(t+s,x)v_n(t,x)dtdx,$$

so that for each test function $\varphi \geq 0$ with compact support

$$\int_{\mathbb{R}\times\mathbb{T}^N} \varphi(t,x)u(t+s,x)dtdx \leq \int_{\mathbb{R}\times\mathbb{T}^N} \varphi(t,x)v(t-s,x)dtdx,\ \forall s \geq 0.$$

Passing to the limit $s \to \infty$ implies that $u^+ \leq v^-$ and this completes the proof of (41). ■

5 Proof of Theorems 1.9 and 1.10

In this section we shall prove Theorem 1.9 and Theorem 1.10. To do so we shall apply Theorem 1.6 and Theorem 1.5 to Problem (9) by showing that under the hypothesis stated in Theorem 1.9 and Theorem 1.10, Assumption 1.1 is satisfied between the two stable stationary points $\psi^-(x) \equiv 0$ and $\psi^+(x) \equiv 1$.

To do so, let us consider the two-dimensional parabolic problem with periodic boundary conditions:

$$\partial_t u = d^2\Delta u + f(x,u),\ t > 0,\ x \in \mathbb{T}^2. \tag{42}$$

Then our first result read as

Lemma 5.1 *Under the assumptions of Theorem 1.9, there exists $d_0 > 0$ such that each $d \in (0, d_0)$, if $\overline{u}_d \in C^2(\mathbb{T}^2)$ is a stationary solution of (42) then one has*

$$0 < \overline{u}_d < 1 \ \Rightarrow\ \Lambda\left(d; q_{\overline{u}_d}\right) > 0.$$

Here we have set $q_u(x) = f_u\left(x, u(x)\right)$ and for each function $q \in L^\infty(\mathbb{T}^2)$ and each $d > 0$, $\Lambda(d;q)$ denotes the principle eigenvalue of the elliptic operator $d^2\Delta + q(x)$ on $\mathbb{T}^2$.

And, using the same notations as described above, our second result reads as

Lemma 5.2 *Under the assumptions of Theorem 1.10, there exists $d^* > 0$ such that each $d > d^*$, if $\overline{u}_d \in C^2(\mathbb{T}^2)$ is a stationary solution of (42) then one has*

$$0 < \overline{u}_d < 1 \ \Rightarrow\ \Lambda\left(d; q_{\overline{u}_d}\right) > 0.$$

Note that these two lemmas imply that Assumption 1.1 holds true between the two stable stationary states $\psi^- = 0$ and $\psi^+ = 1$ when $d > 0$ is small enough and the threshold $a = a(x)$ is uniformly far from $\frac{1}{2}$ or when $d > 0$ is large enough. Then Theorems 1.5 and 1.6 apply depending on the value $\mathcal{I} = \int_{\mathbb{T}^2}\int_0^1 f(x,u)dudx$, and this completes the proof of both Theorems 1.9 and 1.10. (Note that $\mathcal{I} > 0$ under the assumptions of Theorem 1.9 while no assumption on the sign of $\mathcal{I}$ is assumed in Theorem 1.10). Therefore it is sufficient to prove the two lemmas 5.1 and 5.2.

To that aim let us first recall, as in the introduction, the formulation for $\Lambda(d;q)$ in term of Rayleigh quotient: For each $d > 0$ and $q \in L^\infty(\mathbb{T}^2)$ one has

$$\Lambda(d;q) = -\inf_{\phi \in C_c^1(\mathbb{R}^2)\setminus\{0\}} \mathcal{R}(d,\phi),$$

wherein $C^1_c(\mathbb{R}^2)$ denotes the set of C^1 and compactly supported functions while $\mathcal{R}(d,\phi)$ is defined by

$$\mathcal{R}(d,\phi)=\frac{\int_{\mathbb{R}^2}\left[d^2\nabla\phi^2-q(x)\phi^2\right]dx}{\int_{\mathbb{R}^2}\phi^2dx},\ \ \forall\phi\in C^1_c(\mathbb{R}^2)\setminus\{0\}.$$

We can now proceed to the proof of Lemma 5.1.

Proof of Lemma 5.1. In order to prove our lemma, we argue by contradiction by assuming that there exists a sequence $\{d_n\}_{n\geq 0}\subset(0,\infty)$ tending to 0 as $n\to\infty$ and a sequence $0<\overline{u}_n<1$ of stationary solution of (42) with $d=d_n$ such that $\Lambda\left(d_n;q_{\overline{u}_n}\right)\leq 0$. Next we claim that:

Claim 5.3 *For all* $0<\delta<\frac{1}{2}$ *there exists* $N\geq 1$ *large enough such that for all* $n\geq N$*:*

$$\overline{u}_n(x)\in(0,\delta)\cup(1-\delta,1),\ \forall x\in\mathbb{T}^2.$$

We postpone the proof of this claim and we first complete the proof of Lemma 5.1. To do so, note that from the stationary relation, that reads as

$$d_n^2\Delta\overline{u}_n+f\left(x,\overline{u}_n\right)=0,$$

one obtains that

$$\int_{\mathbb{T}^2}f\left(x,\overline{u}_n(x)\right)dx=0,\ \forall n\geq 0.$$

Then recalling the form of the specific function $f\equiv f(x,u)$ in (9) and since $0<\overline{u}_n<1$, the above equality implies that for each $n\geq 0$ there exists $\left(x_n^1,x_n^2\right)\in\mathbb{T}^2\times\mathbb{T}^2$ such that for all $n\geq 0$:

$$\overline{u}_n\left(x_n^1\right)\geq a\left(x_n^1\right)\ \text{ and }\overline{u}_n\left(x_n^2\right)\leq a\left(x_n^2\right).$$

Note that since $0<a<1$ the above inequality contradicts Claim 5.3 since the former ensures that there exists a subsequence $\{n_k\}_{k\geq 0}$ such that u_{n_k} uniformly converges to 0 or 1. As a consequence, to complete the proof of Lemma 5.1 it is sufficient to complete the proof of Claim 5.3. ■

Proof of Claim 5.3. To prove this claim we shall argue by contradiction by assuming that there exist $\delta_0>0$ small enough and a sequence $\{x_n\}_{n\geq 0}\subset\mathbb{T}^2$ such that

$$\delta_0\leq\overline{u}_n\left(x_n\right)\leq 1-\delta_0,\ \forall n\geq 0.$$

Consider now the sequence of function $v_n(y):=\overline{u}_n\left(x_n+yd_n\right)$ so that v_n satisfies

$$\begin{cases}v_n(0)\in\left[\delta_0,1-\delta_0\right],\\ 0=\Delta v_n(y)+f\left(x_n+yd_n,v_n(y)\right),\ n\geq 0.\end{cases}$$

Because of elliptic regularity and the compactness of $\mathbb{T}^2$, one may assume, possibly along a subsequence, that

$$v_n(y)\to v(y)\text{ locally uniformly }y\in\mathbb{R}^2\text{ and }x_n\to x_\infty\text{ in }\mathbb{T}^2.$$

Hence function $0 < v < 1$ satisfies

$$0 = \Delta v(y) + f(x_\infty, v(y)),\ y \in \mathbb{R}^2 \text{ and } v(0) \in [\delta_0, 1-\delta_0].$$

To complete the proof of this claim, let us show that v is weakly stable, in the sense that

$$\int_{\mathbb{R}^2} \left[|\nabla\phi(y)|^2 - f_u\left(x_\infty, v(y)\right)\phi^2(y)\right] dy \geq 0,\ \forall \phi \in C_c^1(\mathbb{R}^2). \tag{43}$$

Indeed one has for each $\psi \in \mathcal{D}(\mathbb{R}^2)$ and each $n \geq 0$:

$$\begin{aligned}&\int_{\mathbb{R}^2} \left[|\nabla\psi(y)|^2 - f_u\left(x_n + d_n y, v_n(y)\right)\psi^2(y)\right] dy \\ =&d_n^2 \int_{\mathbb{R}^2} \left[d_n^2|\nabla\phi|^2(x) - f_u\left(x, \overline{u}_n(x)\right)\phi^2(x)\right] dx,\end{aligned}$$

with $\phi(x) = \psi\left(\frac{x-x_n}{d_n}\right) \in \mathcal{D}(\mathbb{R}^2)$. Hence for each $\psi \in \mathcal{D}(\mathbb{R}^2)$ and each $n \geq 0$ one has

$$\int_{\mathbb{R}^2} \left[|\nabla\psi(y)|^2 - f_u\left(x_n + d_n y, v_n(y)\right)\psi^2(y)\right] dy \geq -\Lambda\left(d_n; q_{\overline{u}_n}\right) \geq 0.$$

Thus passing to the limit $n \to \infty$ ensures that function v is weakly stable. Note that the above weak stability property allows us to make use of Theorem 1 in [9] to conclude that v is a constant function such that

$$f\left(x_\infty, v\right) = 0 \text{ and } f_u\left(x_\infty, v\right) \leq 0.$$

Indeed since $\int_0^1 f\left(x_\infty, s\right) ds \neq 0$, no spatially homogeneous heteroclinic connection do exist between 0 and 1. This means that $v = 0$ or $v = 1$ but these two cases contradict the normalisation condition $v(0) \in [\delta_0, 1-\delta_0]$. This completes the proof of Claim 5.3. ■

Let us now prove Lemma 5.2.

Proof of Lemma 5.2. Similarly as in the proof of the above lemma, we shall argue by contradiction by assuming that there exists a sequence $\{d_n\}_{n\geq 0} \subset (0, \infty)$ tending to ∞ as $n \to \infty$ and a sequence $0 < \overline{u}_n < 1$ of stationary solution of (42) with $d = d_n$ such that $\Lambda\left(d_n; q_{\overline{u}_n}\right) \leq 0$. Next similarly as above, in order to complete the proof of this lemma we claim that:

Claim 5.4 *For all $0 < \delta < \frac{1}{2}$ there exists $N \geq 1$ large enough such that for all $n \geq N$:*

$$\overline{u}_n(x) \in (0, \delta) \cup (1-\delta, 1),\ \forall x \in \mathbb{T}^2.$$

Equipped with this claim, the proof of Lemma 5.2 is similar to the one above. ■

It is therefore sufficient to prove Claim 5.4.

Proof of Claim 5.4. Let us argue by contradiction by assuming that there exist $\delta_0 > 0$ small enough and a sequence $\{x_n\}_{n\geq 0} \subset \mathbb{T}^2$ such that

$$\delta_0 \leq \overline{u}_n(x_n) \leq 1 - \delta_0,\ \forall n \geq 0.$$

Consider the sequence $v_n(x) := \overline{u}_n\left(x_n + d_n x\right)$ that satisfies the equation

$$\begin{cases} v_n(0) \in \left[\delta_0, 1-\delta_0\right], \\ \Delta v_n + f\left(x_n + d_n x, v_n\right) = 0, \ x \in \mathbb{R}^2. \end{cases}$$

Next because elliptic regularity, one may assume up to a subsequence that $v_n \to v$ locally uniformly for $x \in \mathbb{R}^2$. Moreover by setting $\mathcal{M}(g)$ for each function $g \in L^\infty(\mathbb{T}^2)$, the average of g defined by

$$\mathcal{M}(g) = \int_{\mathbb{T}^2} g(x)dx,$$

then due to the result in Appendix A one gets

$$r\left(x_n + d_n x\right) \to \mathcal{M}(r) \text{ weakly}-* \text{ in } L^\infty(\mathbb{R}^2),$$

while

$$r\left(x_n + d_n x\right) a\left(x_n + d_n x\right) \to \mathcal{M}(ra) \text{ weakly}-* \text{ in } L^\infty(\mathbb{R}^2).$$

Hence the function v satisfies:

$$\begin{cases} v(0) \in \left[\delta_0, 1-\delta_0\right], \\ \Delta v + \overline{f}\left(v\right) = 0, \ x \in \mathbb{R}^2, \end{cases} \tag{44}$$

wherein we have set

$$\overline{f}\left(v\right) = \mathcal{M}(r)v(1-v)\left(v - \overline{\theta}\right) \text{ and } \overline{\theta} = \frac{\mathcal{M}(ra)}{\mathcal{M}(r)}.$$

Before going further let us in addition show that the function v is $\mathbb{Z}^2-$periodic, that is

$$v(x+k) = v(x), \ \forall x \in \mathbb{R}^2, \ k \in \mathbb{Z}^2. \tag{45}$$

To see this note that since $\overline{u}_n$ is $\mathbb{Z}^2-$periodic one has for each $n \geq 0$ and $k \in \mathbb{Z}^2$:

$$v_n\left(x+k\right) = \overline{u}_n\left(x_n + d_n x + d_n k\right) = \overline{u}_n\left(x_n + d_n\left(x + \frac{y_n}{d_n}\right)\right) = v_n\left(x + \frac{y_n}{d_n}\right),$$

wherein $y_n \in [0,1]^2$ satisfies $y_n = d_n k \bmod\left(\mathbb{Z}^2\right)$. Hence since $v_n \to v$ locally uniformly and $y_n/d_n \to 0$ as $n \to \infty$, (45) follows.

To complete the proof of this claim, note that since $\Lambda\left(d_n; q_{\overline{u}_n}\right) \geq 0$ for all $n \geq 0$, then the function v is weakly stable as in the proof of Claim 5.3 above. Therefore due to Theorem 1 proved by Dancer in [9] one concludes that either $v(x) \equiv 0$ or 1 and when $\overline{\theta} = \frac{1}{2}$ (so that $\int_0^1 \overline{f}(s)ds = 0$) then either $v(x) \equiv 0$ or 1 or $v(x) = U(\nu \cdot x)$ where $\nu \in \mathbb{S}^2$ is a unit vector while U is a monotone one-dimensional solution of the stationary Allen–Cahn equation:

$$U'' + \overline{f}(U) = 0.$$

As a conclusion all these possible cases contradict either the normalisation condition in (44) or the $\mathbb{Z}^2-$periodicity in (45). This completes the proof of Claim 5.4. $\blacksquare$

To complete this section its remains to prove Corollary 1.12.

Proof of Corollary 1.12. As explained in Remark 1.13, in view of Theorem 1.10 it is sufficient to show that under these conditions, Problem (9) does not admit any almost monotonic stationary transition between $u = 0$ and $u = 1$ in each given direction.

To that aim we shall argue by contradiction by assuming that there exists a sequence $\{d_l\}_{l\geq 0}$ going to infinity and a sequence of direction $\{e_l = (\cos\omega_l, \sin\omega_l)\}_{l\geq 0} \subset \mathbb{S}^1$ for some sequence $\{\omega_l\}_{l\geq 0} \subset \mathbb{R}/2\pi\mathbb{Z}$ such that for each $l \geq 0$ there exists a function $u_l \equiv u_l(x, y)$ satisfying the following set of properties:

$$d_l^2 \Delta u_l(x, y) + f\left(R_{\omega_l}(x, y), u_l(x, y)\right) = 0,\ (x, y) \in \mathbb{R}^2, \tag{46}$$

for all $(k, k') \in \mathbb{Z} \times \mathbb{Z}$:

$$k < k' \ \Rightarrow\ u_l\left(x + k, y\right) < u_l\left(x + k', y\right),\ \forall (x, y) \in \mathbb{R}^2, \tag{47}$$

and the limit behaviour

$$\lim_{x\to-\infty} \sup_{y\in\mathbb{R}} u_l(x, y) = 0,\ \lim_{x\to\infty} \sup_{y\in\mathbb{R}} |1 - u_l(x, y)| = 0. \tag{48}$$

In (46) the operator R_{ω_l} corresponds to the rotation of axis with angle ω_l, that is

$$R_{\omega_l}(x, y) = \left(\cos\omega_l x - \sin\omega_l y, \sin\omega_l x + \cos\omega_l y\right).$$

Now we consider the function $\overline{f}(u) := \mathcal{M}(r)u\left(u - \overline{\theta}\right)(1 - u)$ and let us recall that since $\overline{\theta} \neq \frac{1}{2}$ all the solution $w \equiv w(y)$ of the problem

$$0 < w < 1 \text{ and } w''(y) + \overline{f}(w(y)) = 0,\ \forall y \in \mathbb{R}, \tag{49}$$

are well known based on phase plane analysis (see for instance [1, 13]). They only consist in homoclinic orbit to 0, periodic solution or the constant function $w \equiv \overline{\theta}$. We now fix a value $\delta \in (0, 1)$ sufficiently close to zero such that for each solution w of (49) one has

$$w(y) \leq 1 - 2\delta,\ \forall y \in \mathbb{R}.$$

Next for each $l \geq 0$ we fix a point $x_l \in \mathbb{R}$ such that

$$u_l(x_l, 0) = 1 - \delta,\ \forall l \geq 0.$$

Together with this normalisation point, we consider the sequence of function $v_l \equiv v_l(x, y)$ defined by

$$v_l(x, y) = u_l\left(x_l + d_l x, d_l y\right),\ \ l \geq 0,\ (x, y) \in \mathbb{R}^2.$$

Note that it satisfies the equation

$$\begin{aligned}&\Delta v_l(x,y) + f\left(X_l + d_l R_{\omega_l}(x,y), v_l(x,y)\right) = 0,\ (x,y) \in \mathbb{R}^2,\\&v_l(0,0) = 1-\delta,\end{aligned}$$

and wherein we have set $X_l = R_{\omega_l}(x_l, 0) \in \mathbb{R}^2$.

Now because of elliptic estimates, one assume possibly along a subsequence that $v_l \to v$ locally uniformly where the function v becomes a solution of the homogeneized problem

$$\begin{aligned}&\Delta v + \overline{f}\left(v\right) = 0,\ (x,y) \in \mathbb{R}^2,\\&v(0,0) = 1-\delta,\ \ 0 < v < 1,\end{aligned}$$

We claim that

Claim 5.5 *The function $v \equiv v(x,y)$ described above satisfies*

$$\partial_x v(x,y) \geq 0,\ \ \forall (x,y) \in \mathbb{R}^2.$$

Proof of Claim 5.5. The proof of this claim follows from the almost monotonicity property (47). Let $h > 0$ be given. Then for each $l \geq 0$ let us re-write

$$d_l h = p_l + r_l \text{ with } p_l \in \mathbb{N} \text{ and } r_l \in [0,1).$$

Then for each $l \geq 0$ one has for all $(x,y) \in \mathbb{R}^2$:

$$\begin{aligned}v_l\left(x+h,y\right) &= u_l\left(x_l + p_l + d_l\left(x + \frac{r_l}{d_l}\right), y\right)\\&\geq u_l\left(x_l + d_l\left(x + \frac{r_l}{d_l}, y\right)\right) = v_l\left(x + \frac{r_l}{d_l}, y\right).\end{aligned}$$

Hence passing to the limit $l \to \infty$ yields

$$v(x+h,y) \geq v(x,y),$$

and Claim 5.5 follows. ■

We now complete the proof of the corollary. For that purpose note that because of the normalisation condition $v(0,0) = 1-\delta$, the monotonicity with respect to x stated in Claim 5.5 and the definition of δ one obtains that the function v satisfies

$$\lim_{x \to \infty} v(x,y) = 1 \text{ locally uniformly for } y \in \mathbb{R}. \tag{50}$$

We finally reach a contradiction by considering a suitable compactly supported function v_0 as defined in Aronson and Weinberger in [3] such that

$$v_0(x,y) \leq v(x,y),\ \ \forall (x,y) \in \mathbb{R}^2.$$

Note that the existence of such a function is ensured by (50). Hence the parabolic comparison principle applies and yields

$$\tilde{v}(t,x,y) \leq v(x,y),\ \forall t \geq 0,\ (x,y) \in \mathbb{R}^2, \tag{51}$$

where $\tilde{v}$ is the solution of the equation

$$\begin{cases} \frac{\partial \tilde{v}}{\partial t} = \Delta \tilde{v} + \overline{f}\left(\tilde{v}\right),\ t > 0,\ (x,y) \in \mathbb{R}^2, \\ \tilde{v}(0,.,.) = v_0(.,.). \end{cases}$$

However, recalling that $\overline{\theta} < \frac{1}{2}$ so that $\int_0^1 \overline{f}(s)ds > 0$, Theorem 6.2 in [3] ensures that $\tilde{v}(t,x,y) \to 1$ as $t \to \infty$ (locally uniformly on $\mathbb{R}^2$) so that (51) leads us to $v(x,y) \equiv 1$. This contradicts the normalisation $v(0,0) < 1$ and completes the proof of Corollary 1.12. ■

Appendix A: Weak limit of rapidly oscillating functions

When $g \in L^\infty(\mathbb{T}^N)$ then setting $\mathcal{M}(g) = \int_{\mathbb{T}^N} g(x)dx$, it is well known that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} g\left(\frac{x}{\varepsilon}\right)\phi(x)dx = \mathcal{M}(g)\int_{\mathbb{R}^N} \phi(x)dx,\ \forall \phi \in L^1(\mathbb{R}^N). \tag{52}$$

In this appendix we aim to show that under some regularity conditions, the above convergence is in some sense uniform with respect to $\mathbb{T}^N$−translation. Our result reads as:

Lemma 5.6 *Let $g \in C(\mathbb{T}^N)$ be a given function. Then the following convergence holds true: For all $\phi \in L^1(\mathbb{R}^N)$, for all $\eta > 0$ there exists $\delta > 0$ such that*

$$\forall \varepsilon \in (0,1),\ \forall h \in \mathbb{T}^N\ \left|\int_{\mathbb{R}^N} g\left(h + \frac{x}{\varepsilon}\right)\phi(x)dx - \mathcal{M}(g)\int_{\mathbb{R}^N}\phi(x)dx\right| \leq \eta.$$

Proof. Let $g \in C\left(\mathbb{T}^N\right)$ be given. We denote for each $h \geq 0$ the quantity

$$\omega(h;g) = \sup_{(x,y)\in\mathbb{R}^N, \|x-y\|\leq h} |g(x) - g(y)|\,.$$

Since g is continuous and $\mathbb{Z}^N$−periodic, it is uniformly continuous so that

$$\lim_{h \to 0^+} \omega(g;h) = 0. \tag{53}$$

To proceed to the proof of the lemma, let us argue by contradiction by assuming that there exists $\phi_0 \in L^1(\mathbb{R}^N)$, $\eta_0 > 0$ and a sequence $\{h_n\}_{n\geq 0} \subset [0,1]^N$ and $\{\varepsilon_n\}_{n\geq 0} \subset (0,\infty)$ tending to 0 as $n \to \infty$ such that

$$\left|\int_{\mathbb{R}^N} g\left(h_n + \frac{x}{\varepsilon_n}\right)\phi_0(x)dx - \mathcal{M}(g)\int_{\mathbb{R}^N}\phi_0(x)dx\right| > \eta_0,\ \forall n \geq 0.$$

Since $[0,1]^N$ is compact one may assume that $h_n \to h_\infty \in [0,1]^N$. Now note that one has for each $n \geq 0$:

$$\begin{aligned}
\eta_0 &< \left| \int_{\mathbb{R}^N} g\left(h_n + \frac{x}{\varepsilon_n}\right) \phi_0(x)dx - \mathcal{M}(g) \int_{\mathbb{R}^N} \phi_0(x)dx \right| \\
&\leq \left| \int_{\mathbb{R}^N} \left[g\left(h_n + \frac{x}{\varepsilon_n}\right) - g\left(h_\infty + \frac{x}{\varepsilon_n}\right) \right] \phi_0(x)dx \right| \\
&+ \left| \int_{\mathbb{R}^N} g\left(h_\infty + \frac{x}{\varepsilon_n}\right) \phi_0(x)dx - \mathcal{M}(g) \int_{\mathbb{R}^N} \phi_0(x)dx \right|.
\end{aligned} \tag{54}$$

However on the one hand one has

$$\left| \int_{\mathbb{R}^N} \left[g\left(h_n + \frac{x}{\varepsilon_n}\right) - g\left(h_\infty + \frac{x}{\varepsilon_n}\right) \right] \phi_0(x)dx \right| \leq \omega\left(\|h_n - h_\infty\|; g\right) \|\phi_0\|_{L^1}.$$

Since $\|h_n - h_\infty\| \to 0$ as $n \to \infty$ and recalling (53), one obtains that

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \left[g\left(h_n + \frac{x}{\varepsilon_n}\right) - g\left(h_\infty + \frac{x}{\varepsilon_n}\right) \right] \phi_0(x)dx \right| = 0.$$

On the other hand, recalling that $\varepsilon_n \to 0$ as $n \to \infty$, one gets from (52) that

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^N} g\left(h_\infty + \frac{x}{\varepsilon_n}\right) \phi_0(x)dx &= \mathcal{M}\left(g\left(h_\infty + \cdot\right)\right) \int_{\mathbb{R}^N} \phi_0(x)dx \\
&= \mathcal{M}\left(g\right) \int_{\mathbb{R}^N} \phi_0(x)dx.
\end{aligned}$$

These two limits contradict (54) and this completes the proof of the lemma. ■

As a direct corollary one obtains the following result:

Corollary 5.7 *Let $g \in C\left(\mathbb{T}^N\right)$ be a given function. Let $\{h_n\}_{n\geq 0} \subset \mathbb{T}^N$ be a given sequence and $\{\varepsilon_n\}_{n\geq 0} \subset (0,\infty)$ be a sequence tending to 0 as $n \to \infty$ Then one has:*

$$g\left(h_n + \frac{x}{\varepsilon_n}\right) \rightharpoonup \mathcal{M}(g) \text{ weakly}-* \text{ in } L^\infty(\mathbb{R}^N).$$

Appendix B: A non-existence result of standing transition waves

In this section we will discuss the statement of Remark 1.11. To that aim we consider a rather specific nonlinear diffusion equation of the form

$$\frac{\partial u}{\partial t} = \Delta u + F\left(\tilde{x}, u\right), \tag{55}$$

posed for $x = (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and for some integer $N \geq 2$. In this appendix we derive a sufficient condition ensuring that (55) does not admit any standing transition in the direction $e_0 = (1, 0_{\mathbb{R}^{N-1}}) \in \mathbb{S}^{N-1}$, the direction orthogonal to the heterogeneity. Then this result will be applied to the case of Problem (9) with a periodic row structure to obtain the results stated in Remark 1.11. For that purpose we assume that:

Assumption 5.8 *The function $F \equiv F(\tilde{x}, u) : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}$ is continuous, C^γ (for some $\gamma \in (0,1)$) in x uniformly with respect to $u \in \mathbb{R}$, of the class C^1 in u uniformly with respect to $x \in \mathbb{T}^N$ and F_u is continuous on $\mathbb{T}^N \times \mathbb{R}$. It furthermore satisfies:*

(i) $F(\tilde{x}, 0) \equiv F(\tilde{x}, 1) \equiv 0$;

(ii) for any $u \in \{0, 1\}$, $\sup_{\tilde{x} \in \mathbb{R}^{N-1}} F_u(\tilde{x}, u) < 0$;

(iii) there exists a sequence $R_n \to \infty$ such that

$$\lim_{n \to \infty} \frac{1}{\left|B_{R_n}^{N-1}\right|} \int_{B_{R_n}^{N-1}} W(\tilde{x}, 1)\, d\tilde{x} \neq 0,$$

wherein we have set $W(\tilde{x}, u) = \int_0^u F(\tilde{x}, s)\, ds$. Moreover for each $R > 0$, $B_R^{N-1} \subset \mathbb{R}^{N-1}$ denotes the ball in $\mathbb{R}^{N-1}$ with the radius $R > 0$ and centred at the origin while $\left|B_R^{N-1}\right|$ denotes its measure in $\mathbb{R}^{N-1}$.

Under the above set of assumptions we will prove that the following proposition holds true.

Proposition 5.9 *Let Assumption 5.8 be satisfied. Then Problem* (55) *does not admit any standing transition between $u = 0$ and $u = 1$ in the direction e_0.*

Proof. In order to prove the above proposition, let us assume that there exists a standing transition $u : \mathbb{R}^N \to \mathbb{R}$ connecting $u = 0$ and $u = 1$ in the direction e_0, that is a function $u \equiv u(x_1, \tilde{x})$ that satisfies the equation

$$\Delta u + F(\tilde{x}, u) = 0, \ \ \forall x = (x_1, \tilde{x}) \in \mathbb{R}^N, \tag{56}$$

as well as the following behaviour when $x_1 \to \pm\infty$:

$$\lim_{x_1 \to -\infty} u(x_1, \tilde{x}) = 0, \ \ \lim_{x_1 \to \infty} u(x_1, \tilde{x}) = 1. \tag{57}$$

Here the above limits are uniform with respect to $\tilde{x} \in \mathbb{R}^{N-1}$.

Now because of Assumption 5.8 *(ii)* one obtains the following exponential decay with respect to x_1: there exist some constants $C > 0$ and $\eta > 0$ such that for all $x = (x_1, \tilde{x}) \in \mathbb{R}^N$ one has

$$\begin{aligned} &|u(x)| \leq Ce^{\eta x_1}, \ |1 - u(x)| \leq Ce^{-\eta x_1}, \\ &|\partial_{x_1} u(x)| \leq Ce^{-\eta |x_1|}, \ \ |\nabla_{\tilde{x}} u(x)| \leq C. \end{aligned} \tag{58}$$

We refer for instance to [7, 16] for the derivation of such an exponential decay. Let us also observe that due to elliptic estimates and the uniform limits in (57) one has:

$$\lim_{x_1\to\pm\infty} \nabla_{\tilde{x}} u(x) = 0 \text{ uniformly with respect to } \tilde{x} \in \mathbb{R}^{N-1}. \tag{59}$$

Let $M > 0$ and $R > 0$ be given and fixed. Multiplying (56) by $\partial_{x_1} u$ and integrating over the cylinder $(-M, M) \times B_R^{N-1}$ yields

$$\begin{aligned}
&\frac{1}{2}\int_{B_R^{N-1}} \left(\partial_{x_1} u(M,\tilde{x})\right)^2 d\tilde{x} - \frac{1}{2}\int_{B_R^{N-1}} \left(\partial_{x_1} u(-M,\tilde{x})\right)^2 d\tilde{x} \\
&+ \int_{-M}^{M}\int_{\partial B_R^{N-1}} \nabla_{\tilde{x}} u(x_1,\tilde{x})\cdot\tilde{\nu}(\tilde{x})\partial_{x_1} u(x_1,\tilde{x}) d\sigma\left(\tilde{x}\right) dx_1 \\
&- \frac{1}{2}\int_{B_R^{N-1}} \left[|\nabla_{\tilde{x}} u(M,\tilde{x})|^2 - |\nabla_{\tilde{x}} u(-M,\tilde{x})|^2\right] d\tilde{x} \\
&+ \int_{B_R^{N-1}} W\left(\tilde{x}, u(M,\tilde{x})\right) d\tilde{x} - \int_{B_R^{N-1}} W\left(\tilde{x}, u(-M,\tilde{x})\right) d\tilde{x} \\
&= 0
\end{aligned}$$

In the second line of the above computation, $\tilde{\nu}(\tilde{x}) \in \mathbb{R}^{N-1}$ denotes the outward unit vector to $\partial B_R^{N-1} \subset \mathbb{R}^{N-1}$ at $\tilde{x}$.

Now using the properties stated in (58) and (59) one can let $M \to \infty$ in the above formula to obtain that for each $R > 0$:

$$\int_{-\infty}^{\infty}\int_{\partial B_R^{N-1}} \nabla_{\tilde{x}} u(x_1,\tilde{x})\cdot\tilde{\nu}(\tilde{x})\partial_{x_1} u(x_1,\tilde{x}) d\sigma\left(\tilde{x}\right) dx_1 + \int_{B_R^{N-1}} W\left(\tilde{x}, 1\right) d\tilde{x} = 0.$$

Therefore there exists some constant $K > 0$ such that for all $R > 0$ one has

$$\left|\frac{1}{\left|B_R^{N-1}\right|}\int_{B_R^{N-1}} W\left(\tilde{x}, 1\right) d\tilde{x}\right| \leq K R^{-1}.$$

This former property contradicts Assumption 5.8 (iii) and this completes the proof of Proposition 5.9.

■

We now come back to Problem (9) with $r(x) = r(x_2)$ and $a(x) = a(x_2)$ for all $x = (x_1, x_2) \in \mathbb{T}^1 \times \mathbb{T}^1$. Then the function $f(x,u) = r(x_2) u \left(u - a(x_2)\right)(1-u)$ satisfies the conditions of Assumption 5.8 (i) and (ii) and one has

$$\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R}\int_0^1 f(x,u) du = \mathcal{M}(r)\int_0^1 u\left(u - \overline{\theta}\right)(1-u) du,$$

with $\mathcal{M}(r) = \int_{\mathbb{T}^1} r(x_2) dx_2$ and $\overline{\theta} = \frac{\int_{\mathbb{T}^1} r(x_2) a(x_2) dx_2}{\mathcal{M}(r)}$. Hence in that context Assumption 5.8 (iii) is equivalent to $\overline{\theta} \neq \frac{1}{2}$. This condition is satisfied under the assumptions of Theorem 1.9 while under the assumptions of Theorem 1.10

this condition has to be furthermore assumed. This completes the proof of the statements in Remark 1.11.

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