Spatial propagation for a two component reaction-diffusion system arising in population dynamics

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Abstract

In this work a two component epidemic reaction-diffusion system posed on the whole space $\mathbb{R}^N$ is considered. Uniform boundedness of the solutions is proved using suitable local L^p-estimates. The spatial invasion of a localized introduced amount of infective is studied yielding to the derivation of the asymptotic speed of spread for the infection. This part is achieved using uniform persistence ideas. The state of the population after the epidemic is further investigated using different Lyapunov like arguments. The solution is shown to converge the endemic equilibrium point behind the front in the equi-diffusional case. For general diffusion coefficient unique ergodicity of the tail of invasion is obtained by constructing a suitable sub-harmonic map.

1 Introduction

This work is concerned with the following spatially structured epidemic system of equations

$$\begin{cases} (\partial_t - d\Delta)\, S(t,x) = \Lambda - \gamma S(t,x) - \beta S(t,x) I(t,x), \\ (\partial_t - \Delta)\, I(t,x) = [\beta S(t,x) - (\gamma+\mu)]\, I(t,x), \end{cases} \tag{1.1}$$

posed for $t > 0$ and $x \in \mathbb{R}^N$ for some integer $N \geq 1$ and supplemented together with some initial data

$$S(0,x) = S_0(x), \quad I(0,x) = I_0(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Here both functions S_0 and I_0 are assumed to be bounded, nonnegative and uniformly continuous on $\mathbb{R}^N$.

The above system of equations describes the evolution of an infectious disease within a spatially distributed population of individuals. Here $S(t,x)$ (respectively $I(t,x)$) denotes the density of the susceptible (respectively infected) individuals at time $t \geq 0$ and located at the spatial position $x \in \mathbb{R}^N$. In the absence of the disease, that is when $I(t,x) \equiv 0$, the spatio-temporal evolution of the population satisfies a simple reaction-diffusion equation involving some constant external supply $\Lambda > 0$ and a natural death rate $\gamma > 0$, that reads as

$$(\partial_t - d\Delta) P(t,x) = \Lambda - \gamma P(t,x).$$

Here $d > 0$ describes the spatial mobility of individuals. Coming back to (1.1), the contamination process is assumed to follow the usual mass-action incidence with a contact rate $\beta > 0$. It is also assumed that the disease induces an additional mortality with a given rate $\mu > 0$. Using a rescaling argument we assume, without loss of generality, that the diffusion coefficient for infected equals one (possibly different from d if the disease affects the mobility of individuals). Note that when $\Lambda = \gamma = 0$ the above system reduces to the well-known diffusive Kermack and McKendrick model for which we refer the reader to the original articles of Kermack and McKendrick in [18, 19, 20].

In this work, we shall study some dynamical properties of (1.1)-(1.2). We first investigate the uniform boundedness of the solutions of (1.1)-(1.2). This step is achieved by revisiting the duality arguments developed by Hollis, Martin and Pierre in [15] and by Morgan in [23] to study the boundedness of the solutions of reaction-diffusion systems posed on some bounded domains. Estimates in uniform Lebesgue spaces are obtained to overcome the unboundedness of the whole space $\mathbb{R}^N$.

Then we shall focus upon studying the spatial spread of the infection. Under some survival assumptions, expressed using the so-called basic reproduction number R_0 (see definition (2.3) below) we shall show that the spreading speed of the epidemics is linearly determined. In other word, the infection spreads at the same speed as the one obtained from the linearized equation at the disease free equilibrium.

Finally we shall give some information about the state of the population after the epidemic. We shall more particularly prove that the tail of the propagating solution after the epidemic has a spatial and a temporal averaging property around the so-called endemic stationary state. More refined information is obtained in the equi-diffusional case $d = 1$.

Let us mention that a major difficulty encountered when studying (1.1)-(1.2) is the lack of comparison principle for the system under consideration. Despite the notion of asymptotic speed of spread (spreading speed for short) has been introduced by Aronson et al [1, 2] in the 70's for scalar reaction-diffusion equation, only few results about the asymptotic speed of spread for non-monotone problems have been obtained in the literature. However let us mention the works of Theime [27], Thieme and Zhao [29] and Fang and Zhao [13] (see also the references cited therein) for the study of the asymptotic speed of spread for integral equations, Fang et al [12] for delay lattice equations and Hsu and Zhao [17], Wang and Castillo-Chavez [32] for integrodifference equations and systems. One may also mention the recent work of Wang in [31] who studies a class of non-cooperative reaction-diffusion systems and the work of Ducrot [7] for a study of a specific class of predator-prey reaction-diffusion systems. We also refer to Ducrot et al in [8] for a study of a more general class of predator-prey systems.

This work is organized as follows: Section 2 is concerned with the statement of the main results of this work. Section 3 deals with preliminary duality estimates dedicated to the proof of Theorem 2.2 that is given in Section 4. Section 5 is concerned with the proof of Theorem 2.4 describing the spatial spread of the disease. Finally Section 6 will focus on the proof of Theorem 2.7 and that of ergodicity properties stated in Theorem 2.10.

2 Assumptions and main results

The aim of this section is to state the main results of this work. Coming back to (1.1) we will assume the following set of hypothesis:

Assumption 2.1 *We assume that* $\Lambda > 0$, $\gamma > 0$, $\mu > 0$ *while* $\beta > 0$ *and* $d > 0$.

Before stating our main results, let us introduce the Banach space

$$X = \mathrm{BUC}\left(\mathbb{R}^N, \mathbb{R}^2\right),$$

that denotes the space of bounded and uniformly continuous functions from $\mathbb{R}^N$ into $\mathbb{R}^2$. This space is endowed with the usual supremum norm. We also introduce its positive cone X_+ composed of all functions in X with both nonnegative components.

Our first main result deals with the boundedness of the solution of (1.1).

Theorem 2.2 (Uniform boundedness) *Let Assumption 2.1 be satisfied. System* (1.1)-(1.2) *generates a strongly continuous semiflow* $\{T(t)\}_{t\geq 0}$ *on* X_+. *Moreover, for each* $\kappa > 0$, *there exists* $\widehat{\kappa} > 0$ *such that*

$$\|T(t)U_0\|_X \leq \widehat{\kappa}, \ \ \forall t \geq 0, \ \forall U_0 \in B_X\left(0, \kappa\right) \cap X_+.$$

Here $B_X(0,\kappa)$ *denotes the ball in the Banach space* X *with center* 0 *and radius* κ.

Let us notice that the above result is weaker than the usual semiflow dissipativity. Throughout this work the above property will be referred either as uniform boundedness or weak dissipativity. This simply means that the nonlinear semiflow T maps bounded sets of X_+ into uniformly bounded - in time - sets of X_+.

In order to go further into the description of the dynamical behaviour of Problem (1.1)-(1.2), let us introduce the so-called basic reproduction number $R_0 > 0$ defined by

$$R_0 = \frac{\beta\Lambda}{\gamma(\gamma+\mu)}. \tag{2.3}$$

This parameter acts as a threshold for the dynamics of the spatially homogeneous solutions of System (1.1)-(1.2), namely for the solutions of the underlying ordinary differential equations (see Magal et al [21] and the references therein). Indeed when $R_0 \leq 1$, then the so-called disease free equilibrium $\left(S_F = \frac{\Lambda}{\gamma}, I_F = 0\right)$ is globally stable for the underlying ODE. When $R_0 > 1$ then System (1.1)-(1.2) has a unique positive spatially homogeneous steady state, the so-called endemic equilibrium point, defined by

$$S_E = \frac{S_F}{R_0}, \;\; I_E = \frac{\gamma}{\beta}\left(R_0 - 1\right). \tag{2.4}$$

This equilibrium point describes the asymptotic behaviour of the underlying ODE associated to System (1.1)-(1.2) in the case where $R_0 > 1$.

Our next result considers the dynamics of (1.1)-(1.2) when $R_0 \leq 1$. In such a situation, the epidemic uniformly dies out and our result reads as follows.

Theorem 2.3 (The case $R_0 \leq 1$) *Let Assumption 2.1 be satisfied. If $R_0 \leq 1$ then for each initial data $(S_0, I_0) \in X_+$, the corresponding solution satisfies:*

$$\lim_{t\to\infty} (S, I)(t, x) = \left(\frac{\Lambda}{\gamma}, 0\right),$$

uniformly with respect to $x \in \mathbb{R}^N$.

The situation when $R_0 > 1$ is much more delicate. As mentioned above, in that case the epidemic is sustained and persistent under the semiflow associated to the underlying system of ODE. In the spatially structured situation, we aim at describing the spatial spread of the epidemic. To do so, let us mention that Ducrot and Magal in [10] (see also the references cited therein, [9, 11] for discussions on the case $\Lambda = \mu = 0$ with age structure and [16] without age structure) proved the existence of one-dimensional travelling wave solutions for System (1.1) (including age since infection). These wave solutions connect the disease free equilibrium and the endemic one. Furthermore these special solutions do exist for any wave speed $c > c^*$ where the minimal wave speed $c^* > 0$ is defined by

$$c^* = 2\sqrt{(\gamma+\mu)(R_0 - 1)}. \tag{2.5}$$

We shall now discuss the spatial spread of the epidemic described by System (1.1)-(1.2) by developing a spreading speed approach for this reaction-diffusion problem. Note that System (1.1) does not satisfy the parabolic comparison principle, that turns out to be one of the major difficulty to overcome. The result we shall obtain reads as follows.

Theorem 2.4 (Spreading property) *Let Assumption 2.1 be satisfied and assume furthermore that* $R_0 > 1$. *Then for each* $(S_0, I_0) \in X_+$ *with* $I_0 \not\equiv 0$ *the following spreading properties hold true:*

(i) there exists $\varepsilon > 0$ *such that for each* $c \in (-c^*, c^*)$, *each direction* $e \in \mathbb{S}^{N-1}$ *and any* $x \in \mathbb{R}^N$, *it holds that*

$$\limsup_{t\to\infty} S(t, x + cte) \le \frac{\Lambda}{\gamma} - \varepsilon, \text{ and } \liminf_{t\to\infty} I(t, x + cte) \ge \varepsilon;$$

(ii) if we furthermore assume that I_0 *is compactly supported then the following outer spreading property holds true:*

$$\lim_{t\to\infty} \sup_{|x|\ge c^* t} \left[I(t, x) + \left| S(t, x) - \frac{\Lambda}{\gamma} \right| \right] = 0.$$

The above result ensures the persistence of the disease behind the front, namely on any expanding spheres $\|x\| = ct$ with $c \in [0, c^*)$. In the case where I_0 is compactly supported then the critical propagation speed c^* becomes sharp in the sense that ahead the front, namely for any expanding sphere $\|x\| = ct$ for $c \ge c^*$, the infection dies out.

Remark 2.5 *Here we would like to mention that the methodology we will develop in this work for System* (1.1)-(1.2) *can be extended to the case of epidemic system of the form* (1.1) *with standard (proportionate mixing) incidence. To be more specific, Theorem 2.2, 2.3 and 2.4 also hold true for the following reaction-diffusion system*

$$\begin{cases} (\partial_t - d\Delta) S(t, x) = \Lambda - \gamma S(t, x) - \beta \frac{S(t,x)I(t,x)}{S(t,x)+I(t,x)}, \\ (\partial_t - \Delta) I(t, x) = \left[\beta \frac{S(t,x)}{S(t,x)+I(t,x)} - (\gamma + \mu)\right] I(t, x), \end{cases} \tag{2.6}$$

by replacing R_0 *in* (2.3) *by* $R_0 = \frac{\beta}{\gamma+\mu}$.

In order to obtain a rather complete picture of the solution, we shall now discuss the behaviour of the solution behind the propagating front. To that aim we fix an initial data (S_0, I_0) as in the previous theorem. Consider a time sequence $\{t_n\}_{n\ge0}$ tending to infinity, a given value $c \in [0, c^*)$ and a direction $e \in \mathbb{S}^{N-1}$. Due to parabolic estimates and the uniform boundedness stated in Theorem 2.2, the sequence of maps $(S_n, I_n)(t, x)$ defined by

$$(S_n, I_n)(t, x) = (S, I)(t + t_n, x + c(t + t_n)e),$$

convergences (up to a subsequence) locally uniformly for $(t, x) \in \mathbb{R}\times\mathbb{R}^N$ towards some limit function $\left(\widehat{S}, \widehat{I}\right)(t, x)$. Then the map $(S_\infty, I_\infty)(t, x) := \left(\widehat{S}, \widehat{I}\right)(t, x - cte)$ becomes an entire solution of (1.1) while Theorem 2.4 (i) above ensures that

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I_\infty(t, x) > 0.$$

We refer to Lemma 5.5 for more details on the above construction. Such a class of entire solution will be referred in the sequel as a uniformly persistent entire solution (see Definition 2.6 below). The classification of such solutions will provide information about the solution of the Cauchy problem (1.1)-(1.2) behind the propagating front.

The precise definition of such solutions reads as follows.

Definition 2.6 *A bounded entire solution* (S_∞, I_∞) *of System* (1.1)-(1.2) *is said to be* **uniformly persistent** *if*

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I_\infty(t,x) > 0.$$

As discussed above the properties of uniformly persistent entire solutions provide information on the long term asymptotic of the solution (S, I) of the Cauchy problem (1.1)-(1.2) and more precisely on the quantity $(S, I)(t, cet)$ for $|c| < c^*$, $e \in \mathbb{S}^{N-1}$ and large time.

We conjecture that when $R_0 > 1$ then the set of uniformly persistent entire solutions only consists in the unique endemic equilibrium point (S_E, I_E) (see (2.4) above). However we are not able to prove such a result. We shall focus on giving information about the relationship between uniformly persistent entire solutions and the endemic equilibrium.

Our first result solves the above conjecture in the equi-diffusional case, namely for $d = 1$.

Theorem 2.7 (Equi-diffusional case) *Let Assumption 2.1 be satisfied. Assume furthermore that* $R_0 > 1$ *and* $d = 1$*. Let* (S_∞, I_∞) *be a uniformly persistent entire solution of System* (1.1)-(1.2) *according to Definition 2.6. Then one has*

$$S_\infty(t,x) \equiv S_E, \;\; I_\infty(t,x) \equiv I_E.$$

As a corollary one obtains the following picture for the asymptotic dynamics of (1.1)-(1.2) when the initial amount of infected individual is compactly supported.

Corollary 2.8 (Spreading speed) *Let Assumption 2.1 be satisfied. Assume furthermore that* $R_0 > 1$ *and* $d = 1$*. Let* $(S_0, I_0) \in X_+$ *be given such that* $I_0 \not\equiv 0$ *and* I_0 *compactly supported. Denote by* $(S, I) \equiv (S, I)(t, x)$ *the corresponding solution. Then the following holds true:*

$$\lim_{t\to\infty} (S, I)(t, cet) = (S_E, I_E), \;\; \forall c \in [0, c^*), \; \forall e \in \mathbb{S}^{N-1},$$

and

$$\lim_{t\to\infty} (S, I)(t, cet) = \left(\frac{\Lambda}{\gamma}, 0\right), \;\; \forall c \geq c^*.$$

Remark 2.9 *The second statement directly follows from 2.4* (*ii*)*.*

In the non-equi-diffusional situation (namely $d \neq 1$) we are not able to obtain such a precise dynamical behaviour behind the propagating front. Roughly speaking, we shall show that any uniformly persistent entire solution has an averaging property around the endemic equilibrium. In other words, each such entire solution is uniquely ergodic with respect to time and also with respect to space. Our result reads as follows.

Theorem 2.10 (Averaging property) *Let Assumption 2.1 be satisfied. Assume furthermore that $R_0 > 1$. Let (S_∞, I_∞) be a uniformly persistent entire solution of System* (1.1)-(1.2) *according to Definition 2.6. Then (S_∞, I_∞) has the following averaging properties:*

(i) for each continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and each $t \in \mathbb{R}$, one has

$$\lim_{R\to\infty} (2R)^{-N} \int_{[-R,R]^N} f\big((S_\infty, I_\infty)(t,x)\big)dx = f(S_E, I_E),$$

uniformly with respect to $t \in \mathbb{R}$.

(ii) For each continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and each $x \in \mathbb{R}^N$, one has

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} f\big((S_\infty I_\infty)(t,x)\big)dt = f(S_E, I_E),$$

uniformly with respect to $x \in \mathbb{R}^N$.

3 Preliminary

The aim of this preliminary section is to derive suitable local estimates that will be used in the next section to prove Theorem 2.2. The arguments we shall develop rely on similar duality ideas than the ones proposed by Hollis, Martin and Pierre in [15], and by Morgan in [23]. However in order to deal with the unboundedness of $\mathbb{R}^N$ we shall work with the so-called uniform Lebesgue spaces.

3.1 Uniform Lebesgue spaces and preliminary estimates

Let $p \in [1, \infty]$ be given. The uniform $p-$Lebesgue space, denoted by $L^p_u(\mathbb{R}^N)$, is defined by

$$L^p_u(\mathbb{R}^N) = \left\{\phi \in L^p_{loc}(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^p(B(x,1))} < \infty\right\},$$

wherein for each $x \in \mathbb{R}^N$ and $r > 0$, $B(x,r) \subset \mathbb{R}^N$ denotes the ball of center x and radius r. This space becomes a Banach space when it is endowed with the following norm

$$\|\phi\|_{L^p_u(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^p(B(x,1))}, \ \ \forall \phi \in L^p_u(\mathbb{R}^N).$$

We refer to Arrieta et al [3] and Volpert and Volpert [30] for different equivalent norm formulations for this space.

Consider now the heat semigroup operator $\{T_\Delta(t)\}_{t>0}$ defined by the convolution

$$T_\Delta(t) = K_t * \cdot \text{ with } K_t(x) = (4\pi t)^{-N/2} \exp\left(-\frac{\|x\|^2}{4t}\right).$$

Then the following properties hold true.

Lemma 3.1 *The following holds true:*

(i) There exists some constant $\widehat{M} > 0$ (only depending on $N \geq 1$) such that for each $1 \leq p \leq q \leq \infty$, each $\varphi \in L_u^p\left(\mathbb{R}^N\right)$ and each $t > 0$, we have

$$\|T_\Delta(t)\varphi\|_{L_u^q(\mathbb{R}^N)} \leq \widehat{M}\left(1 + t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right)\|\varphi\|_{L_u^p(\mathbb{R}^N)}.$$

(ii) For any bounded set $A \subset \mathbb{R}^N$ and each $p \in [1, \infty]$, one has

$$\lim_{t \to 0^+} \|T_\Delta(t)\varphi - \varphi\|_{L^p(A)} = 0, \ \ \forall \varphi \in L_u^p\left(\mathbb{R}^N\right).$$

(iii) For each $p \in [1, \infty)$ and each $\varphi \in L_u^p\left(\mathbb{R}^N\right)$, the map $t \mapsto T_\Delta(t)\varphi$ is analytic from $(0, \infty)$ into $\mathrm{BUC}\left(\mathbb{R}^N\right)$.

The proofs of these statements can be found in Proposition 2.1 in [3] and Theorem 3.2 in [25].

In order to deal with Theorem 2.2, we aim to make use of duality like arguments coupled together with local estimates in order to take into account of the unboundedness of the domain. Before proceeding to the proof, we shall need to recall and derive estimates for the heat semigroup.
The first recall the so-called maximal L^p parabolic regularity. We refer for instance to Hieber and Prüss in [14] (see also the references cited therein) for a proof of the following result.

Lemma 3.2 *Let $\nu > 0$ be given. Let $p \in (1, \infty)$ be given. Consider the linear operator Φ acting on the Banach space $L^p\left((0,\infty) \times \mathbb{R}^N\right)$ and defined by the resolution of the heat equation*

$$\Phi[\theta](t,x) = \psi(t,x) \text{ with } \begin{cases} (\partial_t - \Delta + \nu)\psi(t,x) = \theta(t,x), \ t > 0, \ x \in \mathbb{R}^N, \\ \psi(0,x) = 0, \ x \in \mathbb{R}^N. \end{cases}$$

Then Φ is a linear bounded operator on $L^p\left((0,\infty) \times \mathbb{R}^N\right)$. It is represented by the constant variation formula, that reads, for any $\theta \in L^p\left((0,\infty) \times \mathbb{R}^N\right)$, for any $t \geq 0$ and $x \in \mathbb{R}^N$, as

$$\psi(t,x) = \Phi[\theta](t,x) = \int_0^t \int_{\mathbb{R}^N} \frac{e^{-\nu s - \frac{\|y\|^2}{4s}}}{(4\pi s)^{N/2}} \theta(t-s, x-y) dy ds.$$

Moreover there exists some constant $K \equiv K(p,\nu) > 0$ such that for all $\theta \in L^p\left((0,\infty)\times\mathbb{R}^N\right)$, it holds that

$$\|\psi\|_{L^p((0,\infty)\times\mathbb{R}^N)} + \|\partial_t\psi\|_{L^p((0,\infty)\times\mathbb{R}^N)} + \|\Delta\psi\|_{L^p((0,\infty)\times\mathbb{R}^N)} \leq K\|\theta\|_{L^p(0,\infty)\times\mathbb{R}^N)}.$$

We will now derive local estimates that will be used in the sequel.
Let $\nu > 0$ and $p \in (1,\infty)$ be given and fixed. We denote by $B_0 := B(0,1) \subset \mathbb{R}^N$ the unit ball of $\mathbb{R}^N$ centred at the origin. For each function $\theta \in L^p((0,\infty)\times B_0)$, consider the function $\widehat{\theta} \in L^p\left(0,\infty)\times\mathbb{R}^N\right)$ defined by the extension of θ by zero. Then our first estimate reads as follows.

Lemma 3.3 *There exist $M > 0$ large enough and $K > 0$ such that for any $\theta \in L^p((0,\infty)\times B_0)$, one has*

$$\sum_{i\in\mathbb{Z}^N, \|i\|\geq M} \left\|\Phi\left[\widehat{\theta}\right](t,.)\right\|_{L^p(B(i,1))} \leq K\|\theta\|_{L^p((0,\infty)\times B_0)},\ \forall t > 0. \tag{3.7}$$

Proof. Let us first note that for all $i \in \mathbb{Z}^N$ with $\|i\| > 2$ one has

$$x \in B(i,1) \text{ and } y \in B_0 \ \Rightarrow\ \|x-y\| \geq \|i\| - 2. \tag{3.8}$$

Next choose $M > 2$ large enough such that for all $i = (i_1,..,i_N) \in \mathbb{Z}^N$ one has

$$\|i\| \geq M \quad\Rightarrow\quad (\|i\|-2)^2 \geq |i_1| + .. + |i_N|.$$

Next let us set $\psi(t,x) \equiv \Phi\left[\widehat{\theta}\right](t,x)$. Using the representation formula stated in Lemma 3.2 and (3.8), one obtains that for each $t > 0$ and each $i \in \mathbb{Z}^N$ with $\|i\| \geq M > 2$

$$\begin{aligned}\|\psi(t,.)\|^p_{L^p(B(i,1))} &= \int_{B(i,1)}\left|\int_0^t e^{-\nu s}(4\pi s)^{-N/2}\int_{B_0} e^{-\frac{\|x-y\|^2}{4s}}\theta(t-s,y)dyds\right|^p dx \\ &\leq \int_{B(i,1)}\left[\int_0^t e^{-\nu s}(4\pi s)^{-N/2}\int_{B_0} e^{-\frac{(\|i\|-2)^2}{4s}}|\theta|(t-s,y)dyds\right]^p dx \\ &\leq |B_0|\left[\int_0^t e^{-\nu s}(4\pi s)^{-N/2}e^{-\frac{(\|i\|-2)^2}{4s}}\|\theta(t-s,.)\|_{L^1(B_0)}ds\right]^p.\end{aligned}$$

This yield for all $t > 0$ and $i = (i_1,..,i_N) \in \mathbb{Z}^N$ with $\|i\| \geq M$ the following estimate

$$\|\psi(t,.)\|_{L^p(B(i,1))} \leq |B_0|^{\frac{1}{p}}\int_0^t e^{-\nu s}(4\pi s)^{-N/2}e^{-\frac{(|i_1|+...+|i_N|)}{4s}}\|\theta(t-s,.)\|_{L^1(B_0)}ds. \tag{3.9}$$

Now note, since $M > 0$ then for any $z \in [0,1)$ one has

$$\sum_{i\in\mathbb{Z}^N,\ |i\|\geq M} z^{|i_1|+..+|i_N|} \leq \frac{2^N z}{(1-z)^N}.$$

Hence summing-up (3.9) over all index $i = (i_1, .., i_N) \in \mathbb{Z}^N$ with $\|i\| \geq M$ provides for all $t > 0$

$$\sum_{i\in\mathbb{Z}^N,\ |i\|\geq M} \|\psi(t,.)\|_{L^p(B(i,1))} \leq 2^N |B_0|^{\frac{1}{p}} \int_0^t \frac{e^{-\nu s}e^{-\frac{1}{4s}}}{(4\pi s)^{\frac{N}{2}}\left(1-e^{-\frac{1}{4s}}\right)^N} \|\theta(t-s,.)\|_{L^1(B_0)} ds.$$

Note that the function h defined by $h(t) = 2^N \frac{e^{-\nu s}e^{-\frac{1}{4s}}}{(4\pi s)^{\frac{N}{2}}\left(1-e^{-\frac{1}{4s}}\right)^N}$ belongs to $L^q(0,\infty)$ where q is the conjugate exponent of p. Now two successive applications of Hölder inequality yield for all $t > 0$

$$\begin{aligned}\sum_{i\in\mathbb{Z}^N,\ |i\|\geq M} \|\psi(t,.)\|_{L^p(B(i,1))} &\leq |B_0|^{\frac{1}{p}+\frac{1}{q}} \int_0^t h(s)\|\theta(t-s,.)\|_{L^p(B_0)} ds \\ &\leq |B_0| \left(\int_0^t h(s)^q ds\right)^{\frac{1}{q}} \|\theta\|_{L^p((0,t)\times B_0)},\end{aligned}$$

and this completes the proof of Lemma 3.3. ∎

Next using the above notation, the following lemma holds true.

Lemma 3.4 *Let $p \in (1,\infty)$ and $\nu > 0$ be given. Then for each $T > 0$ and $\theta \in L^p((0,T)\times B_0)$, one has, by setting $\psi(t,x) = \Phi\left[\widehat{\theta}\right](t,x)$,*

$$\sup_{t\in[0,T]} \|\psi(t,.)\|_{L^p(\mathbb{R}^N)} \leq (q\nu)^{-\frac{1}{q}} \|\theta\|_{L^p(Q_T)}, \tag{3.10}$$

$$\sup_{t\in[0,T]} \|\psi(t,.)\|_{L^1(\mathbb{R}^N)} \leq |B_0|^{\frac{1}{q}} (q\nu)^{-\frac{1}{q}} \|\theta\|_{L^p(Q_T)}, \tag{3.11}$$

and

$$\|\psi\|_{L^1((0,T)\times\mathbb{R}^N)} \leq \frac{T^{\frac{1}{q}}}{\nu} |B_0|^{\frac{1}{q}} \|\theta\|_{L^p(Q_T)}. \tag{3.12}$$

Here we have set $Q_T = (0,T)\times B_0$, q denotes the conjugate exponent associated to p and $\widehat{\theta}$ is the extension of θ by zero on $(0,\infty)\times\mathbb{R}^N$.

Proof. The proof of the above lemma follows from Young convolution inequality. First for each given $t \in (0,T)$ one has

$$\|\psi(t,.)\|_{L^p(\mathbb{R}^N)} \leq \int_0^t e^{-\nu s}\|\theta(t-s,.)\|_{L^p} ds \leq \|e^{-\nu .}\|_{L^q(0,\infty)} \|\theta\|_{L^p(Q_T)}.$$

This proves (3.10).
Now for each given $t \in (0,T)$, one has

$$\begin{aligned}\|\psi(t,.)\|_{L^1(\mathbb{R}^N)} &\leq \int_0^t e^{-\nu s}\|\theta(t-s,.)\|_{L^1(B_0)} ds \\ &\leq |B_0|^{1/q} \int_0^t e^{-\nu s}\|\theta(t-s,.)\|_{L^p(B_0)} ds \\ &\leq |B_0|^{1/q} \left(\int_0^t e^{-q\nu s} ds\right)^{1/q} \|\theta\|_{L^p(Q_T)},\end{aligned}$$

and estimate (3.11) follows.
Next note that

$$\int_0^T \|\psi(t,.)\|_{L^1(\mathbb{R}^N)}dt \leq \int_0^T \int_0^t e^{-\nu s}\|\theta(t-s,.)\|_{L^1(B_0)}dsdt \leq \frac{1}{\nu}\|\theta\|_{L^1(Q_T)},$$

so that (3.12) follows from Hölder inequality. This completes the proof of Lemma 3.4. ■

As a consequence, coupling the estimate derive in Lemma 3.3 with the estimate (3.10) in Lemma 3.4, the following estimate holds true.

Lemma 3.5 *Let $p \in (1,\infty)$ and $\nu > 0$ be given. Then there exists some constant $\widehat{K} = \widehat{K}(p,\nu,N) > 0$ such that for each $T > 0$ and each function $\theta \in L^p((0,T)\times B_0)$, it holds that*

$$\sum_{i\in\mathbb{Z}^N} \left\|\Phi\left[\widehat{\theta}\right](t,.)\right\|_{L^p(B(i,1))} \leq \widehat{K}\|\theta\|_{L^p(Q_T)},\ \forall t \in (0,T).$$

Next the following estimate holds true.

Lemma 3.6 *Let $p \in (1,\infty)$ and $\nu > 0$ be given. Then there exists some constant $C = C(p,\nu,N) > 0$ such that for each $T > 0$ and each $\theta \in L^p((0,T)\times B_0)$, we have*

$$\int_0^t \int_{\mathbb{R}^N\setminus B(0,2)} \left|\Delta\Phi\left[\widehat{\theta}\right](s,x)\right| dxds \leq Ct^{1-\frac{1}{p}}\|\theta\|_{L^p(Q_T)},\ \forall t \in (0,T). \tag{3.13}$$

Proof. Recalling the definition of the linear operator Φ in Lemma 3.2 and its reformulation, then by setting $\psi(t,x) = \Phi\left[\widehat{\theta}\right](t,x)$, straightforward algebra yields for each $x \in \mathbb{R}^N$ and each $t \in (0,T)$

$$\Delta\psi(t,x) = -\int_0^t e^{-\nu s}\int_{B_0}(4\pi s)^{-\frac{N}{2}}\left[\frac{N}{2s} - \frac{\|x-y\|^2}{4s^2}\right]e^{-\frac{\|x-y\|^2}{4s}}\theta(t-s,y)dyds.$$

Then one obtains for each $t > 0$ that

$$\begin{aligned}&\int_{\|x\|\geq 2}|\Delta\psi(t,x)|dx \\ &\leq \int_0^t e^{-\nu s}\int_{B_0}(4\pi)^{-\frac{N}{2}}s^{-1-\frac{N}{2}}\left[\int_{\|x\|\geq 2}\left[\frac{N}{2} + \frac{\|x-y\|^2}{4s}\right]e^{-\frac{\|x-y\|^2}{4s}}dx\right]|\theta|(t-s,y)dyds.\end{aligned}$$

Now, since $y \in B_0$, we obtain that

$$\int_{\|x\|\geq 2}\left[\frac{N}{2} + \frac{\|x-y\|^2}{4s}\right]e^{-\frac{\|x-y\|^2}{4s}}dx \leq \int_{\|x\|\geq 1}\left[\frac{N}{2} + \frac{\|x\|^2}{4s}\right]e^{-\frac{\|x\|^2}{4s}}dx := G(s).$$

Now, setting $z = \frac{x}{\sqrt{s}}$, yields

$$G(s) = s^{\frac{N}{2}}\int_{\|z\|\geq\frac{1}{\sqrt{s}}}\left[\frac{N}{2} + \frac{\|z\|^2}{4}\right]e^{-\frac{\|z\|^2}{4}}dz.$$

And, the above estimate for $\Delta\psi$ re-writes as

$$\int_{\|x\|\geq 2} |\Delta\psi(t,x)|dx \leq \int_0^t H(s)\|\theta(t-s,.)\|_{L^1(B_0)},$$

wherein the function H is defined by

$$H(s) = (4\pi)^{-\frac{N}{2}} \frac{e^{-\nu s}}{s} \left[\int_{\|z\|\geq \frac{1}{\sqrt{s}}} \left[\frac{N}{2} + \frac{\|z\|^2}{4}\right] e^{-\frac{\|z\|^2}{4}} dz\right].$$

Finally observe that $H \in L^r(0,\infty)$ for each $r \in [1,\infty)$ so that (3.13) follows. Indeed Young convolution inequality implies that

$$\int_0^t \int_{\|x\|\geq 2} |\Delta\psi(s,x)|dxds \leq \|H\|_{L^1(0,\infty)} \|\theta\|_{L^1(Q_t)} \ \forall t \in (0,T),$$

and the result follows using Hölder inequality. ■

Finally, as a consequence of the parabolic maximal regularity recalled in Lemma 3.2 coupled with Lemma 3.6, the following estimate holds true.

Lemma 3.7 *Let $p \in (1,\infty)$ and $\nu > 0$ be given. There exists some constant $C = C(p,\nu,N) > 0$ such that for each $T > 0$ and each $\theta \in L^p\left((0,T)\times B_0\right)$, one has:*

$$\left\|\Delta\Phi\left[\widehat{\theta}\right]\right\|_{L^1(Q_T)} \leq CT^{1-\frac{1}{p}}\|\theta\|_{L^p(Q_T)}. \tag{3.14}$$

3.2 Duality estimates for some parabolic inequality

The aim of this section is to prove the following result.

Theorem 3.8 *Let $T > 0$ be given. Let $h \in L^\infty\left((0,T)\times\mathbb{R}^N\right)$, $(\theta_1,\theta_2) \in \mathbb{R}^2$ and $\nu > 0$ be given. Let $(u,v) \in W^{1,2}_\infty\left((0,T)\times\mathbb{R}^N\right)^2$ be given such that $u \geq 0$ and satisfying for almost every $(t,x) \in (0,T)\times\mathbb{R}^N$*

$$\left(\partial_t - \Delta + \nu\right) u(t,x) \leq h(t,x) + \left(\theta_1\partial_t + \theta_2\Delta\right) v(t,x). \tag{3.15}$$

Then for each $p \in (1,\infty)$, there exists some constant $C(p) > 0$ depending only upon $p \in (1,\infty)$, $\nu > 0$ and $N \geq 1$ such that

$$\left[\int_0^T \|u(t,.)\|^p_{L^p_u(\mathbb{R}^N)}dt\right]^{\frac{1}{p}} \leq K(p,T)\left[1 + \|u(0,.)\|_{L^p_u(\mathbb{R}^N)} + T^{\frac{1}{p}}\right],$$

wherein $K(p,T) > 0$ is defined by

$$K(p,T) = C(p)\left[1 + \|h\|_{L^\infty((0,T)\times\mathbb{R}^N)} + \|v\|_{L^\infty((0,T)\times\mathbb{R}^N)}\left(|\theta_1| + |\theta_2|\right)\right].$$

This result is local version of the results of Hollis et al [15] for bounded domains.

Let $p \in (1, \infty)$ be given and denote by $q \in (1, \infty)$ the conjugate exponent of p. Let $T > 0$ be given and let $\theta \in L^q_+ \left((0,T) \times B(0,1)\right)$ be given. Consider the function $\phi \geq 0$ defined by the backward resolution of the equation:

$$\begin{cases} (\partial_t + \Delta - \nu)\,\phi(t,x) = -\widehat{\theta}(t,x), \quad t \in (0,T),\ x \in \mathbb{R}^N, \\ \phi(T,.) \equiv 0, \end{cases}$$

wherein $\widehat{\theta} \in L^q\left((0,T) \times \mathbb{R}^N\right)$ denotes the extension of θ by zero outside $(0,T) \times B(0,1)$. Note that similar test functions have been successfully used in Hollis et al [15] and Morgan [24] to deal with reaction-diffusion equations in a bounded domain.

Let us first note that using Lemmas 3.4, 3.5 and 3.7, the following estimates hold true.

Lemma 3.9 *There exists some constant $C = C(p, \nu, N)$ such that*

$$\begin{aligned} &\|\phi\|_{L^1((0,T)\times\mathbb{R}^N)} \leq C(p)T^{1/p}\|\theta\|_{L^q(Q_T)}, \\ &\|\phi(0,.)\|_{L^1(\mathbb{R}^N)} \leq C(p)\|\theta\|_{L^q((0,T)\times B(0,1))}, \\ &\sup_{t\in(0,T)} \sum_{i\in\mathbb{Z}^N} \|u(t,.)\|_{L^q(B(i,1))} \leq C(p)\|\theta\|_{L^q((0,T)\times B(0,1))}, \\ &\|\partial_t\phi\|_{L^1((0,T)\times\mathbb{R}^N)} + \|\Delta\phi\|_{L^1((0,T)\times\mathbb{R}^N)} \leq C(p)T^{\frac{1}{p}}\|\theta\|_{L^q((0,T)\times B(0,1))}. \end{aligned}$$

Using these estimates, we shall complete the proof of Theorem 3.8.
To do so, let us multiply (3.15) by the positive function ϕ and integrate over $Q_T = (0,T) \times \mathbb{R}^N$. Then integration by parts yields

$$\begin{aligned} &-\int_{\mathbb{R}^N} \phi(0,x)u(0,x)dx + \iint_{Q_T} u\left[-\phi_t - \Delta\phi + \nu\phi\right]dtdx \leq \iint_{Q_T} \phi(t,x)h(t,x)dtdx \\ &-\theta_1\int_{\mathbb{R}^N} \phi(0,x)v(0,x)dx + \iint_{Q_T} \left[-\theta_1\phi_t + \theta_2\Delta\phi(t,x)\right]v(t,x)dtdx. \end{aligned}$$

Due to the definition of ϕ, this leads us to

$$\begin{aligned} \iint_{Q_T} u(t,x)\theta(t,x)dtdx &\leq \int_{\mathbb{R}^N} \phi(0,x)u(0,x)dx + \iint_{Q_T} \phi(t,x)h(t,x)dtdx \\ &-\theta_1\int_{\mathbb{R}^N} \phi(0,x)v(0,x)dx + \iint_{Q_T} \left[-\theta_1\phi_t + \theta_2\Delta\phi(t,x)\right]v(t,x)dtdx. \end{aligned}$$

Hence we get

$$\begin{aligned} \iint_{Q_T} u(t,x)\theta(t,x)dtdx &\leq \int_{\mathbb{R}^N} \phi(0,x)u(0,x)dx + \|h\|_{L^\infty(Q_T)}\|\phi\|_{L^1(Q_T)} \\ &+\|v\|_{L^\infty(Q_T)}\left[|\theta_1|\|\phi(0,.)\|_{L^1(\mathbb{R}^N)} + |\theta_1|\|\partial_t\phi\|_{L^1(Q_T)} + |\theta_2|\|\Delta\phi\|_{L^1(Q_T)}\right]. \end{aligned} \tag{3.16}$$

Now using Lemma 3.9, let us observe that

$$\begin{aligned}\int_{\mathbb{R}^N}\phi(0,x)u(0,x)dx &= \sum_{i\in\mathbb{Z}^N}\int_{B(i,1)}\phi(0,x)u(0,x)dx\\ &\le \sum_{i\in\mathbb{Z}^N}\|\phi(0,.)\|_{L^q(B(i,1))}\|u(0,.)\|_{L^p(B(i,1))}\\ &\le \left(\sum_{i\in\mathbb{Z}^N}\|\phi(0,.)\|_{L^q(B(i,1))}\right)\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}\\ &\le C(p)\|\theta\|_{L^q(Q_T)}\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}.\end{aligned}$$

As a consequence of Lemma 3.9, if we set

$$K(p,T)=C(p)\left[1+\|h\|_{L^\infty(Q_T)}+\|v\|_{L^\infty(Q_T)}\left(|\theta_1|+|\theta_2|\right)\right],$$

we infer from (3.16) that for each $\theta\in L^q((0,T)\times B(0,1))$, we have

$$\iint_{Q_T}u(t,x)\theta(t,x)dtdx\le K(p,T)\|\theta\|_{L^q(Q_T)}\left[1+\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}+T^{1/p}\right]. \tag{3.17}$$

Now for each $x_0\in\mathbb{R}^N$, the map $\sigma_{x_0}u:=u(.,x_0+.)$ satisfies (3.15) with v and h respectively replaced by $\sigma_{x_0}v:=v(.,x_0+.)$ and $\sigma_{x_0}h:=h(.,x_0+.)$. As a consequence, (3.17) applies to $\sigma_{x_0}u$ and leads for each $x_0\in\mathbb{R}^N$:

$$\iint_{Q_T}u(t,x+x_0)\theta(t,x)dtdx\le K(p,T)\|\theta\|_{L^q(Q_T)}\left[1+\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}+T^{1/p}\right]. \tag{3.18}$$

Now for each $k\ge 1$, let us set $A_k=B(0,k)\subset\mathbb{R}^N$. For each $r'\in(1,\infty)$ and each $k\ge 1$, let us multiply (3.18) by $\varphi\equiv\varphi(x_0)\in L^{r'}_+(A_k)$ and integrate over A_k. This leads us to

$$\begin{aligned}&\iiint_{Q_T\times A_k}u(t,x+x_0)\varphi(x_0)\theta(t,x)dtdxdx_0\\ &\le |A_k|^{1/r}K(p,T)\|\varphi\|_{L^{r'}(A_k)}\|\theta\|_{L^q(Q_T)}\left[1+\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}+T^{1/p}\right].\end{aligned}$$

Hence usual duality argument leads us to

$$\left[\int_0^T\left[\int_{A_k}\|u(t,.)\|^r_{L^p(B(x_0,1))}dx_0\right]^{p/r}\right]^{1/p}\le |A_k|^{1/r}K(p,T)\left[1+\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}+T^{1/p}\right].$$

Letting $r\to\infty$ and using Fatou's Lemma, one obtains that

$$\left[\int_0^T\left[\sup_{x_0\in A_k}\|u(t,.)\|_{L^p(B(x_0,1))}\right]^p\right]^{1/p}\le K(p,T)\left[1+\|u(0,.)\|_{L^p_u(\mathbb{R}^N)}+T^{1/p}\right].$$

Since the left hand side of the above inequality is increasing with respect to k, letting $k \to \infty$ and using Lebesgue monotone theorem, yield

$$\left[\int_0^T \left[\|u(t,.)\|_{L_u^p(\mathbb{R}^N)}\right]^p\right]^{1/p} \leq K(p,T)\left[1+\|u(0,.)\|_{L_u^p(\mathbb{R}^N)}+T^{1/p}\right].$$

This last estimate completes the proof of Theorem 3.8.

4 Proof of Theorem 2.2 and Theorem 2.3

The aim of this section is to prove Theorem 2.2 and its first consequence stated in Theorem 2.3.

Let us first state the following global existence result for the solution of the Cauchy problem (1.1)-(1.2).

Lemma 4.1 *Let Assumption 2.1 be satisfied. Then System* (1.1) *generates a strongly continuous and globally defined semiflow on X_+ denoted by $\{T(t)\}_{t\geq 0}$ or for each $U_0 = (S_0, I_0) \in X_+$ as*

$$\left\{T(t)U_0 = \begin{pmatrix} S(t,.;U_0) \\ I(t,.;U_0)\end{pmatrix}\right\}_{t\geq 0}.$$

For each initial data $U_0 = (S_0, I_0) \in X_+$ the solution $(S,I)(t,x;U_0) = (S,I)(t,x)$ satisfies the following properties:

(i) $(S,I) \in C\left([0,\infty);X_+\right)$,

(ii) For each $t \geq 0$ and $x \in \mathbb{R}^N$, it holds that

$$S(t,x) \leq \frac{\Lambda}{\gamma} + e^{-\gamma t}\left[\|S_0\|_\infty - \frac{\Lambda}{\gamma}\right]. \tag{4.19}$$

and

$$I(t,x) \leq \|I_0\|_\infty e^{\beta\left(\frac{\Lambda}{\gamma}+\|S_0\|_\infty\right)t}. \tag{4.20}$$

(iii) For each $0 < \tau < T$, the following regularity holds true:

$$(S,I) \in C^\infty\left((\tau,T)\times\mathbb{R}^N\right).$$

Note that the above lemma directly follows from standard results on reaction-diffusion equations. Estimate (4.19) is a consequence of the parabolic comparison principle for the $S-$equation while (4.20) is obtained by plugging (4.19) into the $I-$equation and applying the parabolic comparison principle.

We are now able to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. In order to prove Theorem 2.2, we shall apply Theorem 3.8. Indeed, let us first notice that (1.1) leads us to the following differential inequality

$$\partial_t I - \Delta I + (\gamma+\mu)I \leq \Lambda - \partial_t S + d\Delta S.$$

As a consequence of Lemma 4.1, Theorem 3.8 applies and provides that for each $p \in (1, \infty)$, there exists some constant $C(p) > 0$ such that for each $1 \leq \tau < T$, we have

$$\left[\int_\tau^T \|I(t,.)\|^p_{L^p_u(\mathbb{R}^N)} dt\right]^{\frac{1}{p}} \leq \widehat{K}(p,T,\tau)\left[1 + \|I(\tau,.)\|_{L^p_u(\mathbb{R}^N)} + (T-\tau)^{\frac{1}{p}}\right],$$

wherein $\widehat{K}(p,T,\tau) > 0$ is defined by

$$\widehat{K}(p,T,\tau) = C(p)\left[1 + \Lambda + (1+d)\|S\|_{L^\infty((\tau,T)\times\mathbb{R}^N)}\right].$$

Using Lemma 4.1 (ii), one obtains that for each $p \in (1,\infty)$ and each $1 \leq \tau < T$

$$\left[\int_\tau^T \|I(t,.)\|^p_{L^p_u(\mathbb{R}^N)} dt\right]^{\frac{1}{p}} \leq \widetilde{K}(p)\left[1 + \|I(\tau,.)\|_{L^p_u(\mathbb{R}^N)} + (T-\tau)^{\frac{1}{p}}\right],$$

with $\widetilde{K}(p) > 0$ defined by

$$\widetilde{K}(p) = C(p)\left[1 + \Lambda + (1+d)\left(\frac{\Lambda}{\gamma} + \|S_0\|_\infty\right)\right].$$

As a consequence of the above inequality, one can use similar arguments as the ones given by Hollis et al in Lemma 7 in [15] to derive the following lemma.

Lemma 4.2 *Let $\kappa > 0$ be given such that $\|S_0\|_\infty \leq \kappa$ and $\|I_0\|_\infty \leq \kappa$. For each $p \in (1,\infty)$, there exist constants $\Lambda_0(p,\kappa) > 0$, $\Theta_0(p,\kappa) \geq 1$, $\Gamma_0(p,\kappa) > 0$ and a non-decreasing sequence $\{t_k\}_{k\geq 0}$ such that $t_0 = 1$ and such that for each $k \geq 0$, we have*

- *(i)* $1 \leq t_{k+1} - t_k \leq \Lambda_0(p,\kappa)$,
- *(ii)* $\|I(t_k,.)\|_{L^p_u(\mathbb{R}^N)} \leq \Theta_0(p,\kappa)$,
- *(iii)* $\int_{t_k}^{t_{k+1}} \|I(s,.)\|^p_{L^p_u(\mathbb{R}^N)} ds \leq \Gamma_0(p,\kappa)$.

To complete the proof of Theorem 2.2 we shall make use of the $L^p - L^\infty$ estimates for the heat semigroup recalled in Lemma 3.1 (i). Let $p \in (1,\infty)$ be given such that

$$\frac{N}{2(p-1)} < 1.$$

Note that this estimate can be re-formulated as $\frac{qN}{2p} < 1$ if q denotes the conjugate exponent associated to p. Using the notations introduced in Lemma 4.2 together with such a choice of p, let $k \geq 1$ be given. Then for each $t \in [t_k, t_{k+1}]$, one has

$$I(t) = e^{-(\gamma+\mu)(t-t_{k-1})}T_\Delta(t-t_{k-1})\, I(t_{k-1}) + \beta \int_{t_{k-1}}^t e^{-(\gamma+\mu)(t-s)}T_\Delta(t-s)S(s)I(s)ds.$$

Next using Lemma 3.1 (i), one obtains that for each $t \in [t_k, t_{k+1}]$

$$\|I(t)\|_{L_u^\infty} \leq e^{-(\gamma+\mu)(t-t_{k-1})}\widehat{M}\left(1+(t-t_{k-1})^{-\frac{N}{2p}}\right)\|I\left(t_{k-1}\right)\|_{L_u^p(\mathbb{R}^N)}$$
$$+\widehat{M}\beta\int_{t_{k-1}}^t e^{-(\gamma+\mu)(t-s)}\left(1+(t-s)^{-\frac{N}{2p}}\right)\|S(s)\|_\infty\|I(s)\|_{L_u^p}ds.$$

Using (4.19) as well as estimates provided in Lemma 4.2 yields for each $t \in [t_k, t_{k+1}]$:

$$\|I(t)\|_{L_u^\infty} \leq 2\widehat{M}\Theta_0(p,\kappa)+\widehat{M}\beta 2\kappa\left[\int_{t_{k-1}}^t\left(1+(t-s)^{-\frac{N}{2p}}\right)^q ds\right]^{1/q}\left[\int_{t_{k-1}}^{t_{k+1}}\|I(s)\|_{L_u^p}^p ds\right]^{1/p}$$
$$\leq 2\widehat{M}\Theta_0(p,\kappa)+\widehat{M}\beta 2\left(2\Gamma_0(p,\kappa)\right)^{1/p}\kappa\left[\int_0^{2\Lambda_0(p,\kappa)}\left(1+l^{-\frac{N}{2p}}\right)^q dl\right]^{1/q}.$$

This completes the proof of Theorem 2.2. ■

Using the uniform boundedness provided by Theorem 2.2, we are now able to complete the proof of Theorem 2.3 that becomes - using usual limiting arguments and parabolic estimates - a direct consequence of the following result.

Proposition 4.3 *Let Assumption 2.1 be satisfied. Assume furthermore that $R_0 \leq 1$. Let (S,I) be a bounded and positive entire solution of* (1.1). *Then one gets*

$$(S,I)(t,x) \equiv \left(\frac{\Lambda}{\gamma},0\right).$$

Proof. Let us first notice that due to (4.19), the function S satisfies

$$S(t,x) \leq \frac{\Lambda}{\gamma},\ \ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{4.21}$$

Hence recalling that $R_0 \leq 1$, due to the $I-$equation, one gets for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, that

$$\left(\partial_t-\Delta\right)I(t,x)=\left[\beta S(t,x)-(\gamma+\mu)\right]I(t,x)\leq 0.$$

Let $\{(t_n,x_n)\}_{n\geq 0}$ be a sequence such that

$$\lim_{n\to\infty} I(t_n,x_n)=\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I(t,x),$$

and consider the sequence of functions

$$\left(S_n,I_n\right)(t,x):=(S,I)\left(t+t_n,x+x_n\right).$$

Then due to parabolic estimates, possibly up to a subsequence, one may assume that

$$\left(S_n,I_n\right)(t,x)\to\left(\widehat{S},\widehat{I}\right)(t,x),$$

locally uniformly for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and where $\left(\widehat{S},\widehat{I}\right)$ satisfies for each $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, the following system

$$\begin{aligned}(\partial_t - d\Delta)\,\widehat{S}(t,x) &= \Lambda - \gamma\widehat{S}(t,x) - \beta\widehat{S}(t,x)\widehat{I}(t,x),\\ (\partial_t - \Delta)\,\widehat{I}(t,x) &= \left[\beta\widehat{S}(t,x) - (\gamma+\mu)\right]\widehat{I}(t,x).\end{aligned}$$

Moreover due to the definition of (t_n, x_n) one has

$$\widehat{I}(0,0) = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \widehat{I}(t,x) = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I(t,x).$$

Recalling (4.21) note that the condition $R_0 \le 1$ ensures that $\beta\widehat{S}(t,x)-(\gamma+\mu) \le 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. Hence the strong comparison principle applies to the $\widehat{I}-$equation and ensures that

$$\widehat{I}(t,x) \equiv \widehat{I}_0 := \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I(t,x).$$

¿From this we obtains that $\widehat{S}(t,x) \equiv \widehat{S}_0 \in [0,\infty)$ where the pair $\left(\widehat{S}_0,\widehat{I}_0\right)$ satisfies the system of stationary equations

$$\begin{cases}\Lambda - \gamma\widehat{S}_0 - \beta\widehat{S}_0\widehat{I}_0 = 0,\\ \left[\beta\widehat{S}_0 - (\gamma+\mu)\right]\widehat{I}_0 = 0.\end{cases}$$

Now $R_0 \le 1$ leads to $\left(\widehat{S}_0,\widehat{I}_0\right) = \left(\frac{\Lambda}{\gamma},0\right)$. Hence $\widehat{I}_0 = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I(t,x) = 0$ and the result follows. ∎

5 Proof of Theorem 2.4

The aim of this section is to prove Theorem 2.4. The proof of Theorem 2.4 (i) will rely on a dynamical system approach and more specifically on uniform persistence like arguments. This section is split into three parts. The first subsection is devoted to the proof a weak uniform persistence property, the second part is concerned with the proof of Theorem 2.4 (i) while the third part is devoted to the proof of the outer spreading property stated in Theorem 2.4 (ii).

5.1 Weak uniform persistence

This section is concerned with the proof of the following result.

Theorem 5.1 *Recalling the definition of R_0 in (2.3) and c^* in (2.5), assume that $R_0 > 1$. Let $\kappa > 0$ be given. Let $c_0 \in [0, c^*)$ be given. Then there exists*

$\varepsilon = \varepsilon(\kappa, c_0) > 0$ such that for each $x \in \mathbb{R}^N$, each $e \in \mathbb{S}^{N-1}$, each $c \in [-c_0, c_0]$ and any $U_0 \in M^\kappa \times (M^\kappa \setminus \{0\})$, it holds that

$$\limsup_{t\to\infty} I(t, x + cte; U_0) \geq \varepsilon.$$

Here we have set

$$M^\kappa = \left\{\varphi \in \mathrm{BUC}\left(\mathbb{R}^N, \mathbb{R}\right) :\ 0 \leq \varphi \leq \kappa\right\}.$$

Before proving this result let us first state the following lemma that will be used in the proof of the above statement.

Lemma 5.2 *Let $a \in \mathbb{R}$ be given. Let consider for each $R > 0$, each $c \in \mathbb{R}$ and each $e \in \mathbb{S}^{N-1}$, the principle elliptic eigenvalue problem:*

$$\begin{cases} -\Delta u(x) + ce.\nabla u(x) + au(x) = \lambda_R\left[c, e\right] u \text{ for } x \in B(0, R), \\ u(x) = 0, \ \ x \in \partial B(0, R) \\ u(x) > 0 \ \ \forall x \in B(0, R). \end{cases}$$

Then $\lambda_R[c, e]$ does not depend upon $e \in \mathbb{S}^{N-1}$, it is denoted by $\lambda_R[c]$ and one has

$$\lim_{R\to\infty} \lambda_R\left[c\right] = a + \frac{c^2}{4},$$

locally uniformly for $c \in \mathbb{R}$.

Let us mention that similar results have been obtained by Berestycki et al in [5] for more general problems. Here, since we need to have information with respect to c, we will provide a simple proof adapted to this situation.
Proof of Lemma 5.2. Let $R > 0$, $c \in \mathbb{R}$ and $e \in \mathbb{S}^{N-1}$ be given. Consider $u = u_R$ an eigenvector associated to $\lambda_R[c, e]$. Consider now the function $\varphi(x) = u_R(x)e^{-\alpha e.x}$ where $\alpha \in \mathbb{R}$ is some constant that will be specified latter on. Then one has

$$\begin{aligned} \nabla u &= (\nabla\varphi + \alpha e\varphi)e^{\alpha e.x}, \\ \Delta u &= \left(\Delta\varphi + 2\alpha e.\nabla\varphi + \alpha^2\varphi\right) e^{\alpha e.x}. \end{aligned}$$

Thus φ satisfies

$$-\Delta\varphi - 2\alpha e.\nabla\varphi - \alpha^2\varphi + ce.\nabla\varphi + \alpha c\varphi + a\varphi = \lambda_R[c, e]\varphi.$$

Choose now $c = 2\alpha$ so that function φ satisfies

$$\begin{cases} -\Delta\varphi(x) = \left[\lambda_R[c, e] - a - \frac{c^2}{4}\right]\varphi \\ \varphi(x) = 0 \text{ for } x \in \partial B(0, R) \text{ and } \varphi(x) > 0\ \forall x \in B(0, R). \end{cases}$$

This implies that

$$\lambda_R[c, e] - a - \frac{c^2}{4} = \Lambda_R,$$

where Λ_R denotes the principle eigenvalue of $-\Delta$ together with Dirichlet boundary condition on $B(0,R)$. On the other hand, note that

$$\Lambda_R = \inf_{\psi \in H_0^1(B(0,R))} \frac{\|\nabla \psi\|_2^2}{\|\psi\|_2^2} = \frac{1}{R^2}\Lambda_1.$$

As a consequence one obtains that

$$\lambda_R[c,e] = \frac{\Lambda_1}{R^2} + a + \frac{c^2}{4}.$$

This completes the proof of Lemma 5.2. ■

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. In order to prove this result, let us argue by contradiction by assuming that for each $n \geq 0$, there exist $U_0^n = (S_0^n, I_0^n) \in M^\kappa \times M^\kappa \setminus \{0\}$, $x_n \in \mathbb{R}^N$, $c_n \in [-c_0, c_0]$ and $e_n \in \mathbb{S}^{N-1}$ such that the corresponding solution, denoted by (S^n, I^n), satisfies

$$\limsup_{t\to\infty} I^n\left(t, x_n + c_n t e_n\right) \leq \frac{1}{n+1}.$$

As a consequence, for each $n \geq 0$, there exists $t_n > 0$ such that the sequence $t_n \to \infty$ as $n \to \infty$ and

$$I^n(t, x_n + c_n t e_n) \leq \frac{2}{n+1}, \quad \forall t \geq t_n. \tag{5.22}$$

Consider now the sequence of shifted functions u_n and v_n defined by

$$\begin{aligned} u_n(t,x) &= S^n\left(t + t_n, x_n + x + c_n\left(t_n + t\right)e_n\right), \\ v_n(t,x) &= I^n\left(t + t_n, x_n + x + c_n\left(t_n + t\right)e_n\right). \end{aligned} \tag{5.23}$$

Note that (5.22) re-writes as

$$v_n(t,0) \leq \frac{2}{n+1}, \quad \forall t \geq 0, \ n \geq 0. \tag{5.24}$$

We now claim:

Claim 5.3 *One has*

$$\lim_{n\to\infty}\left(u_n, v_n\right)(t,x) = \left(\frac{\Lambda}{\gamma}, 0\right),$$

uniformly with respect to time $t \geq 0$ and x in bounded sets.

Before proving this claim, we complete the proof of Theorem 5.1. Recalling that $R_0 > 1$ and $c_0 < c^*$, let $\eta > 0$ be given small enough such that

$$c_0^2 + 4\beta\eta < \left(c^*\right)^2.$$

Next according to Lemma 5.2, there exists $L = L_\eta > 0$ such that the principal eigenvalue of the problem

$$\begin{cases} -\Delta u(x) + c_n e_n.\nabla u(x) + a_\eta u(x) = \lambda_L[c]u(x) \text{ in } B(0,L), \\ u(x) = 0, \ \ \forall x \in \partial B(0,L) \\ u(x) > 0 \ \ \forall x \in B(0,L), \end{cases} \tag{5.25}$$

wherein we have set

$$a_\eta = -\left[\beta\left(\frac{\Lambda}{\gamma} - \eta\right) - (\gamma+\mu)\right], \tag{5.26}$$

satisfies

$$\lambda_L\left[c_n\right] < 0, \ \ \forall n \geq 1.$$

According to Claim 5.3 we infer that

$$\lim_{n\to\infty} u_n(t,x) = \frac{\Lambda}{\gamma}, \text{ uniformly on } t \geq 0 \text{ and } x \in [-L, L].$$

Therefore, there exists $n_\eta > 0$ such that

$$\frac{\Lambda}{\gamma} - \eta \leq u_n(t,x) \leq \frac{\Lambda}{\gamma} + \eta, \ \ \forall t \geq 0, \ |x| \leq L, \ n \geq n_\eta. \tag{5.27}$$

Recalling (5.23), the function v_n satisfies for each n,

$$\left(\partial_t - c_n e_n.\nabla - \Delta\right) v_n(t,x) = \left[\beta u_n(t,x) - (\gamma+\mu)\right] v_n(t,x).$$

As a consequence of (5.27), one obtains that for each $n \geq n_\eta$, $t \geq 0$ and $x \in [-L, L]$

$$\left(\partial_t - c_n e_n.\nabla - \Delta\right) v_n(t,x) + a_\eta v_n(t,x) \geq 0. \tag{5.28}$$

Let $n \geq n_\eta$ be given and fixed. Consider the function $\Theta : B\left(0, L\right) \to [0, \infty)$ defined as a principal eigenvector of (5.25) with $R = L$, $e = e_n$ and $c = c_n$. Consider also $\delta > 0$ small enough such that

$$v_n\left(0, x\right) \geq \delta\Theta(x), \ \ \forall x \in B\left(0, L\right).$$

Finally, if one introduces the differential operator

$$\mathcal{L} := \partial_t - c_n e_n.\nabla - \Delta + a_\eta,$$

then we get

$$\mathcal{L}[v_n](t,x) \geq 0, \ \forall (t,x) \in [0,\infty) \times B\left(0, L\right),$$

while

$$\mathcal{L}[\Theta](t,x) = \lambda_L[c_n]\Theta \leq 0.$$

Consider the map $\underline{v}(t,x) = \delta e^{-\lambda_L[c_n]t}\Theta(x)$ that satisfies

$$\mathcal{L}[\underline{v}](t,x) = \delta\lambda_L[c_n]e^{-\lambda_L[c_n]t}\Theta - \lambda_L[c_n]e^{-\lambda_L[c_n]t}\Theta = 0.$$

Since one has

$$\begin{aligned}&\underline{v}(0,x)=\delta\Theta(x)\leq v_n(0,x),\ x\in\overline{B}(0,L)\\&\underline{v}(t,x)=0\leq v_n(t,x),\ \ \forall t\geq 0,\ x\in\partial B(0,L),\end{aligned}$$

we infer from the parabolic comparison principle that

$$\delta e^{-\lambda_L[c_n]t}\Theta(x)\leq v_n(t,x),\ \ \forall t\geq 0,\ |x|\leq L.$$

Recalling that $\lambda_L[c_n]<0$, the latter estimate contradicts (5.24). This completes the proof of the result. ■

It remains to prove Claim 5.3.

Proof of Claim 5.3. Due to the uniform bound provided by Theorem 2.2 and parabolic estimates, possibly up to a subsequence, one may assume that $(u_n,v_n)\to(u_\infty,v_\infty)$ locally uniformly for $(t,x)\in\mathbb{R}^2$. Moreover (5.24) implies that

$$v_\infty(t,0)=0,\ \ \forall t\geq 0. \tag{5.29}$$

Since $\mathbb{S}^{N-1}$ is compact, one may assume that $e_n\to e\in\mathbb{S}^{N-1}$. Since $\{c_n\}_{n\geq 0}\subset[-c_0,c_0]$, one may also assume that $c_n\to c\in[-c_0,c_0]$ as $n\to\infty$. Now recalling (5.23), the function (u_∞,v_∞) satisfies

$$\begin{cases}0\leq u_\infty(t,x)\leq\frac{\Lambda}{\gamma},\ \ v_\infty(t,x)\geq 0,\\(\partial_t-ce.\nabla-d\Delta)\,u_\infty(t,x)=\Lambda-\gamma u_\infty(t,x)-\beta u_\infty(t,x)v_\infty(t,x),\\(\partial_t-ce.\nabla-\Delta)\,v_\infty(t,x)=[\beta u_\infty(t,x)-(\gamma+\mu)]\,v_\infty(t,x).\end{cases}$$

Furthermore the strong comparison principle together with (5.29) implies that

$$v_\infty(t,x)\equiv 0\text{ and }u_\infty(t,x)\equiv\frac{\Lambda}{\gamma}.$$

Let $L>0$ be given. Let us assume by contradiction that $v_n\to 0$ as $n\to\infty$ but not uniformly on $[0,\infty)\times\overline{B}(0,L)$. This means that there exist a sequence $(t_n,x_n)\in[0,\infty)\times\overline{B}(0,L)$ and $\varepsilon>0$ such that

$$v_n(t_n,x_n)\geq\varepsilon,\ \ \forall n\geq 0.$$

Without loss of generality, we assume that $x_n\to x_\infty\in\overline{B}(0,L)$ as $n\to\infty$ while $t_n\to\infty$ as $n\to\infty$. Next consider the sequence of map $w_n(t,x)=v_n(t_n+t,x)$. Then due to parabolic estimates, one may assume that it converges locally uniformly to some function w_∞ as $n\to\infty$ where w_∞ satisfies $w_\infty(0,x_\infty)\geq\varepsilon$. Moreover using (5.23) as well as (5.24), one obtains that w_∞ satisfies:

$$\begin{aligned}&w_\infty(0,0)=0,\\&(\partial_t-\Delta-ce.\nabla)\,w_\infty(t,x)=a(t,x)w_\infty(t,x),\end{aligned}$$

wherein $a\equiv a(t,x)$ is some given bounded function. Here again, the strong maximum principle applies and provides that $w_\infty(t,x)\equiv 0$, a contradiction together with $w_\infty(0,x_\infty)\geq\varepsilon$. We deduce from the above argument that

$$\lim_{n\to\infty}\sup_{\substack{t\geq 0\\x\in\overline{B}(0,L)}}v_n(t,x)=0,\ \ \forall L>0.$$

Using the above result, we will show that $u_n \to \frac{\Lambda}{\gamma}$ uniformly for $t \geq 0$ and locally in $x \in \mathbb{R}$. Let $L > 0$ be given and assume that there exist $\varepsilon > 0$ and a sequence $(t_n, x_n) \in [0, \infty) \times \overline{B}(0, L)$ such that

$$\left| \frac{\Lambda}{\gamma} - u_n(t_n, x_n) \right| \geq \varepsilon.$$

Next assume that $x_n \to x_\infty \in \overline{B}(0, L)$ and assume, using parabolic estimates, that $u_n(t_n + ., .) \to u_\infty$ locally uniformly. Then one gets that

$$\left| \frac{\Lambda}{\gamma} - u_\infty(0, x_\infty) \right| \geq \varepsilon, \tag{5.30}$$

while u_∞ is a bounded entire solution of

$$(\partial_t - d\Delta - ce.\nabla)\, u_\infty(t, x) = \Lambda - \gamma u_\infty(t, x).$$

Hence we obtain that $u_\infty(t, x) \equiv \frac{\Lambda}{\gamma}$, a contradiction with (5.30). This completes the proof of Claim 5.3. ■

5.2 Proof of Theorem 2.4 (i)

We now turn to the proof of the inner spreading property stated in Theorem 2.4 (i). It is a direct consequence of the following proposition.

Proposition 5.4 *Let $\kappa > 0$ be given. Let $c \in [0, c^*)$ be given. Then there exists $\widehat{\varepsilon} = \widehat{\varepsilon}(\kappa, c) > 0$ such that for each $U_0 \in M^\kappa \times (M^\kappa \setminus \{0\})$, each $x \in \mathbb{R}^N$ and each $e \in \mathbb{S}^{N-1}$, we have*

$$\liminf_{t\to\infty} I(t, x + cte; U_0) \geq \widehat{\varepsilon}.$$

The arguments used in the following proof are adapted from dynamical system arguments. We refer for instance to Proposition 3.2 derived by Magal and Zhao [22] (see also [28], the monograph [26] and the references cited therein). Here we propose a proof based on parabolic regularity and weak disspativity as stated in Theorem 2.2.

Proof. Consider $\widehat{\kappa} > 0$ the constant defined in Theorem 2.2 associated to κ. Let us argue by contradiction by assuming that there exists a sequence of initial data $\{U_0^m = (S_0^m, I_0^m)\}_{m\geq 0} \subset M^\kappa \times M^\kappa \setminus \{0\}$, $\{x_m\}_{m\geq 0} \subset \mathbb{R}^N$ and $\{e_m\}_{m\geq 0} \subset \mathbb{S}^{N-1}$ such that the sequence of corresponding solution denoted by (S^m, I^m) satisfies

$$\liminf_{t\to\infty} I^m(t, x_m + cte_m) \leq \frac{1}{m+1}, \quad \forall m \geq 0.$$

Let $\varepsilon = \min\left(\varepsilon(\kappa, c), \varepsilon(\widehat{\kappa}, c)\right) > 0$ be the constant provided by Theorem 5.1 with κ and $\widehat{\kappa}$. Recall that for each $U_0 \in M^\kappa \times (M^\kappa \setminus \{0\})$, each $x \in \mathbb{R}^N$ and each $e \in \mathbb{S}^{N-1}$, one has

$$\limsup_{t\to\infty} I(t, x + cte; U_0) \geq \varepsilon. \tag{5.31}$$

Next we set $v^m(t,x) = I^m(t, x_m + x + cte_m; U_0^m)$ and $u^m(t,x) = S^m(t, x_m + x + cte_m; U_0^m)$. Then there exists a sequence $\{t_m\}_{m\geq 0}$ tending to ∞ and a sequence $\{l_m\}_{m\geq 0} \subset (0,\infty)$ such that for each $m \geq 0$, it holds that

$$\begin{aligned} &v^m(t_m, 0) = \frac{\varepsilon}{2}, \\ &v^m(t,0) \leq \frac{\varepsilon}{2} \ \forall t \in [t_m, t_m + l_m], \\ &v^m(t_m + l_m, 0) \leq \frac{1}{m+1}. \end{aligned}$$

Up to a subsequence, one may assume that $v^m(t+t_m, x) \to V_\infty(t,x)$ while $u^m(t+t_m, x) \to U_\infty(t,x)$ locally uniformly. Furthermore the function V_∞ satisfies

$$V_\infty(0,0) = \frac{\varepsilon}{2} \text{ and } V_\infty(t,0) \leq \frac{\varepsilon}{2} \ \ \forall t \in [0,l),$$

wherein we have set $l = \liminf_{m\to\infty} l_m$. On the other hand, one may assume that $e_m \to e \in \mathbb{S}^{N-1}$ as $m \to \infty$ so that the functions U_∞ and V_∞ satisfy the following system of equations

$$\begin{aligned} &(\partial_t - ce.\nabla - d\Delta)\, U_\infty(t,x) = \Lambda - \gamma U_\infty(t,x) - \beta U_\infty(t,x) V_\infty(t,x), \\ &(\partial_t - ce.\nabla - \Delta)\, V_\infty(t,x) = \left[\beta U_\infty(t,x) - (\gamma+\mu)\right] V_\infty(t,x). \end{aligned}$$

If $l < \infty$, one obtains that

$$V_\infty(l, 0) = 0,$$

so that $V_\infty(t,x) \equiv 0$, a contradiction together with the condition $V_\infty(0,0) = \frac{\varepsilon}{2}$.

Thus one obtains that $l = \infty$. This means that $l_m \to \infty$ as $m \to \infty$ and this allows us to conclude that

$$V_\infty(t,0) \leq \frac{\varepsilon}{2} \ \ \forall t \in [0,\infty). \tag{5.32}$$

Now recall that functions (S_∞, I_∞) defined by

$$S_\infty(t,x) = U_\infty(t, x - cte) \text{ and } I_\infty(t,x) = V_\infty(t, x - cte),$$

satisfies the system

$$\begin{cases} (\partial_t - d\Delta)\, S_\infty(t,x) = \Lambda - \gamma S_\infty(t,x) - \beta S_\infty(t,x) I_\infty(t,x), \\ (\partial_t - \Delta)\, I_\infty(t,x) = \left[\beta S_\infty(t,x) - (\gamma+\mu)\right] I_\infty(t,x). \end{cases}$$

Therefore, since $I_\infty(0,x) \not\equiv 0$ and $(S_\infty, I_\infty) \in M^{\widehat{\kappa}} \times M^{\widehat{\kappa}}$, according to Theorem 2.2 we get

$$\limsup_{t\to\infty} I_\infty(t, ct) \geq \varepsilon.$$

Recalling that $I_\infty(t, ct) \equiv V_\infty(t,0)$, we have reached a contradiction together with (5.32). This completes the proof of Proposition 5.4. ■

As a consequence of the above proposition, one obtains the following convergence result.

Lemma 5.5 *Recalling* (2.3) *and* (2.5), *let us assume that* $R_0 > 1$. *Let* $c \in (-c^*, c^*)$, $e \in \mathbb{S}^{N-1}$ *and* $U_0 = (S_0, I_0) \in X_+$ *with* $I_0 \not\equiv 0$ *be given. Let* $\{t_n\}_{n\geq 0}$ *be a given sequence tending to* $+\infty$ *as* $n \to \infty$. *Then there exists a subsequence, still denoted by* $\{t_n\}_{n\geq 0}$, *such that*

$$\lim_{n\to\infty} (S, I)(t + t_n, x + c(t + t_n)e; U_0) = (S^\infty, I^\infty)(t, x - cet),$$

locally uniformly for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ *and where* (S^∞, I^∞) *is a bounded entire solution of* (1.1) *such that*

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I^\infty(t, x) > 0.$$

Proof. Let $\{t_n\}_{n\geq 0}$ be a given sequence tending to ∞. Consider the sequence of map

$$U_n(t, x) = S(t + t_n, x + c(t + t_n)e), \quad V_n(t, x) = I(t + t_n, x + c(t + t_n)e).$$

Using parabolic estimates, one may assume that $\{(U_n, V_n)\}$ converges locally uniformly for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ towards some function pair $\{(U, V)\}$, an entire solution of the system

$$\begin{cases} (\partial_t - ce.\nabla - d\Delta) U(t, x) = \Lambda - \gamma U(t, x) - \beta U(t, x)V(t, x), \\ (\partial_t - ce.\nabla - \Delta) V(t, x) = [\beta U(t, x) - (\gamma + \mu)] V(t, x). \end{cases} \tag{5.33}$$

Moreover, using Theorem 2.4 (i) (see also Proposition 5.4), one obtains that there exists $\varepsilon > 0$ such that

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} V(t, x) \geq \varepsilon. \tag{5.34}$$

The result follows by noticing that $(S^\infty, I^\infty)(t, x) := (U, V)(t, x + cet)$ is an entire solution of (1.1). ∎

The above lemma stresses the importance of bounded entire solutions of (1.1) that are uniformly far from the disease free equilibrium. We refer to Definition 2.6 for the precise definition of uniformly persistent entire solutions. The study of such solutions is the aim of Section 6 below.

5.3 Proof of Theorem 2.4 (ii)

In this section, we prove that the outer spreading property stated in Theorem 2.4 (ii) holds true. To that aim one can note that it becomes a direct consequence of the following lemma.

Lemma 5.6 *Let Assumption 6.1 be satisfied. Let* $U_0 = (S_0, I_0) \in X_+$ *be given such that* $I_0 \not\equiv 0$ *and* I_0 *is compactly supported. Then for each* $\alpha < \frac{N}{2}$ *the following holds true:*

$$\lim_{t\to\infty} \sup_{|x|\geq c^*t-\alpha \ln t} \left[I(t, x) + \left| S(t, x) - \frac{\Lambda}{\gamma} \right| \right] = 0,$$

wherein (S, I) *denotes the solution of* (1.1) *with initial data* U_0.

Proof. The proof of this result relies on estimate (4.19) provided in Lemma 4.1. Indeed the function I satisfies the following differential inequality for all $t \geq 0$ and $x \in \mathbb{R}^N$

$$\partial_t I - \Delta I \leq \left[(\gamma+\mu)(R_0-1) + Ke^{-\gamma t}\right] I,$$

wherein $K > 0$ is some constant depending upon $\|S_0\|_\infty$ (see (4.19)). Next it is easy to check that the map

$$\widehat{I}(t,x) := e^{(\gamma+\mu)(R_0-1)t} \exp\left(\frac{K}{\gamma}\left(1-e^{-\gamma t}\right)\right) \left(T_\Delta(t) I_0\right)(x),$$

satisfies the equation

$$\begin{cases} \left[\partial_t - \Delta - ((\gamma+\mu)(R_0-1) + Ke^{-\gamma t})\right] \widehat{I}(t,x) = 0, \\ \widehat{I}(0,x) = I_0(x). \end{cases}$$

Hence the comparison principle applies and provides that $I(t,x) \leq \widehat{I}(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}^N$.

On the other hand, if $e \in \mathbb{S}^{N-1}$ and $c \in \mathbb{R}$ are given, then for each $t > 0$ and $x \in \mathbb{R}^N$ we have

$$\int_{\mathbb{R}^N} e^{\left(-\frac{\|x+ect-y\|^2}{4t}\right)} I_0(y)dy \leq e^{-\frac{c^2 t}{4}} e^{-\frac{c}{2}(e,x)} J(c),$$

wherein we have set

$$J(c) = \int_{\mathbb{R}^N} e^{\frac{c}{2}|y|} I_0(y)dy.$$

Recalling that I_0 is compactly supported, this ensures that $J(c) < \infty$. As a consequence of the above computations, one obtains that for each $c \in \mathbb{R}$, each $e \in \mathbb{S}^{N-1}$, each $t > 0$ and $x \in \mathbb{R}^N$:

$$I(t, x+cet) \leq \frac{e^{\frac{(c^*)^2-c^2}{4}t}}{(4\pi t)^{N/2}} e^{\frac{K}{\gamma}\left(1-e^{-\gamma t}\right)} e^{-\frac{c}{2}(e,x)} J(c).$$

Hence if $\alpha < \frac{N}{2}$ then one obtains that for each $x \in \mathbb{R}^N$ and each $t > 0$ such that $\|x\| \geq c^* t - \alpha \ln t$, one has

$$I(t,x) \leq (4\pi t)^{-N/2} e^{\frac{K}{\gamma}\left(1-e^{-\gamma t}\right)} t^{\frac{\alpha c^*}{2}} J(c^*),$$

and the result for the convergence of I follows.
The convergence for the $S-$component is a direct consequence from the one of I. Indeed let $\alpha < \frac{N}{2}$ be given and let us assume by contradiction that there exist $\varepsilon > 0$, a sequence $\{t_n\}_{n\geq 0} \subset (0,\infty)$ tending to ∞ and a sequence $\{x_n\}_{n\geq 0} \subset \mathbb{R}^N$ such that

$$\begin{cases} \|x_n\| \geq c^* t_n - \alpha \ln t_n, \quad \forall n \geq 0, \\ \left|S(t_n, x_n) - \frac{\Lambda}{\gamma}\right| \geq \varepsilon, \quad \forall n \geq 0. \end{cases}$$

Let us now consider the sequence of maps $S_n(t,x) = S(t+t_n, x+x_n)$ and $I_n(t,x) = I(t+t_n, x+x_n)$. Due to parabolic estimates, one may assume, up to a subsequence, that they converge to some entire solution of (1.1) (S_∞, I_∞) such that

$$\left| S_\infty(0,0) - \frac{\Lambda}{\gamma} \right| \geq \varepsilon. \tag{5.35}$$

One the other hand, due to the above convergence for I one knows that $I_\infty(0,0) = 0$ and the strong comparison principle ensures that $I_\infty(t,x) \equiv 0$. Finally the function S_∞ becomes an entire solution of equation

$$\left[\partial_t - d\Delta + \gamma\right] S(t,x) = \Lambda,$$

a contradiction together with (5.35). This completes the proof of Lemma 5.6. ■

6 Uniformly persistent entire solutions

The aim of this section is to provide information about uniformly persistent entire solutions of (1.1) (see Definition 2.6). Throughout this section we assume that

Assumption 6.1 *Recalling* (2.3)*, we assume that* $R_0 > 1$.
This allows us to denote by (S_E, I_E)*, the unique strictly positive spatially homogeneous equilibrium point of* (1.1)*, defined in* (2.4).

Observe that under the above assumption, (S_E, I_E) is an example of uniformly persistent entire solution of (1.1).

Before going to the study of such a class of entire solutions, let us first state the following straightforward estimate.

Lemma 6.2 *Let Assumption 6.1 be satisfied. Let* (S, I) *be a given uniformly persistent entire solution of* (1.1)*. Then it satisfies*

$$\frac{\Lambda}{\gamma + \beta \bar{i}} \leq S(t,x) \leq \frac{\Lambda}{\gamma + \beta \underline{i}},\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

wherein we have set

$$\underline{i} = \inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} I(t,x),\ \bar{i} = \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} I(t,x).$$

The proof of this estimate is straightforward.

6.1 The case $d = 1$

We now consider the spatio-temporal uniformly persistent entire solutions of (1.1) in the case when $d = 1$. The following classification holds true.

Proposition 6.3 *Let Assumption 6.1 be satisfied. Assume furthermore that* $d = 1$*. Let* $(S, I) \equiv (S, I)(t,x)$ *be a given uniformly persistent entire solution of the system* (1.1)*. Then we have* $(S, I)(t,x) \equiv (S_E, I_E)$.

Proof. Let $(S, I) \equiv (S, I)(t, x)$ be a given uniformly persistent entire solution of the system (1.1). Due to Lemma 6.2 and Definition 2.6 there exists some constant $\varepsilon > 0$ such that

$$\varepsilon \leq S(t,x) \leq \varepsilon^{-1},\ \varepsilon \leq I(t,x) \leq \varepsilon^{-1},\ \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{6.36}$$

Consider the positive map $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = x - 1 - \ln x$ and let us introduce the function $W : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ defined by

$$W(t,x) = g\left(\frac{S(t,x)}{S_E}\right) + \frac{I_E}{S_E} g\left(\frac{I(t,x)}{I_E}\right).$$

Then one has

$$\begin{aligned}
\partial_t W(t,x) &= \frac{1}{S_E}\left(1 - \frac{S_E}{S(t,x)}\right)\partial_t S(t,x) + \frac{1}{S_E}\left(1 - \frac{I_E}{I(t,x)}\right)\partial_t I(t,x),\\
\nabla W(t,x) &= \frac{1}{S_E}\left(1 - \frac{S_E}{S(t,x)}\right)\nabla S(t,x) + \frac{1}{S_E}\left(1 - \frac{I_E}{I(t,x)}\right)\nabla I(t,x),\\
\Delta W(t,x) &= \frac{1}{S_E}\left(1 - \frac{S_E}{S(t,x)}\right)\Delta S(t,x) + \frac{1}{S_E}\left(1 - \frac{I_E}{I(t,x)}\right)\Delta I(t,x)\\
&\quad + \frac{|\nabla S(t,x)|^2}{S(t,x)^2} + \frac{I_E}{S_E}\frac{|\nabla I(t,x)|^2}{I(t,x)^2}.
\end{aligned}$$

Hence straightforward algebraic manipulations yields for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$:

$$(\partial_t - \Delta) W(t,x) = -[\gamma + \beta I_E] \frac{(S(t,x) - S_E)^2}{S_E S(t,x)} - \frac{|\nabla S(t,x)|^2}{S(t,x)^2} - \frac{I_E}{S_E}\frac{|\nabla I(t,x)|^2}{I(t,x)^2}.$$

Recall now that due to (6.36), function W is uniformly bounded. One can now apply a similar arguments as the ones developed in the proof of Proposition 4.3. Let $\{(t_n, x_n)\}_{n \geq 0} \subset \mathbb{R} \times \mathbb{R}^N$ be a given sequence such that

$$\lim_{n \to \infty} W(t_n, x_n) = \sup_{\mathbb{R} \times \mathbb{R}^N} W.$$

Consider the sequence of maps $U_n(t, x) = S(t + t_n, x_n + x)$, $V_n(t, x) = I(t + t_n, x_n + x)$ and $W_n(t, x) = W(t + t_n, x_n + x)$. Up to a subsequence, one may assume that $U_n \to U$ and $V_n \to V$ locally uniformly for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Moreover one obtains that

$$W_n(t,x) \to \widehat{W}(t,x) \equiv W[U,V](t,x) \text{ locally uniformly,}$$

while $\widehat{W}(0, 0) = \sup_{\mathbb{R} \times \mathbb{R}^N} W = \sup_{\mathbb{R} \times \mathbb{R}^N} \widehat{W}$.

As a consequence, we have

$$(\partial_t - \Delta)\widehat{W}(t,x) \leq 0 \text{ and } \widehat{W}(0,0) = \sup_{\mathbb{R} \times \mathbb{R}^N} \widehat{W}.$$

We therefore infer from the strong comparison principle that $\widehat{W}(t,x) \equiv \sup_{\mathbb{R}\times\mathbb{R}^N} \widehat{W}$. As a consequence of the above computations, we obtain that

$$\nabla U(t,x) \equiv \nabla V(t,x) \equiv 0 \text{ and } U(t,x) \equiv S_E.$$

This implies that $V(t,x) \equiv I_E$ and therefore $\widehat{W}(t,x) \equiv 0$. This means that $\sup_{\mathbb{R}\times\mathbb{R}^N} W = 0$ and since $W(t,x) \geq 0$ we conclude that $W(t,x) \equiv 0$. This re-writes as $S(t,x) \equiv S_E$ while $I(t,x) \equiv I_E$ that completes the proof of the result.

■

As a consequence of Lemma 5.5 and Proposition 6.3, one obtains the following inner propagation result.

Corollary 6.4 *Let Assumption 6.1 be satisfied. Assume that $d = 1$. Let $c \in (-c^*, c^*)$, $e \in \mathbb{S}^{N-1}$ and $U_0 = (S_0, I_0) \in X_+$ with $I_0 \not\equiv 0$ be given. Then the following holds true:*

$$\lim_{t\to\infty} (S,I)(t, x + cte; U_0) = (S_E, I_E),$$

locally uniformly for $x \in \mathbb{R}^N$.

Note that the proof of Corollary 2.8 follows from the above and Lemma 5.6.

6.2 Spatial ergodicity of uniformly persistent entire solutions

We now come back to the general case $d > 0$. However we are not able to prove a similar result than the one obtained in Proposition 6.3 for the particular case $d = 1$.
This section is concerned with the proof of the following result.

Theorem 6.5 (Spatial ergodicity) *Let Assumption 6.1 be satisfied. Let $U = (S, I)$ be a uniformly persistent entire solution of (1.1). Then for each continuous function $f \in C\left(\mathbb{R}^2, \mathbb{R}\right)$, one has*

$$\lim_{R\to\infty} \frac{1}{(2R)^N} \int_{[-R,R]^N} f\left(U(t,h)\right) dh = f\left(U_E\right),$$

wherein we have set $U_E = (S_E, I_E)$ and where the above convergence holds uniformly with respect to $t \in \mathbb{R}$.

The proof of this result relies on the study of the action of the nonlinear semiflow $\{T(t)\}_{t\geq 0}$ - associated to System (1.1) - on an entire solution with spatial shift. Since the semiflow $\{T(t)\}_{t\geq 0}$ is not asymptotically compact when dealing with the Banach space X, we will work with the compact-open topology.

Let $U = (S, I)$ be a given uniformly persistent entire solution of System (1.1). Consider the set

$$\mathcal{O} = \bigcup_{t\in\mathbb{R},\ h\in\mathbb{R}^N} \sigma_h \begin{pmatrix} S(t,.) \\ I(t,.) \end{pmatrix} \subset X_+.$$

Here let us recall that σ denotes the spatial translation operator, namely $\sigma_h\varphi = \varphi(.+h)$ for each $h \in \mathbb{R}^N$ and $\varphi \in C\left(\mathbb{R}^N, \mathbb{R}^2\right)$. Let us now observe that due to parabolic estimates, for each sequence $\{(t_n, h_n)\}_{n\geq 0} \subset \mathbb{R}\times\mathbb{R}^N$, the sequence $\{U_n(t,x) = U(t+t_n, x+h_n)\}_{n\geq 0}$ is relatively compact into $C^k_{loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$ for each $k \geq 0$. Thus, one can consider the $k-$independent set $\mathcal{A} \subset \mathrm{BUC}^\infty\left(\mathbb{R}^N\right)$; the set of $C^\infty-$functions with all derivatives in X; defined by

$$\mathcal{A} = \overline{\mathcal{O}}^{C^k_{loc}\left(\mathbb{R}^N\right)}.$$

The above set is therefore a compact set for the topology of $C_{loc}\left(\mathbb{R}^N, \mathbb{R}^2\right)$ and uniformly bounded. As a consequence, endowed with the compact-open topology, $\mathcal{A}$ is a metrizable compact set. To be more specific, we introduce the distance $d_{\mathcal{A}}$ on $\mathcal{A}$ defined by

$$d_{\mathcal{A}}\left(\varphi, \psi\right) = \sup_{x\in\mathbb{R}^N} e^{-\|x\|}|\varphi(x) - \psi(x)|. \tag{6.37}$$

We consider the metric space $(\mathcal{A}, d_{\mathcal{A}})$ whose topology is equivalent to the compact-open topology. Furthermore, the metric space $(\mathcal{A}, d_{\mathcal{A}})$ is a compact and separable metric space.

We now introduce the set $\mathbb{M}\left(\mathcal{A}\right)$ of probability measures on $\mathcal{A}$ endowed with the $\sigma-$algebra of Borel sets of $(\mathcal{A}, d_{\mathcal{A}})$ denoted by $\mathbb{B}(\mathcal{A})$ in the sequel. This space is endowed with the weak topology of the Banach space of the real valued continuous functions on $\mathcal{A}$ denoted by $C\left(\mathcal{A}\right)$ endowed with the usual supremum norm. Since $\mathcal{A}$ is a separable compact space, the topological space $\mathbb{M}\left(\mathcal{A}\right)$ becomes a compact Polish space. In addition, the so-called dual-bounded Lipschitz metric on $\mathbb{M}(\mathcal{A})$ defined by

$$\pi\left(\mu, \nu\right) = \sup_{f\in C(\mathcal{A}),\ \|f\|_{Lip}\leq 1}\left|\int_{\mathcal{A}} f d\mu - \int_{\mathcal{A}} f d\nu\right|, \tag{6.38}$$

where $\|f\|_{Lip}$ denotes the usual Lipschitz norm, allows to obtain an equivalent topology (we refer for instance to [4]). Note that the compactness of the metric space $(\mathbb{M}(\mathcal{A}), \pi)$ is a consequence of the compactness of $\mathcal{A}$ and the Prokhorov theorem (we also refer to [4]).

With this material and notation, we re-formulate Theorem 6.5 as follows.

Theorem 6.6 *Let Assumption 6.1 be satisfied. Let $U = (S, I)$ be a uniformly persistent entire solution of* (1.1). *Then one has $U_E \in \mathcal{A}$ and*

$$\lim_{R\to\infty} \pi\left(\mu^R_t, \delta_{U_E}\right) = 0,$$

where the above convergence holds uniformly for $t \in \mathbb{R}$ and wherein $\mu^R_t \in \mathbb{M}\left(\mathcal{A}\right)$ is defined by

$$\mu^R_t = \frac{1}{(2R)^N}\int_{[-R,R]^N} \delta_{\sigma_h U(t,.)} dh.$$

In order to prove this theorem we define the semiflow $T^\sharp : [0,\infty) \times \mathbb{M}(\mathcal{A}) \to \mathbb{M}(\mathcal{A})$ by

$$T_t^\sharp \mu (B) = \mu \left(T(t)^{-1}B\right), \ \ \forall t \geq 0, \ \mu \in \mathbb{M}(\mathcal{A}), \ B \in \mathbb{B}(\mathcal{A}).$$

Here $\{T(t)\}_{t\geq 0}$ denote the nonlinear semiflow associated to (1.1).
We also introduce the $\mathbb{R}^N-$translation group action on $\mathbb{M}(\mathcal{A})$, denoted by $\sigma^\sharp$ and defined by

$$\left(\sigma_h^\sharp\right) \mu := \mu \left(\sigma_{-h}.\right), \ \forall h \in \mathbb{R}^N, \ \mu \in \mathbb{M}(\mathcal{A}).$$

Then the following result holds true.

Lemma 6.7 *The following holds true:*

(i) $T^\sharp$ *satisfies:*

$$\begin{aligned} &T_0^\sharp = I_{\mathbb{M}(\mathcal{A})}, \\ &T_{t+s}^\sharp = T_t^\sharp T_s^\sharp, \ \ \forall t, s \geq 0; \end{aligned}$$

(ii) The $\mathbb{R}^N-$*translation action commutes with* $T^\sharp$*, namely*

$$\sigma_h^\sharp \circ T_t^\sharp = T_t^\sharp \circ \sigma_h^\sharp, \ \ \forall t \geq 0, \ h \in \mathbb{R}^N;$$

(iii) The extended semiflow $(t,h,\mu) \to \sigma_h^\sharp \circ T_t^\sharp \mu$ *is continuous from* $[0,\infty) \times \mathbb{R}^N \times \mathbb{M}(\mathcal{A})$ *into* $\mathbb{M}(\mathcal{A})$

The proof of this lemma is straightforward.
Let us define the closed space $\mathbb{P}(\mathcal{A}) \subset \mathbb{M}(\mathcal{A})$ as

$$\mathbb{P}(\mathcal{A}) = \left\{\mu \in \mathbb{M}(\mathcal{A}) : \ \ \sigma_h^\sharp \mu = \mu, \ \ \forall h \in \mathbb{R}^N\right\}.$$

This corresponds to the set of translation invariant probability measures in $\mathbb{M}(\mathcal{A})$.
In order to prove Theorem 6.6 we firstly derive the following lemma.

Lemma 6.8 *Let* $t \in \mathbb{R}$ *be given. Let* $\{R_n\}_{n\geq 0}$ *be a sequence tending to* ∞ *as* $n \to \infty$*. Let* $\{\tau_n\}_{n\geq 0} \subset \mathbb{R}$ *be a given sequence. Then, possibly up to a subsequence, there exists* $\mu^* \in \mathbb{P}(\mathcal{A})$ *such that*

$$\lim_{n\to\infty} \pi \left(\mu_{t+\tau_n}^{R_n}, \mu^*\right) = 0.$$

Proof. Let us first notice that due to Prokhorov theorem, one may assume, that up to a subsequence, $\mu_{t+\tau_n}^{R_n} \to \mu^* \in \mathbb{M}(\mathcal{A})$ in $(\mathbb{M}(\mathcal{A}), \pi)$. To conclude the proof of Lemma 6.8, it is sufficient to prove that $\mu^* \in \mathbb{P}(\mathcal{A})$.

To that aim let $f \in C(\mathcal{A})$ and $x \in \mathbb{R}^N$ be given. On the one hand note that one has

$$\int_{\mathcal{A}} f(u)\left(\sigma_x^\sharp \mu^*\right)(du) = \lim_{n\to\infty} \frac{1}{(2R_n)^N} \int_{[-R_n,R_n]^N} f\left(\sigma_{h+x} U(t+\tau_n,.)\right) dh.$$

On the other hand one has for each $R > 0$ and each $\tau \in \mathbb{R}$:

$$\left|\int_{\mathcal{A}} f(u)\left(\sigma_x^\sharp \mu_\tau^R\right)(du) - \int_{\mathcal{A}} f(u)\left(\mu_\tau^R\right)(du)\right| \leq \|f\|_\infty \frac{\left(x+[-R,R]^N\right)\widehat{\Delta}[-R,R]^N}{(2R)^N}.$$

Here $\widehat{\Delta}$ denotes the usual symmetric difference of sets. Since the right-hand side of the above expression converges to zero as $R \to \infty$ uniformly with respect to $\tau \in \mathbb{R}$, the result follows. ■

In view of Lemma 6.8, the proof of Theorem 6.6 relies on the study of the dynamical system, still denoted by $T^\sharp$, and defined as $T^\sharp : [0,\infty)\times\mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{A})$, the restriction of $T^\sharp$ to $\mathbb{P}(\mathcal{A})$. Our next result provides a classification of the entire solutions for the dynamical system $T^\sharp$ on $\mathbb{P}(\mathcal{A})$. It reads as follows.

Theorem 6.9 *The following holds true:*

(i) The equilibrium point $U_E = \begin{pmatrix} S_E \\ I_E \end{pmatrix}$ *satisfies* $U_E \in \mathcal{A}$.

(ii) Let $\{\mu_t\}_{t\in\mathbb{R}} \subset \mathbb{P}(\mathcal{A})$ *be a given continuous entire solution of* $T^\sharp$*, that is*

$$\mu_{t+s} = T_t^\sharp \mu_s, \ \ \forall s \in \mathbb{R},\ t \geq 0.$$

Then $\mu_t \equiv \delta_{U_E}$.

As a consequence of Theorem 6.9 we obtain the following global asymptotic stability result.

Corollary 6.10 *For each* $\mu \in \mathbb{P}(\mathcal{A})$*, one has*

$$\lim_{t\to\infty} \pi\left(T_t^\sharp \mu, \delta_{U_E}\right) = 0.$$

The proof of Theorem 6.9 is based on the construction of a suitable Lyapunov functional. Such an idea was used by Zelik in [33]for formally gradient systems. Before proceeding to the proof of this result, let us first state the following.

Lemma 6.11 *The map* $(t,U) \to (T(t)U) = \begin{pmatrix} S(t,.;U) \\ I(t,.;U) \end{pmatrix}$ *from* $[0,\infty)\times\mathcal{A}$ *into* $\mathcal{A}$ *is continuous. For each* $U \in \mathcal{A}$*, the map* $(t,x) \to T(t)U(x)$ *belongs to* $C^{1,2}\left([0,\infty)\times\mathbb{R}^N,\mathbb{R}^2\right)$ *and there exists some constant* $M > 0$ *such that for each* $t \geq 0$*, each* $x \in \mathbb{R}^N$ *and each* $U \in \mathcal{A}$*, we have*

$$\left\|\partial_t \begin{pmatrix} S(t,x;U) \\ I(t,x;U) \end{pmatrix}\right\|_\infty + \sum_{|\alpha|\leq 2} \left\|D^\alpha \begin{pmatrix} S(t,x;U) \\ I(t,x;U) \end{pmatrix}\right\|_\infty \leq M.$$

The proof of this lemma is straightforward. It is a direct consequence of usual parabolic estimates by recalling that $\mathcal{A}$ is a bounded subset of $\mathrm{BUC}^3\left(\mathbb{R}^N,\mathbb{R}^2\right)$.

We are now able to deal with the proof of Theorem 6.9. To that aim let us consider the map $V:(0,\infty)\times(0,\infty)\to[0,\infty)$ defined by

$$V\begin{pmatrix} x \\ y \end{pmatrix} = g\left(\frac{x}{S_E}\right) + \frac{I_E}{S_E} g\left(\frac{y}{I_E}\right),$$

wherein function $g:(0,\infty)\to[0,\infty)$ is defined by $g(x)=x-1-\ln x$. Let $x_0\in\mathbb{R}^N$ be given. Next, consider the map $L_{x_0}:\mathbb{P}(\mathcal{A})\to\mathbb{R}$ defined by

$$L_{x_0}[\mu] = \int_{\mathcal{A}} V\left[\delta_{x_0}\circ U\right]\mu(dU),\ \ \forall \mu\in\mathbb{P}(\mathcal{A}).$$

Here δ_{x_0} denotes the Dirac delta function at x_0.
Recalling that $\mathcal{A}$ is constructed using a uniformly persistent entire solution of System (1.1), note that the map $U\mapsto V\left[\delta_{x_0}\circ U\right]$ is continuous from $\mathcal{A}$ into $\mathbb{R}$. So that the map L_{x_0} is continuous on $\mathbb{P}(\mathcal{A})$. Then the following lemma holds true.

Lemma 6.12 *For each $x_0\in\mathbb{R}^N$ and each $\mu\in\mathbb{P}(\mathcal{A})$, the map $t\mapsto L_{x_0}(t)$ defined by*

$$L_{x_0}(t) = L_{x_0}\left[T_t^\sharp\mu\right],\ \ \forall t\geq 0,$$

does not depend on x_0 and it is decreasing with respect to time. Moreover one has for each $0\leq t_1\leq t_2$:

$$\begin{aligned} L\left[T_{t_2}^\sharp\mu\right]-L\left[T_{t_1}^\sharp\mu\right] =& -\int_{t_1}^{t_2} dt\int_{\mathcal{A}}\left[\gamma+\beta I_E\right]\frac{\left(S(t,x_0;U)-S_E\right)^2}{S_E S(t,x_0;U)}\mu(dU) \\ & -\int_{t_1}^{t_2} dt\int_{\mathcal{A}}\left\{d\frac{|\nabla S(t,x_0;U)|^2}{S(t,x_0;U)^2}+\frac{|\nabla I(t,x_0;U)|^2}{I(t,x_0;U)^2}\right\}\mu\left(dU\right). \end{aligned} \tag{6.39}$$

Proof. To see this, let us first re-write $L_{x_0}\left[T_t^\sharp\mu\right]$ as

$$L_{x_0}\left[T_t^\sharp\mu\right] = \int_{\mathcal{A}} V\left[T(t)U(x_0)\right]\mu(dU).$$

Then since $\mu\in\mathbb{P}(\mathcal{A})$, one gets for each $h\in\mathbb{R}^N$:

$$L_{x_0}\left[T_t^\sharp\mu\right] = \int_{\mathcal{A}} V\left[\sigma_h T(t)U(x_0)\right]\mu(dU) = L_{x_0+h}\left[T_t^\sharp\mu\right].$$

This implies that the map $t\mapsto L_{x_0}\left[T_t^\sharp\mu\right]$ does not depend on x_0.

Next let us compute the time derivative of the above quantity. This yields

$$\frac{dL(t)}{dt} = \int_{\mathcal{A}}\left[\frac{1}{S_E}\left(1-\frac{S_E}{S(t,x_0;U)}\right)\partial_t S(t,x_0;U)+\frac{1}{S_E}\left(1-\frac{I_E}{I(t,x_0;U)}\right)\partial_t I(t,x_0;U)\right]\mu\left(dU\right).$$

Using the definition of the semiflow T, simple computations lead us to

$$\begin{aligned}\frac{dL(t)}{dt} &= \frac{d}{S_E}\int_{\mathcal{A}}\left(1-\frac{S_E}{S(t,x_0;U)}\right)\Delta S(t,x_0;U)\mu(dU)\\ &+\frac{1}{S_E}\int_{\mathcal{A}}\left(1-\frac{I_E}{I(t,x_0;U)}\right)\Delta I(t,x_0;U)\mu(dU)\\ &-[\gamma+\beta I_E]\int_{\mathcal{A}}\frac{(S(t,x_0;U)-S_E)^2}{S_E S(t,x_0;U)}\mu(dU)\end{aligned}$$

Let us now show that for each $i=1,..,N$ we have

$$\int_{\mathcal{A}}\frac{1}{S_E}\left(1-\frac{S_E}{S(t,x_0;U)}\right)\partial^2_{x_i}S(t,x_0;U)\mu(dU) = -\int_{\mathcal{A}}\frac{|\partial_{x_i}S(t,x_0;U)|^2}{S(t,x_0;U)^2}\mu(dU).$$

To prove this result, let us recall that μ is invariant with respect to translation, that is $\sigma_h^\sharp\mu\equiv\mu$ for all $h\in\mathbb{R}^N$. Therefore one obtains

$$\begin{aligned}&\int_{\mathcal{A}}\frac{1}{S_E}\left(1-\frac{S_E}{S(t,x_0;U)}\right)\partial^2_{x_i}S(t,x_0;U)\mu(dU)\\ &=\lim_{h\to 0}\int_{\mathcal{A}}\frac{1}{S_E}\left(1-\frac{S_E}{S(t,x_0;U)}\right)\frac{\partial_{x_i}S(t,x_0+he_i;U)-\partial_{x_i}S(t,x_0;U)}{h}\mu\left(dU\right)\\ &=\lim_{h\to 0}\frac{1}{h}\int_{\mathcal{A}}\frac{1}{S_E}\left[\left(1-\frac{S_E}{S(t,x_0-he_i;U)}\right)-\left(1-\frac{S_E}{S(t,x_0)}\right)\right]\partial_{x_i}S(t,x_0;U)\mu\left(dU\right)\\ &=\lim_{h\to 0}\frac{1}{h}\int_{\mathcal{A}}\left[\frac{1}{S(t,x_0;U)}-\frac{1}{S(t,x_0-he_i;U)}\right]\partial_{x_i}S(t,x_0;U)\mu\left(dU\right),\end{aligned}$$

and the result follows. ■

We are now able to complete the proof of Theorem 6.9.

Proof of Theorem 6.9. Using Lemma 6.12, one can first prove Theorem 6.9 (i). To do so, let $\mu\in\mathbb{P}(\mathcal{A})$ be given such that there exists $0<t_1<t_2$ with

$$L\left[T_{t_2}^\sharp\mu\right]=L\left[T_{t_1}^\sharp\mu\right].$$

Then we claim that we have supp $\mu\subset U_E$ and $\mu=\delta_{U_E}$ that completes the proof of Theorem 6.9 (i). To prove this claim, note that due to (6.39), for $(dt\otimes\mu)-$almost every $(t,U)\in(t_1,t_2)\times\mathcal{A}$ we have

$$S(t,x;U)\equiv S_E,\ \ \nabla I(t,x;U)\equiv 0,$$

so that $I(t,x;U)=I_E$ (see [33] for detailed arguments). Hence $U_E\in\mathcal{A}$ and $\mu\left(\{U_E\}\right)=1$. Since $\{U_E\}$ is a closed subset of $\mathcal{A}$, using [6], one concludes that $\mu=\delta_{U_E}$ and the result follows.

To complete the proof of (ii), let us consider a time continuous entire orbit $\{\mu_t\}_{t\in\mathbb{R}}$ of $T^\sharp$ in $\mathbb{P}\left(\mathcal{A}\right)$. Then due to the above computations, the map $t\to L\left[\mu_t\right]$ is decreasing on $\mathbb{R}$. Moreover if $\{s_n\}_{n\geq 0}\subset\mathbb{R}$ is a given non-increasing sequence tending to $-\infty$. Up to a subsequence one may assume that $\mu_{s_n}\to\mu^*$ weakly and the above computations implies that supp $\mu^* \subset \{U_E\}$. As a consequence of the decreasing property of $t \to L[\mu_t]$ one obtains that

$$L[\mu_t] \leq L[\mu^*], \quad \forall t \in \mathbb{R}.$$

Since supp $\mu^* \subset \{U_E\}$ and recalling the definition of V, we obtains that $L[\mu^*] = 0$ so that

$$L[\mu_t] = 0, \quad \forall t \in \mathbb{R}.$$

This re-writes as

$$\int_{\mathcal{A}} V[\delta_x \circ U]\, \mu_t(dU) = 0, \quad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N.$$

Similarly as the above argument, one obtains that $\mu_t = \delta_{U_E}$ is the Dirac delta measure at U_E for all $t \in \mathbb{R}$. This completes the proof of Theorem 6.9. ■

We now come back to the proof of Theorem 6.6 and we consider for each $R > 0$, the family $\{\mu_t^R\}_{t\in\mathbb{R}} \subset \mathbb{M}(\mathcal{A})$ defined by

$$\mu_t^R = \frac{1}{(2R)^N} \int_{[-R,R]^N} \sigma_h^\sharp \delta_{U(t,.)}.$$

We will now study this trajectory and the following lemma holds true.

Lemma 6.13 *The following holds true:*

(i) For each $R > 0$ the map $t \mapsto \mu_t^R$ is an entire solution of $T^\sharp$ in $\mathbb{M}(\mathcal{A})$.

(ii) There exists some constant $K > 0$ such that for each $R > 0$ and $(t,s) \in \mathbb{R}^2$:

$$\pi\left(\mu_t^R, \mu_s^R\right) \leq K|t-s|.$$

Proof. Let $R > 0$ be given. Let $t \geq 0$ and $s \in \mathbb{R}$ be given. Let $f \in C(\mathcal{A};\mathbb{R})$ be given. Then one has

$$\begin{aligned}
\int_{\mathcal{A}} f(U)\left(T_t^\sharp \mu_s^R\right)(dU) &= \int_{\mathcal{A}} f(T(t)U)\, \mu_s^R(dU) \\
&= \frac{1}{(2R)^N} \int_{[-R,R]^N} f(\sigma_h T(t) U(s,.))\, dh \\
&= \frac{1}{(2R)^N} \int_{[-R,R]^N} f(\sigma_h U(t+s,.))\, dh \\
&= \int_{\mathcal{A}} f(U) \mu_{t+s}^R(dU).
\end{aligned}$$

Thus (i) follows.
Now in order to prove (ii), let $(t,s) \in \mathbb{R}$ be given and let $f \in \mathrm{Lip}(\mathcal{A};\mathbb{R})$ be given. Then one has

$$\left|\int_{\mathcal{A}} f(U)\mu_t^R(dU) - \int_{\mathcal{A}} f(U)\mu_s^R(dU)\right| \leq \frac{1}{(2R)^N} \int_{[-R,R]^N} |f(U(t,h+.)) - f(U(s,h+.))|$$
$$\leq \mathrm{Lip}(f) \frac{1}{(2R)^N} \int_{[-R,R]^N} d_{\mathcal{A}}(U(t,h+.), U(s,h+.))\, dh.$$

Recalling the definition of $d_{\mathcal{A}}$ in (6.37), we obtains that

$$d_{\mathcal{A}}\left(U(t,h+.),U(s,h+.)\right)\leq \sup_{x\in\mathbb{R}^N}\left|U(t,x)-U(s,x)\right|.$$

Finally note that Lemma 6.11 provides the existence of some constant $M>0$ such that

$$\|\partial_t U(t,.)\|_\infty\leq M,\ \ \forall t\in\mathbb{R}.$$

Using this last estimates, one obtains

$$\left|\int_{\mathcal{A}} f(U)\mu_t^R(dU)-\int_{\mathcal{A}} f(U)\mu_s^R(dU)\right|\leq \mathrm{Lip}(f)M|t-s|.$$

Recalling the definition of π in (6.38), the results follows. ■

In order to complete the proof of Theorem 6.6, let $\{R_n\}_{n\geq 0}$ be a sequence tending to ∞ and let $\{\tau_n\}_{n\geq 0}\subset\mathbb{R}$ be given. Consider the sequence of function $\{\mu_n:\mathbb{R}\to\mathbb{M}(\mathcal{A})\}_{n\geq 0}$ defined by

$$\mu_n(t)=\mu_{t+\tau_n}^{R_n}.$$

Then due to Lemma 6.13 and Ascoli theorem, one may assume that, up to a subsequence, $\mu_n(t)\to\mu(t)$ as $n\to\infty$, for the topology of $C_{loc}\left(\mathbb{R},\mathbb{M}(\mathcal{A})\right)$. Furthermore due to Lemma 6.8, $t\to\mu(t)$ is an entire orbit of $T^\sharp$ in $\mathbb{P}(\mathcal{A})$. Therefore Theorem 6.9 and provides that $\mu(t)\equiv\delta_{U_E}$. This completes the proof of Theorem 6.6.

6.3 Time-averaging property

The aim of this section is to prove the following result.

Theorem 6.14 (Time ergodicity) *Let Assumption 6.1 be satisfied. Let $U=(S,I)$ be a uniformly persistent entire solution of* (1.1)*. Then for each continuous function $f\in C\left(\mathbb{R}^2,\mathbb{R}\right)$, one has*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f\left(U(t,x)\right)dt=f\left(U_E\right),$$

wherein we have set $U_E=(S_E,I_E)$ and where the above convergence holds uniformly with respect to $x\in\mathbb{R}^N$.

To prove this result we shall use the same notation and materials as in the previous section. Consider the set of time-invariant measures defined by

$$\mathbb{T}(\mathcal{A})=\left\{\mu\in\mathbb{M}(\mathcal{A}):\ \ T_t^\sharp\mu=\mu,\ \ \forall t\geq 0\right\}.$$

Using similar arguments as the ones developed in Section 6.2, in order to prove Theorem 6.14 it is sufficient to prove the following result.

Proposition 6.15 *The following equality holds true:*

$$\mathbb{T}(\mathcal{A}) = \{\delta_{U_E}\}.$$

Before proving this result, let us first recall that due to Theorem 6.6, $U_E \in \mathcal{A}$ and $\{\delta_{U_E}\} \subset \mathbb{T}(\mathcal{A})$. In order to prove the converse inclusion, consider the functional $W : \mathbb{R}^N \times \mathbb{M}(\mathcal{A}) \to \mathbb{R}$ defined by

$$K[x, \mu] = \int_{\mathcal{A}} W(\delta_x \circ U)\, \mu(dU),$$

wherein the function W is defined by

$$W(x, y) = dg\left(\frac{x}{S_E}\right) + \frac{I_E}{S_E} g\left(\frac{y}{I_E}\right),$$

while the function $g : (0, \infty) \to [0, \infty)$ is given by $g(x) = \ln x - x + 1$.

Our first lemma deals with the sub-harmonicity of K as a function of $x \in \mathbb{R}^N$ when $\mu \in \mathbb{T}(\mathcal{A})$.

Lemma 6.16 *Let $\mu \in \mathbb{T}(\mathcal{A})$ be given. Then one for each $t \geq 0$:*

$$K[x, T_t^\sharp \mu] \equiv K[x, \mu]. \tag{6.40}$$

Moreover $x \mapsto K[x, \mu]$ is a C^2 and bounded function on $\mathbb{R}^N$ that satisfies

$$\Delta K[x, \mu] \geq 0, \quad \forall x \in \mathbb{R}^N. \tag{6.41}$$

Proof. Let us first notice that (6.40) directly follows from the definition of $\mathbb{T}(\mathcal{A})$ and of the map K. The regularity of K with respect to x follows from Lemma 6.11, while its boundedness is a consequence of the definition of uniformly persistent entire solution. Thus it remains to prove (6.41). To do so, let $t_0 > 0$ be given. Then one has

$$\begin{aligned}\nabla K\left[x, T_{t_0}^\sharp \mu\right] = &\int_{\mathcal{A}} \frac{d}{S_E} g'\left(\frac{S(t_0, x; U)}{S_E}\right) \nabla S(t_0, x; U) \mu(dU) \\ &+ \int_{\mathcal{A}} \frac{1}{S_E} g'\left(\frac{I(t_0, x; U)}{I_E}\right) \nabla I(t_0, x; U) \mu(dU).\end{aligned}$$

Then one gets

$$\begin{aligned}\Delta K\left[x, T_t^\sharp \mu\right] = &\int_{\mathcal{A}} \frac{d}{S_E^2} g''\left(\frac{S(t_0, x; U)}{S_E}\right) |\nabla S(t_0, x; U)|^2 \mu(dU) \\ &+ \int_{\mathcal{A}} \frac{1}{I_I S_E} g''\left(\frac{I(t_0, x; U)}{I_E}\right) |\nabla I(t_0, x; U)|^2 \mu(dU) \\ &+ \int_{\mathcal{A}} \frac{d}{S_E} g'\left(\frac{S(t_0, x; U)}{S_E}\right) \Delta S(t_0, x; U) \mu(dU) \\ &+ \int_{\mathcal{A}} \frac{1}{S_E} g'\left(\frac{I(t_0, x; U)}{I_E}\right) \Delta I(t_0, x; U) \mu(dU).\end{aligned}$$

Hence this leads us to

$$\begin{aligned}\Delta K\left[x,T_t^\sharp\mu\right] &= \int_{\mathcal{A}}\left[d\frac{|\nabla S|^2}{S^2}+\frac{I_E}{S_E}\frac{|\nabla I|^2}{I^2}\right]\mu(dU)\\ &+\frac{1}{S_E}\int_{\mathcal{A}}\left[\gamma+\beta I_E\right]\frac{\left(S(t,x_0;U)-S_E\right)^2}{S_E S(t,x_0;U)}\mu(dU)\\ &+\frac{1}{S_E}\int_{\mathcal{A}}\left[g'\left(\frac{S}{S_E}\right)\partial_t S+g'\left(\frac{I}{I_E}\right)\partial_t I\right]\mu(dU).\end{aligned}$$

Note that the last two terms in the above expression vanish since $\mu\in\mathbb{T}(\mathcal{A})$. Indeed let us for instance notice that

$$\begin{aligned}&\frac{1}{S_E}\int_{\mathcal{A}} g'\left(\frac{S}{S_E}\right)\partial_t S\mu(dU)=\int_{\mathcal{A}}\partial_t g\left(\frac{S}{S_E}\right)\mu(dU)\\ &=\lim_{h\to 0}\frac{1}{h}\left[\int_{\mathcal{A}} g\left(\frac{S(t_0+h,x;U)}{S_E}\right)-g\left(\frac{S(t_0,x;U)}{S_E}\right)\mu(dU)\right]\\ &=0.\end{aligned}$$

The same argument applies to the other term.

Finally one obtains that

$$\begin{aligned}\Delta K\left[x,T_t^\sharp\mu\right] &= \int_{\mathcal{A}}\left[d\frac{|\nabla S|^2}{S^2}+\frac{I_E}{S_E}\frac{|\nabla I|^2}{I^2}\right]\mu(dU)\\ &+\frac{1}{S_E}\int_{\mathcal{A}}\left[\gamma+\beta I_E\right]\frac{\left(S(t,x_0;U)-S_E\right)^2}{S_E S(t,x_0;U)}\mu(dU),\end{aligned}\tag{6.42}$$

and the result follows. ■

It remains to prove Proposition 6.15.

Proof of Proposition 6.15. Let $\mu\in\mathbb{T}(\mathcal{A})$ be given. Since the map $x\to K[x,\mu]$ is uniformly bounded on $\mathbb{R}^N$, there exists a sequence $\{x_n\}_{n\geq 0}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty} K\left[x_n,\mu\right]=\sup_{x\in\mathbb{R}^N} K[x,\mu].$$

Consider now the sequence of map $K_n[x,\mu]=K\left[x+x_n,\mu\right]$. Then one has

$$K_n[x,\mu]\equiv K\left[x,\sigma_{x_n}^\sharp\mu\right],$$

while

$$\sup_{\mathbb{R}^N} K[.,\mu]=\lim_{n\to\infty}K_n[0,\mu].$$

On the other hand, up to a subsequence, one may assume $\sigma_{x_n}^\sharp\mu\to\widehat{\mu}\in\mathbb{T}(\mathcal{A})$ when $n\to\infty$. This implies that

$$\sup_{\mathbb{R}^N} K[.,\mu]=K[0,\widehat{\mu}]\text{ and }K[x,\widehat{\mu}]\leq K[0,\widehat{\mu}],\ \ \forall x\in\mathbb{R}^N.$$

Now according to Lemma 6.16, the function $x \mapsto K[x, \widehat{\mu}]$ is bounded and subharmonic on $\mathbb{R}^N$ and achieves its maximum value at $x = 0$. The strong elliptic maximum principle implies that $K[x, \widehat{\mu}] \equiv K[0, \widehat{\mu}]$ and therefore that $\Delta K[x, \widehat{\mu}] \equiv 0$. Finally (6.42) completes the proof of Proposition 6.15.

■

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